Hermite Reduction for D-finite Functions via Integral Bases

Shaoshi Chen[a,b], Lixin Du[c], Manuel Kauers[c]
[a]KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[b]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[c]Institute for Algebra, Johannes Kepler University, Linz, A4040, Austria
schen@amss.ac.cn,lixin.du@jku.at,manuel.kauers@jku.at

ABSTRACT

Trager's Hermite reduction solves the integration problem for algebraic functions via integral bases. A generalization of this algorithm to D-finite functions has so far been limited to the Fuchsian case. In the present paper, we remove this restriction and propose a reduction algorithm based on integral bases that is applicable to arbitrary D-finite functions.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Additive decomposition; creative telescoping; symbolic integration

ACM Reference Format:
Shaoshi Chen[a,b], Lixin Du[c], Manuel Kauers[c]. 2023. Hermite Reduction for D-finite Functions via Integral Bases. In *International Symposium on Symbolic and Algebraic Computation 2023 (ISSAC 2023), July 24–27, 2023, Tromsø, Norway.* ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/3597066.3597092

1 INTRODUCTION

Let R be a certain class of functions in one variable x with the derivation D_x. For example, R can be the field of rational functions or algebraic functions. In the context of symbolic integration, the *integrability problem* consists in deciding whether a given element $f \in R$ is of the form $f = D_x(g)$ for some $g \in R$. If such a g exists, we say that f is *integrable* in R. A relaxed form of the integrability problem is the *decomposition problem,* which consists in constructing for a given $f \in R$ elements $g, r \in R$ such that $f = D_x(g) + r$ and r is minimal in a certain sense. Ideally the "certain sense" should be such that $r = 0$ whenever f is integrable. If $f \in R$ depends on a second variable t, one can also consider the *creative telescoping* problem: given an element $f \in R$, the task is to construct $c_0, \ldots, c_r \in R$, not all zero, such that c_i is free of x for all $i \in \{0, \ldots, r\}$ and

$$c_r D_t^r(f) + \cdots + c_0 f = D_x(g) \quad \text{for some } g \in R.$$

The operator $L = c_r D_t^r + \cdots + c_0$, if it exists, is called a *telescoper* for f, and g is called a *certificate* for L.

Zeilberger first showed the existence of telescopers for D-finite functions [22]. Almkvist and Zeilberger [4] solved the integrability problem and the creative telscoping problem for hyperexponential functions. Using the adjoint Ore algebra, Abramov and van Hoeij [2] solved the accurate integration problem for D-finite functions. Chyzak [12] extended the method of creative telescoping from hyperexponential functions to general D-finite functions. During the past ten years, a reduction-based telescoping approach has become popular, which can find a telescoper without computing the corresponding certificate. This approach was first formulated for rational functions [5] and later extended to hyperexponential functions [6], algebraic functions [10], Fuchsian D-finite functions [11] and D-finite functions [7, 21]. The reduction-based telescoping algorithms for algebraic functions and for Fuchsian D-finite functions employ the notion of integral bases, while the known reduction-based telescoping algorithms applicable to arbitrary D-finite functions work differently.

The notion of integrality proposed by Kauers and Koutschan [17] for Fuchsian D-finite functions has recently been generalized by Aldossari [3] to arbitrary D-finite functions, so that the question arises whether there is also a reduction-based telescoping algorithm for arbitrary D-finite functions based on integral bases. The purpose of the present paper is to answer this question affirmatively. This paper is based on the results of Chapter 6 of the second author's Ph.D. thesis [13].

2 INTEGRAL BASES

Below we recall the value functions and integral bases for arbitrary linear differential operators [3, 14, 15, 17]. Let C be a field of characteristic zero and $\bar{C}$ be the algebraic closure of C. Let $C(x)[D]$ be an Ore algebra, where D is the differentiation with respect to x and satisfies the commutation rule $Dx = xD + 1$. For an operator $L = \ell_0 + \ell_1 D + \cdots + \ell_n D^n \in C(x)[D]$ with $\ell_n \neq 0$, we consider the left $C(x)[D]$-module $A = C(x)[D]/\langle L \rangle$, where $\langle L \rangle = C(x)[D]L$. We call the elements of A "functions", even though they are not functions in the usual sense. This is fair because A is isomorphic to a $C(x)[D]$-module containing actual functions. When there is no ambiguity, an equivalence class $f + \langle L \rangle$ in A is also denoted by f. Every element of A can be uniquely represented by $f = f_0 + f_1 D + \cdots + f_{n-1} D^{n-1}$ with $f_i \in C(x)$.

S. Chen was partially supported by the NSFC grants (No. 12271511 and No. 11688101), CAS Project for Young Scientists in Basic Research (Grant No. YSBR-034), the Fund of the Youth Innovation Promotion Association (Grant No. Y2022001), CAS, and the National Key Research and Development Project 2020YFA0712300. L. Du was supported by the Austrian FWF grant P31571-N32. M. Kauers was supported by the Austrian FWF grants P31571-N32 and I6130-N.

ISSAC 2023, July 24–27, 2023, Tromsø, Norway

ACM ISBN 979-8-4007-0039-2/23/07.
https://doi.org/10.1145/3597066.3597092

For each $\alpha \in \bar{C}$, an operator L of order n admits n linearly independent solutions of the form

$$(x-\alpha)^{\mu} \exp(p((x-\alpha)^{-1/s}))b((x-\alpha)^{1/s}, \log(x-\alpha)) \quad (1)$$

for some $s \in \mathbb{N}$, $\mu \in \bar{C}$, $p \in \bar{C}[x]$ and $b \in \bar{C}[[x]][y]$. Such objects are called generalized series solutions at α, see [16, 18]. For $\alpha = \infty$, the operator L admits n linearly independent solutions of the form

$$x^{-\mu} \exp(p(x^{1/s}))b(x^{-1/s}, \log(x^{-1})) \quad (2)$$

for some $s \in \mathbb{N}$, $\mu \in \bar{C}$, $p \in \bar{C}[x]$ and $b \in \bar{C}[[x]][y]$. For each $\alpha \in \bar{C} \cup \{\infty\}$, let $\mathrm{Sol}_\alpha(L)$ be the set of all finite $\bar{C}$-linear combinations of generalized series solutions of L at α. It is called the *solution space* of L. Then $\mathrm{Sol}_\alpha(L)$ is a $\bar{C}$-vector space of dimension n. Throughout the paper, we assume that for each $\alpha \in \bar{C} \cup \{\infty\}$, all series of $\mathrm{Sol}_\alpha(L)$ have $p \in C[x]$, $\mu \in C$ and $b \in C[[x]][y]$ (this can always be achieved by a suitable choice of C). If all series of $\mathrm{Sol}_\alpha(L)$ have $p = 0$ and $s = 1$, then L is called *Fuchsian* at α. The operator L is simply called *Fuchsian* if it is Fuchsian at all $\alpha \in \bar{C} \cup \{\infty\}$.

For simplicity, we assume throughout that C is a subfield of $\mathbb{C}$. If this is not the case, define the valuation as in [17]. Given two complex numbers $a, b \in \mathbb{C}$, we say $a \geq b$ if and only if $\mathrm{Re}(a) \geq \mathrm{Re}(b)$. For each $\alpha \in \bar{C} \cup \{\infty\}$, let $z = x - \alpha$ (or $z = \frac{1}{x}$ if $\alpha = \infty$). The *valuation* $\nu_\alpha(t)$ of a term $t := z^r \exp(p(z^{-1/s})) \log(z)^\ell$ is the real part of the local exponent r. The *valuation* $\nu_\alpha(y)$ of a nonzero generalized series y at α is the minimum of the valuations of all the terms appearing in y (with nonzero coefficients). The *valuation* of 0 is defined as ∞. A generalized series y at α is called *integral* if $\nu_\alpha(y) \geq 0$. Note that in this terminology, it may happen that $\nu_\alpha(y') < \nu_\alpha(y) - 1$. For example, $f = \exp(x^{-2})$ is integral at 0, while the valuation of $y' = -2x^{-3}\exp(x^{-2})$ at 0 is -3, not -1. The valuation of a series only depends on its local exponent and not on its exponential part. An intuition of defining the valuation in such a way is that this valuation is affected by differentiation, while the exponential part keeps unchanged. This valuation is the same as in [3, Definition 5.4].

An operator $P = p_0 + p_1 D + \cdots + p_{n-1}D^{n-1}$ in $C(x)[D]$ acts on a generalized series y via

$$P \cdot y = p_0 y + p_1 y' + \cdots + p_{n-1} y^{(n-1)},$$

where $'$ is the derivation with respect to x. Let $y_1, \ldots, y_n$ be a basis of $\mathrm{Sol}_\alpha(L)$ in the form of (1) (or (2) if $\alpha = \infty$). This is a special basis. The *value function* $\mathrm{val}_\alpha \colon A \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\mathrm{val}_\alpha(f) := \min_{i=1}^{n} \nu_\alpha(P_f \cdot y_i),$$

where $f = P_f + \langle L \rangle$. Then $\mathrm{val}_\alpha(f)$ is the minimum valuation of all series $P_f \cdot y$ at α, where y runs through all series solutions in $\mathrm{Sol}_\alpha(L)$. An element $f \in A$ is called *(locally) integral* at $\alpha \in \bar{C} \cup \{\infty\}$ if $\mathrm{val}_\alpha(f) \geq 0$. An element $f \in A$ is called *(globally) integral* if $\mathrm{val}_\alpha(f) \geq 0$ for all $\alpha \in \bar{C}$, i.e., f is locally integral at all finite places. Although we drop L from the notation of operators in A, everything depends on L. When L is Fuchsian, this notion of integrality falls back to the Fuchsian case discussed in [11, 17].

The set of all globally integral elements $f \in A = C(x)[D]/\langle L \rangle$ forms a $C[x]$-module. A basis of this module is called a *(global) integral basis* for A. Such bases exist and the algorithm for computing integral bases in the Fuchsian case [17] applies to the setting of the non-Fuchsian case literally. More properties can be found in [3].

For a fixed $\alpha \in \bar{C}$, the *valuation* ν_α of a nonzero rational function $f \in C(x)$ is an integer $m \in \mathbb{Z}$ such that $f = (x-\alpha)^m p/q$ with $p, q \in C[x]$, $\gcd(p, q) = 1$ and $(x - \alpha) \nmid pq$. By convention, set $\nu_\alpha(0) = \infty$. The *valuation* ν_∞ of a rational function $f = p/q \in C(x)$ is $\deg_x(q) - \deg_x(p)$. For each $\alpha \in \bar{C} \cup \{\infty\}$, the valuation ν_α of a rational function is the same as the valuation of its Laurent series expansion at α. The set $C(x)_\alpha = \{f \in C(x) \mid \nu_\alpha(f) \geq 0\}$ forms a subring of $C(x)$. The set of all elements $f \in A$ that are locally integral at some fixed $\alpha \in \bar{C} \cup \{\infty\}$ forms a $C(x)_\alpha$-module. A basis of this module is called a *local integral basis* at α of A. Such a basis can also be computed [3, 17]. We write $\mathcal{O}_\alpha$ for the set of all elements in $\bar{C}(x)[D]/\langle L \rangle$ that are locally integral at $\alpha \in \bar{C} \cup \{\infty\}$.

For a rational function $g \in C(x)$ and any $\alpha \in \bar{C}$, if $\nu_\alpha(g) \neq 0$, we have $\nu_\alpha(g') = \nu_\alpha(g) - 1$. So the valuation of a rational function decreases by exactly one under each derivation. In the D-finite case, the valuation decreases by at least one. A lower bound of its valuation under each derivation is given in [3, Lemma 5.7].

Lemma 1. *Let $g \in A$. For any $\alpha \in \bar{C} \cup \{\infty\}$, if $\mathrm{val}_\alpha(g) \neq 0$, then $\mathrm{val}_\alpha(g') \leq \mathrm{val}_\alpha(g) + \mu$, where $\mu = -1$ if $\alpha \in \bar{C}$ and $\mu = 1$ if $\alpha = \infty$.*

Proof. Let y_i be a generalized series solution in $\mathrm{Sol}_\alpha(L)$ such that $\mathrm{val}_\alpha(g) = \nu_\alpha(g \cdot y_i)$. Let $z = x - \alpha$ (or $z = 1/x$ if $\alpha = \infty$). Let $T = z^r \exp(p(z^{-1/s})) \log(z)^\ell$ with $r \neq 0$, $s, \ell \in \mathbb{N}$, $p \in \bar{C}[x]$ be the dominant term of $g \cdot y_i$, i.e., among all terms with minimal r the one with the largest exponent ℓ. Let $k = \deg_x(p)$ and $c = \mathrm{lc}_x(p)$. Then

$$\begin{aligned} D \cdot T = &-r\mu z^{r+\mu} \exp(p(z^{-1/s})) \log(z)^\ell \\ &+ \frac{\mu c k}{s} z^{r - \frac{k}{s} + \mu} \exp(p(z^{-1/s})) \log(z)^\ell + \cdots \\ &- \ell\mu z^{r+\mu} \exp(p(z^{-1/s})) \log(z)^{\ell-1}. \end{aligned}$$

where "$\cdots$" denotes some terms of valuation higher than $r - \frac{k}{s} - 1$. Note that $r - \frac{k}{s} + \mu \leq r + \mu$ and $r \neq 0$ (by the assumption $\mathrm{val}_\alpha(g) \neq 0$). So the valuation of the term $D \cdot T$ in $g' \cdot y_i$ is less than or equal to $r + \mu$, which implies that $\mathrm{val}_\alpha(g') \leq \mathrm{val}_\alpha(g) + \mu$. ∎

An integral basis $\{\omega_1, \ldots, \omega_n\}$ is always a vector space basis of A. Writing an element $f \in A$ as a combination $f = \sum_{i=1}^n f_i \omega_i$ for some $f_i \in C(x)$, we obtain that f has a negative valuation at $\alpha \in \bar{C}$ if and only if at least one of the f_i has a pole there. Furthermore, $\lfloor \mathrm{val}_\alpha(f) \rfloor$ is a lower bound for the valuations of all the f_i's at α.

Lemma 2. *Let $\{\omega_1, \ldots, \omega_n\}$ be a local integral basis of A at some fixed $\alpha \in \bar{C} \cup \{\infty\}$. Let $f \in A$ and $f_1, \ldots, f_n \in C(x)$ be such that $f = \sum_{i=1}^n f_i \omega_i$. Then*

(1) f is integral at α if and only if for each $i \in \{1, \ldots, n\}$, $f_i \omega_i$ is integral at α.
(2) $\lfloor \mathrm{val}_\alpha(f) \rfloor = \min_{i=1}^n \nu_\alpha(f_i)$.

Proof. (1): The direction "⇐" is obvious. To show "⇒", suppose that f is integral at α. Then there exist $\tilde{f}_1, \ldots, \tilde{f}_n \in C(x)_\alpha$ such that $f = \sum_{i=1}^n \tilde{f}_i \omega_i$. Thus $\sum_{i=1}^n (\tilde{f}_i - f_i)\omega_i = 0$, and then $\tilde{f}_i = f_i$ for all i, because $\{\omega_1, \ldots, \omega_n\}$ is a $C(x)$-vector space basis of A. Since $\tilde{f}_i = f_i \in C(x)_\alpha$, it follows that the $f_i\omega_i$'s are integral at α.

(2): Let $\tau := \min_{i=1}^n \nu_\alpha(f_i)$. We have to show that $\tau \in \mathbb{Z}$ is an integer such that

$$\tau \leq \mathrm{val}_\alpha(f) < \tau + 1.$$

Let $z \in \bar{C}(x)$ with $\nu_\alpha(z) = 1$. Since $z^{-\tau} f_i \omega_i$ is integral at α, we have $z^{-\tau} f$ is integral at α. Thus $\mathrm{val}_\alpha(z^{-\tau} f) = \mathrm{val}_\alpha(f) - \tau \geq 0$, which implies $\tau \leq \mathrm{val}_\alpha(f)$. On the other hand, if $\mathrm{val}_\alpha(f) \geq \tau + 1$, then $z^{-(\tau+1)} f$ is integral at α. But $z^{-(\tau+1)} f$ does not belong to the $C(x)_\alpha$-module generated by $\{\omega_1, \ldots, \omega_n\}$ because there is $i \in \{1, \ldots, n\}$ such that $\tau = \nu_\alpha(f_i)$ and $z^{-(\tau+1)} f_i \notin C(x)_\alpha$. This contradicts the fact that $\{\omega_1, \ldots, \omega_n\}$ is a local integral basis at α. ■

Let $W = (\omega_1, \ldots, \omega_n)$ be a vector space basis of A over $C(x)$. For $f \in A$, denote its derivative Df by f'. Let $e \in C[x]$ be a polynomial and $M = (m_{i,j})_{i,j=1}^n \in C[x]^{n \times n}$ be a matrix such that $eW' = MW$ and $\gcd(e, m_{1,1}, m_{1,2}, \ldots, m_{n,n}) = 1$. If W is an integral basis and L is Fuchsian at all finite places, then e must be squarefree, see [11, Lemma 3]. If W is a local integral basis at infinity and L is Fuchsian at infinity, then $\deg_x(m_{i,j}) < \deg_x(e)$ for all i, j, see [11, Lemma 4]. However, these two facts are no longer true in the non-Fuchsian case, as the following examples show:

Example 3. *The operator $L = x^3 D^2 + (3x^2 + 2)D \in \mathbb{C}(x)[D]$ has only one singular point 0 in $\mathbb{C}$, which is an irregular singular point. At the point 0, there are two linearly independent solutions $y_1(x) = 1$ and $y_2(x) = \exp(x^{-2})$ in $\mathrm{Sol}_0(L)$. An integral basis for $A = \mathbb{C}(x)[D]/\langle L \rangle$ is given by $\omega_1 = 1$ and $\omega_2 = x^3 D$, which is also a local integral basis at infinity. Then*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \frac{1}{e} \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

with $e = x^3$. In this example, e is not squarefree.

Example 4. *Let $L = xD^2 - (3x^3 + 2)D \in \mathbb{C}(x)[D]$. Infinity is an irregular singular point. There are two linearly independent solutions $y_1(x) = 1$ and $y_2(x) = \exp(x^3)$ in $\mathrm{Sol}_\infty(L)$. A local integral basis at infinity of $A = \mathbb{C}(x)[D]/\langle L \rangle$ is given by $\omega_1 = 1$ and $\omega_2 = x^{-2} D$. Then*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} 0 & x^2 \\ 0 & 3x^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

In this example, $e = 1$ and the condition $\deg_x(m_{i,j}) < \deg_x(e)$ fails.

A $C(x)$-vector space basis $\{\omega_1, \ldots, \omega_n\}$ of $A = C(x)[D]/\langle L \rangle$ is called *normal* at $\alpha \in \bar{C} \cup \{\infty\}$ if there exist $r_1, \ldots, r_n \in C(x)$ such that $\{r_1 \omega_1, \ldots, r_n \omega_n\}$ is a local integral basis at α. Given an integral basis and a local integral basis at infinity, Trager [20] presented an algorithm for computing an integral basis that is normal at infinity in the algebraic function field. The same procedure also applies in the present situation, see [3, Algorithm 5.20].

3 HERMITE REDUCTION

Hermite reduction, first introduced by Ostrogradsky in 1845 [19], is a classical symbolic integration technique that reduces rational functions to integrands with only simple poles. Hermite reduction was extended by Trager [20] from the field of rational functions to that of algebraic functions via integral bases. Trager's Hermite reduction solved the integration problem for algebraic functions. This work was extended to the case of Fuchsian D-finite functions [11]. We shall further extend Hermite reduction to general D-finite functions, including the non-Fuchsian case. To increase the valuation at infinity, we develop a Hermite reduction at infinity for D-finite functions, which plays the same role as polynomial reduction [6, 8, 10, 11]. This approach was suggested by one of the anonymous referees of [11]. In this section, Hermite reduction at finite places and at infinity are formulated in the same framework. More precisely, we shall use a local integral basis at $\alpha \in \bar{C} \cup \{\infty\}$ to increase the valuation of D-finite functions at α.

For convenience, we introduce some notation for the valuations of a matrix with rational coefficients. For each $\alpha \in \bar{C} \cup \{\infty\}$, the *valuation* of a matrix $T \in C(x)^{n \times n}$ at α, denoted by $\nu_\alpha(T)$, is defined as the minimal valuation at α of all entries in this matrix. The *degree* of $T \in C(x)^{n \times n}$, denoted by $\deg_x(T)$, is defined as $-\nu_\infty(T)$. In particular, the degree of a rational function $f = p/q \in C(x)$ is $\deg_x(p) - \deg_x(q)$.

3.1 The Local Case

Let $L \in C(x)[D]$ be of order n and let $A = C(x)[D]/\langle L \rangle$. For an arbitrary but fixed point $\alpha \in \bar{C} \cup \{\infty\}$, let $W = (\omega_1, \ldots, \omega_n)$ be a local integral basis at α of A. Then there exists a matrix $T \in C(x)^{n \times n}$ such that $W' = TW$. We write $z = x - \alpha$ (or $z = \frac{1}{x}$ if $\alpha = \infty$). Let $\lambda = -\nu_\alpha(T)$. Then $\lambda \in \mathbb{Z}$ and there exists a matrix $M = (m_{i,j})_{i,j=1}^n \in \bar{C}(x)_\alpha^{n \times n}$ such that

$$W' = \frac{1}{z^\lambda} MW \quad \text{and} \quad \nu_\alpha(M) = 0,$$

where $M = z^\lambda T$. Let $f = \frac{1}{z^k} \sum_{i=1}^n a_i \omega_i \in A$ with $k > 1$ (or $k \geq 0$ if $\alpha = \infty$) and $a_1, \ldots, a_n \in \bar{C}(x)_\alpha$. In order to reduce the multiplicity k of the denominator of f at α, we seek $b_1, \ldots, b_n, c_1, \ldots, c_n \in \bar{C}(x)_\alpha$ such that

$$\frac{1}{z^k} \sum_{i=1}^n a_i \omega_i = \left(\frac{1}{z^{k+\mu}} \sum_{i=1}^n b_i \omega_i \right)' + \frac{1}{z^{k-1}} \sum_{i=1}^n c_i \omega_i, \tag{3}$$

where $\mu \in \mathbb{Z}$ is an integer such that $\nu_\alpha(z') = \nu_\alpha(z) + \mu$. In this setting, $\mu = -1$ if $\alpha \in \bar{C}$ (because $(x - \alpha)' = 1$); $\mu = 1$ if $\alpha = \infty$ (because $(\frac{1}{x})' = -\frac{1}{x^2}$). Also $z' = -\mu z^{\mu+1}$.

After expanding the derivative in (3) and multiplying by z^k, we get

$$\sum_{i=1}^n a_i \omega_i = \sum_{i=1}^n \left(\frac{b_i'}{z^\mu} \omega_i + b_i z^k \left(\frac{\omega_i}{z^{k+\mu}} \right)' + c_i z \omega_i \right) \tag{4}$$

$$= \sum_{i=1}^n \left(\frac{b_i'}{z^\mu} \omega_i + \frac{b_i}{z^{\lambda+\mu}} \sum_{j=1}^n m_{i,j} \omega_j + \mu(k+\mu) b_i \omega_i + c_i z \omega_i \right), \tag{5}$$

where $\mu(k + \mu) = z^k (z^{-(k+\mu)})'$. Note that b_i is integral at α. Then $b_i' z^{-\mu} \in z\bar{C}(x)_\alpha$ because $\nu_\alpha(b_i' z^{-\mu}) \geq 1$. For example, if $\alpha \in \bar{C}$, then $(1 + (x - \alpha) + \cdots)'(x - \alpha) = (x - \alpha) + \cdots$; if $\alpha = \infty$, then $(1 + \frac{1}{x} + \cdots)' x = -\frac{1}{x} + \cdots$. So if $-(\lambda + \mu) > 0$, i.e., $\lambda < -\mu$, then Equation (5) can be reduced modulo z:

$$\sum_{i=1}^n a_i \omega_i \equiv \sum_{i=1}^n \mu(k+\mu) b_i \omega_i \mod z, \tag{6}$$

Since $\{\omega_1, \ldots, \omega_n\}$ is a $C(x)$-vector space basis of A, it follows that $b_i \equiv \mu^{-1}(k + \mu)^{-1} a_i \mod z$ is the unique solution of (6) in $\bar{C}(x)_\alpha / \langle z \rangle$. If $\lambda \geq -\mu$, then multiplying (4) by $z^{\lambda+\mu}$ and reducing this equation modulo $z^{\lambda+\mu+1}$ yields

$$\sum_{i=1}^n z^{\lambda+\mu} a_i \omega_i \equiv \sum_{i=1}^n b_i z^{k+\lambda+\mu} \left(\frac{\omega_i}{z^{k+\mu}} \right)' \mod z^{\lambda+\mu+1}. \tag{7}$$

Let $\psi_i := z^{k+\lambda+\mu}\left(\frac{\omega_i}{z^{k+\mu}}\right)'$ for $i = 1, \ldots, n$. To perform Hermite reduction, we have to show that Equation (7) always has a solution $(b_1, \ldots, b_n)$ in $(\bar{C}(x)_\alpha/\langle z^{\lambda+\mu+1}\rangle)^n$.

In the Fuchsian case, Chen et al. [11] proved that $\lambda = 1$. When $\alpha \in \bar{C}$, they showed that $\{\psi_1, \ldots, \psi_n\}$ forms a local integral basis at α and hence (7) has a solution. When $\alpha = \infty$, instead of solving the modular system (7), they introduced the polynomial reduction to reduce the degree in x. We shall show that $\{\psi_1, \ldots, \psi_n\}$ still forms a local integral basis at infinity and provide an alternative method of reducing the degree.

In the non-Fuchsian case, it may happen that $\lambda > 1$, see Examples 3 (for $\alpha = 0$) and 4 (for $\alpha = \infty$). Another difference is that $\{\psi_1, \ldots, \psi_n\}$ may not be a local integral basis at α anymore, see the following Example 5. Fortunately, the linear system (7) still has a solution in $(\bar{C}(x)_\alpha/\langle z^{\lambda+\mu+1}\rangle)^n$ as we shall prove in this section. There are two steps. First we show that $\{\psi_1, \ldots, \psi_n\}$ is linearly independent over $\bar{C}(x)$ and then we find a rational solution $(b_1, \ldots, b_n)$ whose entries admit nonnegative valuation at α. So the b_i's belong to $\bar{C}(x)_\alpha$. Taking b_i modulo $z^{\lambda+\mu+1}$ gives a solution of (7). Once we know that (7) has a solution, equating the coefficients of the ω_i's on both sides and expanding the derivative as (5), we can find its solution $b = (b_1, \ldots, b_n)$ by solving, e.g. with Gaussian elimination, the following linear system of congruence equations:

$$(z^{\lambda+\mu}a_1, \ldots, z^{\lambda+\mu}a_n) \equiv b(M + \mu(k+\mu)z^{\lambda+\mu}I_n) \mod z^{\lambda+\mu+1}, \quad (8)$$

where I_n is the identity matrix in $C[x]^{n\times n}$.

Example 5. *We continue Example 3. For $\alpha = 0$ and $\lambda = 3$, let $\psi_i = x^{k+2}(x^{1-k}\omega_i)'$ for $i = 1, 2$. A direct calculation yields that*

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}-(k-1)x^2 & 1\\ 0 & -(k-1)x^2 - 2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

In this example, ψ_1, ψ_2 are integral elements but do not form a local integral basis at 0*, because $\frac{1}{x^2}(2\psi_1 + \psi_2) = -2(k-1)\omega_1 - (k-1)\omega_2$ is integral at* 0*. In fact, if $k > 1$, then $\{\psi_1, \frac{1}{x^2}(2\psi_1 + \psi_2)\}$ is a local integral basis at* 0*. Now* (8) *becomes*

$$(a_1x^2, a_2x^2) \equiv (b_1, b_2)\begin{pmatrix}-(k-1)x^2 & 1\\ 0 & -(k-1)x^2 - 2\end{pmatrix} \mod x^3.$$

When $k > 1$, even though the coefficient matrix is not invertible over $\bar{C}(x)_0/\langle x^3\rangle$, this equation still has a solution

$$\begin{cases} b_1 \equiv -(k-1)^{-1}a_1 \mod x^3,\\ b_2 \equiv \dfrac{1}{4}\left((k-1)x^2 - 2\right)\left(a_2x^2 + (k-1)^{-1}a_1\right) \mod x^3.\end{cases}$$

Even though $\{\psi_1, \ldots, \psi_n\}$ may not be a local integral basis at α, it is not so far away. In Example 5, we have $\bar{C}(x)_0\psi_1 + \bar{C}(x)_0\psi_2 \subseteq O_0 \subseteq \frac{1}{x^2}(\bar{C}(x)_0\psi_1 + \bar{C}(x)_0\psi_2)$. In general, if we represent a locally integral element at α as a linear combination of $\{\psi_1, \ldots, \psi_n\}$ with coefficients in $\bar{C}(x)$, the pole orders at α of these coefficients are at most $\lambda + \mu$.

Proposition 6. *Let $\alpha \in \bar{C} \cup \{\infty\}$ and $W = \{\omega_1, \ldots, \omega_n\}$ be a local integral basis at α of A. Let $z = x - \alpha$ (or $z = x^{-1}$ if $\alpha = \infty$) and $\mu \in \mathbb{Z}$ be such that $v_\alpha(z') = v_\alpha(z) + \mu$. Let $\lambda \in \mathbb{Z}$ and $M \in \bar{C}(x)_\alpha^{n\times n}$ be such that $z^\lambda W' = MW$ and $v_\alpha(M) = 0$. For some integer $k > 1$ (or $k \geq 0$ if $\alpha = \infty$), we define $\psi_i := z^{k+\lambda+\mu}(z^{-k-\mu}\omega_i)'$. If $\lambda \geq -\mu$, then*

$$\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i \subseteq O_\alpha \subseteq \frac{1}{z^{\lambda+\mu}}\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i.$$

In particular, when $\lambda = -\mu$, we have $\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i = O_\alpha$. In this case, $\{\psi_1, \ldots, \psi_n\}$ forms a local integral basis at α.

Proof. We prove this proposition using the same technique as in [10, Lemma 10]. To show $\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i \subseteq O_\alpha$, we only need to show that for every $i = 1, \ldots, n$, the element ψ_i is integral at α. After expanding, we get $\psi_i = z^\lambda\omega_i' + \mu(d+\mu)z^{\lambda+\mu}\omega_i$. Since $z^\lambda W' = MW$ and $v_\alpha(M) = 0$, it follows that $z^\lambda\omega_i'$ is integral at α. Then ψ_i is integral at α because $\lambda + \mu \geq 0$.

Next we shall prove $O_\alpha \subseteq \frac{1}{z^{\lambda+\mu}}\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i$. Suppose to the contrary that there exists an element $f \in O_\alpha \setminus \frac{1}{z^{\lambda+\mu}}\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i$. Furthermore, we can find such an element f of the form

$$f = \frac{1}{z^{\lambda+\mu+1}}\sum_{i=1}^n c_i\psi_i \quad \text{with } c_i \in \bar{C}(x)_\alpha \text{ and } v_\alpha(c_i) = 0 \text{ for some } i.$$

(Taking $f = 0$, we shall prove that the ψ_i's are linearly independent.) Let $g = z^{-\mu-1}\sum_{i=1}^n c_i'\omega_i$, which is integral for the same reason as between (5) and (6). Then also their sum

$$\begin{aligned} f + g &= z^{k-1}\sum_{i=1}^n\left(c_i\left(z^{-k-\mu}\omega_i\right)' + c_i'z^{-k-\mu}\omega_i\right)\\ &= z^{k-1}\sum_{i=1}^n\left(c_iz^{-k-\mu}\omega_i\right)' = z^{k-1}\left(z^{-k-\mu}h\right)' \end{aligned}$$

must be integral, where $h = \sum_{i=1}^n c_i\omega_i$.

Since $\{\omega_1, \ldots, \omega_n\}$ is an integral basis at α, by Lemma 2 we have $0 \leq \mathrm{val}_\alpha(h) < 1$. Note that

$$\mathrm{val}_\alpha(z^{-k-\mu}h) = -k - \mu + \mathrm{val}_\alpha(h) \leq -1 + \mathrm{val}_\alpha(h) < 0;$$

here we use the assumption that $k > 1$ (resp. $k \geq 0$ if $\alpha = \infty$), because $k = 1$ (resp. $k = -1$) and $\mathrm{val}_\alpha(h) = 0$ imply that $\mathrm{val}_\alpha(z^{-k-\mu}h) = 0$. Since $\mathrm{val}_\alpha(z^{-k-\mu}h) \neq 0$, by Lemma 1 we get

$$\mathrm{val}_\alpha(z^{k-1}(z^{-k-\mu}h)') \leq k-1-k-\mu+\mathrm{val}_\alpha(h)+\mu = \mathrm{val}_\alpha(h)-1 < 0.$$

So $z^{k-1}(z^{-k-\mu}h)' = f + g$ is not integral at α, which contradicts the integrality of f. Hence $O_\alpha \subseteq \frac{1}{z^{\lambda+\mu}}\sum_{i=1}^n \bar{C}(x)_\alpha\psi_i$. ∎

Theorem 7. *Using the same notation as in Proposition 6, let $k > 1$ (or $k \geq 0$ if $\alpha = \infty$). If $\lambda \geq -\mu$, then for any $a_1, \ldots, a_n \in \bar{C}(x)_\alpha$, the linear system*

$$\sum_{i=1}^n z^{\lambda+\mu}a_i\omega_i = \sum_{i=1}^n b_i\psi_i \quad (9)$$

has a solution $(b_1, \ldots, b_n)$ in $(\bar{C}(x)_\alpha/\langle z^{\lambda+\mu+1}\rangle)^n$.

Proof. By Proposition 6, the $\bar{C}(x)_\alpha$-module generated by

$$\left\{\frac{1}{z^{\lambda+\mu}}\psi_1, \ldots, \frac{1}{z^{\lambda+\mu}}\psi_n\right\}$$

contains a submodule O_α of rank n. So $\{\psi_1, \ldots, \psi_n\}$ is linearly independent over $\bar{C}(x)$. Then there exist $t_1, \ldots, t_n \in \bar{C}(x)$ such that $\sum_{i=1}^n z^{\lambda+\mu}a_i\omega_i = \sum_{i=1}^n t_i\psi_i$.

To find a solution b_i, we have to show that $t_i \in \bar{C}(x)_\alpha$ for all $i = 1, \ldots, n$. If so, $b_i \equiv t_i \mod z^{\lambda+\mu+1}$ is a solution of (9). Since $a_i \in \bar{C}(x)_\alpha$ and the ω_i's are integral at α, the element

$$\sum_{i=1}^{n} a_i\omega_i = \frac{1}{z^{\lambda+\mu}} \sum_{i=1}^{n} t_i\psi_i$$

is integral at α. By Proposition 6,

$$\frac{1}{z^{\lambda+\mu}} \sum_{i=1}^{n} t_i\psi_i \in O_\alpha \subseteq \frac{1}{z^{\lambda+\mu}} \sum_{i=1}^{n} \bar{C}(x)_\alpha\psi_i.$$

Then $\sum_{i=1}^n t_i\psi_i \in \sum_{i=1}^n \bar{C}(x)_\alpha\psi_i$. Since $\{\psi_1, \ldots, \psi_n\}$ is linearly independent over $\bar{C}(x)$, we have $t_i \in \bar{C}(x)_\alpha$ for all i. Thus $t_i \in \bar{C}(x)_\alpha$ as claimed. ∎

According to Theorem 7, we can perform one step of Hermite reduction for D-finite functions as described in the beginning of this section. The element b_i in $\bar{C}(x)_\alpha/\langle z^{\lambda+\mu+1}\rangle$ is of the form

$$b_i = b_{i,0} + b_{i,1}z + \cdots + b_{i,\lambda+\mu}z^{\lambda+\mu} \quad \text{with} \quad b_{i,j} \in \bar{C}.$$

So in Equation (3), for $\alpha \in \bar{C}$, if $k > \max\{1, \lambda\}$, then the coefficients of $\frac{1}{z^{k+\mu}} \sum_{i=1}^n b_i\omega_i$ are proper rational functions. For $\alpha = \infty$, if $k \geq \max\{0, \lambda\}$, then the coefficients of $\frac{1}{z^{k+\mu}} \sum_{i=1}^n b_i\omega_i$ are polynomials.

Example 8. *We continue Examples 3 and 5. A local integral basis at $\alpha = 0$ is given by $\omega_1 = 1$ and $\omega_2 = x^3D$. Then $\lambda = 3$. Consider the D-finite function*

$$f = \frac{(-2x^2 - x^4)\omega_1 + (-2 + 3x^2 - 3x^4)\omega_2}{x^4}$$

and use Hermite reduction at 0 to reduce the power of x in its denominator. So we start with $z = x$, $\mu = -1$, $k = 4$, $a_1 = -2x^2 - x^4$ and $a_2 = -2 + 3x^2 - 3x^4$. From (8), we get

$$(a_1x^2, a_2x^2) \equiv (b_1, b_2) \begin{pmatrix} -3x^2 & 1 \\ 0 & -3x^2 - 2 \end{pmatrix} \mod x^3.$$

By Theorem 7, we know that this equation has a solution. Indeed, we find a solution $b_1 = \frac{2}{3}x^2$, $b_2 = \frac{4}{3}x^2$. Then one step of the Hermite reduction at 0 simplifies f to

$$f = \left(\frac{2\omega_1 + 4\omega_2}{3x}\right)' + \frac{(-4 - 3x^2)\omega_1 + (13 - 9x^2)\omega_2}{3x^2}.$$

Example 9. *Let $L = xD^2 - (3x^3 + 2)D \in \mathbb{C}(x)[D]$ be as in Example 4. A local integral basis at $\alpha = \infty$ of $A = \mathbb{C}(x)[D]/\langle L\rangle$ is given by $\omega_1 = 1$ and $\omega_2 = x^{-2}D$. Then*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = x^\lambda \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

with $\lambda = 2$. Consider the D-finite function

$$f = 4x^3 + \frac{1}{x}D = 4x^3\omega_1 + x\omega_2 = x^3\left(4\omega_1 + \frac{1}{x^2}\omega_2\right)$$

and use Hermite reduction at infinity to reduce its degree in x. So we start with $z = \frac{1}{x}$, $\mu = 1$, $k = 3$, $a_1 = 4$ and $a_2 = \frac{1}{x^2}$. From (8), we get

$$(a_1x^{-3}, a_2x^{-3}) \equiv (b_1, b_2) \begin{pmatrix} 4x^{-3} & 1 \\ 0 & 4x^{-3} + 3 \end{pmatrix} \mod x^{-4}$$

This coefficient matrix is not invertible over $C(x)_\infty/\langle x^{-4}\rangle$. However, by Theorem 7, we know this equation has a solution. Indeed, we find a solution $b_1 = 1$, $b_2 = \frac{4}{9x^3} - \frac{1}{3}$. Then one step of the Hermite reduction at infinity simplifies f to

$$f = \left(x^4\omega_1 + \left(\frac{4}{9}x - \frac{1}{3}x^4\right)\omega_2\right)' + \left(x - \frac{4}{9}\right)\omega_2.$$

Let $W = \{\omega_1, \ldots, \omega_n\}$ be a local integral basis at infinity. Let $\lambda \in \mathbb{Z}$ and $M \in C(x)^{n\times n}$ be such that $W' = x^\lambda MW$ and $\nu_\infty(M) = 0$. Then $\lambda = -\nu_\infty(x^\lambda M) = \deg_x(x^\lambda M)$. By repeating the reduction at infinity, we can reduce the degree in x as far as possible and decompose $f \in A$ as

$$f = g' + h \quad \text{with} \quad h = \sum_{i=1}^{n} h_i\omega_i \tag{10}$$

where $g \in A$, $h_i \in C(x)$ with $\deg_x(h_i) < \max\{0, \lambda\}$ for all i and the coefficients of g are polynomials. The following lemma gives an upper bound for the degree of any hypothetical integral of h in A.

Lemma 10. *Let $h \in A$ be as in (10). If h is integrable in A, then $h = (\sum_{i=1}^n b_i\omega_i)'$ with $b_i \in C(x)$ and $\deg_x(b_i) \leq \max\{0, \lambda\}$ for all $i \in \{1, \ldots, n\}$.*

Proof. Suppose h is integrable in A. Then there exists $H = \sum_{i=1}^n b_i\omega_i \in A$ with $b_i \in C(x)$ such that $h = H'$. By (10), we know the coefficients of h satisfy $\deg_x(h_i) < \max\{0, \lambda\}$, which implies $\nu_\infty(h_i) > \min\{0, -\lambda\}$. Since the ω_i's are integral at infinity, it follows that $\operatorname{val}_\infty(h) \geq \min_{i=1}^n\{\nu_\infty(h_i) + \operatorname{val}_\infty(\omega_i)\} > \min\{0, -\lambda\}$.

We want to show that $\deg_x(b_i) \leq \max\{0, \lambda\}$ for all i, which means $\nu_\infty(b_i) \geq \min\{0, -\lambda\}$ for all i. Suppose to the contrary that $\tau := \min_{i=1}^n\{\nu_\infty(b_i)\} < \min\{0, -\lambda\}$. Since $\{\omega_1, \ldots, \omega_n\}$ is a local integral basis at infinity, by Lemma 2 we get $\operatorname{val}_\infty(H) < \tau + 1 \leq \min\{0, -\lambda\}$. So $\operatorname{val}_\infty(H) \neq 0$. By Lemma 1 we have

$$\operatorname{val}_\infty(h) \leq \operatorname{val}_\infty(H) + 1 \leq \min\{0, -\lambda\}.$$

This leads to a contradiction. So $\deg_x(b_i) \leq \max\{0, \lambda\}$ for all i. ∎

3.2 The Global Case

To avoid algebraic extensions of the base field, Hermite reduction can be performed simultaneously at all roots of some squarefree polynomial, which may take advantage of the squarefree decomposition over the base field. Let $W = \{\omega_1, \ldots, \omega_n\}$ be an integral basis of $A = C(x)[D]/\langle L\rangle$. Let $e \in C[x]$ and $M = (m_{i,j})_{i,j=1}^n \in C[x]^{n\times n}$ be such that $eW' = MW$ and $\gcd(e, m_{1,1}, m_{1,2}, \ldots, m_{n,n}) = 1$. Let v be a nontrivial squarefree polynomial and $\lambda \in \mathbb{N}$ be an integer such that $v^\lambda \mid e$ and $\gcd(\frac{e}{v^\lambda}, v) = 1$. Let $f = \frac{1}{uv^k}\sum_{i=1}^n a_i\omega_i \in A$ with $u, a_1, \ldots, a_n \in C[x]$ such that $k > 1$ and $\gcd(u, v) = \gcd(v, v') = \gcd(v, a_1, \ldots, a_n) = 1$. Upon differentiating, the ω_i's may introduce denominators, namely the factors of e. Without loss of generality, we assume $e \mid uv^k$. Suppose $k > \max\{1, \lambda\}$. In order to execute one step of the Hermite reduction to reduce the multiplicity k, we seek $b_1, \ldots, b_n, c_1, \ldots, c_n$ in $C[x]$ such that

$$\frac{\sum_{i=1}^n a_i\omega_i}{uv^k} = \left(\frac{\sum_{i=1}^n b_i\omega_i}{v^{k-1}}\right)' + \frac{\sum_{i=1}^n c_i\omega_i}{uv^{k-1}}. \tag{11}$$

If $\lambda = 0$, then $\gcd(e, v) = 1$. Multiplying (11) by uv^d and reducing this equation modulo v yield

$$\sum_{i=1}^{n} a_i\omega_i \equiv -(k-1)\sum_{i=1}^{n} b_iuv'\omega_i \mod v. \tag{12}$$

Note that $\gcd(u,v) = \gcd(v,v') = 1$. So $b_i \equiv -(k-1)^{-1}(uv')^{-1}a_i \bmod v$ is the unique solution of (12) in $C[x]/\langle v\rangle$. If $\lambda \geq 1$, multiplying (11) by $uv^{k+\lambda-1}$ and reducing this equation modulo v^λ yield

$$\sum_{i=1}^{n} v^{\lambda-1}a_i\omega_i \equiv \sum_{i=1}^{n} b_i uv^{k+\lambda-1}\left(\frac{\omega_i}{v^{k-1}}\right)' \mod v^\lambda. \qquad (13)$$

One can adapt the argument in the local case to show that (13) has a solution $(b_1,\ldots,b_n)$ in $(C[x]/\langle v^\lambda\rangle)^n$. Let $\psi_i := v^{k+\lambda-1}(v^{1-k}\omega_i)'$ for $i = 1,\ldots,n$. As an analog of Proposition 6, for each root $\alpha \in \bar{C}$ of v, we have

$$\sum_{i=1}^{n} \bar{C}(x)_\alpha \psi_i \subseteq O_\alpha \subseteq \frac{1}{v^{\lambda-1}}\sum_{i=1}^{n} \bar{C}(x)_\alpha \psi_i.$$

Thus the linear system $\sum_{i=1}^n v^{\lambda-1}a_i\omega_i = \sum_{i=1}^n ub_i\psi_i$ has a solution $(b_1,\ldots,b_n)$ in $(C[x]/\langle v^\lambda\rangle)^n$. Equating the coefficients of the ω_i's on both sides of (13), the vector $b = (b_1,\ldots,b_n)$ can be found by solving the following linear system of congruence equations:

$$(v^{\lambda-1}a_1,\ldots,v^{\lambda-1}a_n) \equiv b(uv^\lambda e^{-1}M - (k-1)uv^{\lambda-1}v'I_n) \mod v^\lambda,$$

where I_n is the identity matrix in $C[x]^{n\times n}$.

By repeated application of the above Hermite reduction step, we can increase the valuations at finite places as far as possible, i.e., we can decompose any $f \in A$ as

$$f = \tilde{g}' + h \quad \text{with} \quad h = \sum_{i=1}^{n} \frac{h_i\omega_i}{de}, \qquad (14)$$

where $\tilde{g} \in A$, $h_1,\ldots,h_n,d \in C[x]$, d is squarefree, $\gcd(d,e) = 1$ and the coefficients of $\tilde{g}$ are proper rational functions.

Lemma 11. *Let $h \in A$ be as in (14). If h is integrable in A, then $h = \left(\frac{\sum_{i=1}^n q_i\omega_i}{u}\right)'$, where $q_1,\ldots,q_n,u \in C[x]$ and $u \mid \gcd(e,e')$. Furthermore, we have $d \in C$.*

Proof. Suppose h is integrable in A. Then there exists $H = \sum_{i=1}^n b_i\omega_i \in A$ with $b_i \in C(x)$ such that $h = H'$.

If $\alpha \in \bar{C}$ is not a root of e, then b_i has no pole at α for all i. Otherwise, suppose b_i has a pole at α for some $i \in \{1,\ldots,n\}$. Then H has a negative valuation at α, because $\{\omega_1,\ldots,\omega_n\}$ is an integral basis. But then by Lemma 1, h would have a valuation less than -1, which is impossible because $\gcd(d,e) = 1$ and d is squarefree. Therefore b_i is integral at α for all i and hence d is a constant.

If $\alpha \in \bar{C}$ is a root of e, then b_i has a pole at α of order at most $\nu_\alpha(e)-1$. Otherwise, suppose $\tau := \min_{i=1}^n\{\nu_\alpha(b_i)\} \leq -\nu_\alpha(e) \leq -1$. By Lemma 2, $\mathrm{val}_\alpha(H) < \tau + 1 \leq 0$. Then $\mathrm{val}_\alpha(H) \neq 0$. By Lemma 1, we have $\mathrm{val}_\alpha(h) \leq \mathrm{val}_\alpha(H) - 1$. Thus $\mathrm{val}_\alpha(h) < -\nu_\alpha(e)$. But from $h = \sum_{i=1}^n \frac{h_i\omega_i}{de}$, we see $\mathrm{val}_\alpha(h) \geq -\nu_\alpha(e)$ because $\gcd(d,e) = 1$. This leads to a contradiction.

Note that $\nu_\alpha(\gcd(e,e')) = \nu_\alpha(e) - 1$ if α is a root of e and $\nu_\alpha(\gcd(e,e')) = 0$ if α is not a root of e. So $\gcd(e,e')$ is a common multiple of the denominators of the b_i's. ∎

4 ADDITIVE DECOMPOSITIONS

Now we combine the Hermite reduction at finite places and at infinity to decompose a D-finite function f as $f = g' + h$ such that f is integrable if and only if the remainder h is zero. To achieve this goal, first we confine all remainders into a finite-dimensional vector space. Then we find all possible integrable functions in this vector space. This procedure is similar to the hyperexponential case [6], the algebraic case [10], the Fuchsian case [11] and the D-finite case [7, 21]. It provides an alternative method for solving the accurate integration problem for D-finite functions [2].

Since there may not exist a basis of $A = C(x)[D]/\langle L\rangle$ that is a local integral basis at all $\alpha \in \bar{C} \cup \{\infty\}$, we need two bases to perform Hermite reduction at finite places and at infinity, respectively. Let $W = (\omega_1,\ldots,\omega_n) \in A^n$ be an integral basis of A that is normal at infinity. There exists $T = \mathrm{diag}(x^{\tau_1},\ldots,x^{\tau_n}) \in C(x)^{n\times n}$ with $\tau_i \in \mathbb{Z}$ such that $V := TW$ is a local integral basis at infinity. (Theoretically, we can also start with W being a local integral basis at $\bar{C}\setminus\{\alpha\}\cup\{\infty\}$ that is normal at α.) Let $e, a \in C[x]$ and $M, B \in C[x]^{n\times n}$ be such that $eW' = MW$ and $aV' = BV$. Since the derivative of V is $V' = (TW)' = (T' + \frac{1}{e}TM)T^{-1}V$, we may assume that $a = x^\lambda e$ for some $\lambda \in \mathbb{N}$. For $\mu, \delta \in \mathbb{Z}$ with $\mu \leq \delta$, we define a subspace of Laurent polynomials in $C[x,x^{-1}]$ as $C[x]_{\mu,\delta} := \{\sum_{i=\mu}^{\delta} a_ix^i \mid a_i \in C\}$.

Theorem 12. *Let $W, V \in A^n$ be as described above. Then any element $f \in A$ can be decomposed into*

$$f = g' + \frac{1}{d}RW + \frac{1}{x^\lambda e}QV, \qquad (15)$$

where $g \in A$, $d \in C[x]$ is squarefree and $\gcd(d,e) = 1$, $R \in C[x]^n$, $Q \in C[x]^n_{\mu,\delta}$ with $\deg_x(R) < \deg_x(d)$, $\mu = \min\{-\tau_1,\ldots,-\tau_n,0\}$ and $\delta = \max\{\lambda + \deg_x(e), \deg_x(B)\} - 1$. Moreover, f is integrable in A if and only if $R = 0$ and

$$\frac{1}{x^\lambda e}QV \in U' \quad \textit{with} \quad U = \left\{\frac{1}{u}cV \;\middle|\; c \in C[x]^n_{\mu',\delta'}\right\},$$

where $u = \gcd(e,e')$, $\mu' = \min\{-\tau_1,\ldots,-\tau_n,\nu_0(u)\}$ and

$$\delta' = \max\{\deg_x(u), \deg_x(B) - \lambda - \deg_x(e) + \deg_x(u)\}.$$

Proof. Let $h \in A$ be a Hermite remainder as in (14). By the extended Euclidean algorithm, we compute $r_i, s_i \in C[x]$ such that $h_i = r_ie + s_id$ and $\deg_x(r_i) < \deg_x(d)$. Then h decomposes as

$$h = \sum_{i=1}^{n} \frac{h_i}{de}\omega_i = \sum_{i=1}^{n} \frac{r_i}{d}\omega_i + \sum_{i=1}^{n} \frac{s_i}{e}\omega_i.$$

Writing h in vector form, by (14) we decompose $f \in A$ as

$$f = \tilde{g}' + \frac{1}{d}RW + \frac{1}{e}SW, \qquad (16)$$

where $\tilde{g} \in A$, $R = (r_1,\ldots,r_n) \in C[x]^n$, $S = (s_1,\ldots,s_n) \in C[x]^n$. In the next step, we shall reduce the degree of S and confine S to a finite-dimensional vector space over C that is independent of f. We rewrite the last summand in (16) with respect to the basis V:

$$\frac{1}{e}SW = \frac{1}{x^\lambda e}\tilde{S}V,$$

where $\tilde{S} = x^\lambda ST^{-1} \in x^\mu C[x]^n$ with $\mu = \min\{-\tau_1,\ldots,-\tau_n,0\}$. Since V is a local integral basis at infinity, using Hermite reduction at infinity in Section 3.1, we obtain from (10) that

$$\frac{1}{e}SW = (S_1V)' + \frac{1}{x^\lambda e}S_2V, \qquad (17)$$

where $S_1 \in C[x]^n$ and $S_2 \in x^\mu C[x]^n$ satisfies

$$\deg_x\left(\frac{S_2}{x^\lambda e}\right) \leq \max\left\{0, \deg_x\left(\frac{B}{x^\lambda e}\right)\right\} - 1.$$

This implies that $\deg_x(S_2) \le \max\{\lambda + \deg_x(e), \deg_x(B)\} - 1 = \delta$. Thus $S_2 \in C[x]^n_{\mu,\delta}$ and we finally obtain the decomposition (15) by setting $g = \tilde{g} + S_1 V$ and $Q = S_2$.

For the last assertion, assume that f is integrable (the other direction of the equivalence holds trivially). Then Lemma 11 implies that $d \in C$, and therefore R must be zero because $\deg_x(R) < \deg_x(d)$. Hence the last summand in (15) and the left hand side of (17) are also integrable. We want to find its integral by estimating the valuation of this integral at all points in $C \cup \{\infty\}$. Since W is a global integral basis, using Lemma 11 again, we know

$$\frac{1}{e} SW = \left(\frac{1}{u} bW\right)',$$

where $b \in C[x]^n$ and $u = \gcd(e, e')$. Then

$$\frac{1}{x^\lambda e} QV = \frac{1}{e} SW - (S_1 V)' = \left(\left(\frac{bT^{-1}}{u} - S_1\right) V\right)' = \left(\frac{1}{u} cV\right)',$$

where $c = bT^{-1} - uS_1 \in C[x, x^{-1}]^n$. Now we only need to estimate the valuation of c at the remaining two points 0 and ∞. By the expression of c, we get

$$\nu_0(c) \ge \min\{\nu_0(bT^{-1}), \nu_0(uS_1)\} \ge \min\{-\tau_1, \ldots, -\tau_n, \nu_0(u)\} = \mu'.$$

On the other hand, since V is a local integral basis at infinity, it follows from Lemma 10 that $\deg_x\left(\frac{c}{u}\right) \le \max\left\{0, \deg_x\left(\frac{B}{x^\lambda e}\right)\right\}$. Thus

$$\deg_x(c) \le \max\{\deg_x(u), \deg_x(B) - \lambda - \deg_x(e) + \deg_x(u)\} = \delta'.$$

Finally we have $c \in C[x]^n_{\mu',\delta'}$. ∎

The remaining step is to reduce all integrable D-finite functions to zero. Note that in Theorem 12, U is a C-vector space of dimension $n(\delta' - \mu' + 1)$ with a basis

$$\left\{ \frac{x^j v_i}{u} \,\middle|\, i = 1, \ldots, n; j = \mu', \ldots, \delta' \right\},$$

where $V = (v_1, \ldots, v_n)$. Let $K = \left\{ \frac{1}{x^\lambda e} bV \,\middle|\, b \in C[x]^n_{\mu,\delta} \right\}$. Differentiating all elements in the basis of U and using Gaussian elimination, we can find a basis of U' and decompose $K = (U' \cap K) \oplus N_V$ as a direct sum of two subspaces, where N_V is a complement of $U' \cap K$ in K. This means f in (15) can be further decomposed as

$$f = \tilde{g}' + \frac{1}{d} RW + \frac{1}{x^\lambda e} Q_2 V, \tag{18}$$

where $\tilde{g} = g + g_1$ with $g_1' \in U' \cap K$ and $Q_2 \in C[x]^n_{\mu,\delta}$ such that f is integrable in A if and only if $R = Q_2 = 0$. This decomposition (18) is called an *additive decomposition* of f.

In practice, we may choose a fixed complement of $K \cap U'$ in K. To do this, we define a *term over position order* on the set

$$\left\{ x^j v_i \,\middle|\, i = 1, \ldots, n; j \in \mathbb{Z} \right\}$$

such that $x^{j_1} v_{i_1} > x^{j_2} v_{i_2}$ if and only if $j_1 > j_2$ or $j_1 = j_2$ and $i_1 < i_2$. For a nonzero element $p = \sum_{i=1}^n p_i v_i \in A$ with $p_i \in C[x, x^{-1}]$, let $\operatorname{supp}(p)$ denote all terms appearing in p and $\operatorname{lt}(p)$ denote the leading term of p. For example, if $p = 3x^2(v_1 + v_2) + 10xv_1 \in A$, then $\operatorname{supp}(p) = \{x^2 v_1, x^2 v_2, xv_1\}$ and $\operatorname{lt}(p) = x^2 v_1$. Then a *standard complement* of $K \cap U'$ in K is a C-vector space defined by

$$\{h \in K \mid \operatorname{lt}(x^\lambda eg) \notin \operatorname{supp}(x^\lambda eh) \text{ for all } g \in K \cap U'\}.$$

From now on, let N_V denote the standard complement of $K \cap U'$ in K. This definition of N_V is essentially the same as in [11] under the correspondence $\frac{1}{x^\lambda e} bV \mapsto b$.

Note that Q_2 belongs to a C-vector space $C[x]^n_{\mu,\delta}$ of dimension

$$n(\delta - \mu + 1) = \max\{\lambda + \deg_x(e), \deg_x(B)\} + \max\{\tau, 0\}, \tag{19}$$

where $\tau = \max\{\tau_1, \ldots, \tau_n\}$. If L is Fuchsian, by [11, Lemma 4], we know $\deg_x(B) < \lambda + \deg_x(e)$. So Q_2 belongs to a C-vector space of dimension at most $n(\max\{\tau, 0\} + \lambda + \deg_x(e))$. This is a refinement of [11, Proposition 22].

A pseudo code of the algorithm described in this section is given in the appendix.

Example 13. *Let $L = xD^2 - (3x^3 + 2)D \in \mathbb{C}(x)[D]$ be the same operator as in Example 4. Then $W = (\omega_1, \omega_2) = (1, x^{-2}D) = V$. So $e = 1$, $\lambda = 0$ and $M = B = \begin{pmatrix} 0 & x^2 \\ 0 & 3x^2 \end{pmatrix}$. After performing Hermite reduction at infinity in Example 9, we get*

$$f = \left(x^4 \omega_1 + \left(\tfrac{4}{9}x - \tfrac{1}{3}x^4\right) \omega_2\right)' + \left(x - \tfrac{4}{9}\right) \omega_2. \tag{20}$$

Then $\mu = 0$, $\delta = 1$, $u = 1$, $\mu' = 0$, $\delta' = 2$. A basis of U is

$$\{\omega_1,\ \omega_2,\ x\omega_1,\ x\omega_2,\ x^2\omega_1,\ x^2\omega_2\},$$

and hence U' is generated by

$$\{x^2\omega_2,\ 3x^2\omega_2,\ \omega_1 + x^3\omega_2,\ (1 + 3x^3)\omega_2,\ 2x\omega_1 + x^4\omega_2,\ (2x + 3x^4)\omega_2\}.$$

So a basis of $K \cap U'$ is $\{3\omega_1 - \omega_2, 6x\omega_1 - 2x\omega_2\}$. The leading terms of all elements in $K \cap U'$ are ω_1 or $x\omega_1$. Since $\operatorname{lt}((x - \frac{4}{9})\omega_2) = x\omega_2$ is different from all these terms, by Theorem 12 we know f is not integrable in $A = \mathbb{C}(x)[D]/\langle L \rangle$ and (20) is an additive decomposition of f with respect to x.

Example 14. *Let $L = x^3 D^2 + (3x^2 + 2)D \in \mathbb{C}(x)[D]$ be as in Example 3. Then $W = (\omega_1, \omega_2) = (1, x^3 D) = V$. So $e = x^3$, $\lambda = 0$ and $M = B = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$. Combining Hermite reduction at all finite places in Example 8 and Hermite reduction at infinity, we get*

$$f = \left(\left(\tfrac{2}{3x} - x\right) \omega_1 + \left(\tfrac{4}{3x} - 3x\right) \omega_2\right)' - \tfrac{4}{3x^2}\omega_1 - \tfrac{2}{3x^2}\omega_2.$$

Then $\mu = 0$, $\delta = 2$, $u = x^2$, $\mu' = 0$, $\delta' = 2$. A basis of U is

$$\left\{\tfrac{\omega_1}{x^2},\ \tfrac{\omega_2}{x^2},\ \tfrac{\omega_1}{x},\ \tfrac{\omega_2}{x},\ \omega_1,\ \omega_2\right\}.$$

A basis of $K \cap U'$ is $\{-\frac{2}{x^2}\omega_1 - \frac{1}{x^2}\omega_2, \frac{1}{x^3}\omega_2, -\frac{2}{x^3}\omega_2\}$. Therefore f is integrable: $f = \left(\left(\frac{2}{3x} - x + \frac{4}{3x}\right)\omega_1 + \left(\frac{4}{3x} - 3x + \frac{2}{3x}\right)\omega_2\right)'$.

5 APPLICATIONS

Let $K(x)[\partial_t, D_x]$ with $K = C(t)$ be an Ore algebra in which D_x is the differentiation with respect to x and ∂_t is either the differentiation with respect to t or the shift $t \mapsto t + 1$. Let I be a left ideal of $K(x)[\partial_t, D_x]$ such that $A = K(x)[\partial_t, D_x]/I$ is a $K(x)$-vector space of dimension n. Let $\gamma \in A$ be a cyclic vector with respect to D_x. This means that $\{\gamma, D_x\gamma, \ldots, D_x^{n-1}\gamma\}$ is a basis of A over $K(x)$. Then γ is annihilated by L and $\partial_t - u_t$ for some $L, u_t \in K(x)[D_x]$. Every element f in A can be uniquely written as $P_f\gamma + I$ for some $P_f \in K(x)[D_x]$. The map sending f to $P_f + \langle L \rangle$ gives an isomorphism from A to $K(x)[D_x]/\langle L \rangle$ as $K(x)[D_x]$-modules. Using this isomorphism, for any $f \in A$, we can apply our additive decomposition to test whether f is integrable (in x). If $f \in A$ is not integrable, one can ask to find a nonzero operator $T \in C(t)[\partial_t]$ (free of x) such that Tf is integrable. Such an operator T, if it exists, is called a *telescoper* for f. Applying the additive decomposition with respect to x in Section 4 to $\partial_t^i f \in A$ yields that $\partial_t^i f = g_i' + h_i$, where $g_i, h_i \in A$, and $\partial_t^i f$ is integrable in A if and only if $h_i = 0$. If there exist $c_0, c_1, \ldots, c_r \in K$ such that $\sum_{i=0}^r c_i h_i = 0$, then $T = \sum_{i=0}^r c_i \partial_t^i$ is a telescoper for f. This approach is the method of reduction-based telescoping and was developed for various classes of functions [5–7, 9–11, 21]. Similar to the Fuchsian case [11, Lemma 24], for any $i \in \mathbb{N}$, the derivative $D_t^i f$ has an additive decomposition (18) of the form

$$D_t^i f = g_i' + h_i \quad \text{with} \quad h_i = \frac{1}{d} R_i W + \frac{1}{x^\lambda e} Q_i V \tag{21}$$

where $g_i \in A$, $d \in K[x]$, $R_i \in K[x]^n$, $Q_i \in K[x, x^{-1}]^n$, with $\deg_x(R_i) < \deg_x(d)$ and $Q_i \in N_V$. Then by (19) we obtain an upper bound for the order of telescopers, which is a generalization of [11, Corollary 25].

Corollary 15. *Every $f \in A$ has a telescoper of order at most $n \deg_x(d) + \dim_x(N_V)$, which is bounded by*

$$n(\deg_x(d) + \max\{\tau, 0\} + \max\{\lambda + \deg_x(e), \deg_x(B)\}).$$

Example 16. *Let $H = \sqrt{t - 2x}\exp(t^2 x)$, which is annihilated by*

$$L = D_x - \frac{2t^2x - t^3 + 1}{2x - t} \quad \text{and} \quad D_t - \frac{8tx^2 - 4t^2x - 1}{2(2x - t)}.$$

An integral basis of $A = K(x)[D_x]/\langle L\rangle$ with $K = \mathbb{C}(t)$ is $\omega = 1$ and a local integral basis at infinity is $v = x^{-1}\omega$. As the integrand H corresponds to $1 \in A$, its representation in the bases is $f = \omega = xv$. The additive decomposition of f is

$$f = \left(\left(\frac{x}{t^2} - \frac{1}{2t^2}\right)v\right)' - \frac{(t^3+1)x - t}{2t^4 x(2x - t)}v.$$

Next we consider the derivative $D_t f$ which has an additive decomposition

$$D_t f = \left(\left(\frac{2x^2}{t} - \frac{3x}{t^3} - \frac{3t^3 - 6}{4t^5}\right)v\right)' - \frac{3(t^3 - 2)((t^3+1)x - t)}{4t^5 x(2x - t)}v.$$

Now we see the reminders of f and $D_t f$ are linearly dependent over $\mathbb{C}(t)$, which gives rise to a telescoper $2tD_t - 3(t^3 - 2)$. This telescoper was obtained in [6, Example 21] with a different reduction approach.

Example 17. *Let $F_n(x) = x^n J_n(x)$ where J_n denotes the Bessel function of the first kind. Then $F_n(x)$ is annihilated by*

$$L = D_x^2 + (1 - 2n)D_x + x \quad \text{and} \quad P = S_n + xD_x - 2n,$$

where S_n is the shift operator with respect to n. An integral basis of $A = K(x)[D_x]/\langle L\rangle$ with $K = \mathbb{C}(n)$ is $W = (\omega_1, \omega_2) = (1, D_x)$ and a local integral basis at infinity is $V = (v_1, v_2) = (\omega_1, x^{-1}\omega_2)$. As before, $F_n(x)$ is represented by $f = 1 \in A$. The additive decompositions of f and $S_n f = -xD_x + 2n \in A$ are as follows:

$$f = (v_2)' + \frac{(2n-1)x - 1}{x}v_2,$$

$$S_n f = (-xv_1 - (2n+1)v_2)' + \frac{(2n+1)((2n-1)x - 1)}{x}v_2.$$

Now we can find a telescoper $S_n - 2n - 1$. This was obtained by the algorithm in [7].

6 CONCLUSION

In this paper, we present a reduction-based telescoping algorithm for D-finite functions via integral bases. Now we compare our algorithm with van der Hoeven's algorithm [21] and Bostan et al.'s algorithm [7]. A feature of the three methods is constructing local reduction procedures that increase the valuation at various places.

Bostan et. al use the Lagrange identity and develop generalized Hermite reduction. The *adjoint* of $L = \sum_{i=0}^{n-1} \ell_i D^i$with $\ell_i \in C(x)$ is defined as $L^* = \sum_{i=0}^{n-1}(-D)^i \ell_i$. Using the Lagrange identity, the algorithm [7] reduces the integrability problem for $f \in A = C(x)[D]/\langle L\rangle$ to that problem for another element with a rational representative $\tilde{f}$:

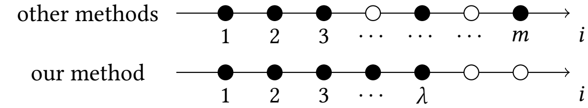
$$f = f_0 + f_1 D + \cdots + f_{n-1}D^{n-1} + \langle L\rangle$$
$$\equiv f_0 - f_1' + \cdots + (-1)^{n-1} f_{n-1}^{(n-1)} + \langle L\rangle =: \tilde{f} + \langle L\rangle \mod D(A).$$

Then f is integrable in A if and only if $\tilde{f} \in L^*(C(x))$. For $\alpha \in \bar{C}$, let $\tilde{f} = \frac{a}{(x-\alpha)^k}$ with $a \in \bar{C}(x)_\alpha$ and $k > 1$. The generalized Hermite reduction chooses g, as in the following table, to reduce the multiplicity k, where σ_α is an integer depending on L and α.

Van der Hoeven's method performs a reduction by finding a "good" basis (not an integral basis) at $\alpha \in \bar{C} \cup \{\infty\}$, which is a so-called tail reduction (or a head reduction if at infinity). This method is a matrix version of Hermite reduction. Let W be a $C(x)$-vector space basis of A. Let $h = \frac{W}{(x-\alpha)^k}$ with k being a parameter. Then for every $b \in \bar{C}^n$, $(bh)' = bh'$. Let $H \in \bar{C}(x)^{n\times n}$ be the coefficient of h' with respect to W. We can view H as a Laurent polynomial in x with coefficients in $\bar{C}(k)^{n\times n}$. In this sense, a "good" basis W at α means that the leading coefficient of H is an invertible matrix over $\bar{C}(k)$. To reduce the multiplicity k in $f = \frac{aW}{(x-\alpha)^k}$ with $a \in \bar{C}(x)_\alpha^n$, the possible choice of g with $\tau_\alpha \in \mathbb{Z}$ is given in the following table.

Method	Tool	Certificate
Bostan et al.'s	Lagrange identity, generalized H.R.	$g = L^*\left(\frac{b}{(x-\alpha)^{k-\sigma_\alpha}}\right)$, $b \in \bar{C}(x)_\alpha$
van der Hoeven's	"good" bases W, tail reduction	$g = \frac{bW}{(x-\alpha)^{k-\tau_\alpha}}$, $b \in \bar{C}^n$
this paper	integral bases W, Hermite reduction	$g = \frac{(\sum_{i=0}^{k-2} b_i (x-\alpha)^i)W}{(x-\alpha)^{k-1}}$, $b_i \in \bar{C}^n$

The black circles in the following figure represent possible terms $\frac{c_i W}{(x-\alpha)^i}$ appearing in a reminder after local reductions, where W is a basis using in the reduction (take $W = 1$ if there is no basis). For the other two methods, local reductions may fail at finitely many k. In our local case, we can reduce the multiplicity k whenever $k > 1$. In our global case, e may introduce denominators. When $k > \max\{1, \nu_\alpha(e)\}$, the corresponding g is given in the above table. An interesting observation is that using integral bases, possible terms in a reminder may be more compact.



Acknowledgement. We thank Shayea Aldossari for sharing his Maple package *integral_bases* and thank the referees for their careful reading and their valuable suggestions.

REFERENCES

[1] https://github.com/LixinDu/HermiteReduction.

[2] Sergei A Abramov and Mark Van Hoeij. Integration of solutions of linear functional equations. *Integral Transforms and Special Functions*, 8(1-2):3–12, 1999.

[3] Shayea Aldossari. *Algorithms for Simplifying Differential Equations*. PhD thesis, Florida State University, 2020.

[4] Gert Almkvist and Doron Zeilberger. The method of differentiating under the integral sign. *Journal of Symbolic Computation*, 10:571–591, 1990.

[5] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li. Complexity of creative telescoping for bivariate rational functions. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, page 203–210, New York, NY, USA, 2010. ACM.

[6] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, Ziming Li, and Guoce Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *Proceedings of the 2013 International Symposium on Symbolic and Algebraic Computation*, pages 77–84, New York, NY, USA, 2013. ACM.

[7] Alin Bostan, Frédéric Chyzak, Pierre Lairez, and Bruno Salvy. Generalized Hermite reduction, creative telescoping and definite integration of D-finite functions. In *Proceedings of the 2018 International Symposium on Symbolic and Algebraic Computation*, pages 95–102, New York, NY, USA, 2018. ACM.

[8] Shaoshi Chen, Lixin Du, and Manuel Kauers. Lazy Hermite reduction and creative telescoping for algebraic functions. In *Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation*, page 75–82, New York, NY, USA, 2021. ACM.

[9] Shaoshi Chen, Hui Huang, Manuel Kauers, and Ziming Li. A modified Abramov-Petkovsek reduction and creative telescoping for hypergeometric terms. In *Proceedings of the 2015 International Symposium on Symbolic and Algebraic Computation*, pages 117–124, New York, NY, USA, 2015. ACM.

[10] Shaoshi Chen, Manuel Kauers, and C. Koutschan. Reduction-based creative telescoping for algebraic functions. In *Proceedings of the 2016 International Symposium on Symbolic and Algebraic Computation*, pages 175–182, New York, NY, USA, 2016. ACM.

[11] Shaoshi Chen, Mark van Hoeij, Manuel Kauers, and Christoph Koutschan. Reduction-based creative telescoping for fuchsian D-finite functions. *Journal of Symbolic Computation*, 85:108–127, 2018.

[12] Frédéric Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Mathematics*, 217:115–134, 2000.

[13] Lixin Du. *Generalized Integral Bases and Applications in Creative Telescoping*. PhD thesis, Johannes Kepler University, 2022.

[14] Erdal Imamoglu. *Algorithms for solving linear differential equations with rational function coefficients*. PhD thesis, Florida State University, 2017.

[15] Erdal Imamoglu and Mark van Hoeij. Computing hypergeometric solutions of second order linear differential equations using quotients of formal solutions and integral bases. *Journal of Symbolic Computation*, 83:254–271, 2017.

[16] Edward L. Ince. *Ordinary Differential Equations*. Dover, 1926.

[17] Manuel Kauers and Christoph Koutschan. Integral D-finite functions. In *Proceedings of the 2015 International Symposium on Symbolic and Algebraic Computation*, pages 251–258, New York, NY, USA, 2015. ACM.

[18] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates*. Springer Publishing Company, Incorporated, 1st edition, 2011.

[19] Mikhail Vasil'evich Ostrogradsky. De l'intégration des fractions rationnelles. *Bull. de la classe physico-mathématique de l'Acad. Impériale des Sciences de Saint-Pétersbourg*, 4:145–167, 286–300, 1845.

[20] Barry M. Trager. *Integration of Algebraic Functions*. PhD thesis, MIT, 1984.

[21] Joris van der Hoeven. Constructing reductions for creative telescoping: the general differentially finite case. *Applicable Algebra in Engineering, Communication and Computing*, 32(5):575–602, nov 2021.

[22] Doron Zeilberger. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics*, 32:321–368, 1990.

APPENDIX

The algorithm of Section 4 can be summarized as follows.

Algorithm. *Input:* $L \in C(x)[D]$ *and* $f \in A = C(x)[D]/\langle L\rangle$; *Output: a certificate* $g \in A$ *and a reminder* $r \in A$ *such that* $f = g' + r$ *is an additive decomposition in* (18).

1 *apply the algorithms in [3] to compute an integral basis* W *of* A *that is normal at infinity and a local integral basis at infinity* $V = TW$ *with* $T = \operatorname{diag}(x^{\tau_1}, \ldots, x^{\tau_n})$ *and* $\tau_i \in \mathbb{Z}$.

2 *compute* $e \in C[x]$, $\lambda \in \mathbb{Z}$ *and* $M, B \in C[x]^{n\times n}$ *such that*

$$eW' = MW \text{ and } x^\lambda eV' = BV.$$

3 *apply Hermite reduction in Section 3.2 to decompose* f *in the form* (14) *and write* f *as in* (16): $f = \tilde{g}' + \frac{1}{d}RW + \frac{1}{e}SW$.

4 *rewrite* $\frac{1}{e}SW = \frac{1}{x^\lambda e}\tilde{S}V$,

5 *apply Hermite reduction at infinity in Section 3.1 to reduce the degree of* $\tilde{S}$ *and obtain* (17): $\frac{1}{e}SW = (S_1V)' + \frac{1}{x^\lambda e}S_2V$.

6 *use Gaussian elimination to decompose* $\frac{1}{x^\lambda e}S_2V = g_1' + \frac{1}{x^\lambda e}Q_2V$, *where* $g_1 \in A$ *and* $\frac{1}{x^\lambda e}Q_2V \in N_V$.

7 *return* $(\tilde{g} + S_1V + g_1, \frac{1}{d}RW + \frac{1}{x^\lambda e}Q_2V)$.

An implementation of this algorithm in Maple and additional examples are available in [1].