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Lazy Hermite Reduction and Creative Telescoping for Algebraic Functions *

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Abstract

We present criteria on the existence of telescopers for trivariate rational functions in four mixed cases, in which discrete and continuous variables appear simultaneously. We reduce the existence problem in the trivariate case to the exactness testing problem, the separation problem and the existence problem in the bivariate case. The existence criteria we present help us determine the termination of Zeilberger's algorithm for the input functions studied in this paper.

1 Introduction

The integration problem for algebraic functions has a long history that can be traced back at least to the work of Euler and others on elliptic integrals [3]. In 1826, Abel initiated the study of general integrals of algebraic functions, which are now called abelian integrals [2]. Liouville in 1833 proved that if the integral

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of an algebraic function is an elementary function, then it must be the sum of an algebraic function and a linear combination of logarithms of algebraic functions (see [29, Chapter IX] for a detailed historical overview). In 1948, Ritt presented an algebraic approach to the problem of integration in finite terms in his book [34]. Based on Liouville's theorem and some developments in differential algebra [35], Risch in 1970 finally solved this classical problem by giving a complete algorithm [33]. After Risch's work, more efficient algorithms have been given due to the emerging developments in symbolic computation [20, 36, 11, 13]. Passing from indefinite integration to definite integration with parameters, the central problem shifts to finding linear differential equations satisfied by the integrals of algebraic functions with parameters. In this direction, the first work was started by Picard in [32] and later a systematical method was developed by Manin in his work on proving the function-field analogue of Mordell's conjecture [30]. In the 1990s, another powerful method was developed by Almkvist and Zeilberger [4] by including the trick of differentiating under the integral sign in the framework of creative telescoping [39].

In a given differential ring R with the derivation D, one can ask two fundamental problems: one is the *integrability problem*, i.e., deciding whether a given element $f \in R$ is of the form $D(g)$ for some $g \in R$, if such a g exists, we say that f is *integrable* in R; another is the *decomposition problem*, i.e., decomposing a given element $f \in R$ into the form $D(g) + r$ with $g, r \in R$ and r is minimal in certain sense and $r = 0$ if and only if f is integrable in R. For algebraic functions, the integrability problem was studied by Liouville in his two memoirs [27, 28]. Hermite reduction [22] solves the decomposition problem for rational functions. Trager in his thesis [36] extended Hermite reduction to the algebraic case. His algorithm requires the computation of an integral bases in the beginning. In order to avoid this expensive step, Bronstein [10] introduced the lazy Hermite reduction that partially solves the decomposition problem. The first contribution in this paper is a sharpened version of the lazy Hermite reduction. We combine the lazy Hermite reduction with a further reduction, namely the polynomial reduction, in order to solve the decomposition problem completely.

When the differential ring R is equipped with two derivations D_1, D_2, one can also consider the *creative telescoping* problem: for a given element $f \in R$, decide whether there exist $c_0, \ldots, c_r \in R$, not all zero, such that $D_2(c_i) = 0$ for all $i \in \{0, \ldots, r\}$ and

$$c_r D_1^r(f) + \cdots + c_0 f = D_2(g) \quad \text{for some } g \in R.$$

The operator $L = c_r D_1^r + \cdots + c_0$ if exists is called a *telescoper* for f. For every algebraic function there exists such a telescoper, and many construction algorithms have been developed in [30, 38, 16, 9, 15]. Our second contribution is an adaption of the reduction-based approach from [15] using the sharpened lazy Hermite reduction.

The remainder of this paper is organized as follows. First we recall Bronstein's idea of lazy Hermite reduction in Section 2. Instead of an integral basis, it uses a so-called "suitable basis" of the function field. In Section 3 we have a closer look at these bases. After developing the polynomial reduction in Section 4, we will present the telescoping algorithm for algebraic functions based on the sharpened lazy Hermite reduction in Section 5. We conclude our paper by some experimental comparisons among several telescoping algorithms in Section 6.

2 Lazy Hermite Reduction

Trager's generalization of Hermite reduction to algebraic functions works as follows [36, 21, 12, 15]. Let K be a field of characteristic zero and $m \in K(x)[y]$ be an irreducible polynomial over $K(x)$. Then $A = K(x)[y]/\langle m\rangle$ is an algebraic extension of $K(x)$. When there is no ambiguity, we also use y to represent the element $y + \langle m\rangle$ in A, which can be viewed as a root of m. Let $W = (\omega_1, \dots, \omega_n)$ be an integral basis of A. Let $f = \frac{1}{uv^d}\sum_{k=1}^n a_k\omega_k \in A$ with $d > 1$ and $u, v, a_1, \dots, a_n \in K[x]$ such that $\gcd(u, v) = \gcd(v, v') = \gcd(v, a_1, \dots, a_n) = 1$. We seek $b_1, \dots, b_n, c_1, \dots, c_n \in K[x]$ such that for $g = \frac{1}{v^{d-1}}\sum_{k=1}^n b_k\omega_k$ and $h = \frac{1}{uv^{d-1}}\sum_{k=1}^n c_k\omega_k$ such that

$$f = g' + h.$$

The g in this equation can be found by solving a certain linear system over $K[x]/\langle v\rangle$, and once g is known, h can be computed as $h = f - g'$. Let $e \in K[x]$ and $M \in K[x]^{n\times n}$ be such that $eW' = MW$, and assume (without loss of generality) that $e \mid uv$. Then the coefficient vector $b = (b_1, \dots, b_n)$ of $v^{d-1}g$ satisfies

$$b\big(uve^{-1}M - (k-1)uv'I_n\big) \equiv (a_1, \dots, a_n) \bmod v \tag{2.1}$$

and using that W is an integral basis, it can be shown that this linear system has a unique solution, see [36, 15] for further details. Applying the process repeatedly, we can eliminate all multiple poles from the integrand, i.e., we can find g and h such that $f = g' + h$ and $h = q^{-1}\sum_{k=1}^n p_k\omega_k$ for some polynomials $p_1, \dots, p_n, q$ with q squarefree.

If W is not an integral basis, the linear system (2.1) may or may not have a unique solution.

Example 1. *Let $m = y^2 - x$ and $f = \frac{y}{(x+1)x^2}$.*

1. *For $W = (x, xy)$ we have $e = 2x$ and $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Applying the reduction to $f = \frac{1}{(x+1)x^3}xy$ leads to the linear system*

$$b\begin{pmatrix} -2(x+1) & 0 \\ 0 & -(x+1) \end{pmatrix} \equiv (0, 2) \bmod x$$

which has a unique solution.

2. *For* $W = (x, y)$ *we have* $e = 2x$ *and* $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Applying the reduction to* $f = \frac{1}{x^2(x+1)} y$ *leads to the linear system*

$$b \begin{pmatrix} 0 & 0 \\ 0 & -(x+1) \end{pmatrix} \equiv (0, 2) \bmod x,$$

which is solvable, but not uniquely.

3. *For* $W = (x, (x+1)y)$ *we have* $e = 2x(x+1)$ *and* $M = \begin{pmatrix} 2(x+1) & 0 \\ 0 & 3x+1 \end{pmatrix}$. *Applying the reduction to*

$$f = \frac{1}{x^2(x+1)^2}(x+1)y$$

leads to the linear system

$$b \begin{pmatrix} -2x & 0 \\ 0 & -(x+1) \end{pmatrix} \equiv (0, 2) \bmod x(x+1)$$

which has no solutions.

Note that none of the bases above in the example above is an integral basis. However, all the bases consist of integral elements and have the property that e is squarefree. Bronstein [10] calls such a basis a *suitable basis* and observes that whenever we apply Hermite reduction to a suitable basis and find that the linear system (2.1) has no solution, then we can construct from any unsolvable system an integral element of A that does not belong to the $K[x]$-module generated by the elements of W. We can then replace W by a suitable basis of an enlarged $K[x]$-module which also includes A and proceed with the reduction.

Example 2. *We continue the previous example.*

1. *For* $W = (x, xy)$, *no basis update is needed because the linear system has a unique solution.*

2. *For* $W = (x, y)$, *the right kernel element* $(1, 0)$ *of the matrix in the linear system translates into the new integral element* $x+1$, *which does not belong to the* $K[x]$ *module generated by* x *and* y *in* A. *A basis of the module generated by* x, y, *and* $x + 1$ *is* $(1, y)$.

3. *For* $W = (x, (x + 1)y)$, *from the lack of solutions of the linear system it can be deduced that* xy *is an integral element not belonging to the module generated by* x *and* y *in* A. *A basis of the module generated by* x, y, *and* xy *is* (x, y).

Starting from a basis W consisting of integral elements, there can be at most finitely many basis updates before we reach an integral basis. Therefore, it takes at most finitely many basis updates (and possibly fewer than needed for reaching an integral basis) to complete the reduction process. This variant of Hermite reduction, which avoids the potentially expensive computation of an integral basis at the beginning, is called lazy Hermite reduction. Its final result is a suitable basis $\tilde{W}$ of A and $g, h \in A$ such that $f = g' + h$ and the coefficients of h with respect to $\tilde{W}$ are rational functions with a squarefree common denominator. In the examples above, we may find $\tilde{W} = (1, y)$, $g = -\frac{2y}{x}$, and $h = -\frac{y}{x(x+1)}$.

One of the key features of Hermite reduction is that we can decide the integrability problem. For example, if we write a rational function $f \in K(x)$ in the form $f = g' + h$ for some $g, h \in K(x)$ where h has a squarefree denominator and numerator degree less than denominator degree, then f admits an integral in $K(x)$ if *and only if* $h = 0$. Trager generalizes this criterion to algebraic functions as follows. By a change of variables, he first ensures that the integrand f has a double root at infinity. Then he performs Hermite reduction with respect to an integral basis that is normal at infinity. If this gives g, h such that $f = g' + h$, then f is integrable in A if *and only if* $h = 0$ [36, 15].

Unfortunately, this criterion does not extend to the lazy version of Hermite reduction. Even if we produce a double root of the integrand at infinity and make the suitable basis normal at infinity (which amounts to a local integral basis computation that we actually would prefer to avoid altogether), a nonzero remainder h does not imply that f is not integrable.

Example 3. *Let $m = y^2 - x$ and $f = \frac{y}{x^3}$. For $W = (1, (x^2 + 1)y)$ we have $e = 2x(x^2 + 1)$ and $M = \begin{pmatrix} 0 & 0 \\ 0 & 5x^2 + 1 \end{pmatrix}$. The lazy Hermite reduction finds $g = -\frac{2(1+x^2)y}{3x^2}$ and $h = \frac{y}{3x}$. Here $f = \left(-\frac{2y}{3x^2}\right)'$ is integrable and has a root of order ≥ 2 at infinity, but the remainder h is nonzero.*

Somewhat surprisingly, Bronstein does not address this issue in his report [10]. In the following sections, we develop a fix using the technique of polynomial reduction.

3 Suitable bases

Let $A = K(x)[y]/\langle m \rangle$ with $m \in K[x, y]$ being an irreducible polynomial over $K(x)$. Let $\bar{K}$ be the algebraic closure of K. If $n = \deg_y(m)$, there are n distinct solutions in the field

$$\bar{K}\langle\langle x - a \rangle\rangle := \bigcup_{r \in \mathbb{N} \setminus \{0\}} \bar{K}(((x - a)^{1/r}))$$

of formal Puiseux series around $a \in \bar{K}$. There are also n distinct solutions in the field

$$\bar{K}\langle\langle x^{-1} \rangle\rangle := \bigcup_{r \in \mathbb{N} \setminus \{0\}} \bar{K}((x^{-1/r}))$$

of formal Puiseux series around ∞. For a fixed $a \in \bar{K} \cup \{\infty\}$, let $y_1, \ldots, y_n$ be all n roots of m in $\bar{K}\langle\langle x - a \rangle\rangle$ (or $\bar{K}\langle\langle x^{-1} \rangle\rangle$ if $a = \infty$). There are n distinct $K(x)$-embeddings $\sigma_1, \ldots, \sigma_n$ of A into $\bar{K}\langle\langle x - a \rangle\rangle$ (or $\bar{K}\langle\langle x^{-1} \rangle\rangle$ if $a = \infty$) such that

$\sigma_i(f(y)) = f(y_i)$ for any $f \in A$. Then for each $a \in \bar{K} \cup \{\infty\}$, we can associate $f \in A$ with n series $\sigma_i(f)$ for $i = 1, \ldots, n$. Moreover, if we equip the fields A, $\bar{K}\langle\langle x-a\rangle\rangle$ and $\bar{K}\langle\langle x^{-1}\rangle\rangle$ with natural differentiations with respect to x, then each embedding σ_i is a differential homomorphism, i.e., $\sigma_i(f') = \sigma_i(f)'$ for any $f \in A$.

A nonzero Puiseux series around $a \in \bar{K}$ can be written in the form

$$P = \sum_{i \geq 0} c_i (x-a)^{r_i},$$

where $c_i \in \bar{K}, c_0 \neq 0$ and $r_i \in \mathbb{Q}$. The *valuation map* ν_a on $\bar{K}\langle\langle x-a\rangle\rangle$ is defined by $\nu_a(P) = r_0$ if P is nonzero and $\nu_a(0) = \infty$. Replacing $x-a$ by $\frac{1}{x}$, we get the *valuation* map ν_∞ on $\bar{K}\langle\langle x^{-1}\rangle\rangle$. A series P in $\bar{K}\langle\langle x-a\rangle\rangle$ or $\bar{K}\langle\langle x^{-1}\rangle\rangle$ is called *integral* if its valuation is nonnegative. The value function $\operatorname{val}_a : A \to \mathbb{Q} \cup \{\infty\}$ is defined by

$$\operatorname{val}_a(f) := \min_{i=1}^{n} \nu_a(\sigma_i(f))$$

for any $f \in A$. An element $f \in A$ is called *(locally) integral* at $a \in \bar{K} \cup \{\infty\}$ if $\operatorname{val}_a(f) \geq 0$, i.e., every series associated to f (around a) is integral. The element $f \in A$ is called *(globally) integral* if $\operatorname{val}_a(f) \geq 0$ at every $a \in \bar{K}$ ("at all finite places"), i.e., f is locally integral at every $a \in \bar{K}$. A basis of the $K[x]$-module of all integral elements of A is called an integral basis of A. The elements of A that are locally integral at some point $a \in K$ form a $K(x)_a$ module, where $K(x)_a$ is the subring of $K(x)$ consisting of all rational functions which do not have a pole at a. In the case $a = \infty$, $K(x)_a$ is the ring of all rational functions p/q with $\deg_x(p) \leq \deg_x(q)$. A basis of the $K(x)_a$-module of locally integral elements of A is called a local integral basis at a.

For a series $P \in \bar{K}\langle\langle x-a\rangle\rangle$, the smallest positive integer r such that $P \in \bar{K}((\,(x-a)^{1/r}))$ is called the *ramification index* of P. If $r > 1$, the series P is said to be *ramified*. For an element $f \in A$, a point $a \in \bar{K}$ is called a *branch point* of f if one of the series associated to f around a is ramified.

Let $W = (\omega_1, \ldots, \omega_n)$ be a $K(x)$-vector space basis of A. Throughout this section, let $e \in K[x]$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n \times n}$ be such that

$$eW' = MW$$

and $\gcd(e, m_{1,1}, m_{1,2}, \ldots, m_{n,n}) = 1$. As already mentioned in the previous section, W is a called a *suitable basis* if e is squarefree and ω_i's are integral for each i. Every integral basis is a suitable basis, see [15, Lemma 3]. Now we explore some further properties of such bases.

Lemma 4. *Let W be an integral basis of A. Let $e \in K[x]$ and $M \in K[x]^{n \times n}$ be such that $eW' = MW$. If $a \in \bar{K}$ is a root of e, there exists $\omega \in W$ such that a is a branch point of ω.*

Proof. Let $W = (\omega_1, \ldots, \omega_n)$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n \times n}$, and let a be a root of e. By $\gcd(e, m_{1,1}, \ldots, m_{n,n}) = 1$, there is $i \in \{1, \ldots, n\}$ such that

$$\omega_i' = \frac{1}{e} \sum_{j=1}^{n} m_{i,j} \omega_j,$$

where a is not a common root of $m_{i,1}, \ldots, m_{i,n}$.

From the above expression of ω_i', we get ω_i' does not belong to the module generated by W over $K[x]$. Since W is a local integral basis at a, it follows that ω_i is not locally integral at a. Thus $\mathrm{val}_a(\omega_i') < 0$.

If a were not a branch point of ω_i, then all Puiseux series at a associated to ω_i were power series, and then all Puiseux series at a associated to ω_i' were power series, implying $\mathrm{val}_a(\omega_i') \geq 0$. As we have seen before that $\mathrm{val}_a(\omega_i') < 0$, it follows that a must be a branch point. ∎

In order to give a converse of Lemma 4, we consider the series associated to an algebraic function. For a ramified Puiseux series $P = \sum_{i \geq 0} c_i (x-a)^{r_i} \in \bar{K}\langle\langle x-a \rangle\rangle$, let

$$\delta(P) := \min\{r_i \mid r_i \in \mathbb{Q} \setminus \mathbb{Z},\, i \geq 0\}$$

be the minimal fractional exponent of P. Define $\delta(P) = \infty$ if the series P is not ramified. Then $\delta(P') = \delta(P) - 1$. Similar as the valuation of a series, the function δ satisfies

$$\delta(P+Q) \geq \min\{\delta(P), \delta(Q)\}$$

for any $P, Q \in \bar{K}\langle\langle x-a \rangle\rangle$.

Lemma 5. *Let W be a $K(x)$-vector space basis of A. Let $e \in K[x]$ and $M \in K[x]^{n\times n}$ be such that $eW' = MW$. Let $a \in \bar{K}$. If there exists some $\omega \in W$ such that a is a branch point of ω, then a is a root of e.*

Proof. Suppose that a is a branch point of some $\omega \in W$. It implies that for such an element ω, there is a $K(x)$-embedding σ of A into the field of Puiseux series (around a) such that the series $\sigma(\omega)$ is ramified. Let $r = \min\{\delta(\sigma(\omega)) | \omega \in W\}$. Then $r \in \mathbb{Q} \setminus \mathbb{Z}$. Choose an element $\omega_i \in W$ such that $\delta(\sigma(\omega_i)) = r$. Then $\sigma(\omega_i)$ must be ramified. After differentiating the series $\sigma(\omega_i)$, its minimal fractional exponent decreases strictly by 1. This means

$$\delta(\sigma(\omega_i)') = \delta(\sigma(\omega_i)) - 1 = r - 1.$$

Let $M = ((m_{i,j}))_{i,j=1}^{n} \in K[x]^{n\times n}$. Since σ is a differential homomorphism and a $K(x)$-embedding, we have

$$\sigma(\omega_i)' = \sigma(\omega_i') = \frac{1}{e} \sum_{j=1}^{n} m_{i,j} \sigma(\omega_j).$$

After multiplying by a polynomial, the minimal fractional exponent of a series will not decrease. So if a is not a root of e, then

$$\delta(\sigma(\omega_i)') \geq \min_{j=1}^{n} \delta(m_{i,j}\sigma(\omega_j)) \geq \min_{j=1}^{n} \delta(\sigma(\omega_j)) = r.$$

This leads to a contradiction. Thus a must be a root of e. ∎

We now show that the polynomial e does not depend on the choice of the basis of A but only on the $K[x]$-submodule it generates. Let U and V be two $K(x)$-vector space bases of A. Let $e_u, e_\nu \in K[x]$ and $M_u, M_\nu \in K[x]^{n\times n}$ be such that $e_u U' = M_u U$ and $e_\nu V' = M_\nu V$. Suppose that U and V generate the same submodule of A over $K[x]$. Then there exists a matrix $T \in K[x]$ such that $U = TV$ and T is an invertible matrix over $K[x]$. Taking derivatives, we get

$$U' = \left(T'T^{-1} + T\frac{1}{e_\nu}M_\nu T^{-1}\right)U = \frac{1}{e_u}M_u U.$$

Since $T, T^{-1} \in K[x]^{n\times n}$, we have e_u divides e_ν. Similarly the fact that $V = RU$ with $R = T^{-1} \in K[x]^{n\times n}$ implies that e_ν divides e_u. Thus $e_u = e_\nu$ when e_u, e_ν are monic.

Lemma 6. *Let W be a suitable basis and U be an integral basis of A. Let $W = TU$ with $T \in K[x]^{n\times n}$. Let $e \in K[x]$ and $M \in K[x]^{n\times n}$ be such that $eW' = MW$. If $a \in \bar{K}$ is a root of $\det(T)$, then a is a root of e. That means if W is not a local integral basis at $a \in \bar{K}$, then a is a root of e.*

Proof. First we shall make a change of bases. Consider the Smith normal form of the matrix T. This means there are two matrices $P, Q \in K[x]^{n\times n}$ such that both P and Q are invertible over $K[x]$ and $PTQ = \Lambda$ for some diagonal matrix $\Lambda \in K[x]^{n\times n}$. Then $W = P^{-1}\Lambda Q^{-1}U$ and hence $PW = \Lambda(Q^{-1}U)$. Replacing PW and $Q^{-1}U$ by W and U, respectively, we may assume

$$W = TU$$

with $T = \operatorname{diag}(r_1, \ldots, r_n) \in K[x]^{n\times n}$. This operation does not change the module generated by W or U, respectively. Note that $\det(PTQ)$ and $\det(T)$ are equal up to a unit in K. It suffices to prove the result for such special bases W and U.

Let $e_u \in K[x]$ and $M_u = ((a_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be such that $e_u U' = M_u U$. Differentiating both sides of $W = TU$ yields that

$$W' = \left(T'T^{-1} + T\frac{1}{e_u}M_u T^{-1}\right)W = \frac{1}{e}MW. \tag{3.1}$$

Substituting $T = \operatorname{diag}(r_1, \ldots, r_n)$ and $M_u = ((a_{i,j}))_{i,j=1}^n$, we get the the i-th diagonal entry of $\frac{1}{e}M$ is $\frac{r_i'}{r_i} + \frac{a_{i,i}}{e_u}$.

Let $a \in \bar{K}$ be a root of $\det(T)$. Since $\det(T) = r_1 r_2 \cdots r_n$, there is $i \in \{1, 2, \ldots, n\}$ such that a is a root of r_i. If a is not a root of e_u, then a must be a pole of the entry $\frac{r_i'}{r_i} + \frac{a_{i,i}}{e_u}$. Since e is a common multiple of the denominator of all the entries of $\frac{1}{e}M$, we have a is a root of e. Now suppose that a is a root of e_u. Since U is an integral basis, Lemma 4 implies that there is $u \in U$ such that a is a branch point of u. Then one of the series associated to u is ramified. In other words, such a series has at least one fractional exponent. Since $W = TU$ for some invertible matrix T in $K(x)^{n\times n}$, there is $\omega \in W$ such that one of the series associated to ω is also ramified. Therefore, a is a branch point of ω. By Lemma 5, a is root of e. ∎

Since the inverse of the matrix T is $T^{-1} = \frac{1}{\det(T)} T^*$, where T^* is the adjoint matrix of T, the least common multiple of the denominator of the entries of T^{-1} is bounded by $\det(T)$. By investigating Equation (3.1), we see a possible root of e must come from $\det(T)$ and e_u. Combining the last paragraph in the argument of Lemma 6, we get those roots of $\det(T)$ and e_u are exactly roots of e. In particular, when W is a suitable basis, the corresponding polynomial e is the squarefree part of the product $\det(T)e_u$. Therefore, if the submodule generated by a suitable basis is larger, then e is smaller.

4 Polynomial Reduction

Polynomial reduction is a postprocessing step for Hermite reduction which was first introduced for hyperexponential terms [6] and has later been formulated for algebraic and fuchsian D-finite functions [15, 17]. For the latter cases, like for Trager's criterion, integral bases that are normal at infinity are employed. Our goal in this section is to relax this requirement to suitable bases, so as to obtain a version of polynomial reduction which can serve as a natural continuation of the lazy Hermite reduction process and provides the feature that the final remainder is zero if and only if the integrand is integrable.

Let $h \in A$ be the remainder of lazy Hermite reduction (see Section 2) with respect to a suitable basis $W = (\omega_1, \dots, \omega_n)$. Then h can be written in the form

$$h = \sum_{i=1}^{n} \frac{h_i}{de} \omega_i \tag{4.1}$$

with $h_i, d \in K[x]$ such that $\gcd(d, e) = \gcd(h_1, \dots, h_n, d) = 1$ and d is squarefree. If h is integrable in A, we shall prove that d is a constant in K. When W is an integral basis, this result was proved in [15, Lemma 9]. The following lemma is a local version.

Lemma 7. *Let $h \in A$ be in the form (4.1). If h is integrable in A and W is a local integral basis at $a \in \bar{K}$, then d has no root at a.*

Proof. Suppose h is integrable in A. Then there exists $H = \sum_{i=1}^{n} b_i \omega_i \in A$ with $b_i \in K(x)$ such that $h = H'$. It suffices to show that every b_i has no pole at a. Otherwise H has a pole at a, because W is a local integral basis at a. Then h has at least a double pole at a. This contradicts the fact that d, e are squarefree. Thus d has no root at a. ∎

Theorem 8. *Let $h \in A$ be in the form (4.1). If h is integrable in A, then d is in K.*

Proof. Suppose that h is integrable in A. In order to show d is a constant, we show that for any $a \in \bar{K}$, a is not a root of d. If W is a local integral basis at a, then the conclusion follows from Lemma 7. Now we assume that W is not a local integral basis at a. By Lemma 6, we know that a is a root of e. Since $\gcd(d, e) = 1$, it follows that a is not a root of d. ∎

To further reduce the lazy Hermite remainder, we give an upper bound for the denominator of its integral if h is integrable in A. This bound does not depend on the integrand h, but only depends on the discriminant of a suitable basis in its representation.

Recall that the *discriminant* of a tuple $W = (\omega_1, \dots, \omega_n)$ of elements of A is defined by the determinant

$$\operatorname{Disc}(W) = \det(\operatorname{Tr}(\omega_i \omega_j)),$$

where Tr is the trace map from A to $K(x)$. If the ω_i's are integral functions, then their traces are polynomials, and thus the discriminant is a polynomial. If $W = TU$ where $T \in K[x]^{n \times n}$ is the change of basis matrix, then $\operatorname{Disc}(W) = \operatorname{Disc}(U) \det(T)^2$.

Lemma 9. *Let $h \in A$ be in the form (4.1). Let U be an integral basis of A and let $T \in K[x]^{n\times n}$ be such that $W = TU$. If h is integrable in A, i.e., there exists $u \in K[x]$ and $q = (q_1, \dots, q_n) \in K[x]^n$ such that*

$$h = \left(\sum_{i=1}^{n} \frac{q_i}{u} \omega_i \right)'$$

and $\gcd(q_1, \dots, q_n, u) = 1$*, then u divides* $\det(T)$*. Hence u^2 divides* $\operatorname{Disc}(W)/\operatorname{Disc}(U)$.

Proof. Since h has a squarefree denominator with respect to W and $W = TU$ with $T \in K[x]^{n\times n}$, it follows that h has a squarefree denominator with respect to U. This means h is also a remainder with respect to the integral basis U. Assume that h is integrable in A. Then Lemma 7 implies that $h = (\sum_{i=1}^{n} a_i u_i)'$ with $a_i \in K[x]$. Write $U = \frac{1}{r} RW$ with $r \in K[x]$, $R = ((b_{i,j}))_{i,j=1}^{n} \in K[x]^{n\times n}$ and $\gcd(r, b_{1,1}, b_{1,2}, \dots, b_{n,n}) = 1$. Then

$$\sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{n} a_i \frac{1}{r} \sum_{j=1}^{n} b_{i,j} \omega_j = \sum_{j=1}^{n} \frac{1}{r} \left(\sum_{i=1}^{n} a_i b_{i,j} \right) \omega_j.$$

Thus u divides r because $a_i, b_{i,j} \in K[x]$ and two antiderivatives of h only differ by a constant.

Since $W = TU$, we have $\frac{1}{r} R = T^{-1} = \frac{1}{\det(T)} T^*$. So r divides $\det(T)$ and hence u also divides $\det(T)$. Moreover, $\operatorname{Disc}(W) = \operatorname{Disc}(U)\det(T)^2$. Since u_i's and ω_i's are integral elements, the discriminants $\operatorname{Disc}(W), \operatorname{Disc}(U)$ are polynomials in $K[x]$. Therefore $\det(T)^2$ divides the polynomial $\operatorname{Disc}(W)/\operatorname{Disc}(U)$, so does u^2. ∎

If we already know that W is an integral basis, then the quotient $\operatorname{Disc}(W)/\operatorname{Disc}(U)$ is a unit in K. By Lemma 9, we see u is a constant. If no integral basis is available, then from the condition u^2 divides $\operatorname{Disc}(W)$, an upper bound of u can be chosen as the product over $p^{\lfloor r/2 \rfloor}$ where p runs through the irreducible factors of $\operatorname{Disc}(W)$ and r is the multiplicity of p in $\operatorname{Disc}(W)$.

Example 10. *Let* $m = y^2 - x$ *and* $h = \frac{y}{x}$.

1. *For* $W = (1, y)$*, we have* $\mathrm{Disc}(U) = 4x$*. So we can choose* $u = 1$*. In fact,* W *is an integral basis and* $h = (2y)'$*.*
2. *For* $W = (1, (x^2+1)y)$*, we have* $\mathrm{Disc}(W) = 4(x^2+1)^2 x$*. So we can choose* x^2+1 *as an upper bound of* u*. In fact,* $h = (\frac{1}{x^2+1} 2(x^2+1)y)'$*.*

If h is integrable in A, we shall reduce h to zero, otherwise we hope to remove all possible integrals whose denominators are bounded by u. Before that we write h as two parts with denominators d and e, respectively. By the extended Euclidean algorithm, there are polynomials $r_i, s_i \in K[x]$ such that $h_i = r_i d + s_i e$ and $\deg_x(r_i) < \deg_x(d)$. Then the lazy Hermite reduction remainder h decomposes as

$$h = \sum_{i=1}^{n} \frac{h_i}{de}\omega_i = \sum_{i=1}^{n} \frac{r_i}{d}\omega_i + \sum_{i=1}^{n} \frac{s_i}{e}\omega_i. \tag{4.2}$$

Our second goal is to confine the s_i's to a finite-dimensional vector space over K. This is a generalization of the *polynomial reduction* in [15]. In this process, we shall rewrite the second term of the remainder h in (4.2) with respect to another basis. This new basis is used to perform the polynomial reduction and obtain the following additive decomposition.

Definition 11. *Let* f *be an element in* A*. Let* W *and* V *be two* $K(x)$*-vector space bases of* A*. Let* $e, a \in K[x]$ *and* $M, B \in K[x]^{n\times n}$ *be such that* $eW' = MW$ *and* $aV' = BV$*. Suppose that* f *can be decomposed into*

$$f = g' + \frac{1}{d}PW + \frac{1}{a}QV, \tag{4.3}$$

where $g \in A$*,* $d \in K[x]$ *is squarefree and* $\gcd(d, e) = 1$*,* $P, Q \in K[x]^n$ *with* $\deg_x(P) < \deg_x(d)$ *and* $Q \in N_V$*, which is a finite-dimensional* K*-vector space. The decomposition in* (4.3) *is called an* additive decomposition *of* f *with respect to* x *if it satisfies the condition that* P, Q *are zero if and only if* f *is integrable in* A*.*

Given an algebraic function, its additive decomposition always exists for some integral basis W and some basis V which is a local integral basis at infinity, see [15, Theorem 14]. We shall show below that we can always find an additive decomposition with respect to certain suitable bases.

Let V be a $K(x)$-vector space basis of A, and let $a \in K[x]$ and $B = ((b_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be such that $aV' = BV$. We do not require that $\gcd(a, b_{1,1}, b_{1,2}, \ldots, b_{n,n}) = 1$. Let $u \in K[x]$ and $p \in K[x]^n$. A direct calculation yields that

$$\left(\frac{p}{u}V\right)' = \left(\frac{p}{u}\right)' V + \frac{p}{u}V' = \frac{aup' - au'p + upB}{u^2 a}V. \tag{4.4}$$

This motivates the following definition.

Definition 12. *For a given polynomial* $u \in K[x]$*, let the map* $\phi_V : K[x]^n \to u^{-2}K[x]^n$ *be defined by*

$$\phi_V(p) = \frac{1}{u^2}(aup' - au'p + upB)$$

for any $p \in K[x]^n$*. We call* ϕ_V *the map for* polynomial reduction *with respect to* u *and* V*, and call the subspace*

$$\operatorname{im}(\phi_V) = \{\phi_V(p) \mid p \in K[x]^n\} \subseteq u^{-2}K[x]^n$$

the subspace for polynomial reduction *with respect to* u *and* V*.*

Note that, by construction, if $q = \phi_V(p)$, then $\frac{q}{a}V = (\frac{p}{u}V)'$. So $\frac{q}{a}V$ is integrable.

We can always view an element of $K[x]^n$ (resp. $K[x]^{n\times n}$) as a polynomial in x with coefficient in K^n (resp. $K^{n\times n}$). In this sense, we use the notation $\operatorname{lt}(\cdot)$ for the leading term of a vector (resp. matrix). For example, if $p \in K[x]^n$ is of the form

$$p = p^{(r)}x^r + \cdots + p^{(1)}x + p^{(0)}, \quad p^{(i)} \in K^n,$$

where $p^{(r)} \neq 0$, then $\deg_x(p) = r$, $\operatorname{lt}(p) = p^{(r)}x^r$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of K^n. Then the $K[x]$-module $K[x]^n$ viewed as K-vector space is generated by

$$\mathcal{S} := \{e_i x^j \mid 1 \leq i \leq n, j \in \mathbb{N}\}.$$

Definition 13. *Let* N_V *be the* K*-subspace of* $K[x]^n$ *generated by*

$$\{t \in \mathcal{S} \mid t \neq \operatorname{lt}(p) \text{ for all } p \in \operatorname{im}(\phi_V) \cap K[x]^n\}$$

Then $K[x]^n = (\operatorname{im}(\phi_V) \cap K[x]^n) \oplus N_V$*. We call* N_V *the* standard complement *of* $\operatorname{im}(\phi_V)$*. For any* $p \in K[x]^n$*, there exists* $p_1 \in K[x]^n$ *and* $p_2 \in N_V$ *such that*

$$\frac{p}{a}V = \left(\frac{p_1}{u}V\right)' + \frac{p_2}{a}V.$$

This decomposition is called the polynomial reduction of p *with respect to* u *and* V*.*

If $u = 1$, then the polynomial reduction with respect to u falls back to the situation discussed in [15].

Proposition 14. *Let* $a \in K[x]$ *and* $B \in K[x]^{n\times n}$ *be such that* $aV' = BV$*, as before. If* $\deg_x(B) \leq \deg_x(a) - 1$*, then* N_V *is a finite dimensional* K*-vector space.*

Proof. Consider the map $\tilde{\phi}_V : K[x]^n \to K[x]^n$ defined by

$$\tilde{\phi}_V(p) = aup' - au'p + upB$$

for any $p \in K[x]^n$. Then $\tilde{\phi}_V(p) = u^2\phi_V(p)$. It is easy to check that $\operatorname{im}(\phi_V) \cap K[x]^n$ and $\operatorname{im}(\tilde{\phi}_V) \cap u^2K[x]^n$ are isomorphic as K-vector spaces via the multiplication by u^2. Considering the codimension of subspaces in $K[x]^n$ over K, we have the formula

$$\begin{aligned}\dim_K(N_V) &= \operatorname{codim}_K\left(\operatorname{im}(\phi_V) \cap K[x]^n\right)\\ &= \operatorname{codim}_K\left(\operatorname{im}(\tilde{\phi}_V) \cap u^2K[x]^n\right)\\ &\leq \operatorname{codim}_K\left(\operatorname{im}(\tilde{\phi}_V)\right) + \operatorname{codim}_K\left(u^2K[x]^n\right).\end{aligned}$$

Let $\mu := \deg_x(a) - 1$, $\ell := \deg_x(u)$ and $s := \deg_x(p)$. Since $\deg_x(aup') = \deg_x(au'p) = s + \ell + \mu$, we have

$$\deg_x(\tilde{\phi}_V(p)) \leq s + \ell + \max\{\mu, \deg_x(B)\}.$$

By an argument analogous to [15, Proposition 12], we distinguish two cases $\deg_x(B) < \mu$ and $\deg_x(B) = \mu$, and get the codimension of $\operatorname{im}(\tilde{\phi}_V)$ is finite. Since $K[x]^n/(u^2K[x]^n) \cong (K[x]/u^2K[x])^n$, the codimension of $u^2K[x]^n$ is also finite. ∎

The condition $\deg_x(B) \leq \deg_x(a) - 1$ is satisfied if V is a local integral basis at infinity, but may not hold for an arbitrary basis. So we introduce a weaker basis that satisfies the degree condition. This is an analogue of suitable basis at infinity.

Definition 15. *A basis V of A is called* suitable at infinity *if for $a \in K[x]$ and $B \in K[x]^{n\times n}$ such that $aV' = BV$ we have $\deg_x(B) < \deg_x(a)$.*

There always exists a basis which is suitable at infinity. We can find such a basis as follows. Start from an arbitrary $K(x)$-basis $V = (v_1, \ldots, v_n)$ of the function field A. We can make its elements $v_1, \ldots, v_n$ integral at infinity by replacing each v_i by $x^{-\tau}v_i$ for a sufficiently large $\tau \in \mathbb{N}$. Consider $a \in K[x]$ and $B \in K[x]^{n\times n}$ be such that $aV' = BV$. If $\deg_x(B) < \deg_x(a)$, we are done. If not, consider a row b in B with $\deg_x(b) \geq \deg_x(a)$ and set $v = xa^{-1}bV$. Then v is integral at infinity (because at infinity differentiating increases the valuation) but it does not belong to the $K(x)_\infty$-module generated by V (because of $\deg_x(b) \geq \deg_x(a)$). Therefore the $K(x)_\infty$-module generated by V and v is strictly larger than the $K(x)_\infty$-module generated by V. Replace V by a basis of this enlarged module, and update a and B such that $aV' = BV$. If we now have $\deg_x(B) < \deg_x(a)$, we are done, otherwise repeat the process just described.

The iteration will terminate because with every update of V the $K(x)_\infty$-module generated by it gets enlarged, and since all these modules are contained in the module of elements of A that are integral at infinity, after at most finitely many updates V will be a local integral basis at infinity. At least then, the desired degree condition must hold, because otherwise there would be an integral element which is not a $K(x)_\infty$-linear combination of the basis elements, in contradiction to the basis being integral at infinity.

Theorem 16. *Let f be an element in A. Let V be a basis of A which is suitable at infinity. Then there exists a suitable basis W of A such that f admits an additive decomposition in (4.3) with respect to the bases W and V.*

Proof. We present a constructive proof to show the existence of an additive decomposition of f. After performing the lazy Hermite reduction on f, we get

$$f = \tilde{g}' + \frac{1}{d}PW + \frac{1}{e}UW$$

where W is a suitable basis, $P = (r_1, \dots, r_n) \in K[x]^n$ and $U = (s_1, \dots, s_n) \in K[x]^n$ with r_i, s_i introduced in (4.2). Let $W = \frac{1}{b}CV$ for some $b \in K[x]$ and $C \in K[x]^{n\times n}$. Let $a \in K[x]$ and $B \in K[x]^{n\times n}$ be such that $aV' = BV$. Multiplying a and B by some polynomial, we may assume that a is a common multiple of e and b. Rewriting the remainder in terms of the new basis V, we get

$$\frac{1}{e}UW = \frac{1}{eb}UCV = \frac{1}{a}\tilde{U}V, \tag{4.5}$$

for some $\tilde{U} \in K[x]^n$. By Theorem 9, if f is integrable, there exist $\tilde{u} \in K[x]$ and $R \in K[x]^n$ such that

$$\frac{1}{e}UW = \left(\frac{RW}{\tilde{u}}\right)' = \left(\frac{RCV}{u}\right)', \tag{4.6}$$

where $u = \tilde{u}b$. Next, we apply the polynomial reduction with respect to the polynomial u and decompose $\tilde{U}$ into $\phi_V(\tilde{U}_1) + \tilde{U}_2$ with $\tilde{U}_1, \tilde{U}_2 \in K[x]^n$ and $\tilde{U}_2 \in N_V$. Then we have

$$\frac{1}{a}\tilde{U}V = \left(\frac{1}{u}\tilde{U}_1V\right)' + \frac{1}{a}\tilde{U}_2V. \tag{4.7}$$

We then get the decomposition (4.3) by setting $g = \tilde{g} + \frac{1}{u}\tilde{U}_1V$ and $Q = \tilde{U}_2$.

Assume that f is integrable. Then Theorem 9 implies that $d \in K$. Since $\deg_x(P) < \deg_x(d)$, we have $P = 0$. Combining (4.7), (4.5) and (4.6) yields that

$$\frac{1}{a}QV = \frac{1}{a}\tilde{U}V - \left(\frac{1}{u}\tilde{U}_1V\right)' = \left(\frac{1}{u}\tilde{Q}V\right)', \tag{4.8}$$

where $\tilde{Q} = RC - \tilde{U}_1$. So $Q = \phi_V(\tilde{Q}) \in \operatorname{im}(\phi_V) \cap K[x]^n$. Since $\operatorname{im}(\phi_V) \cap K[x] \cap N_V = \{0\}$, it follows that $Q = 0$. ∎

Example 17. *We continue with Example 3 by applying the polynomial reduction to the lazy Hermite remainder $h = \frac{1}{3x(x^2+1)}(x^2+1)y$. Since* $\operatorname{Disc}(W) = 4(x^2+1)^2x$*, we choose $u = x^2+1$. Note that W is already a suitable basis at infinity. The map for the polynomial reduction with respect to W and u is $\phi(p) = \frac{1}{u^2}(eup' - eu'p + epM)$ for any $p \in K[x]^n$. Then $h = (\frac{1}{3(x^2+1)}2(x^2+1)y)'$ reduces to 0.*

5 Reduction-Based Telescoping

Lazy Hermite reduction in combination with the polynomial reduction just described can be used for deciding whether a given algebraic function admits an algebraic integral. Most algebraic functions don't. The next question of interest may then be whether the algebraic function at hand can be deformed in some way to a related one that is algebraically integrable. Creative telescoping produces such a deformation. It is applicable to functions f which besides the integration variable x involve some other parameter t. The task of creative telescoping is to find a nonzero operator $L(t, D_t)$ such that $L(t, D_t) \cdot f$ is integrable. Such an operator is called a *telescoper* for f.

Several techniques are known for computing such a telescoper. The so-called reduction based approach is one of them, and it has attracted a lot of attention in recent years. It was first proposed for rational functions [5]. Given $f = p/q \in K(x)$ with $K = C(t)$, we can use Hermite reduction to find $g_i, h_i \in K(x)$ $(i = 0, 1, 2, \dots)$ such that $D_t^i f = D_x g_i + h_i$. If $c_0, \dots, c_r \in K$ are such that $c_0 h_0 + \dots + c_r h_r = 0$, then

$$(c_0 + \dots + c_r D_t^r) \cdot f = D_x \cdot (c_0 g_0 + \dots + c_r g_r),$$

so the operator $L = c_0 + \dots + c_r D_t^r$ is a telescoper for f.

Some recomputation can be avoided by observing that instead of $D_t^i f$ we may as well integrate $D_t h_{i-1}$, because D_x and D_t commute. With this optimization, reduction based telescoping can be summarized as follows.

Algorithm 18. *Input: a function f depending on x and t;*
Output: a telescoper for f

find g_0, h_0 such that $f = g_0' + h_0$.
for $r = 1, 2, 3, \dots$ do:
  if $h_0, \dots, h_{r-1}$ are linearly dependent over K
    return $c_0 + c_1 D_t + \dots + c_{r-1} D_t^{r-1}$ with c_i not all zero and $\sum_{i=0}^{r-1} c_i h_i = 0$.
  find g_r, h_r such that $D_t h_{r-1} = g_r' + h_r$

The termination of this procedure can be secured in two ways. One way is to ensure that the remainders $h_0, h_1, \dots$ belong to a finite dimensional K-vector space. Then the remainders must eventually become linearly dependent. The second way is to ensure that the map which sends f to h such that $f = g' + h$ for some g is K-linear and has the set of all integrable elements as its kernel. It is then guaranteed that the procedure will not miss a telescoper, so it will terminate because we know that for every algebraic function there does exist a telescoper.

Besides for rational functions, both arguments have been worked out for various larger classes of functions [6, 14, 17, 7, 37], including the class of algebraic functions [15]. The version for algebraic functions is uses Trager's Hermite reduction followed by a polynomial reduction, both steps requiring an integral basis of the function field. Using Theorem 16, we will argue that reduction based telescoping also works with lazy Hermite reduction and the variant of polynomial reduction developed in the previous section, with the obvious advantage that no integral bases computation is needed.

For doing so, we must take into consideration that lazy Hermite reduction takes a suitable basis as input but may return the result with respect to an adjusted suitable basis. Let $W_0, W_1, \dots$ denote the suitable bases with respect to which the result of the ith call to lazy Hermite reduction is returned. By supplying W_{i-1} as input to the ith call, we can ensure that the $K[x]$-module generated by W_{i-1} is contained in the $K[x]$-module generated by W_i, for every i. Therefore, if $W_r \neq W_{r-1}$ for some r, we can rewrite all remainders $h_0, \dots, h_{r-1}$ in the bases W_r and V without introducing new denominators. (Note that no update is required for V.) In order to meet the conditions specified in Theorem 16, it may be necessary to rerun the polynomial reduction on the new representations of the old remainders. The termination of the algorithm then follows via the second way indicated above. In order to also justify termination in the first way, it suffices to observe that we can keep V fixed throughout the computation, so the termination follows directly from the finite dimension of N_V.

6 Experiments

For the paper [16] in 2012, we have created a collection of about 100 integration problems, mostly originating from applications in combinatorics [31, 8], and we have compared the performance of six different approaches, including Chyzak's algorithm [18, 19, 26] as implemented by Koutschan [23, 25], Koutschan's ansatz-based method [24], as well as the method based on residue elimination we proposed in [16]. The result of the evaluation was somewhat inconclusive. Most algorithms outperformed the other algorithms at least for some examples. At the same time, the timing differences can be significant.

For the present paper, we have evaluated the performance of reduction based creative telescoping using lazy Hermite reduction on the benchmark set from 2012. The runtime was taken on the same computer (a 64bit Linux server with 24 cores running at 3GHz; in 2012 it had 100G RAM, meanwhile it was upgraded to 700G). The new experiments were performed with a more recent version of Mathematica (Mathematica 12.1.1). Timings and code are available on our website [1].

An interesting observation is that in all examples of the collection (at least those which finished within the specified time limit of 30h), the lazy Hermite reduction procedure never encountered a situation where the linear system (2.1) did not have a unique solution so that a basis change would have been required. In almost all cases, we also found that no basis change was needed before entering the reduction process in order to turn to a suitable basis. Examples where the default basis was not suitable and needed to be adjusted generally took much more time than examples where the default basis was already suitable.

In comparison to the earlier methods, the inclusion of reduction based creative telescoping does not change the general observation that no algorithm is clearly superior to the others. The new approach is faster than the other approaches on some of the examples, while on others it is much worse.

Example 19. *Consider the rational function*

$$f(x,y,t) = (1 - x - y - t + \tfrac{3}{4}(xy + xt + yt))^{-1}.$$

The computation of an annihilating operator for $\operatorname{res}_{x,y} f(x, y, \frac{t}{x^3 y})$ *is completed in about 4.5 seconds by our new approach. The other techniques need at least twice as long and up to 500 seconds. In this example, the telescoper has order 6 and coefficients of degree 15. On the other hand, computing an annihilating operator for* $\operatorname{res}_{x,y} f(x, y, \frac{t}{x^2 y^2})$ *takes about half an hour using the reduction based approach, while all other approaches all need less than 90 seconds. Here the telescoper has order 4 and coefficients of degree 11. It is not clear why the algorithms perform so differently on such closely related input.*

In view of this diverse and unpredictable performance of the various algorithms, it is advantageous to have several independent techniques available. Having more techniques at our disposal increases the chances that a hard instance arising from an application can be completed by at least one of them.

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