Conclusion

Manuel Kauers · Institute for Algebra · JKU

Monday	Tuesday	Wednesday	Thursday	Friday
Intro-duction	Binomial Summation	Creative Telescoping	D-finite univariate	Chyzak's algorithm
Rational Integration Theory	Sister Celine Theory	Gosper's algorithm	D-finite multivariate	Example Session
Rational Integration Coding	Sister Celine Coding	Zeilberger's algorithm	Advanced Closure Properties	Conclusion

Some points to remember:

- What is a telescoper?
- What is it good for?
- How can it be computed?

What did we not cover in this course?

- Liouvillean functions and $\Pi\Sigma$ expressions
- Reduction-based creative telescoping for D-finite functions

Liouvillean functions and $\Pi\Sigma$ expressions

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If K is a field, a function $D\colon K \to K$ is called a derivation if

$$D(a + b) = D(a) + D(b) \quad \text{and} \quad D(ab) = D(a)b + aD(b)$$

for all $a, b \in K$.

Liouvillean functions and ΠΣ expressions

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The field K together with such a D is called a differential field.

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Idea: interpret *complicated expressions* as elements of a suitably constructed differential field.

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Fact: On a rational function field $K = \mathbb{Q}(t_1, t_2, \dots, t_d)$ a derivation D is uniquely determined by $D(t_i)$ for $i = 1, \dots, d$.

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Example: On $K = \mathbb{Q}(t_1, t_2, t_3, t_4)$ we can define a derivation via

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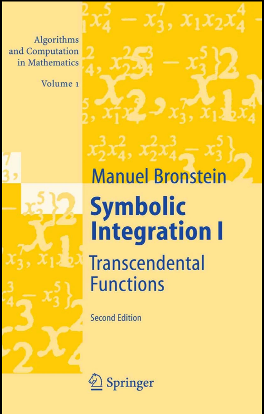
Example: $$\int \frac{1}{1 + \exp(x)} = x - \log(1 + \exp(x))$$

Liouvillean functions and ΠΣ expressions



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- Given: $f_1, \dots, f_r \in K$
- Find: $c_1, \dots, c_r \in C$ and $g \in K$ such that

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or prove that no such things exist.

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Looks familiar?

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We can also use this for evaluating definite integrals of liouvillean functions.

Liouvillean functions and $\Pi\Sigma$ expressions

	hypergeometric summation	liouvillean integration
indefinite	Gosper	Risch
definite (CT)	Zeilberger	Raab

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There is also a summation analog of all this.

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Example: $\displaystyle\sum_{k=1}^{n} \frac{2^k - \sum_{i=1}^{k} \frac{1}{k}}{k! + \sum_{i=1}^{k} \frac{1}{k^2}}$

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Example: $$\sum_{k=1}^{n} \frac{2^k - \sum_{i=1}^{k} \frac{1}{k}}{k! + \sum_{i=1}^{k} \frac{1}{k^2}}$$

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A difference field is a field K together with an automorphism $\sigma\colon K \to K$.

Example: On $K = \mathbb{Q}(t_1, t_2, \dots)$ we can define σ via

$$
\begin{aligned}
\sigma(t_1) &= t_1 + 1 & t_1 &\sim n \\
\sigma(t_2) &= 2t_2 & t_2 &\sim 2^n \\
\sigma(t_3) &= (t_1 + 1)t_3 & t_3 &\sim n! \\
\sigma(t_4) &= t_4 + \frac{1}{t_1 + 1} & t_4 &\sim \sum_{k=1}^{n} \frac{1}{k}, \quad \text{etc.}
\end{aligned}
$$

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Def. A difference field $K = C(t_1, \ldots, t_d)$ is called ΠΣ if the difference subfield $C(t_1, \ldots, t_{d-1})$ is ΠΣ and

- $\sigma(t_d) - t_d \in C(t_1, \ldots, t_{d-1})$ ("t_d is a sum"), or
- $\sigma(t_d)/t_d \in C(t_1, \ldots, t_{d-1})$ ("t_d is a product"),

and $\sigma(r) = r \Leftrightarrow r \in C$ for all $r \in K$.

Liouvillean functions and $\Pi\Sigma$ expressions

Def. A difference field $K = C(t_1, \ldots, t_d)$ is called $\Pi\Sigma$ if the difference subfield $C(t_1, \ldots, t_{d-1})$ is $\Pi\Sigma$ and

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Karr's algorithm solves the summation problem in such fields:

- Given a $\Pi\Sigma$ field K and an element $f \in K$
- Construct a $\Pi\Sigma$ field E with $K \subseteq E$ and an element $g \in E$ such that $\sigma(g) - g = f$, or prove that no such E exists.

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Karr's algorithm solves the summation problem in such fields:

- Given a ΠΣ field K and an element $f \in K$
- Construct a ΠΣ field E with $K \subseteq E$ and an element $g \in E$ such that $\sigma(g) - g = f$, or prove that no such E exists.

Example: $\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{k} = (n+1) \sum_{k=1}^{n} \frac{1}{k} - n$

Liouvillean functions and $\Pi\Sigma$ expressions

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Schneider uses it to do creative telescoping and lots of other things.

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	hypergeometric summation	liouvillean integration	ΠΣ summation
indefinite	Gosper	Risch	Karr
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Reduction-based creative telescoping for D-finite functions

Recall:

- Celine-like algorithms are based on elimination ("k-free recurrence")
- Zeilberger-like algorithms are based on an indefinite summation/integration algorithm
- Apagodu-Zeilberger-like algorithms are based on an ansatz for telescoper and certificate and solving a linear system
- Reduction-based algorithms are based on extracting maximal summable/integrable parts

Reduction-based creative telescoping for D-finite functions

Example: Hermite reduction breaks a given $f \in C(x, y)$ into

$$f = D(g) + h$$

where h is minimal in a certain sense.

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- Hypergeometric terms
  (Abramov, Petkovšek; Chen, Huang, Li, Kauers)

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- Hypergeometric terms
  (Abramov, Petkovšek; Chen, Huang, Li, Kauers)
- D-finite functions
  (Bostan, Brochet, Chen, Du, van Hoeij, van der Hoeven, Lairez, Kauers, Koutschan, Salvy, Wang)

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Example:

$$\sum_{k=1}^{n-1} \frac{k^4 + 5k^3 - k^2 - 5k - 2}{k^2(k+1)^3(k+2)} \binom{2k}{k}$$
$$= -\frac{n^2 - 6}{6n^2(n+1)} \binom{2n}{n} + \frac{5}{6} + \sum_{k=1}^{n-1} \frac{1}{2(k+1)} \binom{2k}{k}.$$

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We obtain reduction-based creative telescoping algorithms.

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We obtain reduction-based creative telescoping algorithms.

These techniques are still subject of ongoing research.

What did we not cover in this course?

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What remains to be done in the future?

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Some Open Problems Related to Creative Telescoping*

CHEN Shaoshi · KAUERS Manuel

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Abstract Creative telescoping is the method of choice for obtaining information about definite sums or integrals. It has been intensively studied since the early 1990s, and can now be considered as a classical technique in computer algebra. At the same time, it is still a subject of ongoing research. This paper presents a selection of open problems in this context. The authors would be curious to hear about any substantial progress on any of these problems.

Keywords Computer algebra, creative telescoping, differential algebra, linear operators, ore algebras, symbolic integration, symbolic summation.

1 Introduction

Summation and integration problems arise in all areas of mathematics, especially in discrete mathematics, special functions, combinatorics, engineering, and physics. Nowadays, many of these problems are solved using computer algebra. The number of applications of summation

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- Bounds on the sizes of telescopers and certificates.
- Predict singularities of telescopers without computing them.
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- Existence of telescopers for differential/difference fields.

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- Software that can handle problems out of reach of available code.