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The Challenge of Computing Geode Numbers

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Abstract. In a fascinating recent American Mathematical Monthly article, Norman Wildberger and Dean Rubine introduced a new kind of combinatorial numbers, that they aptly named the "Geode numbers". While their definition is simple, these numbers are surprisingly hard to compute, in general. While the two-dimensional case has a nice closed-form expression, that make them easy to compute, already the three-dimensional case poses major computational challenges that we do meet, combining experimental mathematics and the holonomic ansatz. Alas, things get really complicated in four and higher dimensions, and we are unable to efficiently compute, for example, the 1000-th term of the four-dimensional diagonal Geode sequence. A donation of 100 US dollars to the OEIS, in honor of the first person to compute this number, is offered.

Maple package: This article is accompanied by the Maple package `Geode.txt`, available at:
https://sites.math.rutgers.edu/~zeilberg/tokhniot/Geode.txt
There are numerous input and output files in the front of this article:
https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/Cgeode.html

1 Introduction

In the May 2025 issue of the *American Mathematical Monthly*, Dean Rubine and Norman Wildberger [7], in the course of stating their beautiful "explicit" solution to the general algebraic equation (with *symbolic* coefficients $c_0, c_1, c_2, \dots, c_k$)

$$0 = c_0 - c_1\alpha + c_2\alpha^2 + \cdots + c_k\alpha^k \quad ,$$

needed the following numbers defined on lists of non-negative integers $[m_1, m_2, \dots, m_k]$, that they called the *hyper-Catalan numbers*:

$$C(m_1, \dots, m_k) := \frac{(2m_1 + 3m_2 + \cdots + (k+1)m_k)!}{(1 + m_1 + 2m_2 + \cdots + km_k)!\, m_1! \cdots m_k!} \quad .$$

Let n and k be positive integers. Define a polynomial, let's call it $P_{n,k}(t_1, \dots, t_k)$, in the k (continuous) variables, $t_1, \dots, t_k$, of total degree n, as follows:

$$P_{n,k}(t_1, \dots, t_k) := \sum_{m_1+m_2+\cdots+m_k=n} C(m_1, \dots, m_k) t_1^{m_1} \cdots t_k^{m_k} \quad .$$

Rubine and Wildberger proved the following surprising theorem ([7, Theorem 12]).

Theorem 1.1 (The Geode Theorem). *For all positive integers n and k, the polynomial $P_{n,k}(t_1, \dots, t_k)$ is divisible by $t_1 + \cdots + t_k$.*

So it makes sense to define another polynomial, of degree n in $t_1, \dots, t_k$:

Definition 1.2 (The Geode polynomial). For positive integers n and k

$$Q_{n,k}(t_1,\ldots,t_k) := \frac{P_{n+1,k}(t_1,\ldots,t_k)}{t_1+\cdots+t_k} \quad .$$

The *geode* numbers are the coefficients of this polynomial. See [2] and [3] for nice combinatorial objects counted by these mysterious numbers.

Definition 1.3 (The Geode numbers). For every list of non-negative integers, $[m_1,\ldots,m_k]$, the coefficient of $t_1^{m_1}\cdots t_k^{m_k}$ in $Q_{m_1+\cdots+m_k,k}(t_1,\ldots,t_k)$ is $G(m_1,\ldots,m_k)$.

Remark 1.4. Procedure `G(m)` in our Maple package uses the above definition to compute these numbers.

For example, in order to find $G(4,7,8)$ type `G([4,7,8]);`, getting 112586144742750300336 00. This works for small arguments, but don't even try to do `G([1000,1000,1000]);` . It will take your computer for ever. Since the computer would first compute the (total) degree-3003 polynomial $P_{3003,3}(t_1,t_2,t_3)$, then divide by $t_1+t_2+t_3$, and then extract the coefficient of $t_1^{1000}t_2^{1000}t_3^{1000}$. As we will soon see, and that is one of the achievements of the present paper, using experimental mathematics and *guessing*, this 3910-digit integer can be computed in 0.231 seconds. Alas, at this time of writing we can't compute $G(1000,1000,1000,1000)$ and one of us (DZ) is pledging to donate \$100 dollars to http://oeisf.org/ in honor of the first person to compute it correctly.

2 The Two-Dimensional Geode Numbers

It turned out that $G(m_1,m_2)$ has a nice **closed-form** expression, very similar to $C(m_1,m_2)$. It was conjectured in [7] and soon proved (along with some other conjectures from that paper) in [1] and [6].

Theorem 2.1 (Conjectured in [7] and proved in [1] and [6]). *We have that*

$$G(m_1,m_2) = \frac{1}{(2m_1+2m_2+3)(m_2+m_3+1)}\cdot\frac{(2m_1+3m_2+3)!}{(m_1+2m_2+2)!m_1!m_2!} \quad .$$

This is implemented in procedure `G2f(m)` in our Maple package, that inputs $m=[m_1,m_2]$ and outputs $G(m_1,m_2)$. For example, computing $G(5000,5000)$ using the definition (i.e. executing `G([5000,5000]);`) takes on our laptop 91 seconds, while using the above explicit formula only takes 0.005 seconds!

3 The Three-Dimensional Geode Numbers

An equivalent way of stating Theorem 2.1, is saying that the discrete bi-variate function, $G(m_1,m_2)$ satisfies the following *pure*, **first-order** linear recurrences

$$G(m_1,m_2) = \frac{(2m_1+2+3m_2)(2m_1+3m_2+3)(2m_1+1+2m_2)(m_1+m_2)}{m_1(2m_1+2m_2+3)(m_1+m_2+1)(m_1+2m_2+2)}\cdot G(m_1-1,m_2) \ ,$$

$$G(m_1, m_2) =$$
$$\frac{(2m_1 + 1 + 3m_2)\,(2m_1 + 2 + 3m_2)\,(2m_1 + 3m_2 + 3)\,(2m_1 + 1 + 2m_2)\,(m_1 + m_2)}{m_2\,(2m_1 + 2m_2 + 3)\,(m_1 + m_2 + 1)\,(1 + m_1 + 2m_2)\,(m_1 + 2m_2 + 2)} G(m_1, m_2 - 1).$$

Together with the *initial condition* $G(1,1) = 16$, this enables one to compute any $G(m_1, m_2)$ very fast, in fact it is more efficient to use the equivalent recurrence than the "closed-form" expression for $G(m_1, m_2)$ given by Theorem 2.1, since you divide very large integers by other very large integers.

While there is no 'closed-form' expression for $G(m_1, m_2, m_3)$, the next-best-thing is true! That is, $G(m_1, m_2, m_3)$ satisfies **second-order**, pure, linear recurrences with polynomial coefficients, in each of its arguments. This also enables very fast computation of $G(m_1, m_2, m_3)$ for large arguments, that would be impractical using the definition.

So we have the next very useful theorem.

Theorem 3.1. *There exist (explicit) rational functions in* m_1, m_2, m_3*,*

- $f_{1,1}(m_1, m_2, m_3)$ *and* $f_{1,2}(m_1, m_2, m_3)$ *both with numerators and denominators of (total) degree* 11
- $f_{2,1}(m_1, m_2, m_3)$ *and* $f_{2,2}(m_1, m_2, m_3)$ *both with numerators and denominators of (total) degree* 14
- $f_{3,1}(m_1, m_2, m_3)$ *and* $f_{3,2}(m_1, m_2, m_3)$ *both with numerators and denominators of (total) degree* 17

such that $G(m_1, m_2, m_3)$ *satisfies the following second-order linear recurrences*

$$G(m_1, m_2, m_3) = f_{1,1}(m_1, m_2, m_3) G(m_1 - 1, m_2, m_3) + f_{1,2}(m_1, m_2, m_3) G(m_1 - 2, m_2, m_3) ,$$

$$G(m_1, m_2, m_3) = f_{2,1}(m_1, m_2, m_3) G(m_1, m_2 - 1, m_3) + f_{2,2}(m_1, m_2, m_3) G(m_1, m_2 - 2, m_3) ,$$

$$G(m_1, m_2, m_3) = f_{3,1}(m_1, m_2, m_3) G(m_1, m_2, m_3 - 1) + f_{3,2}(m_1, m_2, m_3) G(m_1, m_2, m_3 - 2) .$$

Remark 3.2. These rational functions are too complicated to be presented here, but are easily viewable via our Maple package by typing
`Key3(m1,m2,m3)[1][1]`, `Key3(m1,m2,m3)[1][2]`,
`Key3(m1,m2,m3)[2][1]`, `Key3(m1,m2,m3)[2][2]`,
`Key3(m1,m2,m3)[3][1]`, `Key3(m1,m2,m3)[3][2]`, respectively.
Note that together with the initial conditions $G(i, j, k)$ with $1 \le i, j, k \le 2$, these enable very fast computation of $G(m_1, m_2, m_3)$.

If you are only interested in the diagonal sequence $G(n, n, n)$, then we have the following deep theorem.

Theorem 3.3. *There exist rational functions in* n*,* $f_1(n)$ *and* $f_2(n)$*, whose numerators and denominators are polynomials of degree* 35 *in* n*, such that the sequence* $\{G(n, n, n)\}_{n=1}^{\infty}$ *satisfies the second-order linear recurrence*

$$G(n, n, n) = f_1(n)\, G(n-1, n-1, n-1) + f_2(n)\, G(n-2, n-2, n-2) \quad .$$

Again $f_1(n)$ and $f_2(n)$ are too complicated to be displayed here, but you can see them by typing
`Rec3AllPC(n)[1][1][1]`, `Rec3AllPC(n)[1][1][2]`, respectively.
Note that together with the initial conditions $G(1, 1, 1) = 319$ and $G(2, 2, 2) = 669123$ this recurrence allows us to compute, very fast, the values of $G(n, n, n)$ for large n that would he hopeless using the definition.

Remark 3.4. This is implemented in procedure `G3diag(n)`. To see the values of $G(10^i, 10^i, 10^i)$ for $1 \le i \le 5$, see the output file:
`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGeode2.txt`

How did we find these amazing (and useful!) recurrences for the 3D Geode numbers?

By guessing, of course!

We cranked out sufficiently many terms using the definition of $G(m_1, m_2, m_3)$ and then used *linear algebra* and *undetermined coefficients* to guess the recurrences, just like in *machine learning*. We had the advantage that the data was *exact* without noise. Once the computer found the recurrences using the *training data*, we could verify it for many other cases.

Once guessed it is routine (and straightforward) to prove our guesses fully rigorously, as follows. By using the general theory [8] [4], it is readily seen that $G(m_1, m_2, m_3)$, from its definition, is holonomic, i.e. there exist linear recurrences with polynomial coefficients, in each of the directions, that it satisfies. The class of holonomic functions is an *algebra*. Let $G'(m_1, m_2, m_3)$ be the three-variate discrete function *defined* by the above recurrences (of course one would want to verify that the recurrences are compatible, but this is easy). We have to prove that for all $m_1, m_2, m_3 \geq 0$,

$$G(m_1, m_2, m_3) = G'(m_1, m_2, m_3).$$

An equivalent way of defining $G(m_1, m_2, m_3)$ is as the unique function on $\mathbb{N}^3$ satisfying

$$G(m_1 - 1, m_2, m_3) + G(m_1, m_2 - 1, m_3) + G(m_1, m_2, m_3 - 1) \ = \ C(m_1, m_2, m_3).$$

Let $F(m_1, m_2, m_3)$ be defined by

$$F(m_1, m_2, m_3) := G'(m_1-1, m_2, m_3)+G'(m_1, m_2-1, m_3)+G'(m_1, m_2, m_3-1)-C(m_1, m_2, m_3).$$

In order to prove that $G(m_1, m_2, m_3) = G'(m_1, m_2, m_3)$, for all $m_1, m_2, m_3 \geq 0$, we need to prove that

$$F(m_1, m_2, m_3) \ = \ 0 \quad ,$$

for all $m_1, m_2, m_3 \geq 0$. Being a linear combination of holonomic functions $F(m_1, m_2, m_3)$ is automatically also holonomic, and there are standard algorithms, implemented in Christoph Koutschan's *holonomic calculator* Mathematica package [5] that do it. But why bother? Since we know beforehand that $F(m_1, m_2, m_3) = 0$ for all $m_1 \leq K, m_2 \leq K, m_3 \leq K$ for some easily determined K, and after checking that the orders are $\leq K$, it follows immediately that $F(m_1, m_2, m_3)$ is identically zero. Giving a *rigorous* proof to those obtuse readers who insist on it.

4 How about 4 Dimensions and beyond?

We were unable to guess a linear recurrence with polynomial coefficients for $\{G(n, n, n, n)\}$, or pure recurrences for $G(m_1, m_2, m_3, m_4)$. While we know that they **exist** for sure, finding them is a different matter.

One of us (DZ) is offering to donate 100 US dollars to the OEIS for the determination of

$$G(1000, 1000, 1000, 1000) \quad ,$$

and 200 US dollars for

$$G(1000, 1000, 1000, 1000, 1000) \quad .$$

Good luck!

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