

ON TURÁN'S INEQUALITY FOR LEGENDRE POLYNOMIALS

HORST ALZER^a, STEFAN GERHOLD^{b1}, MANUEL KAUERS^{c2}, ALEXANDRU LUPAȘ^d

^a Morsbacher Str. 10, 51545 Waldbröl, Germany
alzerhorst@freenet.de

^b Christian Doppler Laboratory for Portfolio Risk Management, Vienna University of Technology,
Vienna, Austria
sgerhold@fam.tuwien.ac.at

^c Research Institute for Symbolic Computation, J. Kepler University, Linz, Austria
manuel.kauers@risc.uni-linz.ac.at

^d Department of Mathematics, University of Sibiu, 2400 Sibiu, Romania
alexandru.lupas@ulsibiu.ro

Abstract. Let

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x),$$

where P_n is the Legendre polynomial of degree n . A classical result of Turán states that $\Delta_n(x) \geq 0$ for $x \in [-1, 1]$ and $n = 1, 2, 3, \dots$. Recently, Constantinescu improved this result. He established

$$\frac{h_n}{n(n+1)}(1-x^2) \leq \Delta_n(x) \quad (-1 \leq x \leq 1; n = 1, 2, 3, \dots),$$

where h_n denotes the n -th harmonic number. We present the following refinement. Let $n \geq 1$ be an integer. Then we have for all $x \in [-1, 1]$:

$$\alpha_n(1-x^2) \leq \Delta_n(x)$$

with the best possible factor

$$\alpha_n = \mu_{[n/2]} \mu_{[(n+1)/2]}.$$

Here, $\mu_n = 2^{-2n} \binom{2n}{n}$ is the normalized binomial mid-coefficient.

Keywords. Legendre polynomials, Turán's inequality, normalized binomial mid-coefficient.

2000 Mathematics Subject Classification. 26D07, 33C45.

¹S. Gerhold was supported by FWF grant SFB F1305, the Christian Doppler Association (CDG), the Austrian Federal Financing Agency and Bank Austria.

²M. Kauers was supported by FWF grant SFB F1305 and P16613-N12.

1. INTRODUCTION

The Legendre polynomial of degree n can be defined by

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (n = 0, 1, 2, \dots),$$

which leads to the explicit representation

$$P_n(x) = \frac{1}{2^n} \sum_{\nu=0}^{\lfloor n/2 \rfloor} (-1)^\nu \frac{(2n-2\nu)!}{\nu!(n-\nu)!(n-2\nu)!} x^{n-2\nu}.$$

(As usual, $\lfloor x \rfloor$ denotes the greatest integer not greater than x .) The most important properties of $P_n(x)$ are collected, for example, in [1] and [16]. Legendre polynomials belong to the class of Jacobi polynomials, which are studied in detail in [3] and [13]. These functions have various interesting applications. For instance, they play an important role in numerical integration; see [12].

The following beautiful inequality for Legendre polynomials is due to P. Turán [15]:

$$(1.1) \quad \Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0 \quad \text{for } -1 \leq x \leq 1 \text{ and } n \geq 1.^3$$

This inequality has found much attention and several mathematicians provided new proofs, far-reaching generalizations, and refinements of (1.1). We refer to [8, 11, 9, 14] and the references given therein.

In this paper we are concerned with a remarkable result published by E. Constantinescu [7] in 2005. He offered a new refinement and a converse of Turán's inequality. More precisely, he proved that the double-inequality

$$(1.2) \quad \frac{h_n}{n(n+1)}(1-x^2) \leq \Delta_n(x) \leq \frac{1}{2}(1-x^2)$$

is valid for $x \in [-1, 1]$ and $n \geq 1$. Here, $h_n = 1 + 1/2 + \dots + 1/n$ denotes the n -th harmonic number.

It is natural to ask whether the bounds given in (1.2) can be improved. In the next section, we determine the largest number α_n and the smallest number β_n such that we have for all $x \in [-1, 1]$:

$$\alpha_n(1-x^2) \leq \Delta_n(x) \leq \beta_n(1-x^2).$$

We show that the right-hand side of (1.2) is sharp, but the left-hand side can be improved. It turns out that the best possible factor α_n can be expressed in terms of the normalized binomial mid-coefficient

$$\mu_n = 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} \quad (n = 0, 1, 2, \dots).$$

We remark that μ_n has been the subject of recent number theoretic research; see [2] and [5].

In our proof we reduce the desired refinement of Turán's inequality to another inequality, which also depends polynomially on Legendre polynomials. This latter inequality is amenable to a recent computer algebra procedure [10, 11]. The procedure sets up a formula that encodes the induction step of an inductive proof of the inequality and, replacing the quantities $P_n(x), P_{n+1}(x), \dots$ by real variables Y_1, Y_2, \dots , transforms the induction step formula into a polynomial formula in finitely many variables. The recurrence relation of the Legendre polynomials translates into polynomial equations in the Y_k , which are added to the induction step formula. The truth of the resulting formula for all real Y_1, Y_2, \dots can be decided algorithmically and is a sufficient (in general not necessary!) condition for the truth of the initial inequality, if we assume that sufficiently many initial values have been checked.

³A nice anecdote about Turán reveals that he used (1.1) as his 'visiting card'; see [4].

2. MAIN RESULT

The following refinement of (1.2) is valid.

Theorem. *Let n be a natural number. For all real numbers $x \in [-1, 1]$ we have*

$$(2.1) \quad \alpha_n (1 - x^2) \leq P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \leq \beta_n (1 - x^2)$$

with the best possible factors

$$(2.2) \quad \alpha_n = \mu_{\lfloor n/2 \rfloor} \mu_{\lfloor (n+1)/2 \rfloor} \quad \text{and} \quad \beta_n = \frac{1}{2}.$$

Proof. We define for $x \in (-1, 1)$ and $n \geq 1$:

$$f_n(x) = \frac{\Delta_n(x)}{1 - x^2}.$$

We have $f_1(x) \equiv \alpha_1 = \beta_1 = 1/2$. First, we prove that f_n is strictly increasing on $(0, 1)$ for $n \geq 2$. Differentiation yields

$$f'_n(x) = \frac{2x\Delta_n(x) + (1 - x^2)\Delta'_n(x)}{(1 - x^2)^2}.$$

Using the well-known formulas

$$P'_n(x) = \frac{n+1}{1-x^2}(xP_n(x) - P_{n+1}(x))$$

and

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

we obtain the representation

$$(2.3) \quad n(1-x^2)^2 f'_n(x) = (n-1)xP_n(x)^2 - (2nx^2 + x^2 - 1)P_n(x)P_{n+1}(x) + (n+1)xP_{n+1}(x)^2.$$

We prove the positivity of the right-hand side of (2.3) on $(0, 1)$ by typing

In[1]:= << **SumCracker.m**

SumCracker Package by Manuel Kauers - © RISC Linz - V 0.3 2006-05-24

In[2]:= **ProveInequality**[

$$\begin{aligned} & ((n-1)x \text{LegendreP}[n, x]^2 \\ & - (2nx^2 + x^2 - 1)\text{LegendreP}[n, x]\text{LegendreP}[n+1, x] \\ & + (n+1)x \text{LegendreP}[n+1, x]^2) > 0, \end{aligned}$$

From $\rightarrow 2$, **Using** $\rightarrow \{0 < x < 1\}$, **Variable** $\rightarrow n$]

into Mathematica, obtaining, after a couple of seconds, the output

Out[2]=

True

It follows from this that f_n is strictly increasing on $(0, 1)$ for $n \geq 2$. Since

$$P_n(x) = (-1)^n P_n(-x),$$

we conclude that f_n is even. Thus, we obtain

$$(2.4) \quad f_n(0) < f_n(x) < f_n(1) \quad \text{for } -1 < x < 1, x \neq 0.$$

We have

$$P_n(1) = 1 \quad \text{and} \quad P'_n(1) = \frac{1}{2}n(n+1).$$

Therefore,

$$\Delta_n(1) = 0 \quad \text{and} \quad \Delta'_n(1) = -1.$$

Applying l'Hospital's rule gives

$$(2.5) \quad f_n(1) = \lim_{x \rightarrow 1} \frac{\Delta_n(x)}{1-x^2} = -\frac{1}{2} \Delta'_n(1) = \frac{1}{2}.$$

Since

$$P_{2k-1}(0) = 0 \quad \text{and} \quad P_{2k}(0) = (-1)^k \mu_k,$$

we get

$$(2.6) \quad f_{2k-1}(0) = \mu_{k-1} \mu_k \quad \text{and} \quad f_{2k}(0) = \mu_k^2.$$

Combining (2.4)–(2.6) we conclude that (2.1) holds with the best possible factors α_n and β_n given in (2.2). \square

Remarks. (1) The proof of the Theorem reveals that for $n \geq 2$ the sign of equality holds on the left-hand side of (2.1) if and only if $x = -1, 0, 1$ and on the right-hand side if and only if $x = -1, 1$.

(2) The numbers $\mu_p \mu_q$ ($p, q = 0, 1, 2, \dots; p \leq q$) are the eigenvalues of Liouville's integral operator for the case of a planar circular disc of radius 1 lying in \mathbf{R}^3 ; see [6].

(3) The automated proving procedure can be applied to (2.1) directly. However, owing to the computational complexity of the method, we did not obtain any output after a reasonable amount of computation time.

(4) The Mathematica package SumCracker used in the proof of the Theorem contains an implementation of the proving procedure described in [10]. It is available online at

<http://www.risc.uni-linz.ac.at/research/combinat/software>

(5) The normalized Jacobi polynomial of degree n is defined for $\alpha, \beta > -1$ by

$$R_n^{(\alpha, \beta)}(x) = {}_2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1-x)/2).$$

The special case $\alpha = \beta$ leads to the normalized ultraspherical polynomial

$$R_n^{(\alpha, \alpha)}(x) = {}_2F_1(-n, n + 2\alpha + 1; \alpha + 1; (1-x)/2) = \frac{(-1)^n}{2^n (\alpha + 1)_n} \frac{1}{(1-x^2)^\alpha} \frac{d^n}{dx^n} (1-x^2)^{n+\alpha},$$

where $(a)_n$ denotes the Pochhammer symbol. Obviously, we have $R_n^{(0,0)}(x) = P_n(x)$. We conjecture that the following extension of our Theorem holds.

Conjecture. *Let $\alpha > -1/2$ and $n \geq 1$. For all $x \in [-1, 1]$ we have*

$$a_n^{(\alpha)} (1-x^2) \leq R_n^{(\alpha, \alpha)}(x)^2 - R_{n-1}^{(\alpha, \alpha)}(x) R_{n+1}^{(\alpha, \alpha)}(x) \leq b_n^{(\alpha)} (1-x^2)$$

with the best possible factors

$$a_n^{(\alpha)} = \mu_{\lfloor n/2 \rfloor}^{(\alpha)} \mu_{\lfloor (n+1)/2 \rfloor}^{(\alpha)} \quad \text{and} \quad b_n^{(\alpha)} = \frac{1}{2(\alpha + 1)}.$$

Here, $\mu_n^{(\alpha)} = \mu_n / \binom{n+\alpha}{n}$.

(6) Gasper [9] has shown that the normalized Jacobi polynomials satisfy

$$R_n^{(\alpha, \beta)}(x)^2 - R_{n-1}^{(\alpha, \beta)}(x) R_{n+1}^{(\alpha, \beta)}(x) \geq 0 \quad (-1 \leq x \leq 1)$$

if and only if $\beta \geq \alpha > -1$. More general criteria for a family of orthogonal polynomials to satisfy a Turán-type inequality are given by Szwarz [14].

Acknowledgement. We thank the referee for bringing references [9] and [14] to our attention.

REFERENCES

- [1] M. Abramowitz and I. Stegun (eds.), Handbook of Mathematical Functions with Formulas and Mathematical Tables, Dover, New York, 1965.
- [2] H. Alzer and B. Fuglede, Normalized binomial mid-coefficients and power means, *J. Number Th.* **115** (2005), 284–294.
- [3] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.
- [4] R. Askey, Remembering Paul Turán, *J. Approx. Th.* **86** (1996), 253–254.
- [5] T. Bang and B. Fuglede, No two quotients of normalized binomial mid-coefficients are equal, *J. Number Th.* **35** (1990), 345–349.
- [6] C. Berg and B. Fuglede, Liouville's operator for a disc in space, *Manuscr. Math.* **67** (1990), 165–185.
- [7] E. Constantinescu, On the inequality of P. Turán for Legendre polynomials, *J. Inequal. Pure Appl. Math.* **6**(2) (2005), Art. 28. [<http://jipam.vu.edu.au/>]
- [8] D.K. Dimitrov, Higher order Turán inequalities, *Proc. Amer. Math. Soc.* **126** (1998), 2033–2037.
- [9] G. Gasper, An inequality of Turán type for Jacobi polynomials, *Proc. Amer. Math. Soc.* **32** (1972), 435–439.
- [10] S. Gerhold and M. Kauers, A procedure for proving special function inequalities involving a discrete parameter, *Proceedings of ISSAC '05*, ACM Press (2005), 156–162.
- [11] S. Gerhold and M. Kauers, A computer proof of Turán's inequality, *J. Inequal. Pure Appl. Math.* **7**(2) (2006), Art. 42. [<http://jipam.vu.edu.au/>]
- [12] Y.L. Luke, *The Special Functions and Their Approximations*, Vol. 2, Academic Press, New York, 1969.
- [13] G. Szegő, *Orthogonal Polynomials*, 4th ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Rhode Island, 1975.
- [14] R. Szwarc, Positivity of Turán determinants for orthogonal polynomials, In: *Harmonic Analysis and Hypergroups*, (K.A. Ross et al., eds.), Birkhäuser, Boston-Basel-Berlin, 1998, 165–182.
- [15] P. Turán, On the zeros of the polynomials of Legendre, *Časopis Pest. Mat. Fys.* **75** (1950), 113–122.
- [16] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1952.