TARSKI'S IRREDUNDANT BASIS THEOREM DRAFT

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Abstract. The proof of Tarski's Theorem from [BS81, p.33-34] is presented in a special setting.

1. Notation

Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$. For a subset X of A, we define

$$\begin{aligned} E(X) &:= X \cup \{f^{\mathbf{A}}(x_1, \ldots, x_n) \mid n \in \mathbb{N}_0,\ f \in \mathcal{F}_n,\ (x_1, \ldots, x_n) \in X^n\}, \\ E^0(X) &:= X, \\ E^s(X) &:= E(E^{s-1}(X)) \text{ for } s \in \mathbb{N}, \\ \mathrm{Sg}(X) &:= \bigcup\{E^s(X) \mid s \in \mathbb{N}_0\}. \end{aligned}$$

Definition 1.1. Let $\mathbf{A}$ be an algebra. B is a *basis* of $\mathbf{A}$ if $\mathrm{Sg}(B) = A$ and for all $C \subset B$, we have $\mathrm{Sg}(C) \subset A$.

2. The result

Theorem 2.1. Let $n, i, j \in \mathbb{N}_0$, and let $\mathbf{A}$ be an algebra such that all operation symbols of $\mathbf{A}$ have arity at most n. We assume that $i < j$ and that $\mathbf{A}$ has a basis with i and a basis with j elements. Then there is $k \in \mathbb{N}_0$ with $i < k < i + n$ such that $\mathbf{A}$ has a basis with k elements.

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Proof. Let B be a basis with j elements, and let

$$\mathcal{K} := \{C \mid C \text{ is a basis with at most } i \text{ elements}\}.$$

We abbreviate $E^s(B)$ by E^s. We first show that exists $s \in \mathbb{N}_0$ such that

$$\exists C \in \mathcal{K} : C \subseteq E^s. \tag{2.1}$$

By the assumptions, $\mathcal{K}$ is nonempty and contains an element C'. Now $C' \subseteq A = \bigcup\{E^t \mid t \in \mathbb{N}_0\}$. Since C' is finite, there must be $t \in \mathbb{N}_0$ with $C' \subseteq E^t$, and therefore (2.1) holds for $s := t$. Now we choose $s \in \mathbb{N}_0$ minimal such that (2.1) holds. If $s = 0$, then we have $C \in \mathcal{K}$ with $C \subseteq E^0 = B$. Since $|C| \leq i < |B|$, this contradicts the fact that B is a basis. Hence $s \geq 1$.

Next, we choose $C_0 \in \{C \in \mathcal{K} \mid C \subseteq E^s\}$ such that

$$C_0 \cap (E^s \setminus E^{s-1})$$

has a minimal number of elements. If $C_0 \cap (E^s \setminus E^{s-1}) = \emptyset$, we have $C_0 = C_0 \cap E^s = (C_0 \cap (E^s \setminus E^{s-1})) \cup (C_0 \cap E^{s-1}) = C_0 \cap E^{s-1}$. This implies $C_0 \subseteq E^{s-1}$. Then the existence of C_0 contradicts the minimality of s. Therefore, we can pick an element $c_0 \in C_0 \cap (E^s \setminus E^{s-1})$. Since $c_0 \in E^s$, we have $c_0 \in E(E^{s-1})$. Therefore, there exists a function symbol f of arity $m \in \mathbb{N}_0$ with $m \leq n$ and there exist $y_1, \dots, y_m \in E^{s-1}$ such that

$$c_0 = f^{\mathbf{A}}(y_1, \dots, y_m).$$

Let $Y := \{y_1, \dots, y_m\}$, and let

$$C_1 := (C_0 \setminus \{c_0\}) \cup Y.$$

Since $c_0 \in \mathrm{Sg}(C_1)$ and $C_0 \setminus \{c_0\} \subseteq \mathrm{Sg}(C_1)$, we have $C_0 \subseteq \mathrm{Sg}(C_1)$, and thus $\mathrm{Sg}(C_0) \subseteq \mathrm{Sg}(C_1)$. Since C_0 is a basis, this implies $\mathrm{Sg}(C_1) = A$. Therefore C_1 generates A, and therefore there is a subset C_2 of C_1

that is a basis of $\mathbf{A}$. We have $|C_2| \leq |C_1| \leq i - 1 + n$. We will now show that C_2 is the required basis. In the case that $|C_2| > i$, we are done. Hence let us assume that $|C_2| \leq i$. Next, we show

$$C_2 \cap (E^s \setminus E^{s-1}) \subseteq C_0 \cap (E^s \setminus E^{s-1}). \tag{2.2}$$

To this end, let $x \in C_2 \cap (E^s \setminus E^{s-1})$. Then $x \in C_1 = (C_0 \setminus \{c_0\}) \cup Y$. Since $x \notin E^{s-1}$, we have $x \notin Y$, and therefore $x \in C_0 \setminus \{c_0\}$. This proves (2.2). Furthermore, $c_0 \notin C_2$: if $c_0 \in C_2$, then $c_0 \in C_1$, and thus $c_0 \in Y$, and hence $c_0 \in E^{s-1}$, contradicting the choice of c_0. Hence c_0 is not an element of the left hand side of (2.2). However, by its choice, it is an element of the right hand side. This proves that

$$|C_2 \cap (E^s \setminus E^{s-1})| < |C_0 \cap (E^s \setminus E^{s-1})|.$$

Since C_2 is a basis of $\mathbf{A}$ and thus $C_2 \in \mathcal{K}$, this contradicts the choice of C_0. Therefore, the case $|C_2| \leq i$ cannot occur. □

References

[BS81] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer New York Heidelberg Berlin, 1981.