

SOME TOPICS IN EQUATIONAL LOGIC

ABSTRACT. These are additional notes for the course “Universal algebra”.
The course and this presentation mainly follow [BS81].

1. MOTIVATION

These notes are used in a course on universal algebra that follows [BS81] for explaining the two fundamental theorems by Birkhoff from [Bir35]: the HSP-theorem and the completeness of the equational calculus. The proofs are those from [BS81], but some auxiliary material and explanations from this book have been skipped. The material is complemented with some theorems on quasi-identities from universal algebra [Mal54, BS81, Gor98] and universal algebraic geometry [Plo19].

2. THE TERM ALGEBRA

Let $\mathcal{F}$ be an algebraic language and X be a set that is disjoint from $\mathcal{F}$. The set $T(X)$ of terms over X is a subset of $(\mathcal{F} \cup X)^*$, the set of words over the alphabet $\mathcal{F} \cup X$ of positive length. We define $E_0 := X$ and

$$E_n := E_{n-1} \cup \{ft_1 \dots t_m \mid m \in \mathbb{N}_0, f \in \mathcal{F}_m, t_1, \dots, t_m \in E_{n-1}\}$$

for $n \in \mathbb{N}$. Then $T(X) := \bigcup_{n \in \mathbb{N}_0} E_n$.

Lemma 2.1. *Let u, v be terms. If u is a prefix of v , then $u = v$.*

Proof. Let us first consider the case $u \in E_0$. By induction on n , we see that each $w \in E_n$ whose first letter is in X satisfies $w \in E_0$. Hence $v \in E_0$, and thus $u = v$.

We show by induction on $n + m$ that the statement holds for all $u \in E_m, v \in E_n$. The induction basis $m + n = 0$ is covered by the case $u \in E_0$.

Date: January 12, 2024.

Notes for the course “Universal Algebra” at JKU in 2023 by Erhard Aichinger.

PRELIMINARY VERSION.

Now we assume $m + n \geq 1$ and choose $u \in E_m, v \in E_n$ such that u is a prefix of v . The case $u \in E_0$ has already been covered, and so we assume $u \notin E_0$. Then u starts with a letter in $\mathcal{F}$. If $v \in E_0$, then u starts with a letter in $\mathcal{F}$ and v with a letter in X , contradicting that u is a prefix of v . If $v \notin E_0$, we can write $u = ft_1 \dots t_r$ and $v = gu_1 \dots u_s$ with $f \in \mathcal{F}_r, g \in \mathcal{F}_s, t_1, \dots, t_r \in E_{m-1}$ and $u_1, \dots, u_s \in E_{n-1}$. Then $f = g$, and hence $r = s$. Now let i be minimal with $t_i \neq u_i$. Then t_i is a prefix of u_i or u_i is a prefix of t_i . In both cases, the induction hypothesis yields $t_i = u_i$. $\square$

Lemma 2.2. *Let $t_1, \dots, t_r, u_1, \dots, u_s \in T(X)$, $f \in \mathcal{F}_r, g \in \mathcal{F}_s$. If $ft_1 \dots t_r = gu_1 \dots u_s$, then $r = s, f = g$ and $t_i = u_i$ for $i \in \{1, \dots, r\}$.*

Proof. We clearly have $f = g$ and thus $r = s$. Let i be minimal with $t_i \neq u_i$. Then either t_i is a prefix of u_i or u_i is a prefix of t_i , and hence by the previous lemma, $u_i = t_i$. $\square$

We have $T(X) = \emptyset$ if and only if $\mathcal{F}_0 \cup X = \emptyset$. For the case $\mathcal{F}_0 \cup X \neq \emptyset$, we define the term algebra $\mathbf{T}(X)$ by $f^{\mathbf{T}(X)}(t_1, \dots, t_r) := ft_1 \dots t_r$ for all $r \in \mathbb{N}_0, f \in \mathcal{F}_r$.

Theorem 2.3. *Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$, and let X be a set. We assume $\mathcal{F}_0 \cup X \neq \emptyset$. Let $\mathbf{a} \in A^X$. We define a relation $e \subseteq T(X) \times A$ by $e_0 : E_0 \rightarrow A, e_0(x) = \mathbf{a}(x)$ for $x \in X$, and for $n \geq 1$,*

$$e_n = e_{n-1} \cup \{(ft_1, \dots, t_m, f^{\mathbf{A}}(a_1, \dots, a_m)) \mid \\ m \in \mathbb{N}_0, f \in \mathcal{F}_m, \text{ for all } i \in \underline{m} : (t_i, a_i) \in e_{n-1}\}.$$

Let $e := \bigcup_{n \in \mathbb{N}} e_n$. Then e is a homomorphism from $\mathbf{T}(X)$ to $\mathbf{A}$ with $e|_X = \mathbf{a}$.

Proof. It is easy to see that $e \subseteq T(X) \times A$ and that the first projection of e to $T(X)$ is surjective.

Next, we prove by induction on the length of u that for all $u \in E_n$ and $a, b \in A$ with $(u, a) \in e$ and $(u, b) \in e$, we have $a = b$.

If u is of length 1, it is either in X or $\mathcal{F}_0$. In the first case, (u, a) and (u, b) are both elements of e_0 because no other elements of e have a first component in $E_0 = X$. Since e_0 is a function, $a = b$. If $u \in \mathcal{F}_0$ then we see that for all $n \in \mathbb{N}$ with $(u, a) \in e_n$, we have $a = u^{\mathbf{A}}$.

If the length of u is at least 2, then since $(u, a) \in e$, there are $r \in \mathbb{N}_0, f \in \mathcal{F}_r, t_1, \dots, t_r \in T(X)$ and $a_1, \dots, a_r \in A$ such that $(t_i, a_i) \in e$ for all $i \in \underline{r}$,

$u = ft_1 \dots, t_r$ and $a = f^{\mathbf{A}}(a_1, \dots, a_r)$. Since $(u, b) \in e$, there are $s \in \mathbb{N}_0$, $g \in \mathcal{F}_s$, $u_1, \dots, u_s \in T(X)$ and $b_1, \dots, b_s \in A$ such that $(u_i, b_i) \in e$ for all $i \in \underline{s}$, $u = gu_1 \dots, u_s$ and $a = g^{\mathbf{A}}(b_1, \dots, b_r)$. Then by Lemma 2.2, $f = g$, $r = s$ and $t_i = u_i$ for $i \in \underline{r}$. Since $(t_i, a_i) \in e$, $(t_i, b_i) \in e$ and t_i is shorter than u , we have $a_i = b_i$. Thus $a = b$.

From its construction, we see that e is the subuniverse of $\mathbf{T}(X) \times \mathbf{A}$ that is generated by $\mathbf{a}$. Hence e is a function and a subuniverse, and thus a homomorphism. $\square$

For this e , we denote $e(t)$ also by $t^{\mathbf{A}}(\mathbf{a})$. Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $x_1, \dots, x_n \in X$. We write $t(x_1, \dots, x_n)$ to indicate that $t \in T(\{x_1, \dots, x_n\})$ and $t^{\mathbf{A}}(a_1, \dots, a_n)$ for $t^{\mathbf{A}}(\{(x_1, a_1), \dots, (x_n, a_n)\})$. For $u, v \in T(X)$, we write $\mathbf{A} \models u \approx v$ if $u^{\mathbf{A}}(\mathbf{a}) = v^{\mathbf{A}}(\mathbf{a})$ for all $\mathbf{a} \in A^X$. For a class K of similar algebras, we say that u and v are K -equivalent if $\mathbf{A} \models u \approx v$ for all $\mathbf{A}$ from K . In this case, we write $u \sim_K v$ and $K \models u \approx v$.

3. CLASS OPERATORS

Let $\mathcal{F}$ be an algebraic language, and let K be a class of algebras of type $\mathcal{F}$. We use the operators I, H as in [BS81]. By PK we denote the class of all algebras that can be written as $\prod_{i \in I} \mathbf{A}_i$ for some set I and some family $(\mathbf{A}_i)_{i \in I}$ of algebras from K . Deviating from the notation from [BS81], we also allow the empty product $\prod_{i \in \emptyset} \mathbf{A}_i$ to be a member of PK . Hence for every class of similar algebras K , the one-element algebra with universe $\{\emptyset\}$ of type $\mathcal{F}$ belongs to PK . When we restrict I to be finite, then the class is $P_{\text{fin}} K$. By $P_u K$ we denote the class of all ultraproducts of members of K .

4. THE FREE ALGEBRA

Let K be a class of algebras of type $\mathcal{F}$, and let X be a set such that $\mathcal{F}_0 \cup X \neq \emptyset$. The *free algebra* $\mathbf{F}(X, K)$ constructed from X and K is defined as follows: We let

$$\Theta_K := \bigcap \{ \theta \in \text{Con}(\mathbf{T}(X)) \mid \mathbf{T}(X)/\theta \text{ lies in } ISK \},$$

and we define $\mathbf{F}(X, K) := \mathbf{T}(X)/\Theta_K$. We observe that by its definition, $\mathbf{F}(X, K) = \mathbf{T}(X)/\Theta_K$ is isomorphic to a subdirect product of algebras in ISK , and hence $\mathbf{F}(X, K)$ is an element of $ISP K$. This algebra is denoted

by $\mathbf{F}_K(\overline{X})$ in [BS81, Definition II.10.9] and is called the K -free algebra over $\overline{X} = \{x/\Theta_K \mid x \in X\}$.

Lemma 4.1. *Let X be a set, let $s, t \in T(X)$. Then $(s, t) \in \Theta_K$ if and only if s and t are K -equivalent.*

Proof. For the “only if”-direction, we assume that $(s, t) \in \Theta_K$. Let $\mathbf{A}$ be an algebra in K , and let $\mathbf{a} \in A^X$. Then by Theorem 2.3, the mapping e defined by

$$e(u) := u^{\mathbf{A}}(\mathbf{a})$$

is a homomorphism from $\mathbf{T}(X)$ to $\mathbf{A}$. The image of e is the universe of an algebra that lies in $S\{\mathbf{A}\}$, and thus for $\theta := \ker e$, we have $\mathbf{T}(X)/\theta \in ISK$. Hence $\Theta_K \subseteq \theta$ and therefore $(s, t) \in \ker e$, which implies $s^{\mathbf{A}}(\mathbf{a}) = t^{\mathbf{A}}(\mathbf{a})$. Thus $s \sim_K t$.

For the “if”-direction, we assume that $s \sim_K t$. Let $\theta \in \text{Con}(\mathbf{T}(X))$ be such that $\mathbf{T}(X)/\theta$ lies in ISK . Let $x_1, \dots, x_m$ be the variables occurring in s and t . By the definition of $\sim_K$, we have $K \models s(x_1, \dots, x_m) \approx t(x_1, \dots, x_m)$. Since $\mathbf{T}(X)/\theta \in ISK$, we have $\mathbf{T}(X)/\theta \models s(x_1, \dots, x_m) \approx t(x_1, \dots, x_m)$, and therefore

$$\begin{aligned} s/\theta &= s^{\mathbf{T}(X)}(x_1, \dots, x_m)/\theta \\ &= s^{\mathbf{T}(X)/\theta}(x_1/\theta, \dots, x_m/\theta) \\ &= t^{\mathbf{T}(X)/\theta}(x_1/\theta, \dots, x_m/\theta) \\ &= s^{\mathbf{T}(X)}(x_1, \dots, x_m)/\theta \\ &= t/\theta. \end{aligned}$$

Thus $(s, t) \in \theta$. Hence $(s, t) \in \Theta_K$. □

From this property, it is easy to see that $\mathbf{F}(X, K)$ is free for K over $\{x/\Theta_K \mid x \in X\}$ in the sense of [MMT87, Definition 4.107].

5. BIRKHOFF’S HSP-THEOREM

Theorem 5.1. *Let K be a class of similar algebras, and let Y be a countably infinite set. Then $\text{Mod}(\text{Th}_{\text{Id}(Y)}(K)) \subseteq HSP K$.*

Proof. Let $\mathbf{B}$ in $\text{Mod}(\text{Th}_{\text{Id}(Y)}(K))$, and let G be a nonempty subset of B that generates $\mathbf{B}$. Let X be a set of the same cardinality as G , and let $\mathbf{b} = (b_x)_{x \in X}$ be

such that $\{b_x \mid x \in X\} = G$. Let $\mathbf{F} := \mathbf{F}(X, K) = \mathbf{T}(X)/\Theta_K$ be the free algebra constructed from X and K , and let

$$\varphi := \{(t/\Theta_K, t^{\mathbf{B}}(\mathbf{b})) \mid t \in T(X)\}.$$

We first show that φ is a function from F to B . To this end, we suppose that $(s, t) \in \Theta_K$. By the “only if”-direction of Lemma 4.1, we have $s \sim_K t$. Let $x_1, \dots, x_m$ be the variables occurring in s and t . By the definition of $\sim_K$, we have $K \models s(x_1, \dots, x_m) \approx t(x_1, \dots, x_m)$. Since $\mathbf{B} \in \text{Mod}(\text{Th}_{\text{Id}(Y)}(K))$, we then also have $\mathbf{B} \models s(x_1, \dots, x_m) \approx t(x_1, \dots, x_m)$, and therefore $s^{\mathbf{B}}(\mathbf{b}) = t^{\mathbf{B}}(\mathbf{b})$. Hence φ is a function.

Since φ is the image of the homomorphism $\psi : \mathbf{T}(X) \rightarrow \mathbf{T}(X)/\Theta_K \times \mathbf{B}$, $\psi(t) = (t/\Theta_K, t^{\mathbf{B}}(\mathbf{b}))$, φ is a subuniverse of $\mathbf{T}(X)/\Theta_K \times \mathbf{B}$. As a function that is a subuniverse, φ is a homomorphism, and therefore $\mathbf{B} \in H\{\mathbf{F}\}$. Thus $\mathbf{B}$ lies in $HSP K$. $\square$

We call a class K of similar algebras a *variety* if there is a set Y and a set Φ of identities in the variables Y such that $K = \text{Mod}(\Phi)$.

Corollary 5.2. *Let K be a class of similar algebras. Then the variety generated by K is $HSP K$.*

Proof. Let $V(K)$ be the smallest variety containing K . Then there are a set Y and a set Φ of identities in the variables Y such that $V(K) = \text{Mod}(\Phi)$. Let X be a countably infinite set. By renaming the variables in each of the identities in Φ , we obtain identities Φ' in the variables X such that $\text{Mod}(\Phi) = \text{Mod}(\Phi')$. Then we have $\Phi' \subseteq \text{Th}_{\text{Id}(X)}(\text{Mod}(\Phi')) = \text{Th}_{\text{Id}(X)}(V(K)) \subseteq \text{Th}_{\text{Id}(X)}(K)$, and therefore every algebra in $\text{Mod}(\text{Th}_{\text{Id}(X)}(K))$ lies in $\text{Mod}(\Phi') = V(K)$. Since $\text{Mod}(\text{Th}_{\text{Id}(X)}(K))$ is a variety containing K , and $V(K)$ is the smallest variety containing K , $V(K)$ and $\text{Mod}(\text{Th}_{\text{Id}(X)}(K))$ contain the same algebras. Thus by Theorem 5.1, every algebra in $V(K)$ lies in $HSP K$.

Since validity of an identity is preserved by forming products, subalgebras and homomorphic images, every algebra in $HSP K$ satisfies Φ , and therefore every algebra in $HSP K$ lies in $\text{Mod}(\Phi) = V(K)$. $\square$

6. QUASI-IDENTITIES

A *quasi-identity* over X is a formula $(\bigwedge_{i \in \underline{r}} s_i \approx t_i) \rightarrow u \approx v$ with $r \in \mathbb{N}_0$ and $s_1, \dots, s_r, t_1, \dots, t_r, u, v \in T(X)$. $\text{QId}(X)$ is the set of all quasi-identities over X . A *quasivariety* is a class of similar algebras that is axiomatized by a set of quasi-identities.

In [BS81], the following result is proved.

Theorem 6.1 (cf. [BS81, Theorem V.2.25]). *Let K be a class of similar algebras, let X be a countably infinite set, and let $\mathbf{A}$ be an algebra in $\text{Mod}(\text{Th}_{\text{QId}(X)}(K))$. Then $\mathbf{A}$ lies in $\text{ISP}_{\text{u}} P_{\text{fin}} K$.*

Proof. We specialize the proof in [BS81]. Let $\mathbf{A}^*$ be an expansion of $\mathbf{A}$ where for each $a \in A$, we add a nullary operation symbol $\hat{a}$ interpreted by $\hat{a}^{\mathbf{A}} := a$. We let $\mathcal{F}$ be the language of $\mathbf{A}$, and $\mathcal{F}^*$ be the language of $\mathbf{A}^*$. Let $T^*(X)$ be the terms over X in the language $\mathcal{F}^*$, and $T^*(\emptyset)$ be the set of terms using no variables. Each term t' in $T^*(\emptyset)$ can be written as $t(a_1, \dots, a_n)$, where $t \in T(\{x_1, \dots, x_n\})$ is a term of language $\mathcal{F}$, $a_1, \dots, a_n \in A$ and $t(a_1, \dots, a_n)$ is understood as an abbreviation of $t^{\mathbf{A}^*}(\{\hat{a}_1, \dots, \hat{a}_n\})(\hat{a}_1, \dots, \hat{a}_n)$.

We define the set D of formulae in the language $\mathcal{F}^*$ by

$$D := \{s \approx t \mid s, t \in T^*(\emptyset), s^{\mathbf{A}^*} = t^{\mathbf{A}^*}\} \cup \{s \not\approx t \mid s, t \in T^*(\emptyset), s^{\mathbf{A}^*} \neq t^{\mathbf{A}^*}\},$$

and we let

$$F := \{\varphi \mid \varphi \text{ is a finite subset of } D\}.$$

For $\varphi \in F$, we define $\varphi \uparrow := \{\psi \in F \mid \varphi \subseteq \psi\}$. Let

$$\mathcal{A} := \{\varphi \uparrow \mid \varphi \in F\}.$$

This $\mathcal{A}$ is a filter on the set F because $(\varphi_1 \uparrow) \cap (\varphi_2 \uparrow) = (\varphi_1 \cup \varphi_2) \uparrow$. Hence there exists an ultrafilter $\mathcal{U}$ on the set F with $\mathcal{A} \subseteq \mathcal{U}$. Altogether, $\mathcal{U}$ is an ultrafilter on the set F such that for every $\varphi \in F$, we have

$$\varphi \uparrow = \{\psi \in F \mid \varphi \subseteq \psi\} \in \mathcal{U}.$$

Now for every $\varphi \in F$, we construct an $\mathcal{F}^*$ -algebra $\mathbf{B}_\varphi^*$ such that $\mathbf{B}_\varphi^* \models \varphi$ and the $\mathcal{F}$ -reduct of $\mathbf{B}_\varphi^*$ lies in $P_{\text{fin}} K$. Since $\varphi \in F$, there are $k, m, n \in \mathbb{N}_0$, a finite subset $\{x_1, \dots, x_n\}$ of X , $\mathbf{a} = (a_1, \dots, a_n) \in A^n$, and for each $i \in \underline{k}$ and $j \in \underline{m}$ there are

$\mathcal{F}$ -terms $s_i, t_i, u_j, v_j \in T(\{x_1, \dots, x_n\})$ such that

$$\varphi = \{s_1(\mathbf{a}) \approx t_1(\mathbf{a}), \dots, s_k(\mathbf{a}) \approx t_k(\mathbf{a})\} \cup \{u_1(\mathbf{a}) \not\approx v_1(\mathbf{a}), \dots, u_m(\mathbf{a}) \not\approx v_m(\mathbf{a})\}$$

and for $i, j \in \underline{n}$ with $i \neq j$, we have $a_i \neq a_j$. Here, $s_1(\mathbf{a})$ is a shorthand for $s_1^{\mathbf{T}^*(\{x_1, \dots, x_n\})}(\hat{a}_1, \dots, \hat{a}_n)$.

For each $i \in \underline{m}$, we will now construct an algebra $\mathbf{B}_i$ in K with certain properties. We fix $i \in \underline{m}$. Then we have

$$\mathbf{A} \models \exists \mathbf{x} : s_1(\mathbf{x}) \approx t_1(\mathbf{x}) \wedge \dots \wedge s_k(\mathbf{x}) \approx t_k(\mathbf{x}) \wedge u_i(\mathbf{x}) \not\approx v_i(\mathbf{x}).$$

We show that there is $\mathbf{B}_i$ in K such that

$$(6.1) \quad \mathbf{B}_i \models \exists \mathbf{x} : s_1(\mathbf{x}) \approx t_1(\mathbf{x}) \wedge \dots \wedge s_k(\mathbf{x}) \approx t_k(\mathbf{x}) \wedge u_i(\mathbf{x}) \not\approx v_i(\mathbf{x}).$$

Suppose that there is no such $\mathbf{B}_i$. Then

$$(6.2) \quad K \models \forall \mathbf{x} : (s_1(\mathbf{x}) \approx t_1(\mathbf{x}) \wedge \dots \wedge s_k(\mathbf{x}) \approx t_k(\mathbf{x})) \rightarrow u_i(\mathbf{x}) \approx v_i(\mathbf{x}).$$

Since $\mathbf{A} \in \text{Mod}(\text{Th}_{\text{QId}(X)}(K))$, also $\mathbf{A}$ satisfies this quasi-identity. We know that we have $s_1^{\mathbf{A}}(\mathbf{a}) \approx t_1^{\mathbf{A}}(\mathbf{a}), \dots, s_k^{\mathbf{A}}(\mathbf{a}) \approx t_k^{\mathbf{A}}(\mathbf{a})$ and $u_i^{\mathbf{A}}(\mathbf{a}) \neq v_i^{\mathbf{A}}(\mathbf{a})$. This contradicts the fact that $\mathbf{A}$ satisfies the quasi-identity in (6.2). Hence there is $\mathbf{B}_i$ in K with (6.1). Let $\mathbf{b} := (b_1, \dots, b_n) \in B_i^n$ be such that

$$s_1^{\mathbf{B}_i}(\mathbf{b}) = t_1^{\mathbf{B}_i}(\mathbf{b}), \dots, s_k^{\mathbf{B}_i}(\mathbf{b}) = t_k^{\mathbf{B}_i}(\mathbf{b}) \text{ and } u_i^{\mathbf{B}_i}(\mathbf{b}) \neq v_i^{\mathbf{B}_i}(\mathbf{b}).$$

We will now form an $\mathcal{F}^*$ -expansion $\mathbf{B}_i^*$ of $\mathbf{B}_i$. For each $j \in \underline{n}$, set $\hat{a}_j^{\mathbf{B}_i^*} := b_j$, and for $a \in A \setminus \{a_1, \dots, a_n\}$, set $\hat{a}^{\mathbf{B}_i^*}$ to some element of B_i .

We set $\mathbf{B}_\varphi^* := \prod_{i \in \underline{m}} \mathbf{B}_i^*$ and note that in the case $m = 0$, $\mathbf{B}_\varphi^*$ is a one element algebra. Then $\mathbf{B}_\varphi^* \models \varphi$, and the $\mathcal{F}$ -reduct of $\mathbf{B}_\varphi^*$ lies in $P_{\text{fin}} K$.

Next, we show that $\mathbf{C} := \prod_{\varphi \in F} \mathbf{B}_\varphi^* / \mathcal{U}$ satisfies D . To this end, let $\delta \in D$. Now for all φ with $\delta \in \varphi$, we have $\mathbf{B}_\varphi^* \models \delta$ because $\mathbf{B}_\varphi^* \models \varphi$. Hence $\{\varphi \in F \mid \mathbf{B}_\varphi^* \models \delta\} \supseteq \{\delta\}^\uparrow$, and thus $\{\varphi \in F \mid \mathbf{B}_\varphi^* \models \delta\} \in U$. By Łoś's Theorem [BS81, Theorem V.2.9], we therefore have $\mathbf{C} \models \delta$. Thus $\mathbf{C} \models D$.

Now we define a mapping $h : A \rightarrow C$ by $h(a) := \hat{a}^{\mathbf{C}}$. We claim that this mapping is an embedding of $\mathbf{A}$ into the $\mathcal{F}$ -reduct of $\mathbf{C}$. First, if f is an n -ary operation symbol of $\mathbf{A}$, $a_1, \dots, a_n \in A$ and $b = f^{\mathbf{A}}(a_1, \dots, a_n)$, the identity

$$\hat{b} \approx f(\hat{a}_1, \dots, \hat{a}_n)$$

is an element of D . Therefore, since $\mathbf{C} \models D$, $h(b) = \hat{b}^{\mathbf{C}} = f^{\mathbf{C}}(\hat{a}_1^{\mathbf{C}}, \dots, \hat{a}_n^{\mathbf{C}}) = f^{\mathbf{C}}(h(a_1), \dots, h(a_n))$ and thus h is a homomorphism.

Second, if a_1, a_2 are elements of A such that $a_1 \neq a_2$, then $\hat{a}_1 \not\approx \hat{a}_2$ is an element of D . Thus, since $\mathbf{C} \models D$, we have $h(a_1) = \hat{a}_1^{\mathbf{C}} \neq \hat{a}_2^{\mathbf{C}} = h(a_2)$, and therefore h is injective.

Therefore, since $\mathbf{C} \in P_{\mathbf{u}}P_{\mathbf{fin}}K$, we have $\mathbf{A} \in ISP_{\mathbf{u}}P_{\mathbf{fin}}K$. $\square$

Similarly to Corollary 5.2, we obtain:

Corollary 6.2. *Let K be a class of similar algebras. Then the quasi-variety generated by K is $ISP_{\mathbf{u}}P_{\mathbf{fin}}K$.*

For a class K and $m \in \mathbb{N}$, we write P_m for the class of direct products of exactly m algebras from K .

Theorem 6.3. *Let K be a class of similar algebras of finite type, let X be a countably infinite set, and let $\mathbf{A}$ be a finite algebra in $\text{Mod}(\text{Th}_{\text{QId}(X)}(K))$ with n elements. Then $\mathbf{A} \in ISP_{\binom{n}{2}}K$.*

Proof. Let $\mathbf{A}^*$, $\mathcal{F}$ and $\mathcal{F}^*$ be as in the proof of Theorem 6.1. Let $a_1, \dots, a_n$ be the elements of A . Let T (the operation tables) be defined by

$$T := \{(f, (i_0, \dots, i_m)) \mid m \in \mathbb{N}_0, i_0, \dots, i_m \in \underline{n}, \\ f \text{ is an } m\text{-ary function symbol in } \mathcal{F}, a_{i_0} = f^{\mathbf{A}}(a_{i_1}, \dots, a_{i_m})\},$$

and let

$$D^+ := \{\hat{a}_{i_0} \approx f(\hat{a}_{i_1}, \dots, \hat{a}_{i_m}) \mid (f, (i_0, \dots, i_m)) \in T\},$$

and

$$D := D^+ \cup \{\hat{a}_i \not\approx \hat{a}_j \mid i, j \in \underline{n}, i < j, a_i \neq a_j\}.$$

We fix $i, j \in \underline{n}$ with $i < j$. Let

$$\varphi(x_1, \dots, x_n) := \bigwedge \{x_{i_0} = f^{\mathbf{A}}(x_{i_1}, \dots, x_{i_m}) \mid (f, \mathbf{i}) \in T\}.$$

Then

$$\mathbf{A} \models \exists \mathbf{x} : \varphi(x_1, \dots, x_n) \wedge x_i \not\approx x_j.$$

We want to show that there is $\mathbf{B}_{i,j}$ in K such that

$$\mathbf{B}_{i,j} \models \exists \mathbf{x} : \varphi(x_1, \dots, x_n) \wedge x_i \not\approx x_j.$$

Suppose that there is no such $\mathbf{B}_{i,j}$. Then

$$K \models \forall \mathbf{x} : \varphi(x_1, \dots, x_n) \rightarrow x_i \approx x_j.$$

Since $\mathbf{A} \in \text{Mod}(\text{Th}_{\text{QId}(X)}(K))$, we have

$$(6.3) \quad \mathbf{A} \models \forall \mathbf{x} : \varphi(x_1, \dots, x_n) \rightarrow x_i \approx x_j.$$

Setting $(x_1, \dots, x_n) := (a_1, \dots, a_n)$, we see that (6.3) does not hold. This contradiction shows that there is $\mathbf{B}_{i,j}$ in K with

$$\mathbf{B}_{i,j} \models \exists \mathbf{x} : \varphi(x_1, \dots, x_n) \wedge x_i \not\approx x_j.$$

Let $\mathbf{b}$ be an n -tuple witnessing the existence of these x_k 's, and for $k \in \underline{n}$, set $a_k^{\mathbf{B}_{i,j}^*} = b_k$. Now the mapping defined by $h(a_k) := b_k = \hat{a}_k^{\mathbf{B}_{i,j}^*}$ is a homomorphism from $\mathbf{A}^*$ to $\mathbf{B}_{i,j}^*$. To prove this, let f be an m -ary operation symbol in $\mathcal{F}$, and let $i_1, \dots, i_m \in \underline{n}$. We assume that $a_{i_0} = f^{\mathbf{A}}(a_{i_1}, \dots, a_{i_m})$. Then

$$\mathbf{B}_{i,j}^* \models \hat{a}_{i_0} \approx f(\hat{a}_{i_1}, \dots, \hat{a}_{i_m}),$$

and therefore

$$h(a_{i_0}) = \hat{a}_{i_0}^{\mathbf{B}_{i,j}^*} = f^{\mathbf{B}_{i,j}^*}(\hat{a}_{i_1}^{\mathbf{B}_{i,j}^*}, \dots, \hat{a}_{i_m}^{\mathbf{B}_{i,j}^*}) = f^{\mathbf{B}_{i,j}^*}(h(a_{i_1}), \dots, h(a_{i_m})),$$

which concludes the proof that h is a homomorphism. In addition, for $i < j$, we have $h(a_i) \neq h(a_j)$.

Let $\mathbf{B}^* := \prod (\mathbf{B}_{i,j}^*)_{i,j \in \underline{n}, i < j}$. Then $\mathbf{B}^* \models D$, and $\mathbf{A}$ embeds into the $\mathcal{F}$ -reduct $\mathbf{B}$ of $\mathbf{B}^*$. Hence $\mathbf{A}$ embeds into a direct product of exactly $\binom{n}{2}$ algebras in K . $\square$

Corollary 6.4. *Let K be a finite set of similar finite algebras of finite type, let X be a countably infinite set, and let $\mathbf{A}$ be a finite subdirectly irreducible algebra in $\text{Mod}(\text{Th}_{\text{QId}(X)}(K))$. Then $\mathbf{A} \in IS K$.*

Proof. We know that there is a set I and a family $(\mathbf{B}_i)_{i \in I}$ from K such that $\mathbf{A}$ embeds into $\prod_{i \in I} \mathbf{B}_i$. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{A}$ embeds into some $\mathbf{B}_j$, which implies $\mathbf{A} \in IS K$. $\square$

7. GENERALIZED QUASI-IDENTITIES

A *generalized quasi-identity* over Y is a formula $(\bigwedge_{i \in I} s_i \approx t_i) \rightarrow u \approx v$, where I is a (possibly infinite) set and there exists a finite subset X of Y such that $u, v \in T(X)$, and for all $i \in I$, $s_i \in T(X)$ and $t_i \in T(X)$. $\text{GQId}(Y)$ is the class of all generalized quasi-identities over Y . For a class K of similar algebras, let $L K$

be the class of those algebras of the same signature that have the property that every finitely generated subalgebra embeds into some member of K .

Theorem 7.1. *Let K be a class of similar algebras, and let Y be a countably infinite set. If $\mathbf{A}$ lies in $\text{Mod}(\text{Th}_{\text{GQId}(Y)}(K))$, then $\mathbf{A}$ lies in $LSP K$. Hence $\text{Mod}(\text{Th}_{\text{GQId}(Y)}(K)) = LSP K$.*

Proof. Let $\mathbf{A}'$ be a finitely generated subalgebra of $\mathbf{A}$, and let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A$ be such that $\{a_1, \dots, a_n\}$ generates $\mathbf{A}'$.

$$D^+ = \{(s, t) \mid s, t \in T(\{x_1, \dots, x_n\}), s^{\mathbf{A}}(a_1, \dots, a_n) = t^{\mathbf{A}}(a_1, \dots, a_n)\}$$

and

$$D^- = \{(u, v) \mid u, v \in T(\{x_1, \dots, x_n\}), s^{\mathbf{A}}(a_1, \dots, a_n) \neq v^{\mathbf{A}}(a_1, \dots, a_n)\}.$$

For every $(u, v) \in D^-$, we find an algebra $\mathbf{B}_{u,v}$ in K such that $\mathbf{B}_{u,v}$ satisfies

$$\exists x_1, \dots, x_n :$$

$$\left(\bigwedge_{(s,t) \in D^+} s(x_1, \dots, x_n) \approx t(x_1, \dots, x_n) \right) \wedge u(x_1, \dots, x_n) \not\approx v(x_1, \dots, x_n).$$

Then $\mathbf{C} := \prod_{(u,v) \in D^-} \mathbf{B}_{u,v}$ satisfies

$$\exists x_1, \dots, x_n :$$

$$\left(\bigwedge_{(s,t) \in D^+} s(x_1, \dots, x_n) \approx t(x_1, \dots, x_n) \right) \wedge \left(\bigwedge_{(u,v) \in D^-} u(x_1, \dots, x_n) \not\approx v(x_1, \dots, x_n) \right).$$

If we choose $(c_1, \dots, c_n)$ as witnesses for these $x_1, \dots, x_n$, then $h : A' \rightarrow C$, $h(t^{\mathbf{A}}(a_1, \dots, a_n)) = t^{\mathbf{C}}(c_1, \dots, c_n)$ is well-defined because for $\mathbf{c} = (c_1, \dots, c_n)$, we have

$$\bigwedge_{(s,t) \in D^+} s^{\mathbf{C}}(c_1, \dots, c_n) = t^{\mathbf{C}}(c_1, \dots, c_n),$$

and injective because of

$$\left(\bigwedge_{(u,v) \in D^-} u^{\mathbf{C}}(c_1, \dots, c_n) \neq v^{\mathbf{C}}(c_1, \dots, c_n) \right).$$

Therefore $\mathbf{A}' \in LSP K$. Thus $\mathbf{A} \in LSP K$.

Now similar to Corollary 5.2, we can argue that $\text{Mod}(\text{Th}_{\text{GQId}(Y)}(K))$ is equal to $\text{LSP } K$. $\square$

A *fully generalized quasi-identity* over X is a formula $(\bigwedge_{i \in I} s_i \approx t_i) \rightarrow u \approx v$, where I is a (possibly infinite) set, $u, v \in T(X)$, and for all $i \in I$, $s_i \in T(X)$ and $t_i \in T(X)$. $\text{FGQId}(X)$ is the class of all quasi-identities over X . Then we have

Theorem 7.2. *Let K be a class of similar algebras, and let X be a set such that $\mathbf{A}$ can be generated by $|X|$ elements. If $\mathbf{A}$ lies in $\text{Mod}(\text{Th}_{\text{FGQId}(X)}(K))$, then $\mathbf{A}$ lies in $\text{ISP } K$. Thus, $\text{Mod}(\text{Th}_{\text{FGQId}(X)}(K)) = \text{ISP } K$.*

Proof. Let $\mathbf{a} = (a_x)_{x \in X}$ be such that $\{a_x \mid x \in X\}$ generates $\mathbf{A}$.

$$D^+ = \{(s, t) \in T(X) \times T(X) \mid s^{\mathbf{A}}(\mathbf{a}) = t^{\mathbf{A}}(\mathbf{a})\}$$

and

$$D^- = \{(u, v) \in T(X) \times T(X) \mid u^{\mathbf{A}}(\mathbf{a}) \neq v^{\mathbf{A}}(\mathbf{a})\}.$$

For every $(u, v) \in D^-$, $(\bigwedge_{(s,t) \in D^+} s \approx t) \rightarrow u \approx v$ is a fully generalized quasi-identity that does not hold in $\mathbf{A}$. Thus we find an algebra $\mathbf{B}_{u,v}$ in K and $\mathbf{b}_{u,v} \in B_{u,v}^X$ with

$$(\bigwedge_{(s,t) \in D^+} s(\mathbf{b}_{u,v}) \approx t(\mathbf{b}_{u,v})) \wedge u(\mathbf{b}_{u,v}) \not\approx v(\mathbf{b}_{u,v}).$$

Then $\mathbf{B} := \prod_{(u,v) \in D^-} \mathbf{B}_{u,v}$ and $\mathbf{b}$ with $\mathbf{b}_x := ((\mathbf{b}_{u,v})_x)_{(u,v) \in D^-}$ satisfies

$$(\bigwedge_{(s,t) \in D^+} s(\mathbf{b}) \approx t(\mathbf{b})) \wedge (\bigwedge_{(u,v) \in D^-} u(\mathbf{b}) \not\approx v(\mathbf{b})).$$

Now the mapping $h(t^{\mathbf{A}}(\mathbf{a})) = t^{\mathbf{B}}(\mathbf{b})$ is well-defined because of

$$\bigwedge_{(s,t) \in D^+} s^{\mathbf{B}}(\mathbf{b}) = t^{\mathbf{B}}(\mathbf{b}),$$

and injective because of

$$\bigwedge_{(u,v) \in D^-} u^{\mathbf{B}}(\mathbf{b}) \neq v^{\mathbf{B}}(\mathbf{b}).$$

Therefore $\mathbf{A} \in \text{ISP } K$.

Now similar to Corollary 5.2, we can argue that $\text{Mod}(\text{Th}_{\text{FGQId}(Y)}(K))$ is equal to $\text{ISP } K$. $\square$

8. COMPLETENESS OF THE EQUATIONAL CALCULUS

In this section, we see an identity $s \approx t$ as a pair (s, t) .

Theorem 8.1. *Let X be a set, and let $\Sigma \subseteq T(X) \times T(X)$. Then $\text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma))$ is the fully invariant congruence $\Theta_{\text{FI}}(\Sigma)$ generated by Σ .*

Proof. For $\Theta_{\text{FI}}(\Sigma) \subseteq \text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma))$, we observe that $\text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma))$ is a fully invariant congruence of $\mathbf{T}(X)$ that contains Σ as a subset.

For $\Theta_{\text{FI}}(\Sigma) \supseteq \text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma))$, let $\theta := \Theta_{\text{FI}}(\Sigma)$. We first establish

$$(8.1) \quad \mathbf{T}(X)/\theta \models \Sigma.$$

To this end, let $(s(x_1, \dots, x_n), t(x_1, \dots, x_n)) \in \Sigma$, and let $t_1/\theta, \dots, t_n/\theta \in T(X)/\theta$. Since $(s, t) \in \Sigma$, invariance under the endomorphism obtained from extending $\{(x_i, t_i) \mid i \in \underline{n}\}$ yields $(s^{\mathbf{T}(X)}(t_1, \dots, t_n), t^{\mathbf{T}(X)}(t_1, \dots, t_n)) \in \theta$. Thus $s^{\mathbf{T}(X)/\theta}(t_1/\theta, \dots, t_n/\theta) = s^{\mathbf{T}(X)}(t_1, \dots, t_n)/\theta = t^{\mathbf{T}(X)}(t_1, \dots, t_n)/\theta = s^{\mathbf{T}(X)/\theta}(t_1/\theta, \dots, t_n/\theta)$, completing the proof of (8.1).

Now let $(s, t) \in \text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma))$. Then by (8.1), $\mathbf{T}(X)/\theta \models s(x_1, \dots, x_n) \approx t(x_1, \dots, x_n)$, and thus $s/\theta = s^{\mathbf{T}(X)}(x_1, \dots, x_n)/\theta = s^{\mathbf{T}(X)/\theta}(x_1/\theta, \dots, x_n/\theta) = t^{\mathbf{T}(X)/\theta}(x_1/\theta, \dots, x_n/\theta) = t^{\mathbf{T}(X)}(x_1, \dots, x_n)/\theta = t/\theta$, which completes the proof of $\text{Th}_{\text{Id}(X)}(\text{Mod}(\Sigma)) \subseteq \theta$. $\square$

Now the calculus can be obtained by seeing fully invariant congruences as subalgebras of the expansion of $\mathbf{A} \times \mathbf{A}$ constructed in [BS81, Lemma II.14.4], and applying the subalgebra generation process of [BS81, Theorem II.3.2].

REFERENCES

- [Bir35] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc. **31** (1935), 433–454.
- [BS81] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer New York Heidelberg Berlin, 1981.
- [Gor98] V. A. Gorbunov, *Algebraic theory of quasivarieties*, Siberian School of Algebra and Logic, Consultants Bureau, New York, 1998, Translated from the Russian. MR 1654844
- [Mal54] A. I. Mal'cev, *On the general theory of algebraic systems*, Mat. Sb. N.S. **35(77)** (1954), 3–20.

- [MMT87] R. N. McKenzie, G. F. McNulty, and W. F. Taylor, *Algebras, lattices, varieties, volume I*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California, 1987.
- [Plo19] B. Plotkin, *Seven lectures on universal algebraic geometry*, Groups, algebras and identities, Contemp. Math., vol. 726, Amer. Math. Soc., [Providence], RI, [2019] ©2019, pp. 143–215. MR 3937272

INSTITUTE FOR ALGEBRA, JOHANNES KEPLER UNIVERSITY LINZ, LINZ, AUSTRIA