Advanced Closure Properties

Manuel Kauers · Institute for Algebra · JKU

Recall:

Recall:

- A left ideal $I \subseteq C(x,y)[D_x, D_y]$ is called D-finite if

$$\dim_{C(x,y)} C(x,y)[D_x, D_y]/I < \infty.$$

Recall:

- A left ideal $I \subseteq C(x,y)[D_x, D_y]$ is called D-finite if
$$\dim_{C(x,y)} C(x,y)[D_x, D_y]/I < \infty.$$
- A function $f(x,y)$ is called D-finite if its annihilator
$$\operatorname{ann} f(x,y) = \{\, L \in C(x,y)[D_x, D_y] : L \cdot f = 0 \,\}$$
is D-finite.

Recall:

- A left ideal $I \subseteq C(x,y)[D_x, D_y]$ is called D-finite if

$$\dim_{C(x,y)} C(x,y)[D_x, D_y]/I < \infty.$$

- A function $f(x,y)$ is called D-finite if its annihilator

$$\operatorname{ann} f(x,y) = \{\, L \in C(x,y)[D_x, D_y] : L \cdot f = 0 \,\}$$

  is D-finite.
- The notion extends naturally to more variables, and to shifts and other kinds of operators besides derivations.

Recall:

- A left ideal $I \subseteq C(x, y)[D_x, D_y]$ is called D-finite if

$$\dim_{C(x,y)} C(x, y)[D_x, D_y]/I < \infty.$$

- A function $f(x, y)$ is called D-finite if its annihilator

$$\operatorname{ann} f(x, y) = \{ L \in C(x, y)[D_x, D_y] : L \cdot f = 0 \}$$

is D-finite.

- The notion extends naturally to more variables, and to shifts and other kinds of operators besides derivations.

Recall:

- A left ideal $I \subseteq C[x, y][D_x, D_y]$ is called holonomic if

$$I \cap C[U] \neq \emptyset$$

for every $U \subseteq \{x, y, D_x, D_y\}$ with $|U| > 2$.

Recall:

- A left ideal $I \subseteq C[x, y][D_x, D_y]$ is called holonomic if

$$I \cap C[U] \neq \emptyset$$

  for every $U \subseteq \{x, y, D_x, D_y\}$ with $|U| > 2$.
- A function $f(x, y)$ is called holonomic if its annihilator is holonomic.

Recall:

- A left ideal $I \subseteq C[x, y][D_x, D_y]$ is called holonomic if

$$I \cap C[U] \neq \emptyset$$

  for every $U \subseteq \{x, y, D_x, D_y\}$ with $|U| > 2$.
- A function $f(x, y)$ is called holonomic if its annihilator is holonomic.
- The notion extends naturally to more variables, and to shifts and other kinds of operators besides derivations.

Recall:

- If f and g are D-finite/holonomic, then so are $f + g$ and $f \cdot g$.

Recall:

- If f and g are D-finite/holonomic, then so are $f + g$ and $f \cdot g$.
- These are examples of closure properties.

Recall:

- If f and g are D-finite/holonomic, then so are $f+g$ and $f \cdot g$.
- These are examples of closure properties.
- They are based on linear algebra.

Recall:

- If f and g are D-finite/holonomic, then so are $f + g$ and $f \cdot g$.
- These are examples of closure properties.
- They are based on linear algebra.

Next goal:

- Additional closure properties based on creative telescoping.

Recall:

$$\underbrace{P(n, S_n)}_{\text{telescoper (nonzero!)}} - \Delta_k \underbrace{Q(n, k, \Delta_k, S_n)}_{\text{certificate}}$$

Recall:

$$\underbrace{P(x, D_x)}_{\text{telescoper (nonzero!)}} - \Delta_k \underbrace{Q(x, k, \Delta_k, D_x)}_{\text{certificate}}$$

Recall:

$$\underbrace{P(n, S_n)}_{\text{telescoper (nonzero!)}} - D_y \underbrace{Q(n, y, D_y, S_n)}_{\text{certificate}}$$

Recall:

$$\underbrace{P(x, D_x)}_{\text{telescoper (nonzero!)}} - D_y \underbrace{Q(x, y, D_y, D_x)}_{\text{certificate}}$$

Recall:

$$\underbrace{P(x, D_x)}_{\text{telescoper (nonzero!)}} - D_y \underbrace{Q(x, y, D_y, D_x)}_{\text{certificate}}$$

Theorem. If $I \subseteq C[x, y][\partial_x, \partial_y]$ is holonomic, then there exist

- $P \in C[x][\partial_x] \setminus \{0\}$
- $Q \in C[x, y][\partial_x, \partial_y]$

such that

$$P - \partial_y Q \in I.$$

Recall:

$$\underbrace{P(x, D_x)}_{\text{telescoper (nonzero!)}} - D_y \underbrace{Q(x, y, D_y, D_x)}_{\text{certificate}}$$

Theorem. If $I \subseteq C[x, y][\partial_x, \partial_y]$ is holonomic, then there exist

- $P \in C[x][\partial_x] \setminus \{0\}$
- $Q \in C[x, y][\partial_x, \partial_y]$

such that

$$P - \partial_y Q \in I.$$

Given a basis of I, such P and Q can be computed.

Example: $\int_0^\infty \cos(xy)\exp(-y^2/2)\,dy$

Example: $$\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)}\,dy$$

Example: $\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)}\,dy$

- Consider $I = \langle yD_y - xD_x + y^2, D_x^2 + y^2\rangle \subseteq C(x,y)[D_x, D_y]$.

Example: $\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)}\,dy$

- Consider $I = \langle yD_y - xD_x + y^2, D_x^2 + y^2 \rangle \subseteq C(x,y)[D_x, D_y]$.
- This ideal I annihilates $f(x,y)$.

Example: $\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)}\,dy$

- Consider $I = \langle y D_y - x D_x + y^2, D_x^2 + y^2 \rangle \subseteq C(x,y)[D_x, D_y]$.
- This ideal I annihilates $f(x,y)$.
- For $P = D_x + x$; $Q = -x^{-1}(D_y + y)$ we have $P - D_y Q \in I$.

Example: $\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)}\,dy$

- Consider $I = \langle yD_y - xD_x + y^2, D_x^2 + y^2\rangle \subseteq C(x,y)[D_x, D_y]$.
- This ideal I annihilates $f(x,y)$.
- For $P = D_x + x$; $Q = -x^{-1}(D_y + y)$ we have $P - D_yQ \in I$.

Therefore:

$$P \cdot \int_0^\infty f(x,y)dy = \Big[\, Q \cdot f(x,y) \,\Big]_{y=0}^{\infty} = 0.$$

Example: $\int_0^\infty \underbrace{\cos(xy)\exp(-y^2/2)}_{=:f(x,y)} dy$

- Consider $I = \langle yD_y - xD_x + y^2, D_x^2 + y^2 \rangle \subseteq C(x,y)[D_x, D_y]$.
- This ideal I annihilates $f(x,y)$.
- For $P = D_x + x$; $Q = -x^{-1}(D_y + y)$ we have $P - D_y Q \in I$.

Therefore:

$$P \cdot \int_0^\infty f(x,y)dy = \Big[Q \cdot f(x,y) \Big]_{y=0}^{\infty} = 0.$$

P is an annihilating operator for the definite integral.

Example: $\displaystyle\sum_k \sum_i k \binom{n}{i} \binom{i}{k}$

Example: $\sum_{k} \underbrace{\sum_{i} k \binom{n}{i} \binom{i}{k}}_{=:f(n,k)}$

Example: $\sum_k \underbrace{\sum_i k\binom{n}{i}\binom{i}{k}}_{=:f(n,k)}$

- The ideal $I = \langle 2kS_k + (k-n-1)S_n + (n+k+2), (k-n-2)S_n^2 + (3n+5-k)S_n - (2n+2)\rangle \subseteq C(n,k)[S_n, S_k]$ annihilates $f(n,k)$.

Example: $\sum_k \underbrace{\sum_i k\binom{n}{i}\binom{i}{k}}_{=:f(n,k)}$

- The ideal $I = \langle 2kS_k + (k-n-1)S_n + (n+k+2), (k-n-2)S_n^2 + (3n+5-k)S_n - (2n+2)\rangle \subseteq C(n,k)[S_n, S_k]$ annihilates $f(n,k)$.
- For $P = nS_n - (3n+3)$ and $Q = \frac{2kn+k-n-1}{2k}S_n + \frac{n+1-2kn-2k}{k}$ we have $P - \Delta_k Q \in I$

Example: $\sum_k \underbrace{\sum_i k\binom{n}{i}\binom{i}{k}}_{=:f(n,k)}$

- The ideal $I = \langle 2kS_k + (k-n-1)S_n + (n+k+2), (k-n-2)S_n^2 + (3n+5-k)S_n - (2n+2)\rangle \subseteq C(n,k)[S_n, S_k]$ annihilates $f(n,k)$.
- For $P = nS_n - (3n+3)$ and $Q = \frac{2kn+k-n-1}{2k}S_n + \frac{n+1-2kn-2k}{k}$ we have $P - \Delta_k Q \in I$

Therefore:

$$P \cdot \sum_k f(n,k) = \Big[Q \cdot f(n,k) \Big]_{k=-\infty}^{\infty} = 0.$$

Example: $\sum_k \underbrace{\sum_i k\binom{n}{i}\binom{i}{k}}_{=:f(n,k)}$

- The ideal $I = \langle 2kS_k + (k-n-1)S_n + (n+k+2), (k-n-2)S_n^2 + (3n+5-k)S_n - (2n+2)\rangle \subseteq C(n,k)[S_n, S_k]$ annihilates $f(n,k)$.
- For $P = nS_n - (3n+3)$ and $Q = \frac{2kn+k-n-1}{2k}S_n + \frac{n+1-2kn-2k}{k}$ we have $P - \Delta_k Q \in I$

Therefore:

$$P \cdot \sum_k f(n,k) = \Big[Q \cdot f(n,k) \Big]_{k=-\infty}^{\infty} = 0.$$

P is an annihilating operator for the definite sum.

Def. A sum/integral over a holonomic function f is said to have natural boundaries if there is a telescoper/certificate pair (P, Q) such that $Q \cdot f$ evaluates to zero at the boundaries of the summation/integration range.

Def. A sum/integral over a holonomic function f is said to have natural boundaries if there is a telescoper/certificate pair (P, Q) such that $Q \cdot f$ evaluates to zero at the boundaries of the summation/integration range.

Example: regardless of Q, we have $\left[Q \cdot \binom{n}{k}\right]_{k=-\infty}^{\infty} = 0$.

Def. A sum/integral over a holonomic function f is said to have natural boundaries if there is a telescoper/certificate pair (P, Q) such that $Q \cdot f$ evaluates to zero at the boundaries of the summation/integration range.

Example: regardless of Q, we have $\left[Q \cdot \binom{n}{k}\right]_{k=-\infty}^{\infty} = 0$.

Theorem. The class of holonomic functions is closed under definite summation/integration with natural boundaries.

Def. A sum/integral over a holonomic function f is said to have natural boundaries if there is a telescoper/certificate pair (P, Q) such that $Q \cdot f$ evaluates to zero at the boundaries of the summation/integration range.

Example: regardless of Q, we have $\left[Q \cdot \binom{n}{k}\right]_{k=-\infty}^{\infty} = 0$.

Theorem. The class of holonomic functions is closed under definite summation/integration with natural boundaries.

What about non-natural boundaries?

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{(k-2n-1)(k-2n-2)} \binom{2n}{k}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$\sum_{k=0}^{n} \binom{2(n+1)}{k} - 4 \sum_{k=0}^{n} \binom{2n}{k} = \left[\frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k} \right]_{k=0}^{n+1}$$

Example: $\displaystyle\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$\sum_{k=0}^{n} \binom{2(n+1)}{k} - 4 \sum_{k=0}^{n} \binom{2n}{k} = -\frac{4n+3}{4n+2} \binom{2n+2}{n+1}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k - 6n - 5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$\sum_{k=0}^{n} \binom{2(n+1)}{k} - 4 \underbrace{\sum_{k=0}^{n} \binom{2n}{k}}_{=S(n)} = -\frac{4n+3}{4n+2} \binom{2n+2}{n+1}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$\underbrace{\sum_{k=0}^{n} \binom{2(n+1)}{k}}_{=S(n+1)-\binom{2(n+1)}{n+1}} - 4 \underbrace{\sum_{k=0}^{n} \binom{2n}{k}}_{=S(n)} = -\frac{4n+3}{4n+2} \binom{2n+2}{n+1}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$\underbrace{\sum_{k=0}^{n} \binom{2(n+1)}{k}}_{=S(n+1)-\binom{2(n+1)}{n+1}} - 4 \underbrace{\sum_{k=0}^{n} \binom{2n}{k}}_{=S(n)} = -\frac{4n+3}{4n+2} \binom{2n+2}{n+1}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$S(n+1) - 4S(n) = -\frac{1}{4n+2} \binom{2n+2}{n+1}$$

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k - 6n - 5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$S(n+1) - 4S(n) = -\frac{1}{4n+2} \binom{2n+2}{n+1}$$

We can apply the operator $(n+2)S_n - (4n+2)$ to kill the right hand side.

Example: $\sum_{k=0}^{n} \binom{2n}{k}$

We have the following creative telescoping relation:

$$(S_n - 4) \cdot \binom{2n}{k} = \Delta_k \frac{k(2k-6n-5)}{2(2n+1)(n+1)} \binom{2n+2}{k}$$

Summing this equation over $k = 0, \ldots, n$ gives

$$S(n+1) - 4S(n) = -\frac{1}{4n+2} \binom{2n+2}{n+1}$$

We can apply the operator $(n+2)S_n - (4n+2)$ to kill the right hand side. Finally,

$$(n+2)S(n+2) - (8n+10)S(n+1) + (16n+8)S(n) = 0.$$

In general, a sum/integral with non-natural boundaries leads to an inhomogeneous recurrence or differential equation, where the right hand side involves evaluations of the summand/integrand.

In general, a sum/integral with non-natural boundaries leads to an inhomogeneous recurrence or differential equation, where the right hand side involves evaluations of the summand/integrand.

$$P \cdot F = [Q \cdot f]_{\Omega}$$

In general, a sum/integral with non-natural boundaries leads to an inhomogeneous recurrence or differential equation, where the right hand side involves evaluations of the summand/integrand.

$$P \cdot F = [Q \cdot f]_\Omega$$

If the right hand side is annihilated by L, then LP annihilates F.

In general, a sum/integral with non-natural boundaries leads to an inhomogeneous recurrence or differential equation, where the right hand side involves evaluations of the summand/integrand.

$$P \cdot F = [Q \cdot f]_\Omega$$

If the right hand side is annihilated by L, then LP annihilates F.

Question: Does evaluation preserve holonomy?

In general, a sum/integral with non-natural boundaries leads to an inhomogeneous recurrence or differential equation, where the right hand side involves evaluations of the summand/integrand.

$$P \cdot F = [Q \cdot f]_{\Omega}$$

If the right hand side is annihilated by L, then LP annihilates F.

Question: Does evaluation preserve holonomy?

Answer: yes!

Theorem. If $I \subseteq C[x, y][\partial_x, \partial_y]$ is holonomic, then there exist

- $P \in C[x][\partial_x] \setminus \{0\}$
- $Q \in C[x, y][\partial_x, \partial_y]$

such that

$$P - \partial_y Q \in I.$$

Given a basis of I, such P and Q can be computed.

Theorem. If $I \subseteq C[x,y][\partial_x,\partial_y]$ is holonomic, then there exist

- $P \in C[x][\partial_x] \setminus \{0\}$
- $Q \in C[x,y][\partial_x,\partial_y]$

such that

$$P - yQ \in I.$$

Given a basis of I, such P and Q can be computed.

Theorem. If $I \subseteq C[x, y][\partial_x, \partial_y]$ is holonomic, then there exist

- $P \in C[x][\partial_x] \setminus \{0\}$
- $Q \in C[x, y][\partial_x, \partial_y]$

such that

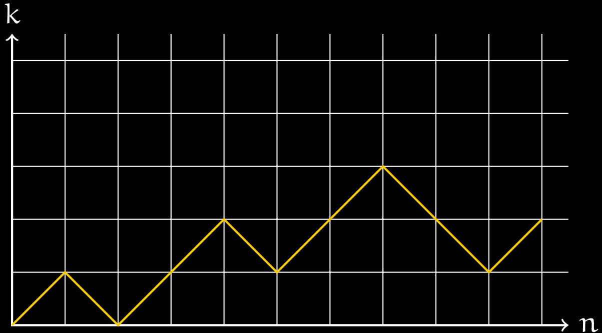
$$P - yQ \in I.$$

Given a basis of I, such P and Q can be computed.

Corollary. If $f(x, y)$ is holonomic, then so is $f(x, 0)$.

Example: $f(n, k) = \#$ Dyck paths of width n ending at height k.

Example: $f(n,k) = \#$ Dyck paths of width n ending at height k.



Example: $f(n,k) = \#$ Dyck paths of width n ending at height k.

- The annihilating ideal of $f(n,k)$ contains the operator

$$(n+4)(n+2)S_n^2 - 4(n+2)(n+1) - k(k+2)S_n^2.$$

Example: $f(n,k) = \#$ Dyck paths of width n ending at height k.

- The annihilating ideal of $f(n,k)$ contains the operator

$$(n+4)(n+2)S_n^2 - 4(n+2)(n+1) - k(k+2)S_n^2.$$

- Therefore,

$$(n+4)(n+2)f(n+2,k) - 4(n+2)(n+1)f(n,k)$$
$$= k(k+2)f(n+2,k).$$

Example: $f(n,k) = \#$ Dyck paths of width n ending at height k.

- The annihilating ideal of $f(n,k)$ contains the operator

$$(n+4)(n+2)S_n^2 - 4(n+2)(n+1) - k(k+2)S_n^2.$$

- Therefore,

$$(n+4)(n+2)f(n+2,k) - 4(n+2)(n+1)f(n,k) \\ = k(k+2)f(n+2,k).$$

- Setting $k=0$ gives

$$(n+4)(n+2)f(n+2,0) - 4(n+2)(n+1)f(n,0) = 0.$$

Closure under evaluation enables non-natural boundaries.

Closure under evaluation enables non-natural boundaries.

Theorem. Let

Closure under evaluation enables non-natural boundaries.

Theorem. Let

- $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ be holonomic,

Closure under evaluation enables non-natural boundaries.

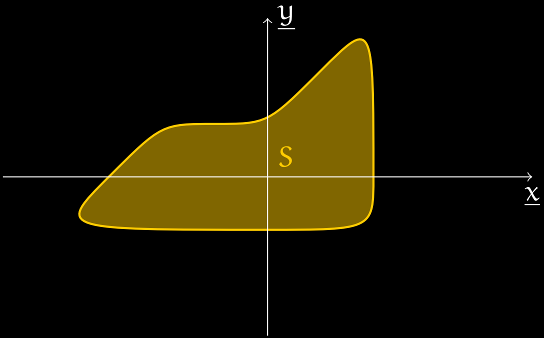
Theorem. Let

- $f(x_1, \dots, x_p, y_1, \dots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,

Closure under evaluation enables non-natural boundaries.

Theorem. Let

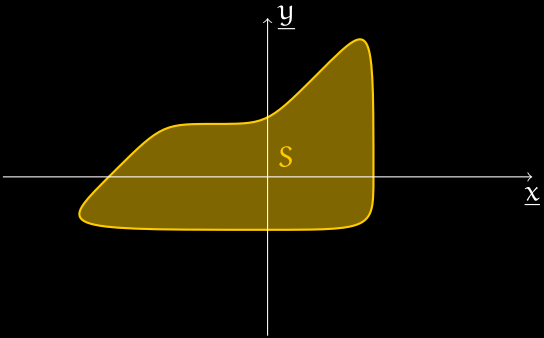
- $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,



Closure under evaluation enables non-natural boundaries.

Theorem. Let

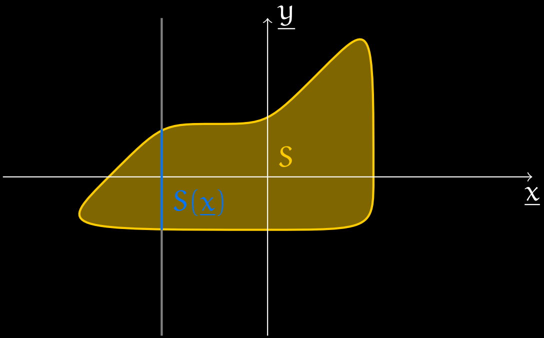
- $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,
- $S(x_1, \ldots, x_p) \subseteq \mathbb{R}^q$ denote the section of S at $(x_1, \ldots, x_q)$,



Closure under evaluation enables non-natural boundaries.

Theorem. Let

- $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,
- $S(x_1, \ldots, x_p) \subseteq \mathbb{R}^q$ denote the section of S at $(x_1, \ldots, x_q)$,



Closure under evaluation enables non-natural boundaries.

Theorem. Let

- $f(x_1, \dots, x_p, y_1, \dots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,
- $S(x_1, \dots, x_p) \subseteq \mathbb{R}^q$ denote the section of S at $(x_1, \dots, x_q)$,
- $F(x_1, \dots, x_p) := \displaystyle\int_{\underline{y} \in S(\underline{x})} f(\underline{x}, \underline{y}) d\underline{y}$.

Closure under evaluation enables non-natural boundaries.

Theorem. Let

- $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,
- $S(x_1, \ldots, x_p) \subseteq \mathbb{R}^q$ denote the section of S at $(x_1, \ldots, x_q)$,
- $F(x_1, \ldots, x_p) := \int\limits_{\underline{y} \in S(\underline{x})} f(\underline{x}, \underline{y}) d\underline{y}.$

Then $F(x_1, \ldots, x_p)$ is holonomic.

Closure under evaluation enables non-natural boundaries.

Theorem. Let

- $f(x_1, \dots, x_p, y_1, \dots, y_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a semi-algebraic set,
- $S(x_1, \dots, x_p) \subseteq \mathbb{R}^q$ denote the section of S at $(x_1, \dots, x_q)$,
- $F(x_1, \dots, x_p) := \int_{\underline{y} \in S(\underline{x})} f(\underline{x}, \underline{y}) d\underline{y}$.

Then $F(x_1, \dots, x_p)$ is holonomic.

Example: $F(x) = \int_{{y_1}^2 + {y_2}^2 \leq x^2} f(x, y_1, y_2)\, dy_1\, dy_2$

There is an analogous result for definite summation.

There is an analogous result for definite summation.

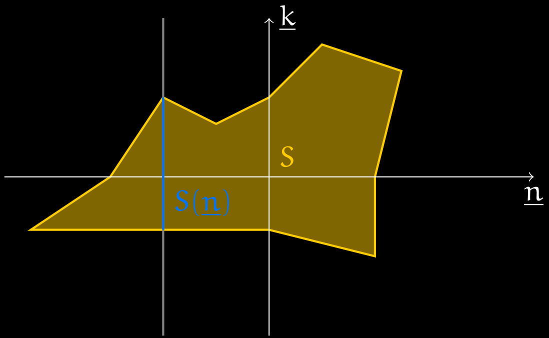
Theorem. Let

- $f(n_1, \ldots, n_p, k_1, \ldots, k_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a rational polygonal set,
- $S(n_1, \ldots, n_p) \subseteq \mathbb{R}^q$ be the section of S at $(n_1, \ldots, n_p)$,

There is an analogous result for definite summation.

Theorem. Let

- $f(n_1, \dots, n_p, k_1, \dots, k_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a rational polygonal set,
- $S(n_1, \dots, n_p) \subseteq \mathbb{R}^q$ be the section of S at $(n_1, \dots, n_p)$,



There is an analogous result for definite summation.

Theorem. Let

- $f(n_1, \dots, n_p, k_1, \dots, k_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a rational polygonal set,
- $S(n_1, \dots, n_p) \subseteq \mathbb{R}^q$ be the section of S at $(n_1, \dots, n_p)$,
- $F(n_1, \dots, n_p) := \sum_{\underline{k} \in S(\underline{n})} f(\underline{n}, \underline{k})$.

There is an analogous result for definite summation.

Theorem. Let

- $f(n_1, \ldots, n_p, k_1, \ldots, k_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a rational polygonal set,
- $S(n_1, \ldots, n_p) \subseteq \mathbb{R}^q$ be the section of S at $(n_1, \ldots, n_p)$,
- $F(n_1, \ldots, n_p) := \sum_{\underline{k} \in S(\underline{n})} f(\underline{n}, \underline{k})$.

Then $F(n_1, \ldots, n_p)$ is holonomic.

There is an analogous result for definite summation.

Theorem. Let

- $f(n_1, \ldots, n_p, k_1, \ldots, k_q)$ be holonomic,
- $S \subseteq \mathbb{R}^{p+q}$ be a rational polygonal set,
- $S(n_1, \ldots, n_p) \subseteq \mathbb{R}^q$ be the section of S at $(n_1, \ldots, n_p)$,
- $F(n_1, \ldots, n_p) := \sum_{\underline{k} \in S(\underline{n})} f(\underline{n}, \underline{k})$.

Then $F(n_1, \ldots, n_p)$ is holonomic.

Example: $F(n_1, n_2) = \sum_{k_1=n_1-n_2}^{5n_1+3n_2} \sum_{k_2=0}^{7n_1+3n_2-k_1} f(n_1, n_2, k_1, k_2)$

Holonomy	D-finiteness

Holonomy	D-finiteness
elegant theory	

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	efficient algorithms

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	efficient algorithms

Can we combine the best of both worlds?

Fact. In the differential case, there is not much difference between holonomy and D-finiteness.

Fact. In the differential case, there is not much difference between holonomy and D-finiteness.

More precisely:

Fact. In the differential case, there is not much difference between holonomy and D-finiteness.

More precisely:

- If $I \subseteq C(x, y)[D_x, D_y]$ is D-finite, then

$$I \cap C[x, y][D_x, D_y]$$

  is holonomic.

Fact. In the differential case, there is not much difference between holonomy and D-finiteness.

More precisely:

- If $I \subseteq C(x,y)[D_x, D_y]$ is D-finite, then

$$I \cap C[x,y][D_x, D_y]$$

  is holonomic.

- If $J \subseteq C[x,y][D_x, D_y]$ is holonomic, then

$$\langle J \rangle \subseteq C(x,y)[D_x, D_y]$$

  is D-finite.

Fact. In the differential case, there is not much difference between holonomy and D-finiteness.

More precisely:

- If $I \subseteq C(x, y)[D_x, D_y]$ is D-finite, then

$$I \cap C[x, y][D_x, D_y]$$

  is holonomic.

- If $J \subseteq C[x, y][D_x, D_y]$ is holonomic, then

$$\langle J \rangle \subseteq C(x, y)[D_x, D_y]$$

  is D-finite.

In particular, telescoper/certificate pairs exist in D-finite ideals.

For formal infinite series, there is no trouble with singularities.

For formal infinite series, there is no trouble with singularities.

Def. Let $f(x,y) := \sum_{n,k\in\mathbb{Z}} a_{n,k} x^n y^k$. Then

$$\operatorname{res}_y f(x,y) := \sum_{n\in\mathbb{Z}} a_{n,-1} x^n$$

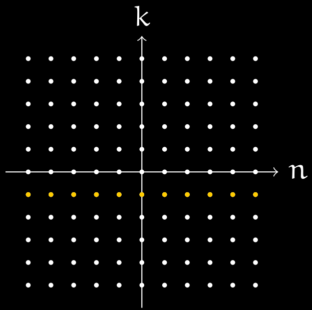
is called the residue of $f(x,y)$ (w.r.t. y).

For formal infinite series, there is no trouble with singularities.

Def. Let $f(x, y) := \sum_{n,k\in\mathbb{Z}} a_{n,k} x^n y^k$. Then

$$\operatorname{res}_y f(x, y) := \sum_{n\in\mathbb{Z}} a_{n,-1} x^n$$

is called the residue of $f(x, y)$ (w.r.t. y).



For formal infinite series, there is no trouble with singularities.

Def. Let $f(x,y) := \sum_{n,k\in\mathbb{Z}} a_{n,k} x^n y^k$. Then

$$\operatorname{res}_y f(x,y) := \sum_{n\in\mathbb{Z}} a_{n,-1} x^n$$

is called the residue of $f(x,y)$ (w.r.t. y).

Note: $\operatorname{res}_y D_y g(x,y) = 0$ for every series $g(x,y)$.

For formal infinite series, there is no trouble with singularities.

Def. Let $f(x,y) := \sum_{n,k\in\mathbb{Z}} a_{n,k} x^n y^k$. Then

$$\operatorname{res}_y f(x,y) := \sum_{n\in\mathbb{Z}} a_{n,-1} x^n$$

is called the residue of $f(x,y)$ (w.r.t. y).

Note: $\operatorname{res}_y D_y g(x,y) = 0$ for every series $g(x,y)$.

Therefore,

$$(P - D_y Q)\cdot f = 0 \quad\Longrightarrow\quad P\cdot \operatorname{res}_y f = 0$$

For formal infinite series, there is no trouble with singularities.

Def. Let $f(x,y) := \sum_{n,k\in\mathbb{Z}} a_{n,k}x^n y^k$. Then

$$\operatorname{res}_y f(x,y) := \sum_{n\in\mathbb{Z}} a_{n,-1}x^n$$

is called the residue of $f(x,y)$ (w.r.t. y).

Note: $\operatorname{res}_y D_y g(x,y) = 0$ for every series $g(x,y)$.

Therefore,

$$(P - D_y Q)\cdot f = 0 \quad\Longrightarrow\quad P\cdot \operatorname{res}_y f = 0$$

In particular, the residue of a D-finite series is D-finite.

For formal infinite series, there is no trouble with singularities.

Def. Let $f(x,y) := \sum_{n,k\in\mathbb{Z}} a_{n,k}x^n y^k$. Then

$$\operatorname{res}_y f(x,y) := \sum_{n\in\mathbb{Z}} a_{n,-1}x^n$$

is called the residue of $f(x,y)$ (w.r.t. y).

Note: $\operatorname{res}_y D_y g(x,y) = 0$ for every series $g(x,y)$.

Therefore,

$$(P - D_y Q)\cdot f = 0 \quad \Longrightarrow \quad P\cdot \operatorname{res}_y f = 0$$

In particular, the residue of a D-finite series is D-finite.

Really?

What is the meaning of $P \cdot f$ if

- $P \in C(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

What is the meaning of $P \cdot f$ if

- $P \in C(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

What is the meaning of $P \cdot f$ if

- $P \in \mathbb{C}(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

- They clearly form a $\mathbb{C}[x, y][D_x, D_y]$-module.

What is the meaning of $P \cdot f$ if

- $P \in \mathbb{C}(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

- They clearly form a $\mathbb{C}[x, y][D_x, D_y]$-module.
- But not a ring.

What is the meaning of $P \cdot f$ if

- $P \in C(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

- They clearly form a $C[x, y][D_x, D_y]$-module.
- But not a ring.

Want:

What is the meaning of $P \cdot f$ if

- $P \in C(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

- They clearly form a $C[x, y][D_x, D_y]$-module.
- But not a ring.

Want:

- Interpretations of rational functions as infinite series

What is the meaning of $P \cdot f$ if

- $P \in \mathbb{C}(x, y)[D_x, D_y]$ and
- f is a bilateral infinite series?

Where do these series live?

- They clearly form a $\mathbb{C}[x, y][D_x, D_y]$-module.
- But not a ring.

Want:

- Interpretations of rational functions as infinite series
- A way to multiply them

Example: $f(x) = \sum_{n \in \mathbb{Z}} x^n$

Example: $f(x) = \sum_{n\in\mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} x^n\right)\left(\sum_{k\in\mathbb{Z}} x^k\right)$$

Example: $f(x) = \sum_{n \in \mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n \in \mathbb{Z}} x^n\right)\left(\sum_{k \in \mathbb{Z}} x^k\right) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x^{n+k}$$

Example: $f(x) = \sum_{n \in \mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n \in \mathbb{Z}} x^n\right)\left(\sum_{k \in \mathbb{Z}} x^k\right) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x^{n+k} = \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} 1\right) x^n$$

Example: $f(x) = \sum_{n\in\mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} x^n\right)\left(\sum_{k\in\mathbb{Z}} x^k\right) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x^{n+k} = \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{k\in\mathbb{Z}} 1\right)}_{\textit{Oups!}} x^n$$

Example: $f(x) = \sum_{n\in\mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} x^n\right)\left(\sum_{k\in\mathbb{Z}} x^k\right) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x^{n+k} = \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{k\in\mathbb{Z}} 1\right)}_{\textit{Oups!}} x^n$$

Recall: The field $\mathbb{C}((x))$ of formal Laurent series consists of all series having a minimal exponent.

$$f(x) = \sum_{n=n_0}^{\infty} a_n x^n$$

Example: $f(x) = \sum_{n \in \mathbb{Z}} x^n$

$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} x^n\right)\left(\sum_{k\in\mathbb{Z}} x^k\right) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x^{n+k} = \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{k\in\mathbb{Z}} 1\right)}_{\textit{Oups!}} x^n$$

Recall: The field $\mathbb{C}((x))$ of formal Laurent series consists of all series having a minimal exponent.

$$f(x) = \sum_{n=n_0}^{\infty} a_n x^n$$

With this restriction, multiplication is well defined.

Example: $f(x) = \sum_{n\in\mathbb{Z}} x^n$

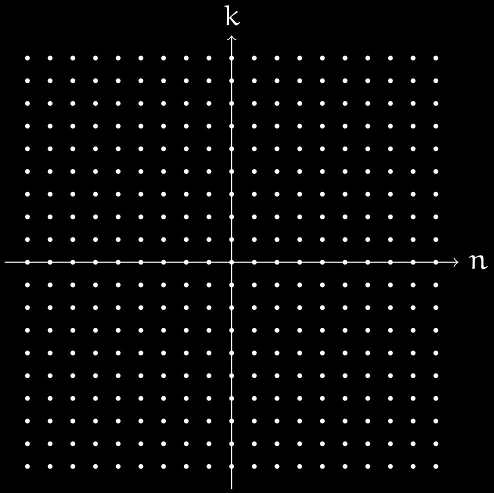
$$f(x)^2 = \left(\sum_{n\in\mathbb{Z}} x^n\right)\left(\sum_{k\in\mathbb{Z}} x^k\right) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x^{n+k} = \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{k\in\mathbb{Z}} 1\right)}_{\textit{Oups!}} x^n$$

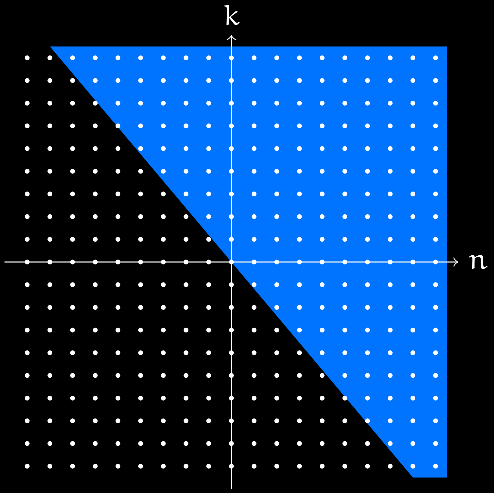
Recall: The field $\mathbb{C}((x))$ of formal Laurent series consists of all series having a minimal exponent.

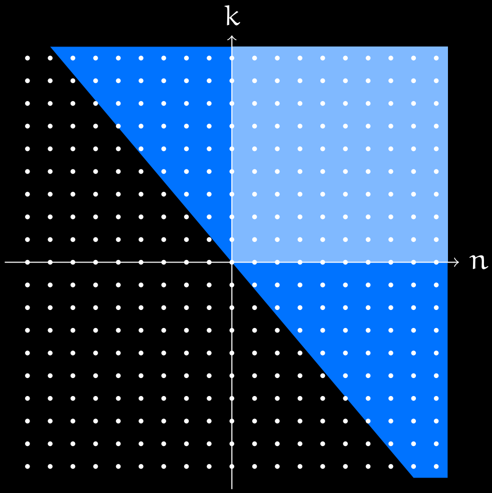
$$f(x) = \sum_{n=n_0}^{\infty} a_n x^n$$

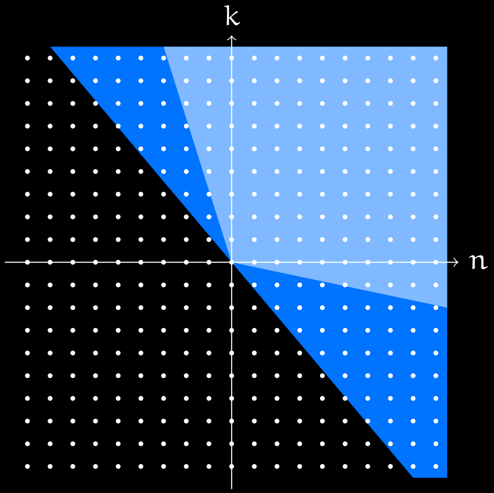
With this restriction, multiplication is well defined.

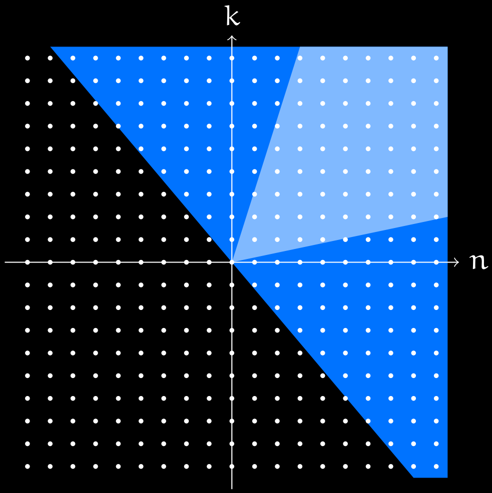
We can apply a similar restriction in the case of several variables.

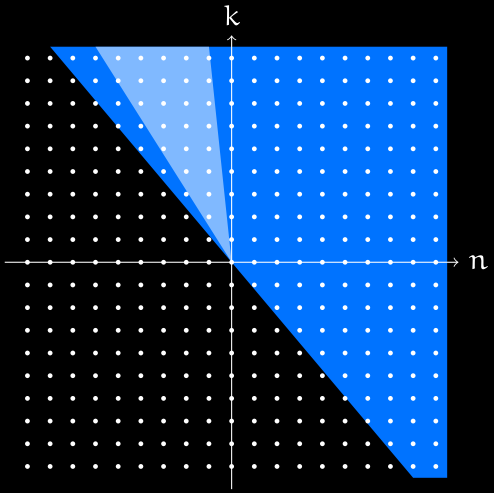

k
n


k
n

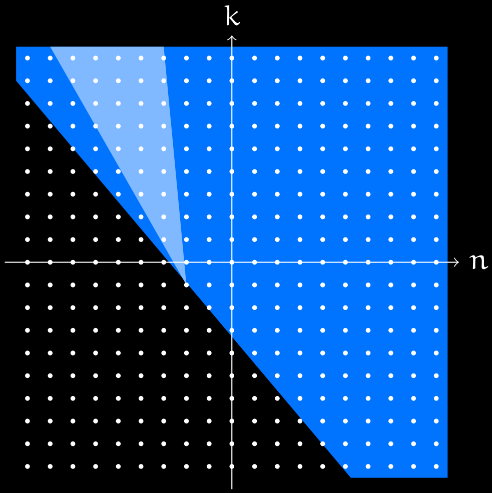

k
n


k
n


k
n



k
n


k
n

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

Fact. This is a field.

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

Fact. This is a field.

Def. It is called the field of bivariate formal Laurent series (w.r.t. H).

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

Fact. This is a field.

Def. It is called the field of bivariate formal Laurent series (w.r.t. H).

Feature: $C((x, y))$ is a $C(x, y)[D_x, D_y]$-module.

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

Fact. This is a field.

Def. It is called the field of bivariate formal Laurent series (w.r.t. H).

Feature: $C((x, y))$ is a $C(x, y)[D_x, D_y]$-module.

We can reasonably talk about elements of $C((x, y))$ being D-finite.

For a fixed hyperplane $H \subseteq \mathbb{R}^2$, let $C((x, y))$ be the set of all series whose support is restricted to some offset of some proper rational subcone of H.

Fact. This is a field.

Def. It is called the field of bivariate formal Laurent series (w.r.t. H).

Feature: $C((x, y))$ is a $C(x, y)[D_x, D_y]$-module.

We can reasonably talk about elements of $C((x, y))$ being D-finite.

Now really: Residues of D-finite formal Laurent series are D-finite.

Example: $f(x, y) = \dfrac{1}{xy^3 + y + 1}$.

Example: $f(x,y) = \dfrac{1}{xy^3 + y + 1}$.

$$(6 + 2(27x+1)D_x + x(27x+4)D_x^2)\cdot f = D_y \cdot \mathsf{rat}(x,y)$$

Example: $f(x,y) = \dfrac{1}{xy^3 + y + 1}$.

$$(6 + 2(27x+1)D_x + x(27x+4)D_x^2)\cdot f = D_y \cdot \mathrm{rat}(x,y)$$
$$(6 + 2(27x+1)D_x + x(27x+4)D_x^2)\cdot \mathrm{res}_y\, f = 0$$

Example: $f(x,y) = \dfrac{1}{xy^3 + y + 1}$.

$$(6 + 2(27x+1)D_x + x(27x+4)D_x^2)\cdot f = D_y \cdot \mathrm{rat}(x,y)$$
$$(6 + 2(27x+1)D_x + x(27x+4)D_x^2)\cdot \mathrm{res}_y\, f = 0$$

Exercise: In general, the residue of a multivariate rational function depends on how we expand it into a multivariate Laurent series, i.e., on the choice of the halfplane H. How does creative telescoping know which H we have in mind?

Why should we care about computing residues?

Let $f(x, y) = \sum_{n,k} a_{n,k} x^n y^k$.

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$.

$\operatorname{diag} f(x,y) = \sum_{n} a_{n,n} x^n$ is called the diagonal of f.

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$.

$\operatorname{diag} f(x,y) = \sum_{n} a_{n,n} x^n$ is called the diagonal of f.

Note:

$$\operatorname{diag} f(x,y) = \operatorname{res}_y y^{-1} f(y, x/y)$$

Let $f(x, y) = \sum_{n,k} a_{n,k} x^n y^k$.

$\operatorname{diag} f(x, y) = \sum_{n} a_{n,n} x^n$ is called the diagonal of f.

Note:

$$\operatorname{diag} f(x, y) = \operatorname{res}_y y^{-1} f(y, x/y)$$

In particular, taking diagonals preserves D-finiteness.

Let $f(x, y) = \sum_{n,k} a_{n,k} x^n y^k$ and $g(x, y) = \sum_{n,k} b_{n,k} x^n y^k$.

Let $f(x, y) = \sum_{n,k} a_{n,k} x^n y^k$ and $g(x, y) = \sum_{n,k} b_{n,k} x^n y^k$.

$f \odot_{x,y} g := \sum_{n,k} a_{n,k} b_{n,k} x^n y^k$ is the Hadamard product of f and g.

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$ and $g(x,y) = \sum_{n,k} b_{n,k} x^n y^k$.

$f \odot_{x,y} g := \sum_{n,k} a_{n,k} b_{n,k} x^n y^k$ is the Hadamard product of f and g.

Note:

$$f \odot_{x,y} g = \mathrm{res}_{x'} \, \mathrm{res}_{y'} (x'y')^{-1} f(x', y') g(x/x', y/y')$$

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$ and $g(x,y) = \sum_{n,k} b_{n,k} x^n y^k$.

$f \odot_{x,y} g := \sum_{n,k} a_{n,k} b_{n,k} x^n y^k$ is the Hadamard product of f and g.

Note:

$$f \odot_{x,y} g = \mathrm{res}_{x'} \, \mathrm{res}_{y'} (x'y')^{-1} f(x', y') g(x/x', y/y')$$

In particular, taking Hadamard products preserves D-finiteness.

Let $f(x, y) = \sum_{n,k} a_{n,k} x^n y^k$.

$[x^> y^>] f(x, y) = \sum_{n,k>0} a_{n,k} x^n y^k$ is called the positive part of f.

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$.

$[x^> y^>] f(x,y) = \sum_{n,k>0} a_{n,k} x^n y^k$ is called the positive part of f.

Note:

$$[x^> y^>] f = \frac{x}{1-x} \frac{y}{1-y} \odot_{x,y} f(x,y)$$

Let $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$.

$[x^> y^>] f(x,y) = \sum_{n,k>0} a_{n,k} x^n y^k$ is called the positive part of f.

Note:

$$[x^> y^>] f = \frac{x}{1-x} \frac{y}{1-y} \odot_{x,y} f(x,y)$$

In particular, taking positive parts preserves D-finiteness.

Example: $f(x, y) = \dfrac{1}{1 - (x + y)}$

Example: $f(x, y) = \dfrac{1}{1-(x+y)}$

y^6	1	7	28	84	210	462	924
y^5	1	6	21	56	126	252	462
y^4	1	5	15	35	70	126	210
y^3	1	4	10	20	35	56	84
y^2	1	3	6	10	15	21	28
y^1	1	2	3	4	5	6	7
y^0	1	1	1	1	1	1	1
	x^0	x^1	x^2	x^3	x^4	x^5	x^6

Example: $\text{diag}\,\dfrac{1}{1-(x+y)}$

y^6	1	7	28	84	210	462	924
y^5	1	6	21	56	126	252	462
y^4	1	5	15	35	70	126	210
y^3	1	4	10	20	35	56	84
y^2	1	3	6	10	15	21	28
y^1	1	2	3	4	5	6	7
y^0	1	1	1	1	1	1	1
	x^0	x^1	x^2	x^3	x^4	x^5	x^6

Example: $[y^0]\dfrac{1}{1-(x/y+y)}$

y^6	1	8	45	220	1001	4368	18564
y^5	1	7	36	165	715	3003	12376
y^4	1	6	28	120	495	2002	8008
y^3	1	5	21	84	330	1287	5005
y^2	1	4	15	56	210	792	3003
y^1	1	3	10	35	126	462	1716
y^0	1	2	6	20	70	252	924
y^{-1}	0	1	3	10	35	126	462
y^{-2}	0	0	1	4	15	56	210
y^{-3}	0	0	0	1	5	21	84
y^{-4}	0	0	0	0	1	6	28
y^{-5}	0	0	0	0	0	1	7
y^{-6}	0	0	0	0	0	0	1
	x^0	x^1	x^2	x^3	x^4	x^5	x^6

Example: $[y^{-1}]\dfrac{1}{y}\dfrac{1}{1-(x/y+y)}$

y^6	1	9	55	286	1365	6188	27132
y^5	1	8	45	220	1001	4368	18564
y^4	1	7	36	165	715	3003	12376
y^3	1	6	28	120	495	2002	8008
y^2	1	5	21	84	330	1287	5005
y^1	1	4	15	56	210	792	3003
y^0	1	3	10	35	126	462	1716
y^{-1}	1	2	6	20	70	252	924
y^{-2}	0	1	3	10	35	126	462
y^{-3}	0	0	1	4	15	56	210
y^{-4}	0	0	0	1	5	21	84
y^{-5}	0	0	0	0	1	6	28
y^{-6}	0	0	0	0	0	1	7
	x^0	x^1	x^2	x^3	x^4	x^5	x^6

Example: $\operatorname{diag} \dfrac{1}{1-(x+y)} = \operatorname{res}_y \dfrac{1}{y} \dfrac{1}{1-(x/y+y)}$

y^6	1	9	55	286	1365	6188	27132
y^5	1	8	45	220	1001	4368	18564
y^4	1	7	36	165	715	3003	12376
y^3	1	6	28	120	495	2002	8008
y^2	1	5	21	84	330	1287	5005
y^1	1	4	15	56	210	792	3003
y^0	1	3	10	35	126	462	1716
y^{-1}	1	2	6	20	70	252	924
y^{-2}	0	1	3	10	35	126	462
y^{-3}	0	0	1	4	15	56	210
y^{-4}	0	0	0	1	5	21	84
y^{-5}	0	0	0	0	1	6	28
y^{-6}	0	0	0	0	0	1	7
	x^0	x^1	x^2	x^3	x^4	x^5	x^6

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	efficient algorithms

Can we combine the best of both worlds?

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	efficient algorithms

Can we combine the best of both worlds?

Yes, for formal series in the differential case.

Holonomy	D-finiteness
elegant theory	trouble with existence and singularities
expensive computations	efficient algorithms

Can we combine the best of both worlds?

Yes, for formal series in the differential case.

What about summation?

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Facts:

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Facts:

- f is hypergeometric but not proper hypergeometric.

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Facts:

- f is hypergeometric but not proper hypergeometric.
- f is D-finite but not holonomic.

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Facts:

- f is hypergeometric but not proper hypergeometric.
- f is D-finite but not holonomic.
- There is no telescoper/certificate pair for f.

Example: $f(n,k) = \dfrac{1}{n^2 + k^2}$

Facts:

- f is hypergeometric but not proper hypergeometric.
- f is D-finite but not holonomic.
- There is no telescoper/certificate pair for f.

Not every D-finite sequence has a telscoper/certificate pair.

Example: $f(n,k) = \binom{n}{k}^2$

Example: $f(n,k) = \binom{n}{k}^2$

$$\Big((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\Big) \cdot f(n,k) = 0$$

Example: $f(n,k) = \binom{n}{k}^2$

$$\left((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\right) \cdot f(n,k) = 0$$

Example: $f(n,k) = \binom{n}{k}^2$

$$\left((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\right) \cdot f(n,k) = 0$$

Who has the courage to sum this equation for $k = 0, \ldots, n$?

Example: $f(n,k) = \binom{n}{k}^2$

$$\Big((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\Big) \cdot f(n,k) = 0$$

Who has the courage to sum this equation for $k = 0, \ldots, n$?

Singularities in the certificate must be inspected by hand.

Example: $f(n,k) = \binom{n}{k}^2$

$$\left((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\right) \cdot f(n,k) = 0$$

Who has the courage to sum this equation for $k = 0, \dots, n$?

Singularities in the certificate must be inspected by hand.

This is bad news for friends of reduction-based algorithms.

Example: $f(n,k) = \binom{n}{k}^2$

$$\Big((n+1)S_n - 2(2n+1) - \Delta_k \frac{k^2(2k-3n-3)}{(n-1-k)^2}\Big) \cdot f(n,k) = 0$$

Who has the courage to sum this equation for $k = 0, \dots, n$?

Singularities in the certificate must be inspected by hand.

This is bad news for friends of reduction-based algorithms.

But there is good news, too.

Journal of Symbolic Computation 80 (2017) 351–386

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

ELSEVIER

Multiple binomial sums

CrossMark

Alin Bostan [a], Pierre Lairez [b], Bruno Salvy [c]

[a] Inria Saclay Île-de-France, Bâtiment Alan Turing, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France
[b] Technische Universität Berlin, Fakultät II, Sekretariat 3-2, Straße des 17. Juni 136, 10623 Berlin, Germany
[c] Inria, LIP (U. Lyon, CNRS, ENS Lyon, UCBL), France

ARTICLE INFO

Article history:
Received 26 October 2015
Accepted 2 April 2016
Available online 23 June 2016

MSC:
05A10

ABSTRACT

Multiple binomial sums form a large class of multi-indexed sequences, closed under partial summation, which contains most of the sequences obtained by multiple summation of products of binomial coefficients and also all the sequences with algebraic generating function. We study the representation of the generating functions of binomial sums by integrals of rational functions. The

Idea: Reduce definite summation to residue extraction.

Idea: Reduce definite summation to residue extraction.

Note:

Idea: Reduce definite summation to residue extraction.

Note:

- $\displaystyle\sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$

Idea: Reduce definite summation to residue extraction.

Note:

- $\displaystyle\sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$
- $\displaystyle\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$

Idea: Reduce definite summation to residue extraction.

Note:

- $$\sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$$
- $$\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$$
- $$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \right) x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

Idea: Reduce definite summation to residue extraction.

Note:

- $\displaystyle\sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$
- $\displaystyle\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$
- $\displaystyle\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k\right) x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$
- $\displaystyle\sum_{n,k=0}^{\infty} a_n b_k x^n y^k = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{k=0}^{\infty} b_k y^k\right)$

Idea: Reduce definite summation to residue extraction.

Using these and similar formulas, translate a given expression into a multivariate rational generating function.

Idea: Reduce definite summation to residue extraction.

Using these and similar formulas, translate a given expression into a multivariate rational generating function.

During this translation, make sums indefinite by introducing new variables.

Idea: Reduce definite summation to residue extraction.

Using these and similar formulas, translate a given expression into a multivariate rational generating function.

During this translation, make sums indefinite by introducing new variables.

In the end, identify variables as needed.

Simple example: $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$

Simple example: $\displaystyle\sum_{k=0}^{n} \boxed{\binom{n}{k}}$

Simple example: $\sum_{k=0}^{n} \boxed{\binom{n}{k}}$

$$\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$$

Simple example: $\sum_{k=0}^{n} \boxed{\binom{n}{k}}$

$$\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$$

Simple example: $\sum_{k=0}^{m} \boxed{\binom{n}{k}}$

$$\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$$

Simple example: $\boxed{\sum_{k=0}^{m} \boxed{\binom{n}{k}}}$

$$\sum_{n,k=0}^{\infty} \binom{n}{k} x^n y^k = \frac{1}{1-(1+y)x}$$

Simple example: $\boxed{\sum_{k=0}^{m} \boxed{\binom{n}{k}}}$

$$\sum_{n,m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{n}{k} \right) x^n z^m = \frac{1}{1-z} \frac{1}{1-(1+z)x}$$

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}}$

$$\sum_{n,m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{n}{k} \right) x^n z^m = \frac{1}{1-z} \frac{1}{1-(1+z)x}$$

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}}$

$$\operatorname{diag} \frac{1}{1-z} \frac{1}{1-(1+z)x}$$

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}}$

$$\operatorname{diag} \frac{1}{1-z}\frac{1}{1-(1+z)x} = \frac{1}{1-2x}$$

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}} = 2^n$

$$\operatorname{diag} \frac{1}{1-z} \frac{1}{1-(1+z)x} = \frac{1}{1-2x}$$

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}} = 2^n$

$$\operatorname{diag} \frac{1}{1-z} \frac{1}{1-(1+z)x} = \frac{1}{1-2x}$$

Expressions that can be handled this way are called binomial sums.

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}} = 2^n$

$$\operatorname{diag} \frac{1}{1-z} \frac{1}{1-(1+z)x} = \frac{1}{1-2x}$$

Expressions that can be handled this way are called binomial sums.

Theorem: Binomial sums are D-finite.

Simple example: $\boxed{\sum_{k=0}^{n} \boxed{\binom{n}{k}}} = 2^n$

$$\operatorname{diag} \frac{1}{1-z} \frac{1}{1-(1+z)x} = \frac{1}{1-2x}$$

Expressions that can be handled this way are called binomial sums.

Theorem: Binomial sums are D-finite.

Note: There is no trouble with singularities.

Summary.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.
- Therefore, holonomy is preserved under evaluation and definite summation and integration.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.
- Therefore, holonomy is preserved under evaluation and definite summation and integration.
- Integration ranges can be any semialgebraic sets, summation ranges can be any rational polygons.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.
- Therefore, holonomy is preserved under evaluation and definite summation and integration.
- Integration ranges can be any semialgebraic sets, summation ranges can be any rational polygons.
- D-finiteness is preserved under residue, diagonal, Hadamard product, and positive part.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.
- Therefore, holonomy is preserved under evaluation and definite summation and integration.
- Integration ranges can be any semialgebraic sets, summation ranges can be any rational polygons.
- D-finiteness is preserved under residue, diagonal, Hadamard product, and positive part.
- In the shift case, D-finite ideals may not contain telescoper/certificate pairs.

Summary.

- Every holonomic ideal contains a telescoper/certificate pair.
- Therefore, holonomy is preserved under evaluation and definite summation and integration.
- Integration ranges can be any semialgebraic sets, summation ranges can be any rational polygons.
- D-finiteness is preserved under residue, diagonal, Hadamard product, and positive part.
- In the shift case, D-finite ideals may not contain telescoper/certificate pairs.
- Nevertheless, at least binomial sums are always D-finite.