Sister Celine's Method

Manuel Kauers · Institute for Algebra · JKU

$$\sum_{k} \binom{n}{k} = 2^n$$

$$\sum_{k} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}$$

$$\sum_{k} \binom{m-r+s}{k} \binom{n+r-s}{n-k} \binom{r+k}{m+n} = \binom{r}{m} \binom{s}{n}$$

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^{l} \binom{n+r}{n+l} \binom{s-r}{m-n-l}$$

$$\sum_{k} f(n,k) = F(n)$$

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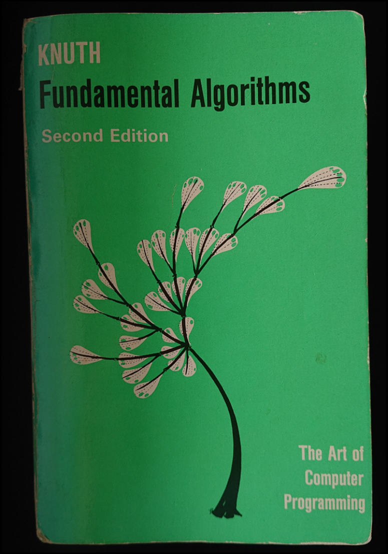
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Fundamental Algorithms
Second Edition
The Art of
Computer
Programming

thereby obtaining a companion formula for Eq. (51).

▸**62.** [*M23*] The text gives formulas for sums involving a product of two binomia coefficients. Of the sums involving a product of three binomial coefficients, the followin one and the identity of exercise 31 seem to be most useful:

$$\sum_k (-1)^k \binom{l+m}{l+k}\binom{m+n}{m+k}\binom{n+l}{n+k} = \frac{(l+m+n)!}{l!\,m!\,n!}, \qquad \text{integer } l, m, n \geq 0.$$

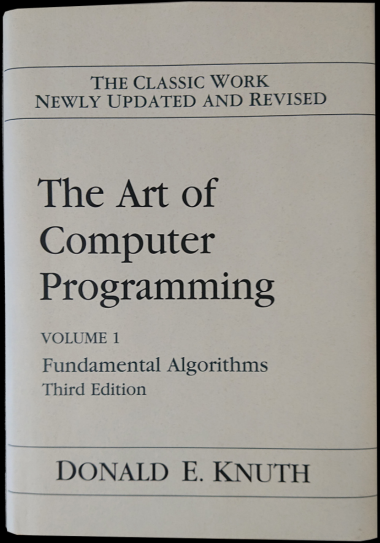
(Note that the sum includes positive and negative values of k.) Prove this identity [*Hint:* There is a very short proof, which begins by applying the result of exercise 31

63. [*46*] Develop computer programs for simplifying sums that involve binomia coefficients.

▸**64.** [*M22*] Show that $\left\{ {n \atop m} \right\}$ is the number of ways to partition a set of n elements int m nonempty disjoint subsets. For example, the set $\{1, 2, 3, 4\}$ can be partitioned int two subsets in $\left\{ {4 \atop 2} \right\} = 7$ ways: $\{1, 2, 3\}\{4\}$; $\{1, 2, 4\}\{3\}$; $\{1, 3, 4\}\{2\}$; $\{2, 3, 4\}\{1\}$ $\{1, 2\}\{3, 4\}$; $\{1, 3\}\{2, 4\}$; $\{1, 4\}\{2, 3\}$. *Hint:* Use the fact that

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}.$$

Note that the result of this exercise provides us with a mnemonic device for remember-ing th... notations for Stirling numbers, since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(The sum includes both positive and negative values of k.) Prove this identity. [*Hint:* There is a very short proof, which begins by applying the result of exercise ...

63. [*M30*] If l, m, and n are integers and $n \geq 0$, prove that

$$\sum_{j,k} (-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j} = (-1)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}$$

64. [*M20*] Show that $\left\{ {n \atop m} \right\}$ is the number of ways to partition a set of n elements into m nonempty disjoint subsets. For example, the set $\{1,2,3,4\}$ can be partitioned into two subsets in $\left\{ {4 \atop 2} \right\} = 7$ ways: $\{1,2,3\}\{4\}$; $\{1,2,4\}\{3\}$; $\{1,3,4\}\{2\}$; $\{2,3,4\}\{1\}$; $\{1,2\}\{3,4\}$; $\{1,3\}\{2,4\}$; $\{1,4\}\{2,3\}$. *Hint:* Use Eq. (46).

65. [*HM35*] (B. F. Logan.) Prove Eqs. (59) and (60).

66. [*HM30*] Suppose x, y, and z are real numbers satisfying

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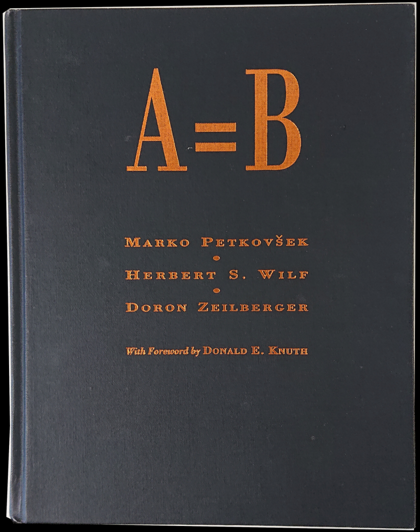
A=B
Marko Petkovšek
Herbert S. Wilf
Doron Zeilberger
With Foreword by Donald E. Knuth

Chapter 4

Sister Celine's Method

4.1 Introduction

The subject of computerized proofs of identities begins with the Ph.D. thesis of Sister Mary Celine Fasenmyer at the University of Michigan in 1945. There she developed a method for finding recurrence relations for hypergeometric polynomials directly from the series expansions of the polynomials. An exposition of her method is in Chapter 14 of Rainville [Rain60]. In his words,

> Years ago it seemed customary upon entering the study of a new set of polynomials to seek recurrence relations ... by essentially a hit-or-miss process. Manipulative skill was used and, if there was enough of it, some relations emerged; others might easily have been lurking around a corner without being discovered ... The interesting problem of the pure recurrence relation for hypergeometric polynomials received probably its first systematic attack at the hands of Sister Mary Celine Fasenmyer ...

The method is quite effective and easily computerized, though it is usually slow in comparison to the methods of Chapter 6. Her algorithm is also important because it has yielded general existence theorems for the recurrence relations satisfied by hypergeometric sums.

We begin by illustrating her method on a simple sum.

Example 4.1.1. Let

$$f(n) = \sum_k k\binom{n}{k} \qquad (n = 0, 1, 2, \ldots).$$

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$$p_0(n)S(n) + p_1(n)S(n+1) + \cdots + p_r(n)S(n+r) = 0$$

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3 Check whether the conjectured identity is true for the first few values of n.

4 Conclude that the identity is true for all n.

Simple example: $\sum_k \binom{n}{k} \stackrel{?}{=} 2^n$

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Note: no algorithm can ever take an "arbitrary function" as input.

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- if $f(n,k)$ is hypergeometric, then so is $f(\alpha n+\beta k+\gamma, \delta n+\epsilon k+\zeta)$ for any $\alpha,\beta,\delta,\epsilon \in \mathbb{Z}$ and any constants γ,ζ.

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- If u and v have has or roots or poles in $\mathbb{Z}^2$, then f is uniquely determined by u,v and $f(0,0)$.
- Typically, u and v do have roots or poles. In this case, manual inspection may be required to check the results of a "formal" computation.

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$$\frac{F(n+1,k)}{F(n,k)} = u(n,k) \quad \text{and} \quad \frac{F(n,k+1)}{F(n,k)} = v(n,k) \quad \text{for all } n,k.$$

Note:

- If u and v have has or roots or poles in $\mathbb{Z}^2$, then f is uniquely determined by u, v and $f(0,0)$.
- Typically, u and v do have roots or poles. In this case, manual inspection may be required to check the results of a "formal" computation.
- We must have $u(n,k+1)v(n,k) = u(n,k)v(n+1,k)$.

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Example:

$$(-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}$$

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Idea: find a recurrence for the summand $f(n, k)$ that can be translated into a recurrence for the sum.

Example: $f(n, k) = \binom{n}{k}$

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$$f(n+1,k+1) - f(n,k) - f(n,k+1) = 0$$

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\begin{aligned}
&(n+1)(2n+1)(4n+7)f(n,k) \\
&+4(n+1)(2n+1)(4n+7)f(n,k+1) + (4n+5)(4n^2+10n+5)f(n+1,k+1) \\
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&+(n+1)(2n+1)(4n+7)f(n,k+4) \\
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&\qquad\qquad + (n+2)(2n+3)(4n+3) \quad S(n+2) \qquad = 0.
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\begin{aligned}
&16(n+1)(2n+1)(4n+7)S(n) \\
&-2(4n+5)(8n^2+20n+11)S(n+1) \\
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Have: two recurrence equations

$$f(n+1,k) = u(n,k)f(n,k) \qquad f(n,k+1) = v(n,k)f(n,k)$$

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Want: a recurrence equation (possibly of higher orders r, s)

$$\begin{aligned} & a_{0,0}(n)f(n,k) + a_{1,0}(n)f(n+1,k) + \cdots + a_{r,0}(n)f(n+r,k) \\ + & a_{0,1}(n)f(n,k+1) + a_{1,1}(n)f(n+1,k+1) + \cdots + a_{r,1}(n)f(n+r,k+1) \\ + & \ldots\ldots\ldots\ldots\ldots\ldots \\ + & a_{0,s}(n)f(n,k+s) + a_{1,s}(n)f(n+1,k+s) + \cdots + a_{r,s}(n)f(n+r,k+s) = 0 \end{aligned}$$

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+&\ldots\ldots\ldots\ldots\ldots\ldots && && \\
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Such a recurrence can be found with linear algebra.

For example, take $r = 1$ and $s = 2$.

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$$\begin{aligned}
&a_{0,0}(n)\frac{f(n,k)}{f(n,k)} && + a_{1,0}(n)\frac{f(n+1,k)}{f(n,k)} \\
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\begin{aligned}
&a_{0,0}(n) && + a_{1,0}(n)u(n,k) \\
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The left hand side is an explicit rational function in n and k whose numerator depends linearly on the unknown coefficients $a_{i,j}(n)$.

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Equate coefficients with respect to k to zero and solve the resulting linear system.

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$$a_{0,0}\frac{\binom{n}{k}}{\binom{n}{k}} \quad + a_{1,0}\frac{\binom{n+1}{k}}{\binom{n}{k}} \qquad + a_{0,1}\frac{\binom{n}{k+1}}{\binom{n}{k}} \qquad + a_{1,1}\frac{\binom{n+1}{k+1}}{\binom{n}{k}} \qquad \stackrel{!}{=} 0$$

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$$a_{0,0} \qquad + a_{1,0}\frac{n+1}{n+1-k} + a_{0,1}\frac{n-k}{k+1} \qquad + a_{1,1}\frac{n+1}{k+1} \qquad \stackrel{!}{=} 0$$

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$$\Big((a_{0,0} - a_{0,1})k^2 - (na_{0,0} - 2na_{0,1} + na_{1,0} - na_{1,1} - a_{0,1} + a_{1,0} - a_{1,1})k - (n+1)(na_{0,1} + na_{1,1} + a_{0,0} + a_{1,0} + a_{1,1})\Big) \Big/ \big((k+1)(k-n-1)\big) \stackrel{!}{=} 0.$$

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&- (n+1)(na_{0,1} + na_{1,1} + a_{0,0} + a_{1,0} + a_{1,1}) \\
&\qquad \stackrel{!}{=} 0.
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$$\begin{pmatrix} -n-1 & -n(n+1) & -n-1 & -(n+1)^2 \\ -n & 2n+1 & -n-1 & n+1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{0,0} \\ a_{0,1} \\ a_{1,0} \\ a_{1,1} \end{pmatrix} \stackrel{!}{=} 0$$

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3 if there is one, translate it to a recurrence for $S(n)$ and return it.

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OUTPUT: a linear recurrence with polynomial coefficients for the sum $S(n) := \sum_k f(n, k)$.

1 choose $r, s \in \mathbb{N}$

2 use linear algebra to search for a k-free recurrence of $f(n, k)$ of order r w.r.t. n and order s w.r.t. k.

3 if there is one, translate it to a recurrence for $S(n)$ and return it.

4 otherwise, increase r and s and try again.

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Fact. Every hypergeometric term $f(n, k)$ can be written in the form

$$q(n, k)\phi^n\psi^k \prod_{m=1}^{M} (a_m n + b_m k + c_m)!^{e_m}$$

for some rational function q, some constants ϕ, ψ, c_m, and some integers a_m, b_m, e_m.

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Example:

$$\binom{n}{k} = n!k!^{-1}(n-k)!^{-1}$$

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Def. A hypergeometric term is called proper if it can be written as above, but with q being a polynomial.

Theorem. Every proper hypergeometric term satisfies a k-free recurrence of some orders r, s.

Questions:

- Does every $f(n, k)$ have a k-free recurrence? No.
- Does every k-free recurrence translate into a nontrivial recurrence for $S(n)$? No.

Questions:

- Does every $f(n,k)$ have a k-free recurrence? Almost.
- Does every k-free recurrence translate into a nontrivial recurrence for $S(n)$? No.

Example. For $f(n,k) = \binom{n}{k}$ we also have the k-free recurrence

$$\begin{aligned} &f(n,k) - f(n+1,k+1) \\ &- f(n,k+2) + f(n+1,k+2) = 0. \end{aligned}$$

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For the sum $S(n) = \sum_k \binom{n}{k}$, it implies

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Oups!

Let's try a bit harder.

$$f(n,k) - \; f(n,k+2) - \; f(n+1,k+1) + \; f(n+1,k+2) = 0.$$

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$$kf(n, k) - kf(n, k+2) - kf(n+1, k+1) + kf(n+1, k+2) = 0.$$

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$$kf(n,k) - kf(n,k+2) - kf(n+1,k+1) + kf(n+1,k+2)$$
$$-(k+1)f(n,k+1) + (k+1)f(n,k+1) = 0.$$

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$$kf(n,k) - kf(n,k+2) - kf(n+1,k+1) + kf(n+1,k+2)$$
$$- (k+1)f(n,k+1) + (k+1)f(n,k+1)$$
$$- 2f(n,k+2) + 2f(n,k+2) = 0.$$

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$$
\begin{aligned}
& kf(n,k) - kf(n,k+2) - kf(n+1,k+1) + kf(n+1,k+2) \\
& - (k+1)f(n,k+1) + (k+1)f(n,k+1) \\
& - 2f(n,k+2) + 2f(n,k+2) \\
& - f(n+1,k+2) + f(n+1,k+2) = 0.
\end{aligned}
$$

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$$
\begin{aligned}
& k f(n, k)-(k+1) f(n, k+1) \\
& +(k+1) f(n, k+1)-(k+2) f(n, k+2) \\
& -k f(n+1, k+1)+(k+1) f(n+1, k+2) \\
& +2 f(n, k+2)-f(n+1, k+2)=0
\end{aligned}
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$$+ 2f(n,k+2) - f(n+1,k+2) = 0 \qquad \Big| \sum_k$$

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Theorem (Wegschaider's Lemma). This works always.

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$$S_x x = (x+1)S_x$$
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$$\sum_k \Delta_k \cdot f(n,k) = 0$$

Write your k-free recurrence in the form

$$\big(P(n, S_n) + \Delta_k Q(n, \Delta_k, S_n)\big) \cdot f(n, k) = 0.$$

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$$-Q(n, \Delta_k, S_n) + \Delta_k k Q(n, \Delta_k, S_n) \quad \cdot f(n, k) = 0.$$

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$$\underbrace{-Q(n, \Delta_k, S_n)}_{} + \Delta_k k Q(n, \Delta_k, S_n) \quad \cdot f(n, k) = 0.$$

$$= \tilde{P}(n, S_n) + \Delta_k \tilde{Q}(n, \Delta_k, S_n)$$

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Iterate if necessary.

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Iterate if necessary.

After at most $\deg_{\Delta_k} Q$ repetitions, the result is nonzero.

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- Every k-free recurrence for $f(n, k)$ gives rise to a linear recurrence for the sum $S(n) = \sum_k f(n, k)$.
- Such a recurrence can be used to prove a conjectural closed form expression for $S(n)$.