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Continued Classification of 3D Lattice Models in the Positive Octant

Axel Bacher[1†], Manuel Kauers[2‡] and Rika Yatchak[2§]

[1] *Laboratoire d'Informatique de Paris Nord, Université Paris 13*
[2] *Institute for Algebra, Johannes Kepler University, Linz, Austria*

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Abstract. We continue the investigations of lattice walks in the three-dimensional lattice restricted to the positive octant. We separate models which clearly have a D-finite generating function from models for which there is no reason to expect that their generating function is D-finite, and we isolate a small set of models whose nature remains unclear and requires further investigation. For these, we give some experimental results about their asymptotic behaviour, based on the inspection of a large number of initial terms. At least for some of them, the guessed asymptotic form seems to tip the balance towards non-D-finiteness.

Keywords: Lattice Walks, D-finiteness, Computer Algebra, Asymptotics

1 Introduction

The past years have seen many contributions to the theory of lattice walks in the first quadrant $\mathbb{N}^2$. For a fixed set $S \subseteq \mathbb{Z}^2$, the quantity of interest is the number $q_{n,i,j}$ of lattice walks of length n from $(0,0)$ to (i,j) not stepping out of $\mathbb{N}^2$. More formally, these walks can be viewed as elements $(s_1, s_2, \dots, s_n) \in S^n$ with the property that $\sum_{\ell=1}^{m} s_\ell \in \mathbb{N}^2$ for $m = 0, \dots, n$ and $\sum_{\ell=1}^{n} s_\ell = (i,j)$. One of the key questions in this context is whether or not the generating function $Q(x,y,t) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{n,i,j} x^i y^j t^n$ is D-finite with respect to t. The answer to this question depends on S.

For all the sets $S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, the nature of the generating function is known. In their seminal paper, Bousquet-Mélou and Mishna (2010) reduced the $2^{3^2-1} = 256$ different models to 79 among which there are no obvious bijections and which are not known to be algebraic for classical reasons (Banderier and Flajolet, 2002; Flajolet and Sedgewick, 2009). They then showed that 22 of these 79 models are D-finite and gave evidence that 56 were not D-finite. The last model required some more work, but we now have several independent proofs that its generating function is algebraic, and hence also D-finite (Bostan and Kauers, 2010; Bostan et al., 2013; Bousquet-Mélou, 2015a,b). For 51 of the remaining models, Kurkova and Raschel (2012) were able to show by analytic means that the corresponding generating functions are not D-finite. Finally, the generating functions of the remaining 5 models are not D-finite because they have too many singularities (Mishna and Rechnitzer, 2009; Melczer and Mishna, 2013).

The case of two dimensions is, in short, well-understood. As it turns out, the D-finiteness of a model is governed by a certain group associated to the model. There is a general theory (Fayolle et al., 1999; Raschel,

†Email: `bacher@lipn.fr`
‡Email: `manuel.kauers@jku.at`. Partially supported by the Austria FWF grants Y464-N18 and F50-04.
§Email: `rika.yatchak@jku.at`. Partially supported by the Austria FWF grant F50-04

2012; Kurkova and Raschel, 2015) that uniformly explains that the generating function is D-finite if and only if the associated group is finite. Among the remaining open problems in 2D are proofs of the non-D-finiteness of the generating functions $Q(1,1,t)$ for the 56 models where $Q(x,y,t)$ is non-D-finite, and the continued classification of models with longer steps.

What happens in three dimensions? Here, for every fixed $S \subseteq \mathbb{Z}^3$ we count the number of walks in $\mathbb{N}^3$ from $(0,0,0)$ to (i,j,k) with exactly n steps. Again, the principal question is whether the generating function $Q(x,y,z,t) = \sum_{n=0}^{\infty} \sum_{i,j,k\in\mathbb{N}} q_{i,j,k,n} x^i y^j z^k t^n$ for the number $q_{i,j,k,n}$ of walks in such a model is D-finite, and again, the answer depends on the choice of S. Restricting S to subsets of $\{-1,0,1\}^3 \setminus \{(0,0,0)\}$, there are now $2^{3^3-1} = 67{,}108{,}864$ models to be considered.

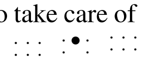
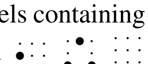
The classification of these models was initiated by Bostan and Kauers (2009). More recently, Bostan et al. (2014a) considered the $\sum_{k=0}^{6} \binom{26}{k} = 313{,}912$ step sets S with $|S| \leq 6$. After discarding symmetric and simple cases, they were left with 20,804 models, of which 151 were recognized as D-finite, and 20,634 were conjectured non-D-finite. No conjecture was made for the remaining 19 cases. Recently, Du et al. (2015) provided non-D-finiteness proofs for most of the 20,634 models that had been conjectured non-D-finite. The 19 models about which nothing is known have a finite group associated to them, which, by analogy to the situation in 2D, would suggest that they are D-finite. On the other hand, Bostan et al. (2014a) were not able to discover any recurrence or differential equations by guessing, which means that either these models are not D-finite, or the equations that they satisfy are very large.

In the present paper, we continue the classification work for octant walks. We apply the techniques of Bostan et al. (2014a) to all 67 million models, and find those models for which D-finiteness can be proved, those which have no reason to be D-finite, and those whose nature is unclear. We give an overview of the finite groups which appear in 3D; the collection turns out to be more diverse than in 2D. In the end we found 170 models that are of a similar nature as the 19 models discovered by Bostan et al. (2014a). For these we computed a large number of terms and tried to come up with reasonable guesses for their asymptotic behaviour.

It seems that the exponent α in the asymptotic growth $c\phi^n n^\alpha$ is rational for some cases and irrational for others. This is interesting because in view of the work of Bostan et al. (2014b), an irrational α can imply non-D-finiteness of the generating functions for these cases. For those where α seems rational, we have computed additional terms modulo a prime and used them to try to guess a recurrence or a differential equation. Several years of computation time has been invested, but no equation was found.

2 Unused Steps, Symmetries, Projectibility, and Decomposibility

A priori there are $2^{3^3-1} = 67{,}108{,}864$ different models. To narrow the number of cases down to a more manageable size, we apply the same techniques as Bostan et al. (2014a).

The **first filter** sorts out cases that are in bijection to others for simple reasons. The bijections in question are the permutations of the coordinates. In this filter we also take care of models containing directions that can never be used. For example, it is clear that the two models  and  consist of exactly the same walks, because no walk of the second model can possibly involve the step $(-1,0,-1)$ without leaving the first octant. Here and below, we depict step sets in the same way as Bostan et al. (2014a): the first block contains the directions $(*,*,-1)$, the second block the directions $(*,*,0)$, and the third block the directions $(*,*,1)$. The diagram on the right therefore refers to the step set

$$\{(-1,0,-1),(-1,-1,0),(0,1,0),(1,-1,0)\}.$$

The number of models surviving this first filter was already worked out by Bostan et al. (2014a) for step sets with arbitrary cardinality. According to their Proposition 5, the generating function $\sum_S u^{|S|}$ where S runs through the essentially different models is given by

$$73u^3 + 979u^4 + 6425u^5 + 28071u^6 + 91372u^7 + 234716u^8 + 492168u^9 + 860382u^{10} + 1271488u^{11} + 1603184u^{12} + 1734396u^{13} + 1614372u^{14} + 1293402u^{15} + 890395u^{16} + 524638u^{17} + 263008u^{18} + 111251u^{19} + 39256u^{20} + 11390u^{21} + 2676u^{22} + 500u^{23} + 73u^{24} + 9u^{25} + u^{26}.$$

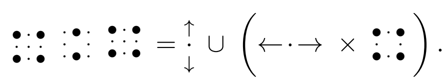
Going beyond this counting result, we have written a program that actually lists these models. The statistics in our listing agrees with this generating function.

The **second filter** sorts out models that are in bijection to some lattice walk model in the quarter plane. For example, for the left model depicted above, all walks stay within the plane corresponding to the first two coordinates, because there is no step that moves towards the third dimension. As pointed out by Bostan et al. (2014a), there are models which have both steps of type $(*, *, -1)$ as well as steps of type $(*, *, 1)$, but where the third dimension nevertheless is immaterial, because the constraint for staying always nonnegative in the third coordinate is implied by the requirements that the first two coordinates are nonnegative. Linear programming can be used to detect whether a given model is in bijection to a two-dimensional model. We have applied this test to all the 11,074,225 models passing the first filter and obtained 10,908,263 models that are inherently three-dimensional. The generating function is

$$u^3 + 220u^4 + 2852u^5 + 17731u^6 + 70590u^7 + 203965u^8 + 457650u^9 + 830571u^{10} + 1251613u^{11} + 1593013u^{12} + 1730461u^{13} + 1613252u^{14} + 1293178u^{15} + 890366u^{16} + 524636u^{17} + 263008u^{18} + 111251u^{19} + 39256u^{20} + 11390u^{21} + 2676u^{22} + 500u^{23} + 73u^{24} + 9u^{25} + u^{26}.$$

It is fair to eliminate the models that are in bijection with models in the plane because we have a classification of the generating functions for the latter. In particular, we have identified (Kauers and Yatchak, 2015) a list of families of 2D models with colored steps whose generating function is D-finite. Evidence was given that this list is complete, at least for models with up to three colors. These are the only models to which a 3D model can possibly be in bijection; see the article of Bostan et al. (2014a) for more details.

As a **third filter,** we determined the models that can be viewed as a direct product of a one-dimensional model and a two-dimensional model. These models were called Hadamard walks by Bostan et al. (2014a). An example is the decomposition



The generating function for the model on the left can be obtained from the two generating functions for the models on the right, and since these are D-finite, the model on the left must also have a D-finite generating function. Conversely, if a model admits a decomposition into two lower-dimensional models, one of which does not have a D-finite generating function, then this does not seem to say much about the nature of the generating function for the three- dimensional model. Nevertheless, we decided to discard all the models which admit a Hadamard decomposition from consideration. We are then left with altogether 10,847,434 models, the statistics according to size of step sets being

$$u^3 + 193u^4 + 2680u^5 + 17238u^6 + 69542u^7 + 202072u^8 + 454485u^9 + 826005u^{10} + 1245615u^{11} + 1585989u^{12} + 1722891u^{13} + 1605940u^{14} + 1286692u^{15} + 885048u^{16} + 520725u^{17} + 260374u^{18} + 109625u^{19} + 38377u^{20} + 10960u^{21} + 2488u^{22} + 436u^{23} + 54u^{24} + 4u^{25}.$$

3 The Associated Group

In their paper on walks in the quarter plane, Bousquet-Mélou and Mishna (2010) make use of a certain group associated to each model, introduced to the combinatorics community by Fayolle et al. (1999). Bostan et al. (2014a) consider the following natural analog of the group for three-dimensional models.

For a fixed step set $S \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$, consider the (Laurent) polynomial

$$P_S(x,y,z) = \sum_{(u,v,w)\in S} x^u y^v z^w = x^{-1} \sum_{(-1,v,w)\in S} y^v z^w + x^0 \sum_{(0,v,w)\in S} y^v z^w + x^1 \sum_{(1,v,w)\in S} y^v z^w.$$

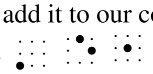
It is easy to see that P_S remains fixed under the rational transformation

$$\phi_x \colon \mathbb{Q}(x,y,z) \to \mathbb{Q}(x,y,z), \qquad \phi_x\Big(x,y,z\Big) := \Big(x^{-1}\frac{\sum_{(-1,v,w)\in S} y^v z^w}{\sum_{(1,v,w)\in S} y^v z^w}, y, z\Big).$$

In the same way, we can define rational transformations ϕ_y and ϕ_z which act on y and z, respectively, and leave the other two variables fixed. The group G associated to the model with step set S is the group generated by ϕ_x, ϕ_y, ϕ_z under composition.

The generators ϕ_x, ϕ_y, ϕ_z are self-inverse, so G can be viewed as a group of words over the alphabet $\{\phi_x, \phi_y, \phi_z\}$ subject to the relations $\phi_x^2 = 1$, $\phi_y^2 = 1$, $\phi_z^2 = 1$ and possibly others. In the case of two dimensions, where the group only has two generators ϕ_x, ϕ_y, the group is finite if and only if $(\phi_x\phi_y)^n = 1$ for some $n \in \mathbb{N}$, so the only finite groups that can appear are the dihedral groups D_{2n} with $2n$ elements.

In 3D, there is more diversity. For $n = 2, 3, \dots$ we consider all the words over ϕ_x, ϕ_y, ϕ_z that are not equivalent to some shorter word modulo a known relation. If there is no such word, the group is finite and we stop. Otherwise, for each word in the list, we check whether the corresponding rational map is the identity. If so, we have found a new relation and add it to our collection.

As an example, consider the model . We have

$$\phi_x\Big(x,y,z\Big) = \Big(\frac{1}{xyz}, y, z\Big), \qquad \phi_y\Big(x,y,z\Big) = \Big(x, \frac{1}{xyz}, z\Big), \qquad \phi_z\Big(x,y,z\Big) = \Big(x, y, \frac{1}{xyz}\Big).$$

The only relations of length two are $\phi_x^2 = \phi_y^2 = \phi_z^2 = 1$. There are no new relations of lengths 3, 4, or 5, but we find the relations $(\phi_x\phi_y)^3 = (\phi_x\phi_z)^3 = (\phi_y\phi_y)^3 = 1$ of length 6. These relations however do not suffice to imply finiteness of the group. After we also find the relations

$$\phi_z\phi_y\phi_z\phi_x\phi_z\phi_y\phi_z\phi_x = \phi_z\phi_y\phi_x\phi_z\phi_x\phi_y\phi_z\phi_x = \phi_z\phi_y\phi_z\phi_x\phi_y\phi_z\phi_y\phi_x = \phi_y\phi_x\phi_y\phi_z\phi_x\phi_y\phi_x\phi_z = 1$$

of length 8, we are able to conclude that no word of length 9 can be formed which does not contain at least one of the found relators as a subword. Thus the group is finite. Its order is 24, the number of words that do not contain any of the relators as a subword.

From a set of relations that completely characterizes a finite group, we can recognize the group using the SmallGroups package in GAP (GAP). Only 243 of the 10,847,434 models surviving the filters described in the previous section have a group with ≤ 400 elements. This is the **fourth filter.** The generating function is

$$8u^4 + 15u^6 + 12u^7 + 21u^8 + 12u^9 + 50u^{10} + 24u^{11} + 15u^{12} + 36u^{13} + 20u^{14} + 6u^{15} + 18u^{16} + 6u^{17}.$$

The groups turn out to be D_{12}, S_4, and $\mathbb{Z}_2 \times S_4$. We believe that the groups with > 400 elements are in fact infinite and expect that the corresponding generating functions are not D-finite. We have also determined the groups of the 60,829 inherently three-dimensional models that admit a Hadamard decomposition. We found that 2,187 of them have a group with ≤ 400 elements, these groups are $\mathbb{Z}_2^3$, D_{12}, and $\mathbb{Z}_2 \times D_8$. See Table 1 for

Group	Hadamard	Non-Hadamard Nonzero O.S.	Non-Hadamard Zero O.S.
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	1852	0	0
D_{12}	253	66	132
$\mathbb{Z}_2 \times D_8$	82	0	0
S_4	0	5	26
$\mathbb{Z}_2 \times S_4$	0	2	12

Tab. 1: Number of models with finite group

more precise statistics. It is noteworthy that we only encounter Coxeter groups. For example, in the example given above, if we replace the generator ϕ_z by $\psi := \phi_z\phi_y\phi_z$, then the group is determined by the relations $\phi_x^2 = \phi_y^2 = \psi^2 = (\phi_x\phi_y)^3 = (\phi_x\psi)^2 = (\phi_y\psi)^3 = 1$.

For models with a finite group, we can form the orbit sum $\sum_{g\in G} \operatorname{sgn}(g)g(xyz) \in \mathbb{Q}(x,y,z)$. As explained by Bousquet-Mélou and Mishna (2010) and Bostan et al. (2014a), a non-zero orbit sum (almost always) implies D-finiteness of the generating function $Q(x,y,z)$ of the model. As a **fifth filter,** we discard those models which have a finite group but a non-zero orbit sum. This leaves 170 models, the generating function marking step set being

$$7u^4 + 12u^6 + 8u^7 + 16u^8 + 8u^9 + 35u^{10} + 16u^{11} + 10u^{12} + 24u^{13} + 14u^{14} + 4u^{15} + 12u^{16} + 4u^{17}.$$

The $7+12 = 19$ models with 4 or 6 steps were already identified by Bostan et al. (2014a). The $243-170 = 73$ models discarded by the fifth filter seem to have a D-finite generating function.

4 Indecomposible and unprojectible models with finite group and zero orbit sum

We now have a closer look at the 170 non-equivalent models which are not projectible to quarter plane models, which do not admit a Hadamard decomposition, whose associated group is finite, and whose orbit sum is zero. It is not known for any of these models whether the associated generating function is D-finite or not. Our goal is to get some idea about this question by looking at the asymptotic behaviour of the counting sequences for excursions and for walks with arbitrary endpoint for each of the models.

4.1 Computation of Terms

To compute terms of the generating functions, we used the straightforward algorithm: we maintain a 3-dimensional array containing all terms $q_{i,j,k,n}$ for a given value of n. Given the step set, we can compute from that the numbers $q_{i,j,k,n+1}$ and iterate over the desired values of n. Unfortunately, this is extremely costly in both time and memory. We used the improvements described below to make the computation more tractable. The C language code for computing the coefficients was automatically generated from the step set by a Sage (The Sage Developers, 2015) script.

Reduction modulo a prime. Instead of computing the whole coefficients, we compute only the residues modulo a prime p. We chose primes satisfying $p \le 2^{15}$, so that the residues fit in a 16-bit integer.

Eliminate known zeros. The next step is to identify the tuples (i,j,k,n) for which the term $q_{i,j,k,n}$ can be nonzero and compute only these terms. From the step set, we can deduce several inequalities that have to be satisfied. These inequalities define a 4-dimensional polytope that we can compute in Sage. Computing only the terms that are in this polytope saves both time and space.

If we are only interested in the number of excursions of length up to N, we can further reduce the number of terms that we need: we only need the terms $q_{i,j,k,n}$ such that there exists an excursion of length $\leq N$ reaching the point (i, j, k) after n steps. This adds more constraints to the polytope.

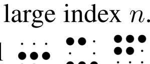
Finally, in many step sets, the quadruple (i, j, k, n) has to satisfy modular constraints (for instance, $i + j + k + n$ has to be even) for $q_{i,j,k,n}$ to be nonzero. This can also be determined automatically. In this case, again, we do not store the coefficients known to be zero and do not compute them.

Vectorization. The next optimization is to use the processor's SIMD instructions (Single Instruction, Multiple Data), that operate on 128-bit vector registers. Such a register can store eight 16-bit integers in a packed fashion, and the processor can operate on all of them in parallel with a single instruction. Moreover, we were able to compute the residues modulo p without using costly integer divisions. To do that, we note that if $p \leq 2^{15}$ and a and b are residues modulo p, their sum modulo p can be computed as $\mathrm{rem}(a + b, p) = \min(a + b, \mathrm{rem}(a + b - p, 2^{16}))$. Modular sums can therefore be computed using the vector addition, subtraction, and minimum instructions present in the SSE4.1 instruction set.

Parallelization. The final optimization is to distribute the computations over multiple processors to save time. This is the easiest step: since all values $q_{i,j,k,n+1}$ can be computed from the values $q_{i,j,k,n}$ independently of each other, we can give each processor a share of them. This was done using the OpenMP interface.

4.2 Guessed Asymptotic Behavior

For all 170 models in question, we calculated the first 2001 terms of the series $Q(0, 0, 0, t)$ counting excursions and of the series $Q(1, 1, 1, t)$ counting walks with arbitrary endpoint. We want to get an idea about the asymptotic behaviour of these sequences as $n \to \infty$. We assume that the asymptotic behaviour of these sequences is given by a linear combination of terms $\phi^n n^\alpha$, and our goal is to determine accurate estimates for the constants ϕ and α. For estimating these constants, it is not enough to know the terms of the counting sequences modulo a prime as computed by the code described above. However, we can apply the code for several primes and reconstruct the integer values from the various modular images using Chinese remaindering. The number of primes needed depends on the size of the integer to be reconstructed. As an upper bound, we can use the observation that in a model with step set S there can be at most $|S|^n$ walks of length n, so if we use primes in the range $2^{14} \dots 2^{15}$, then $\lceil \frac{n \log |S|}{14 \log(2)} \rceil$ primes will always be enough. For our largest step sets, which have 17 elements, we used 584 primes to recover the 2000th term. Alternatively, we could have used Garbit and Raschel (2015)'s Proposition 8, which proves a conjecture of Johnson et al. (2015), to explicitly calculate ϕ.

If a sequence $(a_n)_{n=0}^{\infty}$ grows like $c\phi^n n^\alpha$ for some constants ϕ, α, c, then we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \phi$ and $\lim_{n\to\infty} \frac{n(a_{n+1} - \phi a_n)}{\phi a_n} = \alpha$. We can thus get first estimates for ϕ and α by simply evaluating the respective expressions for some large index n. For example, for the counting sequence $(a_n)_{n=0}^{\infty}$ of walks with arbitrary endpoint in the model  we have

$$\phi \approx \frac{a_{1200}}{a_{1199}} = \frac{25407(\dots 1378 \text{ digits}\dots)93695}{17572(\dots 1377 \text{ digits}\dots)52363} \approx 14.4585690074019.$$

Comparison to

$$\phi \approx \frac{a_{1190}}{a_{1189}} = \frac{63641(\dots 1366 \text{ digits}\dots)06567}{44016(\dots 1365 \text{ digits}\dots)32175} \approx 14.4583480279347$$

suggests that the accuracy is close to 10^{-4}. To get a better estimate, we use a convergence acceleration technique due to Richardson and Gaunt. It is based on the assumption that the convergent sequence $(u_n)_{n=0}^{\infty}$ whose limit u is to be estimated has an asymptotic expansion of the form

$$u_n = u + c_1 n^{-1} + c_2 n^{-2} + O(n^{-3}) \quad (n \to \infty)$$

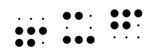
where $c_1, c_2, \ldots$ are some unknown constants. By cancellation of the term $c_1 n^{-1}$, it obtains an auxiliary sequence which converges to the same limit but with speed $1/n^2$ rather than $1/n$. The cancellation can be achieved in several ways. In particular, we have

$$2u_{2n} - u_n = u + 0n^{-1} - \frac{1}{2}c_2 n^{-2} + O(n^{-3}) \quad (n \to \infty)$$

and

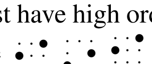
$$(n+1)u_{n+1} - nu_n = u + 0n^{-1} + c_2 \underbrace{\frac{1}{n(n+1)}}_{=n^{-2}+O(n^{-3})} + O(n^{-3}) \quad (n \to \infty).$$

Clearly, both versions can be generalized such as to eliminate further terms in the expansion by taking suitable linear combinations of $u_n, u_{2n}, u_{4n}, \ldots, u_{2^i n}$ or $u_n, u_{n+1}, \ldots, u_{n+i}$, respectively.

For the sequence $(a_n)_{n=0}^{\infty}$ counting walks with arbitrary endpoints in the model  we set $u_n = \frac{a_n}{a_{n-1}}$ and have, for example,

$$\frac{1}{7!}\sum_{k=0}^{7}(-1)^{k+7}\binom{7}{k}(1200-k)^7 u_{1200-k} \approx 14.48528121823356265,$$
$$\frac{1}{7!}\sum_{k=0}^{7}(-1)^{k+7}\binom{7}{k}(1190-k)^7 u_{1190-k} \approx 14.48528121635317802.$$

The expressions on the left are such that they cancel the first six terms in the asymptotic expansion of u_n. These approximations are accurate enough to recover from them, using standard techniques such as LLL or PSLQ, that ϕ is probably $2(1+\sqrt{6})$.

Once a reliable guess for ϕ is available, α can be estimated in the same way. The results of our calculations are given in tables in the appendix. It is noteworthy that for some models, α is easily recognized as a rational number, while for other models, our estimates seem to suggest that these α's are irrational, which in analogy with Bostan et al. (2014b) would imply that the corresponding sequences are not D-finite. We restrict the focus to the models where the counting sequence for excursions as well as the counting sequence for walks with arbitrary endpoint have a (conjecturally) rational α. Of each of these sequences we have computed the first 5127 terms modulo 16381. (It turned out that 5127 was the largest number of terms our C code can compute on a machine with 512G of RAM; we used some twenty such machines, each equipped with 64 processors running in parallel for several days.) Regrettably, we were unable to obtain plausible candidates for a potential recurrence or differential equation for any of the sequences we study here. If such equations exist, they must have high order or degree. We are not entirely convinced that no such equations exist, in view of the example  for which Bostan et al. (2014a) showed that there is a recurrence of order 55 and degree 3815, whose construction requires some 20000 terms.

References

C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281(1–2):37–80, 2002.

A. Bostan and M. Kauers. Automatic classification of restricted lattice walks. In *Proceedings of FPSAC'09*, pages 201–215, 2009.

A. Bostan and M. Kauers. The complete generating function for Gessel walks is algebraic. *Proceedings of the AMS*, 138(9):3063–3078, 2010. with an appendix by Mark van Hoeij.

A. Bostan, I. Kurkova, and K. Raschel. A human proof of Gessel's lattice path conjecture. Technical Report 1309.1023, ArXiv, 2013.

A. Bostan, M. Bousquet-Mélou, M. Kauers, and S. Melczer. On 3-dimensional lattice walks confined to the positive octant. *Annals of Combinatorics*, 2014a. to appear.

A. Bostan, K. Raschel, and B. Salvy. Non-D-finite excursions in the quarter plane. *Journal of Combinatorial Theory Series A*, 121, 2014b.

M. Bousquet-Mélou. An elementary solution of Gessel's walks in the quadrant. Technical Report 1503.08573, ArXiv, 2015.

M. Bousquet-Mélou. Plane lattice walks avoiding a quadrant. Technical Report 1511.02111, ArXiv, 2015.

M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. *Contemporary Mathematics*, 520:1–40, 2010.

D. K. Du, Q.-H. Hou, and R.-H. Wang. Infinite orders and non-d-finite property of 3-dimensional lattice walks. Technical Report 1507.03705, ArXiv, 2015.

G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane*, volume 40 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1999. Algebraic methods, boundary value problems and applications.

P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

S. Johnson, M. Mishna, and K. Yeats. Towards a combinatorial understanding of lattice path asymptotics Technical Report 1305.7418, ArXiv, 2015.

GAP. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*. The GAP Group, 2015. `http://www.gap-system.org`.

R. Garbit and K. Raschel. On the exit time from a cone for random walks with drift. Technical Report 1306.6761v4, ArXiv, 2015.

M. Kauers and R. Yatchak. Walks in the quarter plane with multiple steps. In *Proceedings of FPSAC'15*, DMTCS, pages 35–36, 2015.

I. Kurkova and K. Raschel. On the functions counting walks with small steps in the quarter plane. *Publications Mathematiques de l'IHES*, 116:69–114, 2012.

I. Kurkova and K. Raschel. New steps in walks with small steps in the quarter plane. *Annals of Combinatorics*, 19:461–511, 2015.

S. Melczer and M. Mishna. Singularity analysis via the iterated kernel method. *Combinatorics, Probability and Computing*, 2013. to appear. (Also in Proceedings of FPSAC'13).

M. Mishna and A. Rechnitzer. Two non-holonomic lattice walks in the quarter plane. *Theoretical Computer Science*, 410(38–40):3616–3630, 2009.

K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *Journal of the European Mathematical Society*, 14(3):749–777, 2012.

The Sage Developers. *Sage Mathematics Software (Version 6.8)*, 2015. `http://www.sagemath.org`.

A Tables

idx	step set	(x, y)	Asymptotics	idx	step set	(x, y)	Asymptotics
1		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-8.0256624}$ $6^n n^{-3.2634617}$	2		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-5.9706049}$ $6^n n^{-2.2353017}$ $(*)$
3		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-5.5631102}$ $6^n n^{-2.0315321}$ $(*)$	4		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-4.5566911}$ $6^n n^{-1.5283424}$ $(*)$
5		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-3.5478909}$ $6^n n^{-1.0239354}$ $(*)$	6		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-3.1240844}$ $6^n n^{-0.8120415}$ $(*)$
7		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-8.0256639}$ $8^n n^{-3.2628309}$ $(*)$	8		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-5.9706049}$ $8^n n^{-2.2353022}$ $(*)$
9		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-5.5631088}$ $8^n n^{-2.0315369}$ $(*)$	10		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-4.5566911}$ $8^n n^{-1.5283262}$ $(*)$
11		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-3.5478909}$ $8^n n^{-1.0239455}$ $(*)$	12		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-3.1240844}$ $8^n n^{-0.8120411}$ $(*)$

Tab. 2: Models with group $G = \mathbb{Z}_2 \times S_4$. The notation $[n]_p$ is meant to be 1 if $p \mid n$ and 0 otherwise. For sequences marked with $(*)$ the growth seems to be of the form $c(n)\phi^n n^\alpha$ where $c(n)$ depends on the parity of n. In other cases, the growth looks like $c\phi^n n^\alpha$ for a constant c. A referee pointed out that $\beta \approx \alpha/2 + 3/4$ when α is the exponent for $x = y = 0$ and β is the exponent for $x = y = 1$. Note also that for $x = y = 0$, it seems that each exponent appears for exactly two models.

idx	step set	(x, y)	Asymptotics	idx	step set	(x, y)	Asymptotics
13		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-5.6432110}$ $4^n n^{-2.0750861}$	14		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-5.6432110}$ $4^n n^{-2.0591708}$
15		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-4.7409029}$ $4^n n^{-1.62223}$	16		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-4.7409029}$ $4^n n^{-1.6186}$
17		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-3.6508693}$ $4^n n^{-1.075}$	18		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-3.3257569}$ $4^n n^{-0.9171490}$
19		$(0,0)$ $(1,1)$	$[n]_4\, 4^n n^{-3.3257569}$ $4^n n^{-0.91}$	20		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-5.6432112}$ $6^n n^{-2.0716051}$ $(*)$
21		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-4.7409028}$ $6^n n^{-1.6204494}$ $(*)$	22		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-3.6508694}$ $6^n n^{-1.0754}$
23		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-3.3257568}$ $6^n n^{-0.9128785}$ $(*)$	24		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-5.6432107}$ $8^n n^{-2.0716056}$ $(*)$
25		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-4.7409030}$ $8^n n^{-1.6204500}$ $(*)$	26		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-3.6508692}$ $8^n n^{-1.0754330}$ $(*)$
27		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-3.3257570}$ $8^n n^{-0.9128784}$ $(*)$	28		$(0,0)$ $(1,1)$	$10^n n^{-5.6432111}$ $10^n n^{-2.0716055}$
29		$(0,0)$ $(1,1)$	$10^n n^{-5.6432111}$ $10^n n^{-2.0716054}$	30		$(0,0)$ $(1,1)$	$10^n n^{-4.7409028}$ $10^n n^{-1.6204517}$
31		$(0,0)$ $(1,1)$	$10^n n^{-4.7409028}$ $10^n n^{-1.6204516}$	32		$(0,0)$ $(1,1)$	$10^n n^{-3.6508693}$ $10^n n^{-1.0754347}$
33		$(0,0)$ $(1,1)$	$10^n n^{-3.3257569}$ $10^n n^{-0.9128784}$	34		$(0,0)$ $(1,1)$	$10^n n^{-3.3257569}$ $10^n n^{-0.9128784}$
35		$(0,0)$ $(1,1)$	$14^n n^{-5.6432110}$ $14^n n^{-2.0716054}$	36		$(0,0)$ $(1,1)$	$14^n n^{-4.7409029}$ $14^n n^{-1.6204519}$
37		$(0,0)$ $(1,1)$	$14^n n^{-3.6508693}$ $14^n n^{-1.0754348}$	38		$(0,0)$ $(1,1)$	$14^n n^{-3.3257569}$ $14^n n^{-0.9128784}$

Tab. 3: Models with group $G = S_4$. Same notational conventions as in the previous table. The referee's observation $\beta \approx \alpha/2 + 3/4$ applies also to this table. The referee also noticed that for the case $x = y = 0$ it seems that there are only four different exponents: -5.64321, -4.74090, -3.65086, and -3.32575.

idx	step set	(x,y)	Asymptotics	idx	step set	(x,y)	Asymptotics
39		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-4}$ $6^n n^{-5/4}\ (*)$	40		$(0,0)$ $(1,1)$	$[n]_2\, 6^n n^{-4}$ $6^n n^{-5/4}\ (*)$
41		$(0,0)$ $(1,1)$	$[n]_3\, (2\sqrt{3}+3)^n n^{-4}$ $(2\sqrt{3}+3)^n n^{-2.25}$	42		$(0,0)$ $(1,1)$	$[n]_3\, (2\sqrt{3}+3)^n n^{-4}$ $(2\sqrt{3}+3)^n n^{-2.24}$
43		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{3})^n n^{-4}$ $(4\sqrt{3})^n n^{-11/4}\ (*)$	44		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{3})^n n^{-4}$ $(4\sqrt{3})^n n^{-11/4}\ (*)$
45		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{3})^n n^{-4}$ $7^n n^{-3/4}\ (*)$	46		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{3})^n n^{-4}$ $7^n n^{-3/4}\ (*)$
47		$(0,0)$ $(1,1)$	$[n]_3\, (2\sqrt{3}+3)^n n^{-4}$ $7^n n^{-3/4}$	48		$(0,0)$ $(1,1)$	$[n]_3\, (2\sqrt{3}+3)^n n^{-4}$ $7^n n^{-3/4}$
49		$(0,0)$ $(1,1)$	$7^n n^{-4}$ $7^n n^{-9/4}$	50		$(0,0)$ $(1,1)$	$7^n n^{-4}$ $7^n n^{-9/4}$
51		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-4}$ $8^n n^{-5/4}\ (*)$	52		$(0,0)$ $(1,1)$	$[n]_2\, 8^n n^{-4}$ $8^n n^{-5/4}\ (*)$
53		$(0,0)$ $(1,1)$	$7^n n^{-4}$ $8^n n^{-3/4}$	54		$(0,0)$ $(1,1)$	$7^n n^{-4}$ $8^n n^{-3/4}$
55		$(0,0)$ $(1,1)$	$[n]_2\, (6\sqrt{2})^n n^{-4}$ $(6\sqrt{2})^n n^{-11/4}\ (*)$	56		$(0,0)$ $(1,1)$	$[n]_2\, (6\sqrt{2})^n n^{-4}$ $(6\sqrt{2})^n n^{-11/4}\ (*)$
57		$(0,0)$ $(1,1)$	$9^n n^{-4}$ $9^n n^{-5/4}$	58		$(0,0)$ $(1,1)$	$9^n n^{-4}$ $9^n n^{-5/4}$
59		$(0,0)$ $(1,1)$	$9^n n^{-4}$ $9^n n^{-5/4}$	60		$(0,0)$ $(1,1)$	$9^n n^{-4}$ $9^n n^{-5/4}$
61		$(0,0)$ $(1,1)$	$[n]_2\, (6\sqrt{2})^n n^{-4}$ $9^n n^{-3/4}$	62		$(0,0)$ $(1,1)$	$[n]_2\, (6\sqrt{2})^n n^{-4}$ $9^n n^{-3/4}$
63		$(0,0)$ $(1,1)$	$(2\sqrt{6}+3)^n n^{-4}$ $(2\sqrt{6}+3)^n n^{-2.2502770}$	64		$(0,0)$ $(1,1)$	$(2\sqrt{6}+3)^n n^{-4}$ $(2\sqrt{6}+3)^n n^{-2.2502612}$
65		$(0,0)$ $(1,1)$	$[n]_2\, (2\sqrt{21})^n n^{-4}$ $(2\sqrt{21})^n n^{-11/4}\ (*)$	66		$(0,0)$ $(1,1)$	$[n]_2\, (2\sqrt{21})^n n^{-4}$ $(2\sqrt{21})^n n^{-11/4}\ (*)$
67		$(0,0)$ $(1,1)$	$(2\sqrt{3}+6)^n n^{-4}$ $(2\sqrt{3}+6)^n n^{-2.2530695}$	68		$(0,0)$ $(1,1)$	$(2\sqrt{3}+6)^n n^{-4}$ $(2\sqrt{3}+6)^n n^{-2.2530524}$
69		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{6})^n n^{-4}$ $(4\sqrt{6})^n n^{-11/4}\ (*)$	70		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{6})^n n^{-4}$ $(4\sqrt{6})^n n^{-11/4}\ (*)$
71		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3138908}$	72		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3139180}$
73		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3138703}$	74		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3139010}$
75		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3138550}$	76		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $(4\sqrt{3}+3)^n n^{-2.3138788}$
77		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{6})^n n^{-4}$ $10^n n^{-3/4}$	78		$(0,0)$ $(1,1)$	$[n]_2\, (4\sqrt{6})^n n^{-4}$ $10^n n^{-3/4}$
79		$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-3/4}$	80		$(0,0)$ $(1,1)$	$(2\sqrt{6}+3)^n n^{-4}$ $10^n n^{-3/4}$

81	$(0,0)$ $(1,1)$	$(2\sqrt{3}+6)^n n^{-4}$ $10^n n^{-3/4}$	82	$(0,0)$ $(1,1)$	$(2\sqrt{3}+6)^n n^{-4}$ $10^n n^{-3/4}$
83	$(0,0)$ $(1,1)$	$[n]_2\,(2\sqrt{21})^n n^{-4}$ $10^n n^{-3/4}$	84	$(0,0)$ $(1,1)$	$(2\sqrt{6}+3)^n n^{-4}$ $10^n n^{-3/4}$
85	$(0,0)$ $(1,1)$	$[n]_2\,(2\sqrt{21})^n n^{-4}$ $10^n n^{-3/4}$	86	$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-0.7501789}$
87	$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-0.7486131}$	88	$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-0.7484231}$
89	$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-0.7481541}$	90	$(0,0)$ $(1,1)$	$(4\sqrt{3}+3)^n n^{-4}$ $10^n n^{-0.7504432}$
91	$(0,0)$ $(1,1)$	$(2\sqrt{7}+3)^n n^{-4}$ $(2\sqrt{7}+3)^n n^{-2.2502209}$	92	$(0,0)$ $(1,1)$	$(2\sqrt{7}+3)^n n^{-4}$ $(2\sqrt{7}+3)^n n^{-2.2502015}$
93	$(0,0)$ $(1,1)$	$10^n n^{-4}$ $10^n n^{-9/4}$	94	$(0,0)$ $(1,1)$	$10^n n^{-4}$ $10^n n^{-9/4}$
95	$(0,0)$ $(1,1)$	$[n]_2\,(4\sqrt{7})^n n^{-4}$ $(4\sqrt{7})^n n^{-11/4}$ $(*)$	96	$(0,0)$ $(1,1)$	$[n]_2\,(4\sqrt{7})^n n^{-4}$ $(4\sqrt{7})^n n^{-11/4}$ $(*)$
97	$(0,0)$ $(1,1)$	$11^n n^{-4}$ $11^n n^{-5/4}$	98	$(0,0)$ $(1,1)$	$11^n n^{-4}$ $11^n n^{-5/4}$
99	$(0,0)$ $(1,1)$	$11^n n^{-4}$ $11^n n^{-5/4}$	100	$(0,0)$ $(1,1)$	$11^n n^{-4}$ $11^n n^{-5/4}$
101	$(0,0)$ $(1,1)$	$[n]_2\,(4\sqrt{7})^n n^{-4}$ $11^n n^{-3/4}$	102	$(0,0)$ $(1,1)$	$10^n n^{-4}$ $11^n n^{-3/4}$
103	$(0,0)$ $(1,1)$	$(2\sqrt{7}+3)^n n^{-4}$ $11^n n^{-3/4}$	104	$(0,0)$ $(1,1)$	$[n]_2\,(4\sqrt{7})^n n^{-4}$ $11^n n^{-3/4}$
105	$(0,0)$ $(1,1)$	$10^n n^{-4}$ $11^n n^{-3/4}$	106	$(0,0)$ $(1,1)$	$(2\sqrt{7}+3)^n n^{-4}$ $11^n n^{-3/4}$
107	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $(6\sqrt{2}+3)^n n^{-2.2546223}$	108	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $(6\sqrt{2}+3)^n n^{-2.2546131}$
109	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $(6\sqrt{2}+3)^n n^{-2.2546063}$	110	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $(6\sqrt{2}+3)^n n^{-2.2545976}$
111	$(0,0)$ $(1,1)$	$12^n n^{-4}$ $12^n n^{-5/4}$	112	$(0,0)$ $(1,1)$	$12^n n^{-4}$ $12^n n^{-5/4}$
113	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $12^n n^{-3/4}$	114	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $12^n n^{-3/4}$
115	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $12^n n^{-3/4}$	116	$(0,0)$ $(1,1)$	$(6\sqrt{2}+3)^n n^{-4}$ $12^n n^{-3/4}$
117	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $(2\sqrt{21}+3)^n n^{-2.2523243}$	118	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $(2\sqrt{21}+3)^n n^{-2.2523199}$
119	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $(2\sqrt{21}+3)^n n^{-2.2523167}$	120	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $(2\sqrt{21}+3)^n n^{-2.2523126}$
121	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $(4\sqrt{6}+3)^n n^{-2.2721995}$	122	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $(4\sqrt{6}+3)^n n^{-2.2721868}$
123	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $(4\sqrt{6}+3)^n n^{-2.2721586}$	124	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $(4\sqrt{6}+3)^n n^{-2.2721471}$
125	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $(4\sqrt{3}+6)^n n^{-2.3215931}$	126	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $(4\sqrt{3}+6)^n n^{-2.3215822}$
127	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $(4\sqrt{3}+6)^n n^{-2.3215670}$	128	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $(4\sqrt{3}+6)^n n^{-2.3215560}$

129	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $13^n n^{-3/4}$	130	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $13^n n^{-3/4}$
131	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $13^n n^{-3/4}$	132	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $13^n n^{-3/4}$
133	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $13^n n^{-3/4}$	134	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $13^n n^{-3/4}$
135	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $13^n n^{-3/4}$	136	$(0,0)$ $(1,1)$	$(2\sqrt{21}+3)^n n^{-4}$ $13^n n^{-3/4}$
137	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $13^n n^{-0.6802255}$	138	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $13^n n^{-0.6795702}$
139	$(0,0)$ $(1,1)$	$(4\sqrt{3}+6)^n n^{-4}$ $13^n n^{-0.6785484}$	140	$(0,0)$ $(1,1)$	$(4\sqrt{6}+3)^n n^{-4}$ $13^n n^{-0.7499959}$
141	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $(4\sqrt{7}+3)^n n^{-2.2583992}$	142	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $(4\sqrt{7}+3)^n n^{-2.2583717}$
143	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $(4\sqrt{7}+3)^n n^{-2.2583622}$	144	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $(4\sqrt{7}+3)^n n^{-2.2583888}$
145	$(0,0)$ $(1,1)$	$14^n n^{-4}$ $14^n n^{-5/4}$	146	$(0,0)$ $(1,1)$	$14^n n^{-4}$ $14^n n^{-5/4}$
147	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $14^n n^{-3/4}$	148	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $14^n n^{-3/4}$
149	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $14^n n^{-3/4}$	150	$(0,0)$ $(1,1)$	$(4\sqrt{7}+3)^n n^{-4}$ $14^n n^{-3/4}$
151	$(0,0)$ $(1,1)$	$(6\sqrt{2}+6)^n n^{-4}$ $(6\sqrt{2}+6)^n n^{-2.2567151}$	152	$(0,0)$ $(1,1)$	$(6\sqrt{2}+6)^n n^{-4}$ $(6\sqrt{2}+6)^n n^{-2.2566922}$
153	$(0,0)$ $(1,1)$	$(6\sqrt{2}+6)^n n^{-4}$ $15^n n^{-3/4}$	154	$(0,0)$ $(1,1)$	$(6\sqrt{2}+6)^n n^{-4}$ $15^n n^{-3/4}$
155	$(0,0)$ $(1,1)$	$(2\sqrt{21}+6)^n n^{-4}$ $(2\sqrt{21}+6)^n n^{-2.2533372}$	156	$(0,0)$ $(1,1)$	$(2\sqrt{21}+6)^n n^{-4}$ $(2\sqrt{21}+6)^n n^{-2.2533261}$
157	$(0,0)$ $(1,1)$	$(4\sqrt{6}+6)^n n^{-4}$ $(2\sqrt{42}+3)^n n^{-2.2792897}$	158	$(0,0)$ $(1,1)$	$(4\sqrt{6}+6)^n n^{-4}$ $(4\sqrt{6}+6)^n n^{-2.2793307}$
159	$(0,0)$ $(1,1)$	$(2\sqrt{42}+3)^n n^{-4}$ $(2\sqrt{42}+3)^n n^{-2.3617}$	160	$(0,0)$ $(1,1)$	$(2\sqrt{42}+3)^n n^{-4}$ $(2\sqrt{42}+3)^n n^{-2.15}$
161	$(0,0)$ $(1,1)$	$(4\sqrt{6}+6)^n n^{-4}$ $16^n n^{-3/4}$	162	$(0,0)$ $(1,1)$	$(2\sqrt{21}+6)^n n^{-4}$ $16^n n^{-3/4}$
163	$(0,0)$ $(1,1)$	$(4\sqrt{6}+6)^n n^{-4}$ $16^n n^{-3/4}$	164	$(0,0)$ $(1,1)$	$(2\sqrt{21}+6)^n n^{-4}$ $16^n n^{-3/4}$
165	$(0,0)$ $(1,1)$	$(2\sqrt{42}+3)^n n^{-4}$ $16^n n^{-0.7859779}$	166	$(0,0)$ $(1,1)$	$(2\sqrt{42}+3)^n n^{-4}$ $16^n n^{-0.7859706}$
167	$(0,0)$ $(1,1)$	$(4\sqrt{7}+6)^n n^{-4}$ $(4\sqrt{7}+6)^n n^{-2.2614300}$	168	$(0,0)$ $(1,1)$	$(4\sqrt{7}+6)^n n^{-4}$ $(4\sqrt{7}+6)^n n^{-2.2613989}$
169	$(0,0)$ $(1,1)$	$(4\sqrt{7}+6)^n n^{-4}$ $17^n n^{-3/4}$	170	$(0,0)$ $(1,1)$	$(4\sqrt{7}+6)^n n^{-4}$ $17^n n^{-3/4}$

Tab. 4: Models with group $G = D_{12}$. Same notational conventions as in the previous table. It could be that all exponents belong to $\frac{1}{4}\mathbb{Z}$, but where our estimates did not agree to at least four decimal digits from these numbers, we refrained from rounding.