Creative Telescoping

Manuel Kauers · Institute for Algebra · JKU

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- We want a recurrence for the sum $S(n) = \sum_k f(n,k)$.
- Such a recurrence is obtained from a k-free recurrence for $f(n,k)$.
- More precisely, we construct an annihilating operator for $f(n,k)$ of the form

$$P(n, S_n) - \Delta_k Q(n, k, \Delta_k, S_n)$$

where

  - P is nonzero and free of k, S_k, Δ_k.
  - Q may be zero and may involve any variables or operators.

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  where
  - P is nonzero and free of k, S_k, Δ_k.
  - Q may be zero and may involve any variables or operators.
- Then $P(n, S_n)$ is an annihilating operator for $S(n)$.

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$$\downarrow$$

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- The application of a telescoper turns $f(n, k)$ into a telescoping term. (Hence the name.)
- If we only have P, it is not obvious whether $P \cdot f$ telescopes, i.e., whether P is really a telescoper.
- If we have both P and Q, then checking $(P - \Delta_k Q) \cdot f \stackrel{?}{=} 0$ is easy. Thus Q certifies that P is a telescoper.

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$$\big((n+1)S_n - (4n+2)\big) \cdot \binom{n}{k}^2 = \Delta_k \cdot \frac{k^2(2k-3n-3)}{(n+1)^2} \binom{n+1}{k}^2$$

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The operator $\frac{k^2(2k-3n-3)}{(n+1)^2}S_n$ is a certificate for the telescoper.

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Therefore:

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This differential equation together with the initial value $S(0) = 1$ implies the claimed identity.

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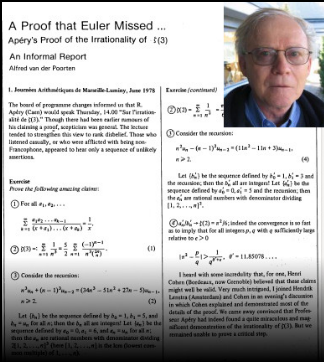

A Proof that Euler Missed ...

Apéry's Proof of the Irrationality of $\zeta(3)$

An Informal Report

Alfred van der Poorten

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

Exercise
Prove the following amazing claims:

(1) For all $a_1, a_2, \ldots$

$$\sum_{k=1}^{\infty} \frac{a_1 a_2 \cdots a_{k-1}}{(x+a_1)\cdots(x+a_k)} = \frac{1}{x}.$$

(2) $$\zeta(3) =: \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \qquad (1)$$

(3) Consider the recursion:

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geq 2. \qquad (2)$$

Let $\{b_n\}$ be the sequence defined by $b_0 = 1$, $b_1 = 5$, and $b_n = u_n$ for all n; then the b_n all are integers! Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 6$, and $a_n = u_n$ for all n; then the a_n are rational numbers with denominator dividing $2[1, 2, \ldots, n]^3$ (here $[1, 2, \ldots, n]$ is the lcm (lowest common multiple) of $1, \ldots, n$).

Exercise *(continued)*

(2') $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \ldots$

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(4') $a'_n/b'_n \to \zeta(2) = \pi^2/6$; indeed the convergence is so fast as to imply that for all integers p, q with q sufficiently large relative to $\epsilon > 0$

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{\theta'+\epsilon}}, \quad \theta' = 11.85078\ldots.$$

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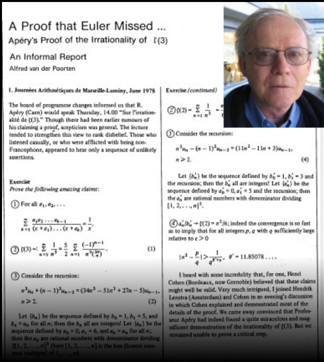

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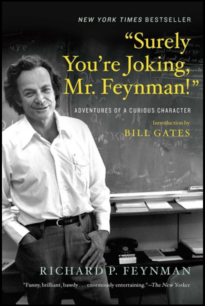

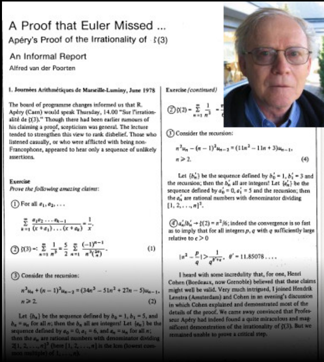

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Let $\{b_n\}$ be the sequence defined by $b_0 = 1, b_1 = 5$, and $b_n = u_n$ for all n; then the b_n all are integers! Let $\{a_n\}$ be the sequence defined by $a_0 = 0, a_1 = 6$, and $a_n = u_n$ for all n; then the a_n are rational numbers with denominator dividing $2[1, 2, \ldots, n]^3$ (here $[1, 2, \ldots, n]$ is the lcm (lowest common multiple) of $1, \ldots, n$).

Exercise *(continued)*

②' $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} =$

③' Consider the recursion:

$$n^2 u_n - (n-1)^2 u_{n-2} = (11n^2 - 11n + 3)u_{n-1}, \quad n \geq 2. \qquad (4)$$

Let $\{b_n'\}$ be the sequence defined by $b_0' = 1, b_1' = 3$ and the recursion; then the b_n' all are integers! Let $\{a_n'\}$ be the sequence defined by $a_0' = 0, a_1' = 5$ and the recursion; then the a_n' are rational numbers with denominator dividing $[1, 2, \ldots, n]^2$.

④' $a_n'/b_n' \to \zeta(2) = \pi^2/6$; indeed the convergence is so fast as to imply that for all integers p, q with q sufficiently large relative to $\epsilon > 0$

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{\theta'+\epsilon}}, \quad \theta' = 11.85078\ldots.$$

I heard with some incredulity that, for one, Henri Cohen (Bordeaux, now Grenoble) believed that these claims might well be valid. Very much intrigued, I joined Hendrik Lenstra (Amsterdam) and Cohen in an evening's discussion in which Cohen explained and demonstrated most of the details of the proof. We came away convinced that Professeur Apéry had indeed found a quite miraculous and magnificent demonstration of the irrationality of $\zeta(3)$. But we remained unable to prove a critical step.

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In the present segment, let's focus on the differential case, and rational functions.

Task.

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- Given: a rational function $f \in C(x, y)$
- Find: operators $P(x, D_x) \neq 0$ and $Q(x, y, D_x, D_y)$ such that

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telescoper certificate

Equivalently:

- Find: an operator $P(x, D_x) \neq 0$ and a rational function $g(x, y) \in C(x, y)$ such that

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f is integrable in $C(y)$ if and only if $h = 0$.

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Now consider a bivariate rational function $f \in C(x, y)$.

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Idea: find a $C(x)$-linear relation among $h_0, h_1, h_2, \dots$

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D_x^2 \cdot f &= D_y \cdot \tfrac{-162x^3y^5+\cdots-8y^2-16y-8}{x^2(27x+4)^2(xy^3+y+1)^2} &&+ \tfrac{6(54x-27xy-y-1)}{x(27x+4)^2(xy^3+y+1)}
\end{aligned}
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Example. $f = \dfrac{1}{xy^3 + y + 1}$.

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&+ x(27x+4)\,\tfrac{6(54x-27xy-y-1)}{x(27x+4)^2(xy^3+y+1)} = \mathbf{0}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left(6 + 2(27x+1)D_x + x(27x+4)D_x^2\right) \cdot f\\
&= D_y \cdot \tfrac{2(3x^2y^6+3xy^6-21xy^4-21xy^3-y^4-y^3+3y^2+6y+3)}{(xy^3+y+1)^3} + \mathbf{0}
\end{aligned}
$$

Reduction-based creative telescoping for rational functions

INPUT: $f \in C(x, y)$

OUTPUT: a telescoper for f.

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3 if there are $c_0, \ldots, c_r \in C(x)$, not all zero, such that

$$c_0 h_0 + \cdots + c_r h_r = 0$$

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Exercise. In step 2, we can use $D_x \cdot h_{r-1}$ instead of $D_x^r \cdot f$. Why is this better?

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D_x^2 \cdot f &= \frac{\qquad\qquad\qquad\; q^{r-2}}{q^{3}\,q^{r-2}} && \leftarrow \deg_y \le r\deg_y q + \deg_y p \\
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Consider again a rational function $f(x,y) = \frac{p(x,y)}{q(x,y)} \in C(x,y)$.

For every $a_0, a_1, \dots, a_r \in C(x)$ (free of y), we have

$$(a_0 + a_1 D_x + \cdots + a_r D_x^r) \cdot f = \frac{\qquad\qquad}{q^{r+1}} \qquad \leftarrow \deg_y \le r \deg_y q + \deg_y p$$

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For $b_0, \dots, b_s \in C(x)$ (also free of y), consider the rational function

$$g = \frac{b_0 + b_1 y + \cdots + b_s y^s}{q^r}.$$

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$$D_y \cdot g = \frac{\boxed{}}{q^{r+1}} \qquad \leftarrow \deg_y \le \deg_y q + s - 1$$

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For $b_0, \ldots, b_s \in C(x)$ (also free of y), consider the rational function

$$g = \frac{b_0 + b_1 y + \cdots + b_s y^s}{q^r}.$$

Then, with $s = (r-1)\deg_y q + \deg_y p + 1$,

$$D_y \cdot g = \frac{\qquad\qquad}{q^{r+1}} \qquad \leftarrow \deg_y \leq r \deg_y q + \deg_y p$$

For undetermined $a_0, \ldots, a_r$ and $b_0, \ldots, b_s$, enforce

$$(a_0 + a_1 D_x + \cdots + a_r D_x^r) \cdot f \overset{!}{=} D_y g$$

For undetermined $a_0, \ldots, a_r$ and $b_0, \ldots, b_s$, enforce

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Comparing coefficients with respect to y leads to a linear system with

- $(r+1)+(s+1)$ variables
- $1 + r\deg_y q + \deg_y p$ equations

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Comparing coefficients with respect to y leads to a linear system with

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It will have a nontrivial solution as soon as $r \geq \deg_y q - 1$.

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Comparing coefficients with respect to y leads to a linear system with

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Exercise. Does every nonzero solution give rise to a nonzero telescoper?

Example. $f = \dfrac{1}{xy^3 + y + 1}$

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$$
\begin{aligned}
&(a_0 + a_1 D_x + a_2 D_x^2) \cdot f \\
&\quad = \tfrac{(a_0x^2 - a_1x + 2a_2)y^6 + (2a_0x - a_1)y^4 + (2a_0x - a_1)y^3 + a_0y^2 + 2a_0y + a_0}{(xy^3 + y + 1)^3}.
\end{aligned}
$$

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$$(a_0 + a_1 D_x + a_2 D_x^2) \cdot f$$
$$= \frac{(a_0x^2 - a_1x + 2a_2)y^6 + (2a_0x - a_1)y^4 + (2a_0x - a_1)y^3 + a_0y^2 + 2a_0y + a_0}{(xy^3+y+1)^3}.$$

The y-derivative of $g = \dfrac{b_0 + b_1y + b_2y^2 + b_3y^3 + b_4y^4}{(xy^3 + y + 1)^2}$ is

$$\frac{-2b_4xy^6 - 3b_3xy^5 + (2b_4 - 4b_2x)y^4 + (b_3 - 5b_1x + 4b_4)y^3 + (3b_3 - 6b_0x)y^2 + (2b_2 - b_1)y - 2b_0 + b_1}{(xy^3+y+1)^3}$$

Example. $f = \dfrac{1}{xy^3 + y + 1}$

$$\begin{aligned} &(a_0x^2 - a_1x + 2a_2 + 2b_4x)y^6 \\ &+ 3b_3xy^5 \\ &+ (2a_0x - a_1 + 4b_2x - 2b_4)y^4 \\ &+ (2a_0x - a_1 + 5b_1x - b_3 - 4b_4)y^3 \\ &+ (a_0 + 6b_0x - 3b_3)y^2 \\ &+ (2a_0 + b_1 - 2b_2)y \\ &+ a_0 + 2b_0 - b_1 = 0 \end{aligned}$$

Example. $f = \dfrac{1}{xy^3 + y + 1}$

$$
\begin{aligned}
(a_0 x^2 - a_1 x + 2a_2 + 2b_4 x) &= 0 \\
+\, 3b_3 x &= 0 \\
+\, (2a_0 x - a_1 + 4b_2 x - 2b_4) &= 0 \\
+\, (2a_0 x - a_1 + 5b_1 x - b_3 - 4b_4) &= 0 \\
+\, (a_0 + 6b_0 x - 3b_3) &= 0 \\
+\, (2a_0 + b_1 - 2b_2) &= 0 \\
+\, a_0 + 2b_0 - b_1 &= 0
\end{aligned}
$$

Example. $f = \dfrac{1}{xy^3 + y + 1}$

$$\begin{pmatrix} x^2 & -x & 2 & 0 & 0 & 0 & 0 & 2x \\ 0 & 0 & 0 & 0 & 0 & 0 & 3x & 0 \\ 2x & -1 & 0 & 0 & 0 & 4x & 0 & -2 \\ 2x & -1 & 0 & 0 & 5x & 0 & -1 & -4 \\ 1 & 0 & 0 & 6x & 0 & 0 & -3 & 0 \\ 2 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 0$$

Example. $f = \dfrac{1}{xy^3 + y + 1}$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \left\langle \begin{pmatrix} 6x \\ 2x(27x+1) \\ x^2(27x+4) \\ -1 \\ 2(3x-1) \\ 9x-1 \\ 0 \\ -3x(x+1) \end{pmatrix} \right\rangle$$

Example. $f = \dfrac{1}{xy^3 + y + 1}$

$$\left(6x + 2x(27x+1)D_x + x^2(27x+4)D_x^2\right) \cdot f$$
$$= D_y \cdot \frac{-1 + 2(3x-1)y + (9x-1)y^2 - 3x(x+1)y^4}{(xy^3+y+1)^2}$$

The Apagodu-Zeilberger Algorithm

INPUT: $f \in C(x, y)$

OUTPUT: a telescoper and a certificate for f

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$s = (r-1)\deg_y q + \deg_y p + 1$

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5 equate the y-coefficients of $\operatorname{den}(f)^{r+1}(P \cdot f - D_y \cdot g)$ to zero and solve the resulting linear system

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2. $s = (r-1)\deg_y q + \deg_y p + 1$
3. make an ansatz $P = a_0 + a_1 D_x + \cdots + a_r D_x^r$
4. make an ansatz $g = (b_0 + b_1 y + \cdots + b_s y^s)/\operatorname{den}(f)^r$
5. equate the y-coefficients of $\operatorname{den}(f)^{r+1}(P \cdot f - D_y \cdot g)$ to zero and solve the resulting linear system
6. for a solution vector $(a_0, \ldots, a_r, b_0, \ldots, b_s)$ with at least one nonzero a_i, return P and g

Summary.

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$$P(n, S_n) - \Delta_k \, Q(n, y, \Delta_k, S_n)$$

Summary.

$$\underbrace{P(n, S_n)}_{\text{telescoper (nonzero!)}} - \Delta_k \underbrace{Q(n, y, \Delta_k, S_n)}_{\text{certificate}}$$

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- Sister Celine's algorithm is a creative telescoping algorithm for hypergeometric terms.

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- Creative telescoping is the search for a telescoper (with or without a corresponding certificate).
- Sister Celine's algorithm is a creative telescoping algorithm for hypergeometric terms.
- For rational functions in the differential case, we have two creative telescoping algorithms:

Reduction-based telescoping	Apagodu-Zeilberger algorithm
Hermite reduction + small linear system	large linear system