Apparent Singularities of D-Finite Systems

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Abstract

We generalize the notions of ordinary points and singularities from linear ordinary differential equations to D-finite systems. Ordinary points and apparent singularities of a D-finite system are characterized in terms of its formal power series solutions. We also show that apparent singularities can be removed like in the univariate case by adding suitable additional solutions to the system at hand. Several algorithms are presented for removing and detecting apparent singularities. In addition, an algorithm is given for computing formal power series solutions of a D-finite system at apparent singularities.

Key words: D-finite system, Gröbner basis, Ordinary point, Formal power series, Apparent singularity, Desingularization

1. Introduction

Linear ordinary differential equations allow easy access to the singularities of their solutions: every point α which is a singularity of some solution f of a differential equation must be a zero of the coefficient of the highest order derivative appearing in the equation,

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or a singularity of one of the other coefficients. For example, x^{-1} is a solution of the equation $xf'(x) + f(x) = 0$, and the singularity at 0 is reflected by the root of the polynomial x in front of the term $f'(x)$ in the equation. Unfortunately, the converse is not true: there may be roots of the leading coefficient which do not indicate solutions that are singular there. For example, all the solutions of the equation $xf'(x) - 5f(x) = 0$ are constant multiples of x^5, and none of these functions is singular at 0.

For a differential equation $p_0(x)f(x) + \cdots + p_r(x)f^{(r)}(x) = 0$ with polynomial coefficients $p_0, \ldots, p_r$ and $p_r \neq 0$, the roots of p_r are called the singularities of the equation. Those roots α of p_r such that the equation has no solution that is singular at α are called *apparent*. In other words, a root α of p_r is apparent if the differential equation admits r linearly independent formal power series solutions in $x - \alpha$. Deciding whether a singularity is apparent is therefore the same as checking whether the equation admits a fundamental system of formal power series solutions at this point. This can be done by inspecting the so-called *indicial polynomial* of the equation at α: if there exists a power series solution of the form $(x - \alpha)^\ell + c_{\ell+1}(x - \alpha)^{\ell+1} + \cdots$, then ℓ is a root of this polynomial.

When some singularity α of an ODE is apparent, then it is always possible to construct a second ODE whose solution space contains all the solutions of the first ODE, and which does not have α as a singularity. This process is called desingularization. The idea can be easily explained as follows. The key observation is that a point α is not a singularity if and only if the indicial polynomial at α is equal to $n(n-1)\cdots(n-r+1)$ and the ODE admits r linearly independent formal power series solutions in $x-\alpha$. As the indicial polynomial at an apparent singularity has only nonnegative integer roots, we can bring it into the required form by adding a finite number of new factors. Adding a factor $n-s$ to the indicial polynomial amounts to adding a solution of the form $(x-\alpha)^s + \cdots$ to the solution space, and this is an easy thing to do using well-known arithmetic of differential operators. See (Abramov, et al, 2006; Barkatou and Maddah, 2015; Chen, et al, 2016; Ince, 1926; Jaroschek, 2013) for an expanded version of this argument and (Abramov, et al, 2006; Abramov and van Hoeij, 1999; Barkatou and Jaroschek, 2018) for analogous algorithms for recurrence equations.

The purpose of the present paper is to generalize the two facts sketched above to the multivariate setting. Instead of a linear ODE, we consider a special class of systems of linear PDEs known as D-finite systems. For such systems, we define the notion of a singularity in terms of the polynomials appearing in them (Definition 3.1). We show in Theorem 3.7 that a point is a singularity of the system unless it admits a basis of formal power series solutions in which the starting terms are as small as possible with respect to some term order. Then a singularity is apparent if the system admits a full basis of power series solutions, the starting terms of which are not as small as possible. We then prove in Theorem 4.7 that apparent singularities can be removed like in the univariate case by adding suitable additional solutions to the system at hand. The operators in the resulting system will be contained in the Weyl closure of the original ideal, but unlike Tsai (2000) we cannot prove that they form a basis of the Weyl closure. Based on Theorems 3.7 and 4.7, we show how to remove a given apparent singularity (Algorithms 5.11 and 5.21), and how to detect whether a given point is an apparent singularity (Algorithm 5.14). At last, we present an algorithm for computing formal power series solutions of a D-finite system at apparent singularities. Part of materials in this paper was presented in the PhD thesis of the fourth author (Zhang, 2017).

2. Preliminaries

In this section, we recall some notions and results concerning linear partial differential operators, Gröbner bases, formal power series, solution spaces and Wronskians for D-finite systems. We also specify notation to be used in the rest of this paper.

2.1. *Rings of differential operators*

Throughout the paper, $\mathbb{N}$ stands for the set of non-negative integers and $\mathbb{Z}^+$ for the set of positive integers. For a finite set S, its cardinality is denoted by $|S|$. For a vector $(v_1, \ldots, v_n)$, its transpose is denoted by $(v_1, \ldots, v_n)^t$. We assume that $\mathbb{K}$ is a field of characteristic zero. For instance, $\mathbb{K}$ is the field of complex numbers. Moreover, $\mathbf{0}$ denotes the zero vector of a finite-dimensional vector space.

Let $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n]$ be the ring of usual commutative polynomials over $\mathbb{K}$. The quotient field of $\mathbb{K}[\mathbf{x}]$ is denoted by $\mathbb{K}(\mathbf{x})$. The *ring of differential operators with rational function coefficients* is denoted by $\mathbb{K}(\mathbf{x})[\partial_1, \ldots, \partial_n]$, in which addition is coefficient-wise and multiplication is defined by associativity via the commutation rules

(i) $\partial_i \partial_j = \partial_j \partial_i$;

(ii) $\partial_i f = f \partial_i + \frac{\partial f}{\partial x_i}$ for each $f \in \mathbb{K}(\mathbf{x})$,

where $\frac{\partial f}{\partial x_i}$ is the usual derivative of f with respect to x_i, $1 \leq i, j \leq n$. This ring is an Ore algebra (Chyzak and Salvy, 1998) and denoted by $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ for brevity.

Another ring is $\mathbb{K}[\mathbf{x}][\boldsymbol{\partial}] := \mathbb{K}[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$, which is a subring of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. We call it the *ring of differential operators with polynomial coefficients* or the *Weyl algebra* (Saito et al, 1999, Section 1.1).

A left ideal I in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ is said to be *D-finite* if the quotient $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/I$ is a finite-dimensional vector space over $\mathbb{K}(\mathbf{x})$. The dimension of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/I$ as a vector space over $\mathbb{K}(\mathbf{x})$ is called the *rank* of I and denoted by $\operatorname{rank}(I)$. For a subset S of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, the left ideal generated by S is denoted by $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]S$. For instance, let

$$I = \mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]\ \{\partial_1 - 1, \partial_2 - 1\}.$$

Then I is D-finite because the quotient $\mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]/I$ is a vector space of dimension one over $\mathbb{Q}(x_1, x_2)$. Thus, $\operatorname{rank}(I) = 1$.

2.2. *Gröbner bases*

Gröbner bases in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ are well known (Chyzak and Salvy, 1998, Section 1.5) and implementations for them are available for example in the Maple package `Mgfun` by Chyzak (2008) and in the Mathematica package `HolonomicFunctions.m` by Koutschan (2010). We briefly summarize some facts about Gröbner bases in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$.

We denote by $\mathrm{T}(\boldsymbol{\partial})$ the commutative monoid generated by $\partial_1, \ldots, \partial_n$. An element of $\mathrm{T}(\boldsymbol{\partial})$ is called a *term*. For a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, the symbol $\boldsymbol{\partial}^{\mathbf{u}}$ stands for the term $\partial_1^{u_1} \cdots \partial_n^{u_n}$, and $\mathbf{u}$ is called its *exponent*. The *order* of $\boldsymbol{\partial}^{\mathbf{u}}$ is defined to be $|\mathbf{u}| := u_1 + \cdots + u_n$. For a nonzero operator $P \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, the *order* of P is defined to be the highest order of the terms that appear in P effectively.

Let $\prec$ be a monomial ordering on $\mathbb{N}^n$ (Cox, et al, 2015, Definition 1, page 55). Since there is a one-to-one correspondence between terms in $\mathrm{T}(\boldsymbol{\partial})$ and elements in $\mathbb{N}^n$, the ordering $\prec$ on $\mathbb{N}^n$ induces an ordering on $\mathrm{T}(\boldsymbol{\partial})$ with $\boldsymbol{\partial}^{\mathbf{u}} \prec \boldsymbol{\partial}^{\mathbf{v}}$ if and only if $\mathbf{u} \prec \mathbf{v}$. For brevity, we fix an ordering $\prec$ on $\mathbb{N}^n$ in the rest of this paper, and use the graded lexicographic order with $\partial_1 \prec \cdots \prec \partial_n$ in examples, unless otherwise stated.

For a nonzero element $P \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, the *head term* of P, denoted by $\mathrm{HT}(P)$, is the highest term appearing in P. The coefficient of $\mathrm{HT}(P)$ is called the *head coefficient* of P and is denoted by $\mathrm{HC}(P)$. For a subset S of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, we denote by $\mathrm{HT}(S)$ and $\mathrm{HC}(S)$ the sets of head terms and head coefficients of nonzero elements in S, respectively. For a left ideal $I \subset \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, a term is said to be *parametric* if it does not belong to $\mathrm{HT}(I)$. The set of exponents of all parametric terms of I is referred to as the set of *parametric exponents* of I and denoted by $\mathrm{PE}(I)$. If I is D-finite, then its rank is equal to $|\mathrm{PE}(I)|$.

Given a Gröbner basis G of I, an exponent $\mathbf{u}$ belongs to $\mathrm{PE}(I)$ if and only if $\boldsymbol{\partial}^{\mathbf{u}}$ is not divisible by any term in $\mathrm{HT}(G)$. We say that G is *reduced* if $\mathrm{HT}(g)$ does not divide any term appearing in g' for all $g, g' \in G$ with $g \neq g'$.

Let $P \in \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be in the form $P = a_0\boldsymbol{\partial}^{\mathbf{u}_0} + a_1\boldsymbol{\partial}^{\mathbf{u}_1} + \cdots + a_m\boldsymbol{\partial}^{\mathbf{u}_m}$, where $a_0, \ldots, a_m$ are nonzero elements of $\mathbb{K}[\mathbf{x}]$ and $\mathbf{u}_0, \ldots, \mathbf{u}_m$ are distinct. We say that P is *primitive* if $\gcd(a_0, a_1, \ldots, a_m) = 1$. A Gröbner basis G in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ is said to be *primitive* if it is reduced and its elements are primitive in $\mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$. Every nontrivial left ideal in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ has a primitive Gröbner basis.

Remark 2.1. Assume that G and G' are two primitive Gröbner bases of a left ideal. Then $\mathrm{HT}(G) = \mathrm{HT}(G')$, because both G and G' are reduced. For $g \in G$ and $g' \in G'$ with the same head term, g and g' are linearly dependent over $\mathbb{K}$, because they are primitive.

2.3. Formal power series

Let $\mathbb{K}[[\mathbf{x}]]$ be the ring of formal power series in $x_1, \ldots, x_n$. For an operator $P \in \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ and a series $f \in \mathbb{K}[[\mathbf{x}]]$, the usual partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ induce a natural action of P on f, which is denoted by $P(f)$. In particular,

$$PQ(f) = P(Q(f)) \tag{1}$$

with $Q \in \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$. For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, the product $(u_1!) \cdots (u_n!)$ is denoted by $\mathbf{u}!$, and $x_1^{u_1} \cdots x_n^{u_n}$ by $\mathbf{x}^{\mathbf{u}}$. A formal power series can always be written in the form

$$f = \sum_{\mathbf{u} \in \mathbb{N}^n} \frac{c_{\mathbf{u}}}{\mathbf{u}!}\mathbf{x}^{\mathbf{u}},$$

where $c_{\mathbf{u}} \in \mathbb{K}$. Such a form is convenient for differentiation.

Taking the constant term $c_{\mathbf{0}}$ of a formal power series f gives rise to a ring homomorphism, which is denoted by ϕ. A direct calculation yields

$$\phi\left(\boldsymbol{\partial}^{\mathbf{u}}(f)\right) = c_{\mathbf{u}}, \tag{2}$$

which allows us to determine whether a formal power series is zero by differentiating and taking constant terms, as stated in the next lemma.

Lemma 2.2. Let $f \in \mathbb{K}[[\mathbf{x}]]$. Then $f = 0$ if and only if $\phi\left(\boldsymbol{\partial}^{\mathbf{u}}(f)\right) = 0$ for all $\mathbf{u} \in \mathbb{N}^n$.

The fixed ordering $\prec$ on $\mathbb{N}^n$ also induces an ordering on the monoid $\mathrm{T}(\mathbf{x})$ generated by $x_1, \ldots, x_n$ in the following manner: $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$ if and only if $\mathbf{u} \prec \mathbf{v}$. The induced ordering enables us to characterize ordinary points of a D-finite ideal by formal power series.

A nonzero element $f \in \mathbb{K}[[\mathbf{x}]]$ can be written as

$$f = \frac{c_{\mathbf{u}}}{\mathbf{u}!}\mathbf{x}^{\mathbf{u}} + \text{ higher monomials with respect to } \prec,$$

where $c_{\mathbf{u}}$ is a nonzero element of $\mathbb{K}$. We call $\mathbf{u}$ the *initial exponent* of f.

For brevity, we refer to formal power series as power series in the sequel.

2.4. Solutions and Wronskians

We recall some basic facts about solutions of linear partial differential polynomials in (Kolchin, 1973, Chapter IV, Section 5). The first proposition is a special case of Proposition 2 in (Kolchin, 1973, page 152).

Proposition 2.3. For a left ideal $I \subset \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ with rank d, there exists a differential field $\mathbb{E}$ containing $\mathbb{K}[[\mathbf{x}]]$ such that the set of solutions of I in $\mathbb{E}$ is a d-dimensional vector space over $C_{\mathbb{E}}$, where $C_{\mathbb{E}}$ stands for the subfield of constants in $\mathbb{E}$.

Such differential fields can also be constructed by a Picard-Vessiot approach given in (van der Put and Singer, 2003, Appendix D) or (Bronstein, et al, 2005). In the rest of this paper, we assume that $\mathbb{E}$ is a differential field as described in the above proposition. For a D-finite ideal I, the solution space of I in $\mathbb{E}$ is denoted by $\mathrm{sol}_{\mathbb{E}}(I)$.

The next proposition is a differential analog of the Nullstellensatz for D-finite ideals. It is an easy consequence of Corollary 1 in (Kolchin, 1973, page 152).

Proposition 2.4. Let $V \subset \mathbb{E}$ be a d-dimensional linear subspace over $C_{\mathbb{E}}$. Then there exists a unique left ideal $I \subset \mathbb{E}[\boldsymbol{\partial}]$ of rank d such that $V = \mathrm{sol}_{\mathbb{E}}(I)$. Furthermore, an operator P belongs to I if and only if P annihilates every element of V.

Linear dependence over the constants can be determined by Wronskian-like determinants (Kolchin, 1973, Chapter II, Theorem 1), which implies that a finite number of elements in $\mathbb{K}[[\mathbf{x}]]$ are linearly independent over $\mathbb{K}$ if and only if they are linearly independent over any field of constants that contains $\mathbb{K}$.

Wronskian-like determinants are expressed by elements of $\mathrm{T}(\boldsymbol{\partial})$ via wedge notation in (Li, et al, 2002). For $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell \in \mathbb{N}^n$ and $\ell \in \mathbb{Z}^+$, the exterior product

$$\boldsymbol{\partial}^{\mathbf{v}_1} \wedge \boldsymbol{\partial}^{\mathbf{v}_2} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{v}_\ell}$$

is defined as a $C_{\mathbb{E}}$-multilinear function from $\mathbb{E}^\ell$ to $\mathbb{E}$ that maps $(z_1, \ldots, z_\ell) \in \mathbb{E}^\ell$ to:

$$\begin{vmatrix} \boldsymbol{\partial}^{\mathbf{v}_1}(z_1) & \boldsymbol{\partial}^{\mathbf{v}_1}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{v}_1}(z_\ell) \\ \boldsymbol{\partial}^{\mathbf{v}_2}(z_1) & \boldsymbol{\partial}^{\mathbf{v}_2}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{v}_2}(z_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\partial}^{\mathbf{v}_\ell}(z_1) & \boldsymbol{\partial}^{\mathbf{v}_\ell}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{v}_\ell}(z_\ell) \end{vmatrix}.$$

It follows from Theorem 1 in (Kolchin, 1973, Chapter II) that $z_1, \ldots, z_\ell$ are linearly independent over $C_{\mathbb{E}}$ if there exist $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in \mathbb{N}^n$ such that

$$(\boldsymbol{\partial}^{\mathbf{v}_1} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{v}_\ell})(z_1, \ldots, z_\ell) \neq 0.$$

Lemma 2.5. Let $f_1, \ldots f_\ell \in \mathbb{K}[[\mathbf{x}]]$ be nonzero power series with initial exponents $\mathbf{u}_1$, $\ldots$, $\mathbf{u}_\ell$. If $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ are mutually distinct, then $(\boldsymbol{\partial}^{\mathbf{u}_1} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{u}_\ell})(f_1, \ldots, f_\ell)$ is invertible in $\mathbb{K}[[\mathbf{x}]]$. In particular, $f_1, \ldots, f_\ell$ are linearly independent over $\mathbb{K}$.

Proof. Let $g(\mathbf{x}) = (\boldsymbol{\partial}^{\mathbf{u}_1} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{u}_\ell})(f_1, \ldots, f_\ell)$. Without loss of generality, we assume that $\mathbf{u}_1 \prec \cdots \prec \mathbf{u}_\ell$. It follows from (2) that $g(\mathbf{0})$ is an upper triangular determinant whose diagonal consists of the respective coefficients of $\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_\ell}$ in $f_1, \ldots, f_\ell$. So $g(\mathbf{0})$ is a nonzero element of $\mathbb{K}$. Accordingly, $g(\mathbf{x})$ is invertible in $\mathbb{K}[[\mathbf{x}]]$. $\square$

The following proposition is Lemma 4 by Li, et al (2002) in slightly different notation.

Proposition 2.6. Let I be a D-finite ideal in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ and $\mathrm{PE}(I) = \{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$. Let

$$w_I = \boldsymbol{\partial}^{\mathbf{u}_1} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{u}_d}.$$

Assume that $z_1, \ldots, z_d \in \mathrm{sol}_{\mathbb{E}}(I)$.

(i) The elements $z_1, \ldots, z_d$ are linearly independent over $C_{\mathbb{E}}$ if and only if $w_I(z_1, \ldots, z_d)$ is nonzero.

(ii) Let G be a reduced Gröbner basis of I, and $\boldsymbol{\partial}^{\mathbf{v}}$ be the head term of an element g of G. Assume further that $z_1, \ldots, z_d$ are linearly independent over $C_{\mathbb{E}}$. Set $\mathbf{z}$ to be $(z_1, \ldots, z_d)$ and

$$(w_I \wedge \boldsymbol{\partial}^{\mathbf{v}})(\mathbf{z}, \cdot) = \begin{vmatrix} \boldsymbol{\partial}^{\mathbf{u}_1}(z_1) & \boldsymbol{\partial}^{\mathbf{u}_1}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{u}_1}(z_d) & \boldsymbol{\partial}^{\mathbf{u}_1} \\ \boldsymbol{\partial}^{\mathbf{u}_2}(z_1) & \boldsymbol{\partial}^{\mathbf{u}_2}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{u}_2}(z_d) & \boldsymbol{\partial}^{\mathbf{u}_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \boldsymbol{\partial}^{\mathbf{u}_d}(z_1) & \boldsymbol{\partial}^{\mathbf{u}_d}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{u}_d}(z_d) & \boldsymbol{\partial}^{\mathbf{u}_d} \\ \boldsymbol{\partial}^{\mathbf{v}}(z_1) & \boldsymbol{\partial}^{\mathbf{v}}(z_2) & \cdots & \boldsymbol{\partial}^{\mathbf{v}}(z_d) & \boldsymbol{\partial}^{\mathbf{v}} \end{vmatrix},$$

in which the elements of $\mathrm{T}(\boldsymbol{\partial})$ are placed on the right-hand side of a product. Then

$$w_I(\mathbf{z})^{-1} \left(w_I \wedge \boldsymbol{\partial}^{\mathbf{v}}(\mathbf{z}, \cdot) \right) = \mathrm{HC}(g)^{-1} g.$$

The above two results will be used to reconstruct a Gröbner basis from its solutions.

3. Ordinary points and singularities

The goal of this section is to define ordinary points and singularities of a D-finite ideal by primitive Gröbner bases, and to characterize ordinary points in terms of power series.

3.1. Ordinary points and singularities

Our definitions of ordinary points and singularities are motivated by the studies of the singular locus of a differential system in (Saito et al, 1999, page 36).

Definition 3.1. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, and let G be a primitive Gröbner basis of I. A point $\alpha \in \overline{\mathbb{K}}^n$ is called an *ordinary point* of I with respect to $\prec$ if none of the elements in $\mathrm{HC}(G)$ vanishes at α. Otherwise, it is called a *singularity* of I with respect to $\prec$.

The above definition is independent of the choices of primitive Gröbner bases by Remark 2.1, and it is compatible with that by Abramov, et al (2006) and Chen, et al (2016) in the univariate case. For brevity, the phrase "with respect to $\prec$" will be omitted when we speak about ordinary points and singularities, unless we examine them with respect to different orderings.

Example 3.2. Let the D-finite ideal I be generated by $G = \{\partial_2 - \partial_1, \partial_1^2 + 1\}$, which is a primitive Gröbner basis. Since $\mathrm{HC}(G) = \{1\}$, the ideal I has no singularity.

Example 3.3. Let the D-finite ideal I be generated by the primitive Gröbner basis

$$G = \{x_1\partial_1^2 - (x_1x_2 - 1)\partial_1 - x_2,\, x_2\partial_2 - x_1\partial_1\}$$

in $\mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]$. Then $\mathrm{HC}(G) = \{x_1, x_2\}$. So the singularities of I are the points in $\{(a, b) \in \overline{\mathbb{Q}}^2 \mid a = 0 \text{ or } b = 0\}$. In particular, the origin is a singularity.

The next example illustrates that an ordinary point with respect to a term order may be a singularity with respect to another one. Such singularities are shown to be apparent (see Definition 4.1 and Remark 4.4).

Example 3.4. Let $G_1 = \partial_1 + x_2\partial_2 - x_2 - 1$, $G_2 = \partial_2^2 - 2\partial_2 + 1$ be two linear differential operators in $\mathbb{K}(x_1, x_2)[\partial_1, \partial_2]$, and $I = \mathbb{K}(x_1, x_2)[\partial_1, \partial_2]\{G_1, G_2\}$.

Assume that $\prec$ is the graded lexicographic order with $\partial_2 \prec \partial_1$. Then G_1 and G_2 form a primitive Gröbner basis of I with $\mathrm{HC}(\{G_1, G_2\}) = \{1\}$. Thus, every point in $\mathbb{K}^2$ is an ordinary point with respect to $\prec$. On the other hand, assume that $\prec$ is the graded lexicographic order with $\partial_1 \prec \partial_2$. Then $\{G_1, G_2\}$ is also a primitive Gröbner basis of I with respect to $\prec$. But $\mathrm{HC}(\{G_1, G_2\}) = \{x_2, 1\}$. So all points on the line $x_2 = 0$ are singularities of I with respect to $\prec$.

3.2. Characterization of ordinary points

From now on, we focus on power series solutions of a D-finite ideal around the origin, as a point in $\mathbb{K}^n$ can always be translated to the origin, and we may assume that $\mathbb{K}$ is algebraically closed when necessary. The next proposition is a linear version of the Cauchy-Kawalevskii theorem, which is also mentioned in (Saito et al, 1999, Theorem 1.4.19). The proof below is based on (Wu, 1989, Section 11).

Proposition 3.5. Let I be a left ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, and $G \subset \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be a Gröbner basis of I. If none of the elements in $\mathrm{HC}(G)$ vanishes at the origin, then I has a power series solution in $\mathbb{K}[[\mathbf{x}]]$ with initial exponent $\mathbf{u}$ for each $\mathbf{u} \in \mathrm{PE}(I)$.

Proof. For $\mathbf{v} \in \mathbb{N}^n$, let $N_{\mathbf{v}}$ be the normal form of $\boldsymbol{\partial}^{\mathbf{v}}$ with respect to G. Then

$$N_{\mathbf{v}} = \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{v}}(\mathbf{x})\boldsymbol{\partial}^{\mathbf{u}}, \tag{3}$$

with $a_{\mathbf{u},\mathbf{v}} \in \mathbb{K}(\mathbf{x})$. Since any element of $\mathrm{HC}(G)$ does not vanish at the origin, nor does the denominator of $a_{\mathbf{u},\mathbf{v}}(\mathbf{x})$. It follows that each $a_{\mathbf{u},\mathbf{v}}$ can be viewed as elements in $\mathbb{K}[[\mathbf{x}]]$.

We associate to each tuple $\mathbf{u} \in \mathrm{PE}(I)$ an arbitrary constant $c_{\mathbf{u}} \in \mathbb{K}$. For a non-parametric exponent $\mathbf{v}$, set

$$c_{\mathbf{v}} = \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{v}}(\mathbf{0})c_{\mathbf{u}}. \tag{4}$$

Note that $c_{\mathbf{v}}$ is well-defined, since there are only finitely many $a_{\mathbf{u},\mathbf{v}}$'s unequal to zero in (3). Furthermore, let f be the power series $\sum_{\mathbf{v} \in \mathbb{N}^n}(c_{\mathbf{v}}/\mathbf{v}!)\mathbf{x}^{\mathbf{v}}$. We are going to show that f is a solution of I. Assume $G = \{G_1, \ldots, G_k\}$. By (1) and Lemma 2.2, it suffices to show the claim that, for all $\mathbf{w} \in \mathbb{N}^n$ and $i \in \{1, \ldots, k\}$,

$$\phi\left(\boldsymbol{\partial}^{\mathbf{w}} G_i(f)\right) = 0, \tag{5}$$

where ϕ is the ring homomorphism from $\mathbb{K}[[\mathbf{x}]]$ to $\mathbb{K}$ that takes constant terms, as described in Section 2.3.

Set $\ell_i = \mathrm{HC}(G_i)$ and $\boldsymbol{\partial}^{\mathbf{v}_i} = \mathrm{HT}(G_i)$ for all i with $1 \le i \le k$. Note that $\ell_i(\mathbf{x})^{-1} \in \mathbb{K}[[\mathbf{x}]]$ because $\ell_i(\mathbf{0}) \neq 0$. Moreover, assume that $\mathbf{v}_1 \prec \cdots \prec \mathbf{v}_k$.

We proceed by Noetherian induction on $S = \{\mathrm{HT}(\boldsymbol{\partial}^{\mathbf{w}} G_i) \,|\, \mathbf{w} \in \mathbb{N}^n,\, i \in \{1, \ldots, k\}\}$ according to the fixed term ordering. Our starting point is to prove $\phi(G_1(f)) = 0$, since $\boldsymbol{\partial}^{\mathbf{v}_1}$ is the minimal element of S.

Note that $\boldsymbol{\partial}^{\mathbf{v}_1} = \ell_1(\mathbf{x})^{-1} G_1 + N_{\mathbf{v}_1}$. By (3), $\boldsymbol{\partial}^{\mathbf{v}_1} = \ell_1(\mathbf{x})^{-1} G_1 + \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{v}_1}(\mathbf{x}) \boldsymbol{\partial}^{\mathbf{u}}$. Applying this equality to f and then applying ϕ to the result, we get

$$c_{\mathbf{v}_1} = \ell_1(\mathbf{0})^{-1} \phi(G_1(f)) + \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{v}_1}(\mathbf{0}) c_{\mathbf{u}}$$

by (2). Thus, $\ell_1(\mathbf{0})^{-1}\phi(G_1(f)) = 0$ by (4). So $\phi(G_1(f)) = 0$. The claim (5) holds for $\mathbf{v}_1$.

Let $\mathbf{w} \in \mathbb{N}^n$ and let $i \in \{1, \ldots, k\}$. Assume that, for all $\mathbf{v} \in \mathbb{N}^n$ and $j \in \{1, \ldots, k\}$,

$$\phi\left(\boldsymbol{\partial}^{\mathbf{v}} G_j(f)\right) = 0 \tag{6}$$

whenever $\mathrm{HT}(\boldsymbol{\partial}^{\mathbf{v}} G_j) \prec \mathrm{HT}(\boldsymbol{\partial}^{\mathbf{w}} G_i)$. Reducing $\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i}$ modulo G, we have

$$\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i} = \ell_i(\mathbf{x})^{-1} \boldsymbol{\partial}^{\mathbf{w}} G_i + \left(\sum_{\mathrm{HT}(\boldsymbol{\partial}^{\mathbf{v}} G_j) \prec \mathrm{HT}(\boldsymbol{\partial}^{\mathbf{w}} G_i)} p_{\mathbf{v},j}(\mathbf{x}) \boldsymbol{\partial}^{\mathbf{v}} G_j \right) + N_{\mathbf{w}+\mathbf{v}_i},$$

where $p_{\mathbf{v},j}(\mathbf{x}) \in \mathbb{K}[[\mathbf{x}]]$. This is because every term in $\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i} - (\ell_i(\mathbf{x}))^{-1}(\boldsymbol{\partial}^{\mathbf{w}} G_i)$ is lower than $\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i}$ and because none of $\ell_1, \ldots, \ell_k$ vanishes at the origin. Letting the above equality act on f, and then applying ϕ to the result, we see that $\phi\left(\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i}(f)\right)$ equals

$$\ell_i(\mathbf{0})^{-1} \phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f)) + \sum_{\mathrm{HT}(\boldsymbol{\partial}^{\mathbf{v}} G_j) \prec \mathrm{HT}(\boldsymbol{\partial}^{\mathbf{w}} G_i)} p_{\mathbf{v},j}(\mathbf{0}) \phi(\boldsymbol{\partial}^{\mathbf{v}} G_j(f)) + \phi\left(N_{\mathbf{w}+\mathbf{v}_i}(f)\right).$$

Then the induction hypothesis (6) implies that

$$\phi\left(\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i}(f)\right) = \ell_i(\mathbf{0})^{-1} \phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f)) + \phi\left(N_{\mathbf{w}+\mathbf{v}_i}(f)\right).$$

It follows from (3) that

$$\phi\left(\boldsymbol{\partial}^{\mathbf{w}+\mathbf{v}_i}(f)\right) = \ell_i(\mathbf{0})^{-1} \phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f)) + \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{w}+\mathbf{v}_i}(\mathbf{0}) \phi\left(\boldsymbol{\partial}^{\mathbf{u}}(f)\right),$$

which, together with (2), implies that

$$c_{\mathbf{w}+\mathbf{v}_i} = \ell_i(\mathbf{0})^{-1} \phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f)) + \sum_{\mathbf{u} \in \mathrm{PE}(I)} a_{\mathbf{u},\mathbf{w}+\mathbf{v}_i}(\mathbf{0}) c_{\mathbf{u}}.$$

Therefore, $\ell_i(\mathbf{0})^{-1} \phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f))$ is equal to zero by (4). So is $\phi(\boldsymbol{\partial}^{\mathbf{w}} G_i(f))$. Consequently, the claim given in (5) holds.

For each $\mathbf{w} \in \mathrm{PE}(I)$, let $f_{\mathbf{w}} = \sum_{\mathbf{v} \in \mathbb{N}^n} (c_{\mathbf{v}}/\mathbf{v}!) \mathbf{x}^{\mathbf{v}}$ be the power series defined by (4) with $c_{\mathbf{w}} = 1$ and $c_{\mathbf{u}} = 0$ for all $\mathbf{u} \in \mathrm{PE}(I) \setminus \{\mathbf{w}\}$. Then $f_{\mathbf{w}}$ is a solution of I by (5).

It remains to show that $\mathbf{w}$ is the initial exponent of $f_{\mathbf{w}}$. Let $\mathbf{v} \in \mathbb{N}^n$. By the definition of normal forms, $a_{\mathbf{u},\mathbf{v}}(\mathbf{x})$ in (3) is equal to zero whenever $\mathbf{u} \succ \mathbf{v}$. It follows that $a_{\mathbf{u},\mathbf{v}}(\mathbf{0})$ in (4) is equal to zero whenever $\mathbf{u} \succ \mathbf{v}$. Assume now that $\mathbf{v} \prec \mathbf{w}$. Then the above discussion and (4) imply

$$c_{\mathbf{v}} = \sum_{\mathbf{u} \in \mathrm{PE}(I),\, \mathbf{u} \preceq \mathbf{v}} a_{\mathbf{u},\mathbf{v}}(\mathbf{0}) c_{\mathbf{u}},$$

which is zero in $f_{\mathbf{w}}$, because all the $c_{\mathbf{u}}$'s in $f_{\mathbf{w}}$ are set to be zero whenever $\mathbf{u} \prec \mathbf{w}$. So $\mathbf{w}$ is the initial exponents of $f_{\mathbf{w}}$ according to the induced ordering. $\square$

Note that the left ideal I in the above proposition is not necessarily D-finite. We prove its converse under two additional assumptions.

Proposition 3.6. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, $\mathrm{PE}(I) = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$, and G be a primitive Gröbner basis of I. If, for every j with $1 \le j \le r$, there exists a power series $f_j \in \mathbb{K}[[\mathbf{x}]]$ with initial exponent $\mathbf{u}_j$ such that $f_j \in \mathrm{sol}_{\mathbb{E}}(I)$, then any element of $\mathrm{HC}(G)$ does not vanish at the origin.

Proof. Let $w_I = \boldsymbol{\partial}^{\mathbf{u}_1} \wedge \cdots \wedge \boldsymbol{\partial}^{\mathbf{u}_r}$ and $\mathbf{f} = (f_1, \ldots, f_r)$. Then $w_I(\mathbf{f})$ is invertible in $\mathbb{K}[[\mathbf{x}]]$ by Lemma 2.5. Let $G = \{G_1, \ldots, G_k\}$. For all i with $1 \le i \le k$, let

$$G_i = \ell_i \boldsymbol{\partial}^{\mathbf{v}_i} + \sum_{j=1}^{r} \ell_{ij} \boldsymbol{\partial}^{\mathbf{u}_j},$$

where $\boldsymbol{\partial}^{\mathbf{v}_i} = \mathrm{HT}(G_i)$, $\ell_i = \mathrm{HC}(G_i)$ and $\ell_{ij} \in \mathbb{K}[\mathbf{x}]$. Moreover, let $F_i = (w_I \wedge \boldsymbol{\partial}^{\mathbf{v}_i})(\mathbf{f}, \cdot)$.

By Proposition 2.6, $(1/\ell_i)G_i = w_I(\mathbf{f})^{-1} F_i$, which, together with $(1/\ell_i)G_i \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ and $w_I(\mathbf{f})^{-1} F_i \in \mathbb{K}[[\mathbf{x}]][\boldsymbol{\partial}]$, implies that $\ell_{ij}/\ell_i \in \mathbb{K}(\mathbf{x}) \cap \mathbb{K}[[\mathbf{x}]]$. Set $\ell_{ij}/\ell_i = p_{ij}/q_{ij}$, where p_{ij} and q_{ij} belong to $\mathbb{K}[\mathbf{x}]$ with $\gcd(p_{ij}, q_{ij}) = 1$. Then q_{ij} does not vanish at the origin by Theorem 1 in (Gessel, 1981). Since G_i is primitive, each ℓ_i is a factor of the product of $q_{i1}, \ldots, q_{ir}$. Hence, it does not vanish at the origin either. $\square$

We are ready to characterize ordinary points in terms of power series.

Theorem 3.7. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Then the origin is an ordinary point of I if and only if I has a power series solution with initial exponent $\mathbf{u}$ for each $\mathbf{u} \in \mathrm{PE}(I)$.

Proof. Let G be a primitive Gröbner basis of I. If the origin is an ordinary point of I, then any element of $\mathrm{HC}(G)$ does not vanish at the origin. By Proposition 3.5, I has a power series solution in $\mathbb{K}[[\mathbf{x}]]$ with initial exponent $\mathbf{u}$ for each $\mathbf{u} \in \mathrm{PE}(G)$. The converse is immediate from Proposition 3.6. $\square$

The next corollary and example indicate that it appears an optimal choice to define the notion of ordinary points via primitive Gröbner bases.

Corollary 3.8. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, and $G \subset \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be a Gröbner basis of I. If any element of $\mathrm{HC}(G)$ does not vanish at the origin, then the origin is an ordinary point of I.

Proof. By Proposition 3.5, I has a power series solution with initial exponent $\mathbf{u}$ for each $\mathbf{u} \in \mathrm{PE}(I)$. By Theorem 3.7, the origin is an ordinary point of I. $\square$

Example 3.9. Let $G_1 = x_2\partial_2 + \partial_1$ and $G_2 = \partial_1$ in $\mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]$. Let I be the left ideal generated by G_1 and G_2. Then $\{G_1, G_2\}$ is a Gröbner basis of I, which is not reduced. Although $\mathrm{HC}(G_1)$ vanishes at the origin, the origin is an ordinary point of I because I has a primitive Gröbner basis $\{\partial_1, \partial_2\}$.

Theorem 3.7 can also be stated in terms of solution spaces.

Corollary 3.10. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ with $\mathrm{PE}(I) = \{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$. Then the origin is an ordinary point of I if and only if there exist $f_1, \ldots, f_d \in \mathbb{K}[[\mathbf{x}]]$ with respective initial exponents $\mathbf{u}_1, \ldots, \mathbf{u}_d$ such that $f_1, \ldots, f_d$ form a $C_{\mathbb{E}}$-basis of $\mathrm{sol}_{\mathbb{E}}(I)$.

Proof. Note that d is the dimension of $\mathrm{sol}_{\mathbb{E}}(I)$ over $C_{\mathbb{E}}$ by Proposition 2.3. The corollary is immediate from Theorem 3.7 and Lemma 2.5. □

4. Apparent singularities

The goal of this section is to define the notion apparent singularities in the D-finite case, and to characterize them by intersections of D-finite ideals.

Definition 4.1. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ and $d = \mathrm{rank}(I)$. Assume that the origin is a singularity of I. We call the origin an *apparent singularity* of I if I has d linearly independent power series solutions in $\mathbb{K}[[\mathbf{x}]]$.

The above definition is compatible with the univariate case in (Abramov, et al, 2006, Definition 5).

Example 4.2. Let the left ideal I be generated by the primitive Gröbner basis

$$G = \{x_2\partial_2 + \partial_1 - x_2 - 1, \partial_1^2 - \partial_1\}$$

in $\mathbb{K}(x_1, x_2)[\partial_1, \partial_2]$. Then $\mathrm{rank}(I) = 2$ and $\mathrm{HC}(G) = \{x_2, 1\}$. So the origin is a singularity of I. As I has two $\mathbb{K}$-linearly independent power series solutions $\exp(x_1 + x_2)$ and $x_2 \exp(x_2)$, the origin is an apparent singularity.

Example 4.3. The solution space of the primitive Gröbner basis

$$G = \{x_2^2\partial_2 - x_1^2\partial_1 + x_1 - x_2, \partial_1^2\}$$

is generated by $\{x_1+x_2, x_1x_2\}$. In this case, $\mathrm{HC}(G) = \{x_2^2, 1\}$. So the origin is an apparent singularity of $\mathbb{K}(x_1, x_2)[\partial_1, \partial_2]G$.

Remark 4.4. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Assume that the origin is an ordinary point with respect to the preselected ordering. Then $\mathrm{sol}_{\mathbb{E}}(I)$ has a basis contained in $\mathbb{K}[[\mathbf{x}]]$. Hence, the origin is an apparent singularity of I if it is a singularity of I with respect to another ordering by Definition 4.1.

For a subset S of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, we denote by $\mathrm{IE}_{\mathbf{0}}(S)$ the set of initial exponents of nonzero elements in $\mathrm{sol}_{\mathbb{E}}(S) \cap \mathbb{K}[[\mathbf{x}]]$ and call it the set of initial exponents of S at the origin. Then $|\mathrm{IE}_{\mathbf{0}}(S)|$ is the dimension of $\mathrm{sol}_{\mathbb{E}}(S) \cap \mathbb{K}[[\mathbf{x}]]$ by Lemma 2.5. For a D-finite ideal I, the origin is an ordinary point if and only if $\mathrm{IE}_{\mathbf{0}}(I) = \mathrm{PE}(I)$ by Theorem 3.7. It is an apparent singularity if and only if $\mathrm{IE}_{\mathbf{0}}(I) \neq \mathrm{PE}(I)$ but $|\mathrm{IE}_{\mathbf{0}}(I)| = |\mathrm{PE}(I)|$ by Definition 4.1.

Before characterizing apparent singularities, we prove two technical lemmas.

Lemma 4.5. Let I and J be two D-finite ideals in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Then

(i) $\mathrm{rank}(I \cap J) + \mathrm{rank}(I + J) = \mathrm{rank}(I) + \mathrm{rank}(J)$.

(ii) $\dim \mathrm{sol}_{\mathbb{E}}(I \cap J) + \dim \mathrm{sol}_{\mathbb{E}}(I + J) = \dim \mathrm{sol}_{\mathbb{E}}(I) + \dim \mathrm{sol}_{\mathbb{E}}(J)$.

(iii) $\mathrm{sol}_{\mathbb{E}}(I \cap J) = \mathrm{sol}_{\mathbb{E}}(I) + \mathrm{sol}_{\mathbb{E}}(J)$.

Proof. By Proposition 10 in (Bourbaki, 1998, page 207), we have an exact sequence

$$0 \to \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/(I \cap J) \to \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/I \oplus \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/J \to \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]/(I+J) \to 0$$

of $\mathbb{K}$-linear spaces. The first assertion holds. The second one follows from the first and Proposition 2.3. It is evident that $\mathrm{sol}_{\mathbb{E}}(I) + \mathrm{sol}_{\mathbb{E}}(J) \subset \mathrm{sol}_{\mathbb{E}}(I \cap J)$. On the other hand,

$$\begin{aligned}
\dim(\mathrm{sol}_{\mathbb{E}}(I) + \mathrm{sol}_{\mathbb{E}}(J)) &= \dim(\mathrm{sol}_{\mathbb{E}}(I)) + \dim(\mathrm{sol}_{\mathbb{E}}(J)) - \dim(\mathrm{sol}_{\mathbb{E}}(I) \cap \mathrm{sol}_{\mathbb{E}}(J)). \\
&= \dim(\mathrm{sol}_{\mathbb{E}}(I)) + \dim(\mathrm{sol}_{\mathbb{E}}(J)) - \dim(\mathrm{sol}_{\mathbb{E}}(I+J)) \\
&\quad (\text{since } \mathrm{sol}_{\mathbb{E}}(I) \cap \mathrm{sol}_{\mathbb{E}}(J) = \mathrm{sol}_{\mathbb{E}}(I+J)) \\
&= \dim(\mathrm{sol}_{\mathbb{E}}(I \cap J)) \quad (\text{by the second assertion}).
\end{aligned}$$

Hence, $\mathrm{sol}_{\mathbb{E}}(I \cap J) = \mathrm{sol}_{\mathbb{E}}(I) + \mathrm{sol}_{\mathbb{E}}(J)$. The last assertion holds. □

Our characterization of apparent singularities is based on the fact that there are at most finitely many terms lower than a given term. So *the fixed monomial ordering $\prec$ is assumed to be graded* from now on.

As a matter of notation, we define $\mathbb{N}^n_m = \{\mathbf{u} \in \mathbb{N}^n \mid |\mathbf{u}| \le m\}$ for $m \in \mathbb{N}$. The next lemma illustrates a connection between parametric exponents and initial ones.

Lemma 4.6. Let J be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Assume that $\mathrm{sol}_{\mathbb{E}}(J)$ has a basis in $\mathbb{K}[[\mathbf{x}]]$ and $\mathrm{IE}_{\mathbf{0}}(J) = \mathbb{N}^n_m$ for some $m \in \mathbb{N}$. Then $\mathrm{PE}(J) = \mathbb{N}^n_m$. Consequently, the origin is an ordinary point of J.

Proof. Let G be a Gröbner basis of J and $\bar{J}$ be the left ideal generated by J in $\mathbb{E}[\boldsymbol{\partial}]$. Then G is also a Gröbner basis of $\bar{J}$ by Buchberger's algorithm. It follows that $\mathrm{PE}(\bar{J})$ is equal to $\mathrm{PE}(J)$. So it suffices to prove $\mathrm{PE}(\bar{J}) = \mathbb{N}^n_m$.

Assume that $f_1, \ldots f_\ell \in \mathbb{K}[[\mathbf{x}]]$ form a basis of $\mathrm{sol}_{\mathbb{E}}(J)$ and their initial exponents are distinct. Then $\ell = |\mathbb{N}^n_m|$. Let $\mathbf{f} = (f_1, \ldots, f_\ell)$ and $w_J = \bigwedge_{\mathbf{u} \in \mathrm{IE}_{\mathbf{0}}(J)} \boldsymbol{\partial}^{\mathbf{u}}$. Then $w_J(\mathbf{f})$ is a nonzero element in $\mathbb{K}[[\mathbf{x}]]$ by Lemma 2.5.

For every $\mathbf{v} \in \mathbb{N}^n \setminus \mathbb{N}^n_m$, we let $F_{\mathbf{v}} = (w_J \wedge \boldsymbol{\partial}^{\mathbf{v}})(\mathbf{f}, \cdot)$, which belongs to $\mathbb{K}[[\mathbf{x}]][\boldsymbol{\partial}]$. Then $\mathrm{HT}(F_{\mathbf{v}}) = \boldsymbol{\partial}^{\mathbf{v}}$ because $w_J(\mathbf{f})$ is nonzero and the ordering $\prec$ is graded. Furthermore, $F_{\mathbf{v}}$ vanishes on $\mathrm{sol}_{\mathbb{E}}(J)$ because $(w_J \wedge \boldsymbol{\partial}^{\mathbf{v}})(\mathbf{f}, f_i) = 0$ for all i with $1 \le i \le \ell$. It follows from Proposition 2.4 that $F_{\mathbf{v}}$ belongs to $\bar{J}$. Therefore, $\boldsymbol{\partial}^{\mathbf{v}}$ is not a parametric term of $\bar{J}$. Accordingly, $\mathrm{PE}(\bar{J}) \subset \mathbb{N}^n_m$. Hence, $\mathrm{PE}(\bar{J}) = \mathbb{N}^n_m$ because $|\mathrm{PE}(\bar{J})| = \ell$ and $\ell = |\mathbb{N}^n_m|$. The origin is an ordinary point by Corollary 3.10. □

We are ready to characterize apparent singularities.

Theorem 4.7. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Assume that the origin is a singularity of I. Then the origin is an apparent singularity of I if and only if it is an ordinary point of some D-finite ideal contained in I.

Proof. Assume that the origin is an apparent singularity of I. Let $m = \max_{\mathbf{u} \in \mathrm{IE}_0(I)} |\mathbf{u}|$. For every $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n$, we denote by $I_{\mathbf{v}}$ the left ideal generated by $x_1\partial_1 - v_1$, $\ldots$, $x_n\partial_n - v_n$ in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Then $\mathrm{sol}_{\mathbb{E}}(I_{\mathbf{v}})$ is spanned by $\mathbf{x}^{\mathbf{v}}$.

Let $J = \bigcap_{\mathbf{v} \in \mathbb{N}^n_m \setminus \mathrm{IE}_0(I)} I_{\mathbf{v}}$. By construction, the two left ideals I and J have no nonzero solution in common, which, together with Lemma 4.5 (iii), implies that

$$\mathrm{sol}_{\mathbb{E}}(I \cap J) = \mathrm{sol}_{\mathbb{E}}(I) \oplus \mathrm{sol}_{\mathbb{E}}(J).$$

In particular, the dimension of $\mathrm{sol}_{\mathbb{E}}(I \cap J)$ is equal to $|\mathbb{N}^n_m|$, because $\mathrm{sol}_{\mathbb{E}}(I)$ and $\mathrm{sol}_{\mathbb{E}}(J)$ have respective dimensions $|\mathrm{IE}_0(I)|$ and $|\mathbb{N}^n_m| - |\mathrm{IE}_0(I)|$. So $\mathrm{IE}_0(I \cap J) = \mathbb{N}^n_m$. Then the origin is an ordinary point of $I \cap J$ by Lemma 4.6.

Conversely, assume that $J \subset I$ is a D-finite ideal such that the origin is an ordinary point of J. Then $\mathrm{sol}_{\mathbb{E}}(I) \subset \mathrm{sol}_{\mathbb{E}}(J)$.

Assume that $\{f_1, \ldots, f_\ell\} \subset \mathbb{K}[[\mathbf{x}]]$ is a basis of $\mathrm{sol}_{\mathbb{E}}(J)$. Since $\mathrm{sol}_{\mathbb{E}}(I)$ is contained in $\mathrm{sol}_{\mathbb{E}}(J)$, every element of $\mathrm{sol}_{\mathbb{E}}(I)$ is a linear combination of $f_1, \ldots, f_\ell$ over $C_{\mathbb{E}}$. Therefore, $\mathrm{sol}_{\mathbb{E}}(I)$ is contained in $C_{\mathbb{E}}[[\mathbf{x}]]$. It remains to show that $\mathrm{sol}_{\mathbb{E}}(I)$ has a basis in $\mathbb{K}[[\mathbf{x}]]$.

Let $G \subset \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be a finite basis of I, and $f = z_1 f_1 + \cdots + z_\ell f_\ell$, where $z_1, \ldots, z_\ell \in C_{\mathbb{E}}$ are to be determined. Then $f \in \mathrm{sol}_{\mathbb{E}}(I)$ if and only if $P(f) = 0$ for all $P \in G$, which is equivalent to

$$z_1 P(f_1) + \cdots + z_\ell P(f_\ell) = 0 \quad \text{for all } P \in G.$$

By comparing the coefficients of $\mathbf{x}^{\mathbf{w}}$ ($\mathbf{w} \in \mathbb{N}^n$) in both sides of the above equations, we derive a linear system $A\mathbf{z} = \tilde{\mathbf{0}}$, where A is a matrix with infinitely many rows but ℓ columns, $\mathbf{z} = (z_1, \ldots, z_\ell)^t$, and $\tilde{\mathbf{0}}$ is a column vector consisting of infinitely many zeros. Let $\ker(A)$ be the solution space of $A\mathbf{z} = \tilde{\mathbf{0}}$ contained in $C_{\mathbb{E}}^\ell$. Then

$$\mathrm{sol}_{\mathbb{E}}(I) = \left\{ c_1 f_1 + \cdots + c_\ell f_\ell \mid (c_1, \ldots, c_\ell)^t \in \ker(A) \right\}. \tag{7}$$

Let $\mathbf{f} = (f_1, \ldots, f_\ell)$ and $\mathbf{c}_1, \ldots, \mathbf{c}_m \in \ker(A)$. The $C_{\mathbb{E}}$-linear independence of $f_1, \ldots, f_\ell$ implies that $\mathbf{f}\mathbf{c}_1, \ldots, \mathbf{f}\mathbf{c}_m$ are $C_{\mathbb{E}}$-linearly independent if and only if $\mathbf{c}_1, \ldots, \mathbf{c}_m$ are $C_{\mathbb{E}}$-linearly independent. In particular, $\dim_{C_{\mathbb{E}}} \ker(A) = \dim_{C_{\mathbb{E}}} \mathrm{sol}_{\mathbb{E}}(I)$, which is denoted by d. Then $\mathrm{rank}(A)$ is equal to $\ell - d$. Since $I \subset \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ and $f_1, \ldots, f_\ell \in \mathbb{K}[[\mathbf{x}]]$, the matrix A is over $\mathbb{K}$. It follows that $\ker(A) \cap \mathbb{K}^\ell$ contains d linearly independent vectors over $C_{\mathbb{E}}$. By (7), those vectors give rise to a basis of $\mathrm{sol}_{\mathbb{E}}(I)$, which is contained in $\mathbb{K}[[\mathbf{x}]]$. The origin is an apparent singularity of I by Definition 4.1. $\square$

Assume that the origin is an apparent singularity of I. By desingularizing the origin, we mean computing a D-finite ideal $J \subset I$ such that the origin is an ordinary point of J.

Definition 4.8. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ and S be a finite subset of $\mathbb{N}^n$. Then the left ideal

$$I \cap \left(\bigcap_{(v_1, \ldots, v_n) \in S} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\{x_1\partial_1 - v_1, \ldots, x_n\partial_n - v_n\} \right)$$

is called the *sub-ideal of I with respect to S.*

It is clear that the sub-ideal of a D-finite ideal with respect to any finite subset of $\mathbb{N}^n$ is again D-finite. The next corollary helps us to desingularize an apparent singularity. Its proof is immediate from the first paragraph in the proof of the above theorem.

Corollary 4.9. Let I be a D-finite ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Assume that the origin is an apparent singularity of I. Set $m = \max_{\mathbf{u} \in \mathrm{IE}_0(I)} |\mathbf{u}|$. Then the origin is an ordinary point of the sub-ideal of I with respect to $(\mathbb{N}^n_m \setminus \mathrm{IE}_0(I))$.

5. Desingularization and applications

We are going to apply Corollary 4.9 to desingularize an apparent singularity.

5.1. Indicial ideals

We extend the notion of indicial polynomials for linear ordinary differential operators to the D-finite case.

For an element a in a ring and a positive integer m, the m-th falling factorial of a is defined as $(a)^{\underline{m}} = a(a-1)\cdots(a-m+1)$. Let $\delta_i = x_i\partial_i$ be the Euler operator with respect to x_i, $i = 1, \ldots, n$. The commutation rules in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ imply that, for all $i, j \in \{1, \ldots, n\}$,

$$\delta_i\delta_j = \delta_j\delta_i \quad \text{and} \quad \delta_i x_i = x_i(\delta_i + 1).$$

For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, the symbol $\boldsymbol{\delta}^{\mathbf{u}}$ stands for the product $\delta_1^{u_1}\cdots\delta_n^{u_n}$. We recall the following well-known facts on Euler operators.

Proposition 5.1. (i) $x_i^m\partial_i^m = (\delta_i)^{\underline{m}}$ for each $m \in \mathbb{Z}^+$ and $i \in \{1, \ldots, n\}$.
(ii) $p(\boldsymbol{\delta})(\mathbf{x}^{\mathbf{u}}) = p(\mathbf{u})\mathbf{x}^{\mathbf{u}}$ for each $p \in \mathbb{K}[\mathbf{x}]$ and $\mathbf{x}^{\mathbf{u}} \in \mathrm{T}(\mathbf{x})$.

Set $\mathbb{K}[\mathbf{y}] = \mathbb{K}[y_1, \ldots, y_n]$ to be the ring of usual commutative polynomials with indeterminates $y_1, \ldots, y_n$.

Definition 5.2. Let a nonzero operator $P \in \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be of order m. Write

$$\mathbf{x}^{\mathbf{m}}P = \sum_{\mathbf{v}\in S}\mathbf{x}^{\mathbf{v}}\left(\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}}\boldsymbol{\delta}^{\mathbf{u}}\right), \tag{8}$$

where $\mathbf{m} = (m, \ldots, m) \in \mathbb{N}^n$, S is a finite subset of $\mathbb{N}^n$ and $c_{\mathbf{u},\mathbf{v}} \in \mathbb{K}$. Let $\mathbf{x}^{\mathbf{v}_0}$ be the minimal term among $\{x^{\mathbf{v}} \mid \mathbf{v} \in S\}$ such that $\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}_0}\boldsymbol{\delta}^{\mathbf{u}}$ is nonzero. We call

$$\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}_0}\mathbf{y}^{\mathbf{u}} \in \mathbb{K}[\mathbf{y}]$$

the *indicial polynomial* of P, and denote it by $\mathrm{ind}(P)$. We further define $\mathrm{ind}(0) := 0$.

By Proposition 5.1 (i), we may always write $\mathbf{x}^{\mathbf{m}}P$ in the form (8). The above definition is compatible with the univariate case in (Jaroschek, 2013; Saito et al, 1999), and was already used in the multivariate setting by Aroca and Cano (2001, Definition 11).

Proposition 5.3. Let P be a nonzero element of $\mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ and f a power series solution of P with initial exponent $\mathbf{w}$. Then $\mathbf{w}$ is a zero of $\mathrm{ind}(P)$.

Proof. Assume that P is of order m and in the form (8). Moreover, let $\mathbf{v}_0$ be the same as in Definition 5.2. By Proposition 5.1 (ii), we have

$$\begin{aligned}(\mathbf{x}^{\mathbf{m}}P)(f) &= \left[\textstyle\sum_{\mathbf{v}\in S}\mathbf{x}^{\mathbf{v}}\left(\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}}\boldsymbol{\delta}^{\mathbf{u}}\right)\right](\mathbf{x}^{\mathbf{w}} + \text{ higher monomials in } \mathbf{x})\\ &= \mathbf{x}^{\mathbf{v}_0}\left(\textstyle\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}_0}\boldsymbol{\delta}^{\mathbf{u}}\right)(\mathbf{x}^{\mathbf{w}}) + \text{ higher monomials in } \mathbf{x}\\ &= \left(\textstyle\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}_0}\mathbf{w}^{\mathbf{u}}\right)\mathbf{x}^{\mathbf{v}_0+\mathbf{w}} + \text{ higher monomials in } \mathbf{x}\\ &= 0.\end{aligned}$$

Thus, $\sum_{|\mathbf{u}|\le m} c_{\mathbf{u},\mathbf{v}_0}\mathbf{w}^{\mathbf{u}} = 0$, that is, $\mathrm{ind}(P)(\mathbf{w}) = 0$. □

Example 5.4. Consider the D-finite ideal I generated by

$$G_1 = x_1x_2\partial_2 - x_1x_2\partial_1 + x_2 - x_1 \quad \text{and} \quad G_2 = x_1^2\partial_1^2 - 2x_1\partial_1 + x_1^2 + 2$$

in $\mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]$. Recall that we assume a term order with $\partial_2 \succ \partial_1$ and $x_2 \succ x_1$. A straightforward calculation yields that $\mathrm{ind}(G_1) = y_2 - 1$ and $\mathrm{ind}(G_2) = (y_1 - 1)(y_1 - 2)$. Note that I has two solutions $x_1x_2 \sin(x_1 + x_2)$ and $x_1x_2 \cos(x_1 + x_2)$. Their respective initial exponents are $(2, 1)$ and $(1, 1)$, which are common zeros of $\mathrm{ind}(G_1)$ and $\mathrm{ind}(G_2)$.

Definition 5.5. Let I be a left ideal of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. We call

$$\{\mathrm{ind}(P) \mid P \in I \cap \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]\}$$

the *indicial ideal* of I, and denote it by $\mathrm{ind}(I)$.

Theorem 5.6. Let I be a left ideal in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Then $\mathrm{ind}(I)$ is an ideal in $\mathbb{K}[\mathbf{y}]$. Moreover, I is zero-dimensional if it is D-finite.

Proof. For $a, b \in \mathrm{ind}(I)$, there exist $P, Q \in I$ such that $a = \mathrm{ind}(P)$ and $b = \mathrm{ind}(Q)$. Let p and q be the respective orders of P and Q. Set $\mathbf{p} = (p, \ldots, p)$ and $\mathbf{q} = (q, \ldots, q)$. Expressing $\mathbf{x^p}P$ and $\mathbf{x^q}Q$ as polynomials in $\mathbf{x}$ with coefficients in $\mathbb{K}[\boldsymbol{\delta}]$ placed on the right-hand side of the powers of $\mathbf{x}$, we find $\mathbf{s}, \mathbf{t} \in \mathbb{N}^n$ such that

$$\begin{aligned} \mathbf{x^p}P &= \mathbf{x^s}\left(\textstyle\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}}\right) + \text{ higher terms}, \\ \mathbf{x^q}Q &= \mathbf{x^t}\left(\textstyle\sum_{|\mathbf{v}|\le q} c_{\mathbf{v},\mathbf{t}}\boldsymbol{\delta}^{\mathbf{v}}\right) + \text{ higher terms}. \end{aligned}$$

Thus, $a = \sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\mathbf{y^u}$ and $b = \sum_{|\mathbf{v}|\le q} c_{\mathbf{v},\mathbf{t}}\mathbf{y^v}$. Let $L = \mathbf{x^t}(\mathbf{x^p}P) + \mathbf{x^s}(\mathbf{x^q}Q)$, which belongs to I. Then

$$L = \mathbf{x}^{\mathbf{s}+\mathbf{t}}\left(\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}} + \sum_{|\mathbf{u}|\le q} c_{\mathbf{v},\mathbf{t}}\boldsymbol{\delta}^{\mathbf{v}}\right) + \text{ higher terms}.$$

Let m be the order of L and $\mathbf{m} = (m, \ldots, m)$. Then

$$\mathbf{x^m}L = \mathbf{x}^{\mathbf{s}+\mathbf{t}+\mathbf{m}}\left(\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}} + \sum_{|\mathbf{v}|\le q} c_{\mathbf{v},\mathbf{t}}\boldsymbol{\delta}^{\mathbf{v}}\right) + \text{ higher terms}.$$

Thus, $a + b$ is either zero or equal to $\mathrm{ind}(L)$. Consequently, $a + b \in \mathrm{ind}(I)$.

Next, we prove that $ra \in \mathrm{ind}(I)$ for all $r \in \mathbb{K}[\mathbf{y}]$. Since r is a sum of monomials in y_1, $\ldots$, y_n, it suffices to prove that $ra \in \mathrm{ind}(I)$ for each monomial r. Assume that $r = \mathbf{y^w}$, where $\mathbf{w} = (w_1, \ldots, w_n)$. Let $\mathbf{s}$ in the expression of $\mathbf{x^p}P$ be $(s_1, \ldots, s_n)$. Furthermore, let $H = \left(\prod_{i=1}^n (\delta_i - s_i)^{w_i}\right)\mathbf{x^p}P$. Then H belongs to I. The commutation rules for the δ_i's yield $(\delta_i - k)x_i^k = x_i^k\delta_i$ for all $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} H &= \left(\textstyle\prod_{i=1}^n (\delta_i - s_i)^{w_i}\right)\mathbf{x^s}\left(\textstyle\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}}\right) + \text{ higher terms} \\ &= \left(\textstyle\prod_{i=1}^n (\delta_i - s_i)^{w_i}x_i^{s_i}\right)\left(\textstyle\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}}\right) + \text{ higher terms} \\ &= \left(\textstyle\prod_{i=1}^n x_i^{s_i}\delta_i^{w_i}\right)\left(\textstyle\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}}\right) + \text{ higher terms} \\ &= \mathbf{x^s}\left(\boldsymbol{\delta}^{\mathbf{w}}\textstyle\sum_{|\mathbf{u}|\le p} c_{\mathbf{u},\mathbf{s}}\boldsymbol{\delta}^{\mathbf{u}}\right) + \text{ higher terms}. \end{aligned}$$

Let $\tilde{m}$ be the order of H and $\tilde{\mathbf{m}} = (\tilde{m}, \ldots, \tilde{m})$. Then

$$\mathbf{x}^{\tilde{\mathbf{m}}} H = \mathbf{x}^{\mathbf{s}+\tilde{\mathbf{m}}} \left(\boldsymbol{\delta}^{\mathbf{w}} \sum_{|\mathbf{u}| \le p} c_{\mathbf{u},\mathbf{s}} \boldsymbol{\delta}^{\mathbf{u}} \right) + \text{ higher terms.}$$

Thus, $ra = \mathrm{ind}(H)$, which belongs to $\mathrm{ind}(I)$. Consequently, $\mathrm{ind}(I)$ is an ideal in $\mathbb{K}[\mathbf{y}]$.

Assume further that I is D-finite. Then there exists a nonzero operator H of some order m such that $H \in I \cap \mathbb{K}[\mathbf{x}][\partial_1]$. By Proposition 5.1 (i), we have

$$\begin{aligned} (x_1^m \cdots x_n^m) H &= (x_2 \cdots x_n)^m x_1^m \left(h_0 + h_1 \partial_1 + \cdots + h_m \partial_1^m \right) \\ &= (x_2 \cdots x_n)^m \left(h_0 x_1^m + h_1 x_1^{m-1} \delta_1^{\underline{1}} + \cdots + h_m \delta_1^{\underline{m}} \right), \end{aligned}$$

where $h_0, \ldots, h_m \in \mathbb{K}[\mathbf{x}]$. Thus, $\mathrm{ind}(H) \in \mathbb{K}[y_1] \setminus \{0\}$. In the same vein, $\mathrm{ind}(I) \cap \mathbb{K}[y_i]$ is nontrivial for all i with $2 \le i \le n$. By Theorem 6 in (Cox, et al, 2015, page 251), $\mathrm{ind}(I)$ is zero-dimensional. □

The last paragraph of the proof of the above theorem enables us to construct a nontrivial zero-dimensional ideal contained in $\mathrm{ind}(I)$ when I is D-finite. However, this does not necessarily give access to a basis of $\mathrm{ind}(I)$.

Definition 5.7. Let I be a D-finite ideal in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Assume that M is a zero-dimensional ideal contained in $\mathrm{ind}(I)$. The set of nonnegative integer solutions of M is called a set of *initial exponent candidates* for I.

By Proposition 5.3, the set of initial exponents of power series solutions of I must be contained in a set of initial exponent candidates for I. Such a candidate set can be obtained by computing nonnegative integer solutions of some zero-dimensional algebraic system over $\mathbb{K}$.

Example 5.8. Consider the D-finite ideal I from Example 5.4. Then $\mathrm{ind}(G_1) = y_2 - 1$ and $\mathrm{ind}(G_2) = (y_1 - 1)(y_1 - 2)$. A set of initial exponent candidates for I is $\{(2,1), (1,1)\}$. Actually, $(2,1)$ and $(1,1)$ are the initial exponents of the solutions $x_1 x_2 \sin(x_1 + x_2)$ and $x_1 x_2 \cos(x_1 + x_2)$, respectively.

The following example indicates that initial candidates for I do not necessarily give rise to power series solutions of I.

Example 5.9. Consider the D-finite ideal I generated by the Gröbner basis

$$\begin{aligned} G_1 &= x_1 x_2 \partial_2 + (-x_1^2 + 2x_1 x_2)\partial_1 - 2x_2, \\ G_2 &= (x_1^3 - x_1^2 x_2)\partial_1^2 + 2x_1 x_2 \partial_1 - 2x_2 \end{aligned}$$

in $\mathbb{Q}(x_1, x_2)[\partial_1, \partial_2]$. A direct calculation yields $\mathrm{ind}(G_1) = y_2 - y_1$ and $\mathrm{ind}(G_2) = (y_1 - 1)y_1$. Thus, a set of initial exponent candidates for I is $S = \{(0,0), (1,1)\}$. Actually, $\mathrm{sol}_{\mathbb{E}}(I)$ is spanned by $\{x_1/(x_1 - x_2), x_1 x_2\}$. In this case, $(1,1)$ is the initial exponent of $x_1 x_2$. However, $(0,0)$ is not the initial exponent of any power series solution of I.

Applying Corollary 4.9 amounts to determining a basis of the intersection of several D-finite ideals. The next proposition is implicitly used in brief descriptions on how to compute the sum of two ∂-finite objects in (Chyzak and Salvy, 1998, Section 2.2.1). We prove it formally for completeness.

Proposition 5.10. Let I and J be two left ideals of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, and $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]$ be the ring of polynomials over $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ with the commutation rule $Pt = tP$ for all $P \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Let H be the left ideal generated by $tI + (1-t)J$ in $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]$. Then $I \cap J = H \cap \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$.

Proof. There are two ways to prove the proposition. Observe that 0, 1 and t belong to the center of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]$. So the two substitutions given by $t \mapsto 0$ and $t \mapsto 1$ induce two ring homomorphisms from $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]$ to $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$, respectively. We can then proceed by imitating the proof in of Theorem 11 in (Cox, et al, 2015, page 187). Below is a self-contained proof.

From $P = tP + (1-t)P$, we see that $P \in H \cap \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ for all $P \in I \cap J$. Conversely, assume $P \in H \cap \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$. Then there exist $Q \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]I$ and $R \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}][t]J$ such that $P = tQ + (1-t)R$ by the commutation rule for t. The same rule enables us to write

$$P = t\left(\sum_{i=0}^{m} q_i t^i\right) + (1-t)\left(\sum_{i=0}^{m} r_i t^i\right)$$

for some $m \in \mathbb{N}$, $q_i \in I$ and $r_i \in J$ for all i with $0 \le i \le m$. Therefore,

$$P = r_0 + \left(\sum_{i=1}^{m} (q_{i-1} - r_{i-1} + r_i) t^i\right) + (q_m - r_m) t^{m+1}.$$

Since P is free of t, we have $P = r_0$. It suffices to prove $r_0 \in I \cap J$. Since $r_0 \in J$, it remains to prove $r_0 \in I$. From the above equality, we see that

$$r_m = q_m,\, r_{m-1} - r_m = q_{m-1}, \ldots, r_1 - r_2 = q_1,\, r_0 - r_1 = q_0.$$

It follows that $r_0 = q_m + \cdots + q_0$, which belongs to I. $\square$

The above result allows us to determine the basis of the intersection of two left ideals by contraction, which can be handled by noncommutative elimination via Gröbner bases. For two D-finite ideals, one may avoid computing Gröbner bases naively by a noncommutative version of the FGLM algorithm. Please see (Chyzak and Salvy, 1998, Section 2.2.2) and (Koutschan, 2010, Section 2.3) for more details.

Next, we present two algorithms: one is for removing an apparent singularity, and the other is for detecting whether a singularity is apparent. In what follows, by "given a D-finite ideal I", we mean that a finite basis of I is given.

Algorithm 5.11. Given a D-finite ideal I with the origin being an apparent singularity, compute a primitive Gröbner basis $M \subset I$ such that the origin is an ordinary point of the D-finite ideal $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$.

(1) Compute the rank d of I.
(2) Compute a set of initial exponent candidates S for I by the algorithm that is implicit in Theorem 5.6.
(3) For each $B \subset S$ with $|B| = d$,

(3.1) set $m := \max_{\mathbf{u}\in B} |\mathbf{u}|$;
(3.2) compute a primitive Gröbner basis M_B of the sub-ideal of I with respect to $(\mathbb{N}_m^n \setminus B)$.
(3.3) if the origin is an ordinary point of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M_B$, then return M_B.

We have that $\mathrm{IE}_0(I) \subset S$ by Proposition 5.3, and $|\mathrm{IE}_0(I)| = d$ by Definition 4.1. So one of the B's in loop (3) is equal to $\mathrm{IE}_0(I)$. By Corollary 4.9, the above algorithm terminates within loop (3). However, it may not return the smallest possible M in the sense that $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M \subset I$ is a D-finite ideal of the smallest rank such that the origin is an ordinary point.

Example 5.12. Consider the D-finite ideal I in Example 4.2. Note that the solution space of I is spanned by $\exp(x_1+x_2)$ and $x_2\exp(x_2)$ whose initial exponents are $(0,0)$ and $(0,1)$, respectively. The origin is an apparent singularity of I. Let $B = \{(0,0),(0,1)\}$. Then $\left(\mathbb{N}_1^2 \setminus B\right) = \{(1,0)\}$. Compute a primitive Gröbner basis M of the sub-ideal of I with respect to $\{(1,0)\}$. We find that $\mathrm{HC}(M) = \{1-x_1-x_1x_2\}$. The origin is an ordinary point of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$.

Example 5.13. Consider the D-finite ideal I in Example 4.3. Note that the solution space of I is spanned by $\{x_1+x_2, x_1x_2\}$. The origin is an apparent singularity of I. By the two generators of I given in Example 4.3, $\mathrm{ind}(I)$ contains a zero-dimensional ideal generated by $y_1 - 1$ and $y_2(y_2-1)$. So a candidate set B of indicial exponents is equal to $\{(1,0),(1,1)\}$. We find that $M = \{\partial_1^3, \partial_1^2\partial_2, \partial_1\partial_2^2, \partial_2^3\}$ is a primitive Gröbner basis of the sub-ideal of I with respect to $\{(1,0),(1,1)\}$. The origin is an ordinary point of the sub-ideal $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$.

The next algorithm is a direct application of Algorithm 5.11.

Algorithm 5.14. Given a D-finite ideal I with the origin being a singularity, determine whether the origin is an apparent one, and return a primitive Gröbner basis $M \subset I$ such that the origin is an ordinary point of the D-finite ideal $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$ when it is apparent.

(1) Compute the rank d of I.
(2) Compute a set of initial exponent candidates S for I by the algorithm that is implicit in Theorem 5.6.
(3) If $|S| < d$, then return "the origin is not an apparent singularity".
(4) For each $B \subset S$ with $|B| = d$,
  (4.1) set $m := \max_{\mathbf{u}\in B} |\mathbf{u}|$;
  (4.2) compute a primitive Gröbner basis M_B of the sub-ideal of I with respect to $(\mathbb{N}_m^n \setminus B)$.
  (4.3) if the origin is an ordinary point, then return M_B.
(5) Return "the origin is not an apparent singularity".

The above algorithm clearly terminates. The solution space of I cannot be spanned by power series if the candidate set S has less than d elements. So the origin is not an apparent singularity in this case. The rest is correct by Algorithm 5.11.

Example 5.15. Consider the left ideal I from Example 5.9. The ideal is of rank two. The origin is a singularity of I. Moreover, $\mathrm{ind}(G_1) = y_2 - y_1$ and $\mathrm{ind}(G_2) = (y_1-1)y_1$, where G_1 and G_2 are given in Example 5.9. Thus, $S = \{(0,0),(1,1)\}$ is a set of initial exponent candidates for I.

Since the rank of I is equal to two, S is the only subset B in step 2 of the above algorithm, and $\left(\mathbb{N}_2^2 \setminus B\right) = \{(1,0),(0,1),(2,0),(0,2)\}$. Computing a primitive Gröbner basis M of the sub-ideal of I with respect to $\{(1,0),(0,1),(2,0),(0,2)\}$, we find that

$$\mathrm{HC}(M) = \left\{x_1(x_1-x_2)^3, -(x_1-x_2)^3\right\}.$$

The origin is a singularity of the D-finite ideal generated by M. So it is not an apparent singularity of I. Actually, $\mathrm{sol}_{\mathbb{E}}(I)$ is spanned by $x_1/(x_1-x_2)$ and x_1x_2.

Example 5.16. Let $I = \mathbb{Q}(x_1,x_2)[\partial_1,\partial_2]G$, where G consists of

$$G_1 = (x_1-x_2)\partial_1^2 - x_1x_2\partial_2 + x_1x_2\partial_1 + (x_1-x_2),$$

$$G_2 = (x_1-x_2)\partial_1\partial_2 - (1+x_1x_2)\partial_2 + (1+x_1x_2)\partial_1 + (x_1-x_2),$$

and

$$G_3 = (x_1-x_2)\partial_2^2 - x_1x_2\partial_2 + x_1x_2\partial_1 + (x_1-x_2).$$

Then $\mathrm{rank}(I) = 3$ and the origin is a singularity of I.

From the three indicial polynomials $\mathrm{ind}(G_1) = (y_1-1)y_1$, $\mathrm{ind}(G_2) = y_2(y_1-1)$ and $\mathrm{ind}(G_3) = (y_2-1)y_2$, we see that $S = \{(0,0),(1,0),(1,1)\}$ is a set of initial exponent candidates for I. Let $B = S$. Then $\left(\mathbb{N}_2^2 \setminus B\right) = \{(0,1),(2,0),(0,2)\}$. A primitive Gröbner basis M of the sub-ideal of I with respect to $\{(0,1),(2,0),(0,2)\}$ asserts that the origin is an ordinary point of the sub-ideal. By Theorem 4.7, the origin is an apparent singularity of I. Actually, $\mathrm{sol}_{\mathbb{E}}(I)$ is spanned by $\{\sin(x_1+x_2), \cos(x_1+x_2), x_1x_2\}$.

Remark 5.17. Given a D-finite ideal I, we can determine whether $\mathrm{sol}_{\mathbb{E}}(I)$ is spanned by power series in $\mathbb{K}[[\mathbf{x}]]$ by Algorithm 5.14. This is because $\mathrm{sol}_{\mathbb{E}}(I)$ is spanned by power series in $\mathbb{K}[[\mathbf{x}]]$ if and only if the origin is either an ordinary point or an apparent singularity of I by Theorems 3.7 and 4.7.

5.3. A heuristic method for desingularization

For a nonzero operator $L \in \mathbb{K}[x_1][\partial_1]$ with apparent singularities, the randomized algorithm by Chen, et al (2016) computes a desingularized operator for L by taking the least common left multiple of L with a random operator of appropriate order with constant coefficients. This algorithm has been proved to obtain a correct desingularized operator for L with probability one, and is more efficient than deterministic algorithms. We now extend this randomized technique to the case of several variables. To this end, we need two lemmas about determinants.

Lemma 5.18. Let $U = (u_{i,j})$ be a $(k+d)\times d$ matrix of full rank over $\mathbb{K}$, and $Y = (Y_{i,m})$ be a $(k+d)\times k$ matrix whose entries are distinct indeterminates. Then

$$\det(U,Y) = \sum_{(i_1,\ldots,i_k)\in S} \alpha_{i_1,\ldots,i_k} Y_{i_1,1}\cdots Y_{i_k,k},$$

where S is a nonempty subset of $\mathbb{N}_{k+d}^k$, and every coefficient $\alpha_{i_1,\ldots,i_k}$ is nonzero.

Proof. Since U is of full rank, it contains a $d\times d$ nonzero minor. Without loss of generality, we assume that the minor consists of the first d rows and the first d columns in U. Setting $Y_{i,m}=0$ for all i,j with $1\le i\le d$ and $1\le m\le k$, we map $\det(U,Y)$ to

$$\begin{vmatrix} u_{1,1} & \cdots & u_{1,d} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{d,1} & \cdots & u_{d,d} & 0 & \cdots & 0 \\ u_{d+1,1} & \cdots & u_{d+1,d} & Y_{d+1,1} & \cdots & Y_{d+1,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{d+k,1} & \cdots & u_{d+k,d} & Y_{d+k,1} & \cdots & Y_{d+k,k} \end{vmatrix},$$

which is nonzero. So $\det(U,Y)$ is also nonzero. Collecting the like terms of $\det(U,Y)$, we prove the lemma. $\square$

Lemma 5.19. Let U be the same matrix as given in Lemma 5.18, and let $Z_1,\ldots,Z_k$ be mutually disjoint sets of indeterminates. Let $Z_{1,m},\ldots,Z_{d+k,m}$ be distinct monomials in the indeterminates belonging to Z_m, $m=1,\ldots,k$. Denote by Z the $(k+d)\times k$ matrix $(Z_{i,m})$. Then $\det(U,Z)$ is a nonzero polynomial in $\mathbb{K}[Z_1\cup\cdots\cup Z_n]$.

Proof. By Lemma 5.18, we have

$$\det(U,Z)=\sum_{(i_1,\ldots,i_k)\in S}\alpha_{i_1,\ldots,i_k}Z_{i_1,1}\cdots Z_{i_k,k},$$

where S is a nonempty subset of $\mathbb{N}_{k+d}^k$ and every $\alpha_{i_1,\ldots,i_k}$ is nonzero. For two distinct elements $(i_1,\ldots,i_k),(j_1,\ldots,j_k)\in S$, the two terms $Z_{i_1,1}\cdots Z_{i_k,k}$ and $Z_{j_1,1}\cdots Z_{j_k,k}$ are also distinct by the definition of $Z_{i,j}$'s. Hence, there are no like terms to be collected in the right-hand side of the above equality. $\square$

Theorem 5.20. Let $I\subset\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ be a D-finite ideal of rank d. Assume that the origin is an apparent singularity of I, and that $f_1,\ldots,f_d$ are power series solutions of I with distinct initial exponents $\mathbf{u}_1,\ldots,\mathbf{u}_d$, respectively. Set $\ell=|\mathbb{N}_m^n|$,

$$m=\max_{1\le i\le d}|\mathbf{u}_i|\quad\text{and}\quad\mathbb{N}_m^n\setminus\mathrm{IE}_0(I)=\{\mathbf{u}_{d+1},\ldots,\mathbf{u}_\ell\}.$$

For each $j\in\{1,\ldots,\ell-d\}$, let f_{d+j} be the power series expansion of

$$\exp\left(z_{1,j}x_1+\cdots+z_{n,j}x_n\right)$$

around the origin, where $z_{1,j},\ldots,z_{n,j}$ are distinct constant indeterminates. Furthermore, let $A=(a_{i,j})$ be the $\ell\times\ell$ matrix, where $a_{i,j}$ is equal to the constant term of $\boldsymbol{\partial}^{\mathbf{u}_i}(f_j)$ for all $i,j\in\{1,\ldots,\ell\}$. Then

(i) $\det(A)$ is a nonzero polynomial in $\mathbb{K}[z_{1,1},\ldots,z_{n,1},\ldots,z_{1,\ell-d},\ldots,z_{n,\ell-d}]$.

(ii) Let $c_{i,j}$ be an element of $\mathbb{K}$ for all i with $1\le i\le n$ and j with $1\le j\le\ell-d$. If $\det(A)$ does not vanish at $(c_{1,1},\ldots,c_{n,1},\ldots,c_{1,\ell-d},\ldots,c_{n,\ell-d})$, then the origin is an ordinary point of $I\cap\left(\bigcap_{j=1}^{\ell-d}\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\left\{\partial_1-c_{1,j},\ldots,\partial_n-c_{n,j}\right\}\right)$.

Proof. (i) Without loss of generality, we order the initial exponents $\mathbf{u}_1, \dots, \mathbf{u}_d$ increasingly with respect to $\prec$. Then the submatrix consisting of the first d rows and first d columns in A is in an upper triangular form whose elements in the diagonal are all nonzero by (9). Thus, the first d columns of A are linearly independent over $\mathbb{K}$. Let $\mathbf{z}_j = (z_{1,j}, \dots, z_{n,j})$ for all j with $1 \le j \le \ell - d$. Then the $(d+j)$th column of A consists of $\mathbf{z}_j^{\mathbf{u}_1}, \dots, \mathbf{z}_j^{\mathbf{u}_\ell}$, which are distinct monomials in $\mathbf{z}_j$. Thus, $\det(A)$ is nonzero by Lemma 5.19.

(ii) Two ring homomorphisms are needed for the proof of the second assertion.

Let $R = \mathbb{K}[z_{1,1}, \dots, z_{n,1}, \dots, z_{1,\ell-d}, \dots, z_{n,\ell-d}]$. We define ϕ to be the homomorphism from $R[[\mathbf{x}]]$ to R that takes the constant term of a power series in $\mathbf{x}$, which extends the homomorphism from $\mathbb{K}[[\mathbf{x}]]$ to $\mathbb{K}$ defined in Section 2.3. By (2),

$$\forall\, \mathbf{w} \in \mathbb{N}^n \text{ with } \mathbf{w} \prec \mathbf{v},\ \phi\left(\boldsymbol{\partial}^{\mathbf{w}}(f)\right) = 0 \quad \text{and} \quad \phi\left(\boldsymbol{\partial}^{\mathbf{v}}(f)\right) \neq 0 \tag{9}$$

for every nonzero power series $f \in R[[\mathbf{x}]]$ with initial exponent $\mathbf{v}$.

Let $\psi : R \to \mathbb{K}$ be the substitution that maps z_{ij} to c_{ij} for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, \ell - d\}$. Then ψ is a ring homomorphism. We extend ψ to a homomorphism from R to $R[[\mathbf{x}]]$ by the rule $\psi(x_i) = x_i$. $i = 1, \dots, n$. The extended homomorphism is also denoted by ψ.

Since the determinant of A does not vanish at $(c_{1,1}, \dots, c_{n,1}, \dots, c_{1,\ell-d}, \dots, c_{n,\ell-d})$, the determinant of $\psi(A)$ is nonzero. Let $g_j = \psi(f_j)$ for $j = 1, \dots, \ell$ and let $B = (b_{ij})$ be the $\ell \times \ell$ matrix with $b_{ij} = \boldsymbol{\partial}^{\mathbf{u}_i} g_j$ for all $i, j \in \{1, \dots, \ell\}$. Then $\phi(b_{ij}) = \psi(a_{ij})$ for all $i, j \in \{1, \dots, \ell\}$, because $\phi \circ \psi = \psi \circ \phi$ and $\psi \circ \partial_k = \partial_k \circ \psi$ for all k with $1 \le k \le n$. It follows that $\phi(B) = \psi(A)$. Thus, $\det(\phi(B))$ is nonzero, and so is $\det(B)$. Accordingly, g_1, $\dots$, g_ℓ are linearly independent over $\mathbb{K}$ by Lemma 2.5.

Set $I_j = \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\,\{\partial_1 - c_{1,j}, \dots, \partial_n - c_{n,j}\}$ for all j in $\{1, \dots, \ell - d\}$. Then $g_1, \dots, g_d$ form a basis of $\mathrm{sol}_{\mathbb{E}}(I)$, because $g_i = f_i$, $i = 1, \dots, d$, and g_{d+j} spans $\mathrm{sol}_{\mathbb{E}}(I_j)$, because g_{d+j} corresponds to the power series expansion of $\exp\left(c_{1,j}x_1 + \cdots + c_{n,j}x_n\right)$ at the origin for all $j \in \{1, \dots, \ell - d\}$. It follows from Lemma 4.5 (iii) that $g_1, \dots, g_\ell$ form a basis of $\mathrm{sol}_{\mathbb{E}}(J)$, where $J = I \cap I_1 \cap \cdots \cap I_{\ell-d}$.

To prove that the origin is an ordinary point of J, it suffices to find a basis of $\mathrm{sol}_{\mathbb{E}}(J)$ in $\mathbb{K}[[\mathbf{x}]]$ whose initial exponents are exactly the elements of $\mathbb{N}^n_m$ by Lemma 4.6. Since the power series $g_1, \dots, g_\ell$ are linearly independent over $\mathbb{K}$, there exists an $\ell \times \ell$ invertible matrix C over $\mathbb{K}$ such that $(h_1, \dots, h_\ell) = (g_1, \dots, g_\ell)C$, in which $h_1, \dots, h_\ell$ have distinct initial exponents. Set $H = BC$. Then H is the $\ell \times \ell$ matrix whose element at the ith row and jth column is equal to $\boldsymbol{\partial}^{\mathbf{u}_i} h_j$. Moreover, $\phi(H)$ is of full rank because $\phi(H)$ is equal to $\phi(B)C$. Suppose that there exists $h_j \in \{h_1, \dots, h_\ell\}$ such that its initial exponent $\mathbf{v}$ does not belong to $\mathbb{N}^n_m$. Then $\mathbf{v}$ is higher than any element of $\mathbb{N}^n_m$, because $\prec$ is graded. In other words, $\mathbf{u}_i \prec \mathbf{v}$ for all i with $1 \le i \le \ell$. It follows from (9) that the jth column of $\phi(H)$ is a zero vector, a contradiction. Therefore, the initial exponents of h_1, $\dots$, h_ℓ are exactly the elements of $\mathbb{N}^n_m$. $\square$

Algorithm 5.21. Given a D-finite ideal I with the origin being an apparent singularity, compute a primitive Gröbner basis M such that $M \subset I$ and the origin is an ordinary point of the D-finite ideal $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$, or return "fail".

(1) Set $d := \mathrm{rank}(I)$.

(2) Compute a set of initial exponent candidates S for I by the algorithm that is implicit in Theorem 5.6.

(3) For each $B \subset S$ with $|B| = d$,
   (3.1) set $m := \max_{\mathbf{u} \in B} |\mathbf{u}|$ and $\ell := |\mathbb{N}_m^n|$;
   (3.2) choose a point $\mathbf{c} = (c_{1,1}, \ldots, c_{n,1}, \ldots, c_{1,\ell-d}, \ldots, c_{n,\ell-d}) \in \mathbb{K}^{n(\ell-d)}$;
   (3.3) compute the primitive Gröbner basis M_B of the D-finite ideal

$$I \cap \left(\bigcap_{j=1}^{\ell-d} \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\{\partial_1 - c_{1,j}, \ldots, \partial_n - c_{n,j}\} \right);$$

   (3.4) if the origin is an ordinary point, then return M_B.

(4) return "fail".

The above algorithm clearly terminates. If B in step 3 coincides with $\mathrm{IE}_0(G)$ and $\det(A)$ given in Theorem 5.20 does not vanish at $\mathbf{c}$, then the origin is an ordinary point of the D-finite ideal $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M_B$ by Theorem 5.20. So it does not return "fail" unless $\mathbf{c}$ lies on the variety defined by $\det(A) = 0$. In this sense, the above algorithm succeeds outside of a variety which is not the full space. A feature of the above algorithm is that it is more efficient to compute a Gröbner basis of the intersection of several left ideals, most of which are generated by first-order operators with constant coefficients. Another advantage is that this algorithm is likely to remove all apparent singularities, not just the origin, because almost all choices of $c_{i,j}$ will also work for apparent singularities at almost any other point. On the other hand, it is not convenient to apply Theorem 5.20 to determine whether the origin is an apparent singularity, because the above algorithm will always return "fail" if the origin is a singularity but not an apparent one.

Example 5.22. Consider the left ideal from Example 4.2. Then $n = 2$, $\mathrm{rank}(I) = 2$ and the origin is an apparent singularity. A set of initial exponent candidates for I is $S = \{(0,0), (0,1)\}$. Set $B = S$ and $\ell = |\mathbb{N}_1^2| = 3$. Choose $\mathbf{c} = (19, 23) \in \mathbb{K}^{n(\ell-d)} = \mathbb{K}^2$. Let M_B be the primitive Gröbner basis of the left ideal $I \cap \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\{\partial_1 - 19, \partial_2 - 23\}$. We find that $\mathrm{HC}(M_B) = \{9 + 11x_2\}$. It follows from Definition 3.1 that $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M_B \subset I$ for which the origin is an ordinary point.

5.4. Truncating power series solutions at apparent singularities

When the origin is an apparent singularity of a D-finite ideal, the solution space of the ideal has a basis of power series. We show how to truncate such a basis by the proof of Theorem 4.7. To this end, we introduce some new notation.

For a nonzero element $P \in \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]$ with $\mathrm{HT}(P) = \boldsymbol{\partial}^{\mathbf{p}}$, we call $\mathbf{p}$ the *exponent* of P. Let $f = \sum_{\mathbf{u} \in \mathbb{N}^n} (c_{\mathbf{u}}/\mathbf{u}!)\mathbf{x}^{\mathbf{u}} \in \mathbb{K}[[\mathbf{x}]]$. For each $\mathbf{m} \in \mathbb{N}^n$, set

$$[f]_{\mathbf{m}} = \sum_{\mathbf{u} \preceq \mathbf{m}} \frac{c_{\mathbf{u}}}{\mathbf{u}!} \mathbf{x}^{\mathbf{u}} \in \mathbb{K}[\mathbf{x}],$$

which is called the *truncation* of f at $\mathbf{m}$. The next lemma enables us to truncate the application of an operator to a power series.

Lemma 5.23. Let $P \in \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ with exponent $\mathbf{p}$. Then $[P(f)]_{\mathbf{m}} = [P([f]_{\mathbf{m}+\mathbf{p}})]_{\mathbf{m}}$ for all $f \in \mathbb{K}[[\mathbf{x}]]$ and $\mathbf{m} \in \mathbb{N}^n$, *Proof.* For $\mathbf{v} \in \mathbb{N}^n$, let $H_{\mathbf{v}}$ denote the ideal generated by all powers $\mathbf{x}^{\mathbf{u}}$ with $\mathbf{u} \succ \mathbf{v}$ in $\mathbb{K}[[\mathbf{x}]]$. Then $P(g) \in H_{\mathbf{m}}$ for all $g \in H_{\mathbf{m}+\mathbf{p}}$ by a direct calculation. Since $f = [f]_{\mathbf{m}+\mathbf{p}} + g$ for some $g \in H_{\mathbf{m}+\mathbf{p}}$, we have $P(f) = P([f]_{\mathbf{m}+\mathbf{p}}) + P(g)$. Truncating both sides of the above equality yields the lemma. □

Recall some notation and results given in the proof of Theorem 4.7. Let I be a D-finite ideal of rank d and $G \subset \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$ be a finite basis of I. Moreover, let G contain k elements. Assume that the origin is an apparent singularity of I. We desingularize the origin to obtain another D-finite ideal J contained in I. Assume that $\ell = \operatorname{rank}(J)$ and that M is a Gröbner basis of J. With the help of M and formula (4), we can compute any truncations of a basis $f_1, \ldots, f_\ell$ of $\operatorname{sol}_{\mathbb{E}}(J)$.

For $z_1, \ldots, z_\ell \in \mathbb{K}$, we have that $z_1 f_1 + \cdots + z_\ell f_\ell \in \operatorname{sol}_{\mathbb{E}}(I)$ if and only if

$$z_1 P(f_1) + \cdots + z_\ell P(f_\ell) = 0 \quad \text{for all } P \in G. \tag{10}$$

The above equations result in a matrix A with infinitely many rows and ℓ columns over $\mathbb{K}$ such that the equations hold if and only if $A\mathbf{z} = \tilde{\mathbf{0}}$, where $\mathbf{z} = (z_1, \ldots, z_\ell)^t$. Moreover, the rank of A is equal to $\ell - d$.

To compute a truncation of power series solutions, we need some submatrices of A. For every $\mathbf{u} \in \mathbb{N}^n$, let $B_{\mathbf{u}}$ be the $k \times \ell$ matrix obtained by equating the coefficients of $\mathbf{x}^{\mathbf{u}}$ in both sides of (10). By Lemma 5.23, the matrix $B_{\mathbf{u}}$ can be obtained from the equations

$$z_1 P([f_1]_{\mathbf{u}+\mathbf{p}}) + \cdots + z_\ell P([f_\ell]_{\mathbf{u}+\mathbf{p}}) = 0 \quad \text{for all } P \in G, \tag{11}$$

where $\mathbf{p}$ is the exponent of P. For $\mathbf{v} \in \mathbb{N}^n$, we denote by $A_{\mathbf{v}}$ the matrix obtained by stacking submatrices $B_{\mathbf{u}}$ for all $\mathbf{u}$ with $\mathbf{u} \preceq \mathbf{v}$. Since $\operatorname{rank}(A) = \ell - d$, there exists a vector $\mathbf{w} \in \mathbb{N}^n$ such that $A_{\mathbf{w}}$ is of rank $\ell - d$. We can find a basis $\mathbf{c}_1, \ldots, \mathbf{c}_d$ of the right kernel of $A_{\mathbf{w}}$. Then $g_1 = (f_1, \ldots, f_\ell)\mathbf{c}_1, \ldots, g_d = (f_1, \ldots, f_\ell)\mathbf{c}_d$ form a basis of $\operatorname{sol}_{\mathbb{E}}(I)$ according to the proof of Theorem 4.7. It follows that

$$[g_1]_{\mathbf{m}} = ([f_1]_{\mathbf{m}}, \ldots, [f_\ell]_{\mathbf{m}})\,\mathbf{c}_1, \ \ldots, \ [g_d]_{\mathbf{m}} = ([f_1]_{\mathbf{m}}, \ldots, [f_\ell]_{\mathbf{m}})\,\mathbf{c}_d.$$

This idea is encoded in the following algorithm.

Algorithm 5.24. Given $\mathbf{m} \in \mathbb{N}^n$ and a left ideal I of rank d with the origin being an apparent singularity, compute polynomials $g_1, \ldots, g_d \in \mathbb{K}[\mathbf{x}]$ such that there exist $\mathbb{K}$-linearly independent power series solutions $h_1, \ldots, h_d$ of I with the property

$$g_1 = [h_1]_{\mathbf{m}}, \ldots, g_d = [h_d]_{\mathbf{m}}.$$

(1) By Algorithm 5.11, compute a primitive Gröbner basis M such that the origin is an ordinary point of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$. And set ℓ to be the rank of $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]M$.
(2) Compute $[f_1]_{\mathbf{m}}, \ldots, [f_\ell]_{\mathbf{m}}$ by M according to (4), where $f_1, \ldots, f_\ell$ stand for a basis of $\operatorname{sol}_{\mathbb{E}}(M)$.
(3) Construct a matrix $A_{\mathbf{w}}$ of rank $\ell - d$ by (11) incrementally with respect to $\prec$.
(4) Find a basis $\mathbf{c}_1, \ldots, \mathbf{c}_d$ of the right kernel of $A_{\mathbf{w}}$.
(5) Return $([f_1]_{\mathbf{m}}, \ldots, [f_\ell]_{\mathbf{m}})\,\mathbf{c}_1, \ldots, ([f_1]_{\mathbf{m}}, \ldots, [f_\ell]_{\mathbf{m}})\,\mathbf{c}_d$.

Note that the algorithm avoids the (possibly expensive) computation of the Weyl closure $I \cap \mathbb{K}[\mathbf{x}][\boldsymbol{\partial}]$, which might lead to an alternative method for computing the power series solutions of I.

Example 5.25. Consider the D-finite ideal I generated by $G_1 = x_2\partial_2 + \partial_1 - x_2 - 1$ and $G_2 = \partial_1^2 - \partial_1$. Then $\operatorname{rank}(I) = 2$ and the origin is an apparent singularity of I. We compute a power series basis of $\operatorname{sol}_{\mathbb{E}}(I)$ truncated at $\mathbf{m} = (0, 2)$. Recall that the term order is graded lexicographic with $x_1 \prec x_2$.

(1) Let M be the primitive Gröbner basis of the left ideal J, where J is the intersection of $I \cap \mathbb{K}(\mathbf{x})[\boldsymbol{\partial}]\{x_1\partial_1 - 1, \partial_2\}$. The origin is an ordinary point of J and $\operatorname{rank}(J) = 3$.

(2) By formula (4), we obtain a power series basis of $\operatorname{sol}_{\mathbb{E}}(J)$ truncated at $\mathbf{m}$, which consists of $p_1 = [\exp(x_1 + x_2) - x_1 - x_2 \exp(x_2)]_{\mathbf{m}}$, $p_2 = x_1$ and $p_3 = [x_2 \exp(x_2)]_{\mathbf{m}}$.

(3) A straightforward calculation yields

$$A_{\mathbf{0}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is of rank one. A basis of its right kernel is $\{(1, 1, 0)^t, (0, 0, 1)^t\}$. It follows that a power series basis of $\operatorname{sol}_{\mathbb{E}}(I)$ truncated at $\mathbf{m}$ is

$$\{[\exp(x_1 + x_2) - x_2 \exp(x_2)]_{\mathbf{m}}, [x_2 \exp(x_2)]_{\mathbf{m}}\}.$$

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