WALKS WITH SMALL STEPS IN THE 4D-ORTHANT

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Abstract. We provide some first experimental data about generating functions of restricted lattice walks with small steps in $\mathbb{N}^4$.

1. Introduction

For given sets $S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, many people have studied the number of lattice walks in $\mathbb{N}^2$ starting at $(0,0)$ and consisting of n steps each of which is taken from S. If $a_{i,j,n}$ denotes the number of such walks ending at the point $(i,j) \in \mathbb{N}^2$, then it is interesting to study the nature of the formal power series $f(x,y,t) = \sum_{n,i,j=0}^{\infty} a_{i,j,n} x^i y^j t^n$, i.e., whether it is algebraic, or, if not, whether it is at least D-finite. It turns out that the nature of f depends on the choice of S, and a lot of work has been done in order to classify all step sets S according to the nature of the corresponding generating function f, and to understand the deeper reasons that lead to the different types of series. This line of research was initiated by Bousquet-Mélou and Mishna [10], and many other researchers have been contributing to it during the past years, see [14, 10, 8, 7, 16, 5, 11, 2, 12] for some of the milestones.

As there are not many remaining open questions for the classical setting described above, people are now more and more turning to the study of variations and generalizations. One such generalization concerns the situation in higher dimensions. A first step was taken by Bostan and Kauers in [6], who used automated guessing to identify potentially D-finite step sets of size up to 5 in three dimensions. This work was extended by Bostan, Bousquet-Melou, Kauers, and Melczer [4] to step sets of size up to 6. They introduced the notion of a dimension of a lattice walk model, and the so-called Hadamard decomposition of a step set, and they used these new concepts as well as the classical orbit sum method for proving the D-finiteness in certain cases. Bacher, Kauers and Yatchak [1] have extended this work to step sets of arbitrary size, Du, Hou, and Wang provided non-D-finiteness results for many cases [13], and most recently, Bogosel, Perrollaz, Raschel, and Trotignon [3] have systematically explored the asymptotic behaviour of counting sequences for walks in the octant and observed a striking relation between the nature of the generating function and the angles of certain triangles on the sphere. Despite all this progress, there are still many open questions related to walks in the octant. In particular, there is a list of 170 models whose nature remains unclear. For example, this list includes the 3D version of the classical 2D Kreweras model [15, 9, 10], the step set $\{(-1,0,0),(0,-1,0),(0,0,-1),(1,1,1)\}$. Although the 2D version has an algebraic generating function, the current asymptotic estimates suggest (without proof) that the 3D version is not D-finite.

In this short note, we have nothing new to say about the 3D cases. Instead, our aim is to open the discussion for 4D. When the dimension of the lattice increases, the classification problem becomes more difficult in two ways. First, and most importantly, the total number of models explodes. For dimension D, there are 2^{3^D-1} step sets, which evaluates to more than 10^{24} when $D = 4$. There is no way to go through all of them in a reasonable time, even if we spend only a tiny amount of computation time per model. The second problem is that it won't be enough to spend only a tiny amount of computation time per model, because with increasing dimension it also becomes more costly to analyze a particular model. For example, computing the first N terms of a counting sequence using the standard recurrence requires $\mathrm{O}(N^{D+1})$ time and $\mathrm{O}(N^D)$ memory. For $D = 4$, this means that on a computer with 1 Tb of main memory, we were only able to compute $N = 700$ terms of a counting sequence.

2. Search Procedure

In order to identify potentially interesting models, we have applied a similar search procedure as Bacher, Kauers, and Yatchak [1] did in their search for interesting models in 3D. The procedure can be summarized as follows:

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- **Only step sets $S \subseteq \{-1,0,1\}^4 \setminus \{(0,0,0,0)\}$ with $|S| \le 7$ or $|S| \ge 73$ were considered.** This restriction has no combinatorial motivation but was only made to reduce the computational cost to a manageable amount, similar as it was done in [6, 4] for 3D.
- **Step sets containing unused steps were discarded.** Recall from [4] that an element s of S is called unused if it cannot appear in any walk of the model. For example, the step set $S = \{(1,0,-1,0),(0,1,0,-1),(1,1,0,0)\}$ leads to the same generating function as the step set $\{(1,1,0,0)\}$, because any use of $(1,0,-1,0)$ or $(0,1,0,-1)$ would lead the walk out of $\mathbb{N}^4$, which is not allowed.
- **Only one step set from each symmetry class was considered.** Permuting the coordinates of the steps corresponds to a permutation of the variables of the generating function. For example, if $f(x_1,x_2,x_3,x_4,t)$ is the generating function for the model with step set $\{(1,0,1,1),(-1,1,0,0),(0,0,0,1)\}$, then $f(x_2,x_4,x_1,x_3,t)$ is the generating function for the model with step set $\{(0,1,1,1),(1,0,-1,0),(0,1,0,0)\}$. Since permutation of variables preserves algebraicity and D-finiteness, it suffices to consider one model per equivalence class.
- **Step sets admitting a Hadamard decomposition were discarded.** Recall from [4] that a step set S is said to admit a (d_1,d_2)-Hadamard decomposition for some positive d_1,d_2 with $d_1+d_2 = D$ if it can be written as $S = (V \times \{0\}) \cup (U \times W)$ with $V,U \subseteq \mathbb{Z}^{d_1}$ and $W \subseteq \mathbb{Z}^{d_2}$. If this is the case, the generating function for the lattice walk model for S can be expressed in terms of the Hadamard product of the generating functions associated to the lower dimensional models corresponding to $U \cup V$ and W.
- **Step sets with dimension less than 4 were discarded.** Recall from [4] that the dimension of a model is defined as the number of coordinates for which the nonnegativity restriction is not redundant. For example, for the step set $\{(1,1,1),(1,-1,0),(1,0,-1)\}$, the number of walks in $\mathbb{N}^3$ is the same as the number of walks in $\mathbb{Z}\times\mathbb{N}^2$, because there is no way to get a negative first coordinate with the available steps. As the restriction on the other two coordinates is essential, the dimension is 2 in this case. Since lattice walk models in $\mathbb{N}^4$ whose dimension is less than 4 are equivalent to models in $\mathbb{N}^3$ (possibly with multiple steps), it is fair to discard them.
- **Step sets whose associated group has more than 800 elements were discarded.** Recall from [10, 4] that to every model of maximal dimension we can associate a certain group. Given a step set $S \subseteq \{-1,0,1\}^D \setminus \{(0,\dots,0)\}$, the group is constructed as follows. For $i = 1,\dots,D$, let Φ_i be the rational maps that sends x_j to itself for $j \neq i$ and x_i to $x_i^{-1}A_{i,-}/A_{i,+}$, where $A_{i,+} = \sum_{(s_1,\dots,s_D)\in S: s_i=1} x_1^{s_1}\cdots x_D^{s_D}/x_i$ and $A_{i,-} = \sum_{(s_1,\dots,s_D)\in S: s_i=-1} x_1^{s_1}\cdots x_D^{s_D}/x_i^{-1}$. The group associated to S is the group generated by $\Phi_1,\dots,\Phi_D$ under composition. A main result about the case $D=2$ is that this group is finite if and only if the generating function is D-finite [10, 4, 5]. While the experimental results for $D=3$ suggest that there may be non-D-finite cases with finite group, we are not aware of any (conjectured) D-finite case with an infinite group. For this reason, and also because a finite group gives the chance to apply the so-called orbit sum method for proving D-finiteness, we have decided to restrict the search to models with finite group.

Out of the 7005847194 step sets with cardinality at most 7 or at least 73, there were 58 step sets which survived all these filters, the last filter being the by far strongest one. The surviving models are listed in the next section. They all have cardinality 5 or 7.

3. Results

For models with a finite group, the orbit sum method is one approach to showing that the generating function is D-finite. It rests on the observation that, when certain technical conditions are satisfied, the generating function for a model can be expressed as

$$f(x_1,\dots,x_D,t) = \frac{1}{x_1\cdots x_D}[x_1^{>}\dots x_D^{>}]\frac{1}{1-tP_S}\sum_{g\in G} g(x_1\cdots x_D),$$

where G is the group, $P_S := \sum_{(s_1,\dots,s_D)\in S} x_1^{s_1}\cdots x_D^{s_D}$ is the step set polynomial (also called the inventory by some authors), and $[x_1^{>}\dots x_D^{>}]$ is the positive part extraction operator. Note that the expression to which the positive part extraction operator is applied is a rational function. By the closure of D-finiteness under taking positive parts, the formula above implies that the generating function is D-finite.

For 50 of the 58 step sets identified by the procedure of Section 2, the orbit sum $\sum_{g\in G} g(x_1\cdots x_D)$ happens to be zero. In this case, the "technical conditions" alluded to above are not satisfied and we cannot directly conclude D-finiteness. In the other eight cases, we have checked with Yatchak's algorithm [17] that the technical conditions are satisfied, so the generating functions of these models are D-finite.

For the 50 cases whose orbit sum is zero, we have tried to detect recurrence equations or differential equations via automated guessing, as systematically done in [6] for 3D models. As remarked in the introduction, we were only able to compute 700 terms for each of these counting sequences, which only in one case (number 13 in the listing below) was enough to find equations. For the generating function of walks with arbitrary endpoint, $f(1,\dots,1,t)$, we found a linear differential equation of order 12 with polynomial coefficients of degree up to 135. Its coefficient sequence appears to satisfy a linear recurrence of order 18 with polynomial coefficients of degree up to 113.

We suspect that further models are D-finite but only satisfy equations that are too large to be recovered from 700 sequence terms, and we invite the lattice walk counting community to have a closer look at these models. In the tables below, we write $\bar{1}$ instead of -1 for better readability. We also use a pictorial description of the step sets, extending similar descriptions used in the literature for lower dimensions. A step $(s_1,s_2,s_3,s_4)\in\{-1,0,1\}^4$ is represented by a bullet at position (s_1,s_2,s_3,s_4), where s_1 is the column block (-1 = left, 0 = middle, 1 = right), s_2 is the row block (1 = top, 0 = middle, -1 = bottom), and s_3, s_4 are the column and row, respectively, within the block specified by s_1, s_2. Models with nonzero orbit sum are highlighted. The orbit sums are stated in a separate table.

Table 1. Models with a group isomorphic to $C_2 \times C_2 \times S_3$.

1 $\bar1\bar1\bar10, \bar1\bar10\bar1, \bar1\bar111, \bar1100, 10\bar1\bar1, 1001, 1010.$ $1, 2, 6, 18, 84, 340, \ldots$	2 $\bar1\bar1\bar10, \bar1\bar101, \bar1\bar11\bar1, \bar1100, 10\bar11, 100\bar1, 1010.$ $1, 1, 3, 8, 33, 122, \ldots$	3 $\bar1\bar1\bar1\bar1, \bar1\bar101, \bar1\bar110, \bar1100, 10\bar10, 100\bar1, 1011.$ $1, 1, 4, 13, 58, 245, \ldots$	4 $\bar10\bar10, \bar1001, \bar101\bar1, 1\bar1\bar11, 1\bar10\bar1, 1100, 1\bar110.$ $1, 1, 3, 9, 35, 125, \ldots$
5 $1\bar1\bar1\bar1, \bar100\bar1, \bar10\bar10, 1100, 1\bar110, 1\bar101, \bar1011.$ $1, 1, 4, 14, 60, 238, \ldots$	6 $\bar10\bar1\bar1, 1\bar1\bar10, 1\bar10\bar1, 1100, \bar1001, \bar1010, 1\bar111.$ $1, 1, 4, 14, 63, 241, \ldots$	7 $\bar1\bar100, \bar11\bar10, \bar1101, \bar111\bar1, 10\bar11, 100\bar1, 1010.$ $1, 1, 4, 12, 62, 255, \ldots$	8 $\bar1\bar100, \bar11\bar10, \bar110\bar1, \bar1111, 10\bar1\bar1, 1001, 1010.$ $1, 2, 8, 30, 166, 764, \ldots$
9 $\bar11\bar1\bar1, \bar1\bar100, \bar1101, \bar1110, 10\bar10, 100\bar1, 1011.$ $1, 1, 6, 21, 126, 581, \ldots$	10 $\bar10\bar10, \bar100\bar1, \bar1011, 11\bar1\bar1, 1\bar100, 1101, 1110.$ $1, 2, 10, 46, 260, 1402, \ldots$	11 $\bar10\bar1\bar1, \bar1001, \bar1010, 11\bar10, 110\bar1, 1\bar100, 1111.$ $1, 1, 7, 33, 197, 1065, \ldots$	12 $\bar10\bar10, \bar1001, \bar101\bar1, 11\bar11, 1\bar100, 110\bar1, 1110.$ $1, 1, 5, 20, 102, 496, \ldots$

Table 2. Models with a group isomorphic to $S_3 \times S_3$.

13 $\bar10\bar10, 0\bar1\bar11, \bar1001, 0\bar10\bar1, 1100, \bar101\bar1, 0\bar110.$ $1, 1, 3, 9, 27, 117, \ldots$	14 $\bar10\bar1\bar1, 0\bar1\bar10, 0\bar10\bar1, 1100, \bar1001, \bar1010, 0\bar111.$ $1, 1, 4, 14, 45, 223, \ldots$	15 $\bar1\bar1\bar10, \bar1\bar10\bar1, \bar1\bar111, 01\bar1\bar1, 0101, 0110, 1000.$ $1, 3, 9, 37, 169, 759, \ldots$	16 $\bar1\bar1\bar10, \bar1\bar101, \bar1\bar11\bar1, 01\bar11, 010\bar1, 0110, 1000.$ $1, 2, 5, 18, 72, 295, \ldots$
17 $\bar1\bar1\bar1\bar1, \bar1\bar101, \bar1\bar110, 01\bar10, 010\bar1, 0111, 1000.$ $1, 2, 6, 26, 118, 548, \ldots$	18 $0\bar1\bar10, 0\bar101, 0\bar11\bar1, 1000, \bar11\bar11, \bar110\bar1, \bar1110.$ $1, 1, 2, 5, 14, 47, \ldots$	19 $\bar11\bar1\bar1, 0\bar1\bar10, 0\bar10\bar1, 1000, \bar1101, \bar1110, 0\bar111.$ $1, 1, 3, 9, 27, 103, \ldots$	20 $0\bar1\bar1\bar1, \bar110\bar1, \bar11\bar10, 1000, 0\bar110, 0\bar101, \bar1111.$ $1, 1, 2, 6, 19, 73, \ldots$
21 $\bar1\bar100, 01\bar10, 0101, 011\bar1, 10\bar11, 100\bar1, 1010.$ $1, 2, 8, 36, 184, 978, \ldots$	22 $\bar1\bar100, 01\bar1\bar1, 0101, 0110, 10\bar10, 100\bar1, 1011.$ $1, 3, 14, 74, 425, 2515, \ldots$	23 $\bar11\bar10, \bar110\bar1, \bar1111, 0\bar100, 10\bar1\bar1, 1001, 1010.$ $1, 2, 8, 34, 176, 908, \ldots$	24 $\bar11\bar1\bar1, 10\bar10, 100\bar1, 0\bar100, \bar1101, \bar1110, 1011.$ $1, 1, 6, 24, 133, 695, \ldots$
25 $\bar11\bar10, \bar1101, \bar111\bar1, 0\bar100, 10\bar11, 100\bar1, 1010.$ $1, 1, 4, 14, 66, 309, \ldots$	26 $\bar10\bar10, \bar100\bar1, \bar1011, 0\bar100, 11\bar1\bar1, 1101, 1110.$ $1, 2, 10, 46, 244, 1358, \ldots$	27 $\bar10\bar1\bar1, \bar1001, \bar1010, 0\bar100, 11\bar10, 110\bar1, 1111.$ $1, 1, 7, 33, 181, 1025, \ldots$	28 $\bar10\bar10, \bar1001, \bar101\bar1, 0\bar100, 11\bar11, 110\bar1, 1110.$ $1, 1, 5, 20, 94, 478, \ldots$
29 $\bar10\bar1\bar1, 01\bar10, 010\bar1, 1\bar100, \bar1001, \bar1010, 0111.$ $1, 1, 4, 16, 65, 299, \ldots$	30 $\bar10\bar10, \bar100\bar1, \bar1011, 01\bar1\bar1, 0101, 0110, 1\bar100.$ $1, 2, 6, 22, 94, 414, \ldots$	31 $\bar10\bar10, \bar1001, \bar101\bar1, 01\bar11, 010\bar1, 0110, 1\bar100.$ $1, 1, 3, 10, 37, 151, \ldots$	

Table 3. Models with a group isomorphic to S_5.

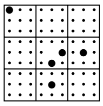
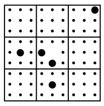
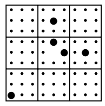
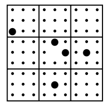
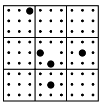
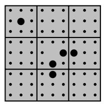
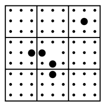
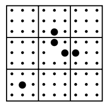
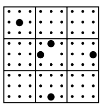
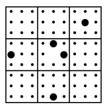
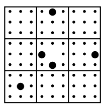
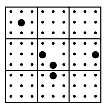
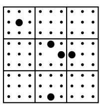
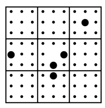
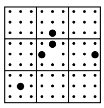
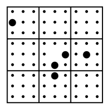
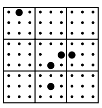
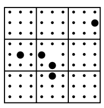
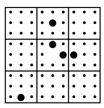
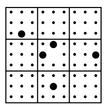
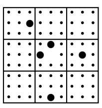
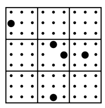
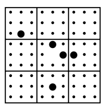
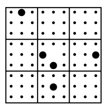
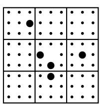
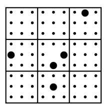
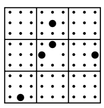
32  $\bar{1}1\bar{1}1, 0\bar{1}00, 000\bar{1}, 0010, 1000.$ $1, 2, 4, 10, 30, 98, \dots$	33  $\bar{1}000, 0\bar{1}00, 000\bar{1}, 00\bar{1}0, 1111.$ $1, 1, 5, 21, 81, 325, \dots$	34  $\bar{1}\bar{1}\bar{1}\bar{1}, 0010, 0001, 0100, 1000.$ $1, 4, 16, 64, 256, 1048, \dots$	35  $\bar{1}1\bar{1}\bar{1}, 0\bar{1}00, 0001, 0010, 1000.$ $1, 3, 9, 27, 87, 303, \dots$
36  $0\bar{1}00, 00\bar{1}0, 000\bar{1}, 1000, \bar{1}111.$ $1, 1, 2, 6, 21, 73, \dots$	37  $000\bar{1}, 10\bar{1}0, \bar{1}100, 0010, 0\bar{1}01.$ $1, 1, 2, 4, 10, 26, \dots$	38  $000\bar{1}, 00\bar{1}0, 1100, \bar{1}010, 0\bar{1}01.$ $1, 1, 3, 9, 29, 99, \dots$	39  $\bar{1}100, 0010, 0001, 010\bar{1}, 10\bar{1}0.$ $1, 2, 6, 18, 60, 206, \dots$
40  $0\bar{1}0\bar{1}, 00\bar{1}0, 1010, \bar{1}100, 0001.$ $1, 2, 6, 20, 71, 269, \dots$	41  $\bar{1}0\bar{1}0, 0\bar{1}0\bar{1}, 0001, 0010, 1100.$ $1, 3, 9, 31, 117, 467, \dots$	42  $\bar{1}100, 000\bar{1}, 00\bar{1}0, 1010, 0101.$ $1, 2, 6, 20, 80, 318, \dots$	43  $00\bar{1}0, 000\bar{1}, \bar{1}100, 0\bar{1}01, 1010.$ $1, 1, 3, 8, 24, 78, \dots$
44  $0\bar{1}0\bar{1}, 10\bar{1}0, 0010, \bar{1}100, 0001.$ $1, 2, 5, 14, 42, 136, \dots$	45  $\bar{1}0\bar{1}0, 000\bar{1}, 0\bar{1}01, 0010, 1100.$ $1, 2, 5, 16, 57, 209, \dots$	46  $\bar{1}100, 00\bar{1}0, 0001, 010\bar{1}, 1010.$ $1, 2, 6, 18, 63, 229, \dots$	47  $\bar{1}1\bar{1}0, 000\bar{1}, 0\bar{1}01, 0010, 1000.$ $1, 2, 4, 10, 28, 82, \dots$
48  $0\bar{1}00, 000\bar{1}, 10\bar{1}0, 0010, \bar{1}101.$ $1, 1, 2, 4, 11, 31, \dots$	49  $\bar{1}000, 00\bar{1}0, 000\bar{1}, 0\bar{1}01, 1110.$ $1, 1, 4, 14, 49, 183, \dots$	50  $\bar{1}10\bar{1}, 0010, 10\bar{1}0, 0001, 0100.$ $1, 3, 10, 35, 126, 474, \dots$	51  $\bar{1}10\bar{1}, 00\bar{1}0, 0\bar{1}00, 0001, 1010.$ $1, 2, 5, 15, 52, 185, \dots$
52  $0\bar{1}0\bar{1}, 00\bar{1}0, 1000, 0001, \bar{1}110.$ $1, 2, 5, 14, 45, 159, \dots$	53  $0\bar{1}0\bar{1}, 0010, 1000, \bar{1}1\bar{1}0, 0001.$ $1, 3, 9, 29, 99, 355, \dots$	54  $\bar{1}10\bar{1}, 0\bar{1}00, 0010, 10\bar{1}0, 0001.$ $1, 2, 5, 13, 38, 119, \dots$	55  $0\bar{1}00, 000\bar{1}, 00\bar{1}0, 1010, \bar{1}101.$ $1, 1, 3, 9, 31, 109, \dots$
56  $00\bar{1}0, 000\bar{1}, 1000, 0\bar{1}01, \bar{1}110.$ $1, 1, 2, 5, 15, 47, \dots$	57  $\bar{1}0\bar{1}0, 0\bar{1}00, 000\bar{1}, 0010, 1101.$ $1, 2, 6, 22, 88, 358, \dots$	58  $\bar{1}10\bar{1}, 00\bar{1}0, 0001, 0100, 1010.$ $1, 3, 10, 35, 132, 534, \dots$	

Table 4. Nonzero orbit sums.

idx	orbit sum
4	$\frac{(w^2-z)(wz-1)(w-z^2)(w^2+z-wy^2z+wz^2)(w^2x^2-wy+x^2z-w^2yz+wx^2y^2z+wx^2z^2-yz^2)}{w^3xyz^3(w^2+z+wy^2z+wz^2)}$
2	$\frac{(w^2-z)(wz-1)(w-z^2)(w+w^2z-wy^2z+z^2)(w^2x^2y-w^2z+x^2yz-wy^2z-z^2+wx^2yz^2-w)}{w^3xy^2z^3(w^2+z+wz^2)}$
7	$\frac{(w^2-z)(1-wz)(w-z^2)(w^2x^2y-wy^2-wz+x^2yz-w^2y^2z+wx^2yz^2-y^2z^2)(wy^2-wz+w^2y^2z+y^2z^2)}{w^2xy^2z^2(w+w^2z+z^2)(w^2+z+wz^2)}$
12	$\frac{(w^2-z)(1-wz)(w-z^2)(w^2y^2-wz+y^2z+wy^2z^2)(w^2x^2y^2+wx^2z-w^2yz+x^2y^2z-yz^2+wx^2y^2z^2-wy)}{w^2xyz^2(w^2+z+wz^2)(w^2y^2+wz+y^2z+wy^2z^2)}$
18	$\frac{(w^2-z)(1-wz)(w-z^2)(w+w^2z-wxyz+z^2)(w^2x+xz-wy^2z+wxz^2)(w^2x^2-wy+x^2z-w^2yz+wx^2z^2-yz^2)}{w^4x^2y^2z^4(w^2+z+wz^2)}$
25	$\frac{(w^2-z)(1-wz)(w-z^2)(w^2x^2-wy+x^2z-w^2yz+wx^2z^2-yz^2)(w^2xy-wz+xyz+wxyz^2)(wy^2-wxz+w^2y^2z+y^2z^2)}{w^2x^2y^2z^2(w+w^2z+z^2)(w^2+z+wz^2)^2}$
31	$\frac{(w^2-z)(wz-1)(w-z^2)(wy-wx^2z+w^2yz+yz^2)(w^2xy-w^2z+xyz-z^2+wxyz^2-w)(w^2y^2-wxz+y^2z+wy^2z^2)}{w^3x^2y^2z^3(w^2+z+wz^2)^2}$
37	$\frac{(w^2-y)(wy-x)(wx-y^2)(wx-z)(wz-1)(wz-xy)(xz-y)(w-yz)(x^2-yz)(x-z^2)}{w^4x^4y^4z^4}$

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