# SOME TOPICS IN EQUATIONAL LOGIC

Abstract. These are additional notes for the course "Universal algebra". The course and this presentation mainly follow [BS81].

## 1. Motivation

These notes are used in a course on universal algebra that follows [BS81] for explaining the two fundamental theorems by Birkhoff from [Bir35]: the HSP-theorem and the completeness of the equational calculus. The proofs are those from [BS81], but some auxiliary material and explanations from this book have been skipped. The material is complemented with some theorems on quasi-identities from universal algebra [Mal54, BS81, Gor98] and universal algebraic geometry [Plo19].

## 2. The term algebra

Let $\mathcal{F}$ be an algebraic language and $X$ be a set that is disjoint from $\mathcal{F}$. The set $T(X)$ of terms over $X$ is a subset of $(\mathcal{F} \cup X)^*$, the set of words over the alphabet $\mathcal{F} \cup X$ of positive length. We define $E_0 := X$ and

$$E_n := E_{n-1} \cup \{ft_1 \dots t_m \mid m \in \mathbb{N}_0, f \in \mathcal{F}_m, t_1, \dots, t_m \in E_{n-1}\}$$

for $n \in \mathbb{N}$. Then $T(X) := \bigcup_{n \in \mathbb{N}_0} E_n$.

**Lemma 2.1.** *Let $u, v$ be terms. If $u$ is a prefix of $v$, then $u = v$.*

*Proof.* Let us first consider the case $u \in E_0$. By induction on $n$, we see that each $w \in E_n$ whose first letter is in $X$ satisfies $w \in E_0$. Hence $v \in E_0$, and thus $u = v$.

We show by induction on $n+m$ that the statement holds for all $u \in E_m$, $v \in E_n$. The induction basis $m + n = 0$ is covered by the case $u \in E_0$.

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Notes for the course "Universal Algebra" at JKU in 2023 by Erhard Aichinger.

**PRELIMINARY VERSION**.

Now we assume $m+n \geq 1$. For $m = 0$, we have already established $u = v$. Hence we assume $m \geq 1$. If $n = 0$, then $u$ starts with a letter in $\mathcal{F}$ and $v$ with a letter in $X$, contradicting that $u$ is a prefix of $v$. If $n \geq 1$, we can write $u = ft_1 \dots t_r$ and $v = gu_1 \dots u_s$ with $f \in \mathcal{F}_r$, $g \in \mathcal{F}_s$, $t_1, \dots, t_r \in E_{m-1}$ and $u_1, \dots, u_s \in E_{n-1}$. Then $f = g$, and hence $r = s$. Now let $i$ be minimal with $t_i \neq u_i$. Then $t_i$ is a prefix of $u_i$ or $u_i$ is a prefix of $t_i$. In both cases, the induction hypothesis yields $t_i = u_i$. □

**Lemma 2.2.** *Let $t_1, \dots, t_r, u_1, \dots, u_s \in T(X)$, $f \in \mathcal{F}_r$, $g \in \mathcal{F}_s$. If $ft_1 \dots t_r = gu_1 \dots u_s$, then $r = s$, $f = g$ and $t_i = u_i$ for $i \in \{1, \dots, r\}$.*

*Proof.* We clearly have $f = g$ and thus $r = s$. Let $i$ be minimal with $t_i \neq u_i$. Then either $t_i$ is a prefix of $u_i$ or $u_i$ is a prefix of $t_i$, and hence by the previous lemma, $u_i = t_i$. □

We have $T(X) = \varnothing$ if and only if $\mathcal{F}_0 \cup X = \varnothing$. For the case $\mathcal{F}_0 \cup X \neq \varnothing$, we define the term algebra $\mathbf{T}(X)$ by $f^{\mathbf{T}(X)}(t_1, \dots, t_r) := ft_1 \dots t_r$ for all $r \in \mathbb{N}_0$, $f \in \mathcal{F}_r$.

**Theorem 2.3.** *Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$, and let $X$ be a set. We assume $\mathcal{F}_0 \cup X \neq \varnothing$. Let $\boldsymbol{a} \in A^X$. We define a relation $e \subseteq T(X) \times A$ by $e_0 : E_0 \to A$, $e_0(x) = \boldsymbol{a}(x)$ for $x \in X$, and for $n \geq 1$,*

$$e_n = e_{n-1} \cup \{(ft_1, \dots, t_m, f^{\mathbf{A}}(a_1, \dots, a_m)) \mid \\ m \in \mathbb{N}_0, f \in \mathcal{F}_m, \text{ for all } i \in \underline{m} : (t_i, a_i) \in e_{n-1}\}.$$

*Let $e := \bigcup_{n \in \mathbb{N}} e_n$. Then $e$ is a homomorphism from $\mathbf{T}(X)$ to $\mathbf{A}$ with $e|_X = \boldsymbol{a}$.*

*Proof.* It is easy to see that $e \subseteq T(X) \times A$ and that the first projection of $e$ to $T(X)$ is surjective.

Next, we prove by induction on the length of $u$ that for all $u \in E_n$ and $a, b \in A$ with $(u, a) \in e$ and $(u, b) \in e$, we have $a = b$.

If $u$ is of length 1, it is either in $X$ or $\mathcal{F}_0$. In the first case, $(u, a)$ and $(u, b)$ are both elements of $e_0$ because no other elements of $e$ have a first component in $E_0 = X$. Since $e_0$ is a function, $a = b$. If $u \in \mathcal{F}_0$ then we see that for all $n \in \mathbb{N}$ with $(u, a) \in e_n$, we have $a = u^{\mathbf{A}}$.

If the length of $u$ is at least 2, then since $(u, a) \in e$, there are $r \in \mathbb{N}_0$, $f \in \mathcal{F}_r$, $t_1, \dots, t_r \in T(X)$ and $a_1, \dots, a_r \in A$ such that $(t_i, a_i) \in e$ for all $i \in \underline{r}$,

$u = ft_1 \ldots, t_r$ and $a = f^{\mathbf{A}}(a_1, \ldots, a_r)$. Since $(u, b) \in e$, there are $s \in \mathbb{N}_0$, $g \in \mathcal{F}_s$, $u_1, \ldots, u_s \in T(X)$ and $b_1, \ldots, b_s \in A$ such that $(u_i, b_i) \in e$ for all $i \in \underline{s}$, $u = gu_1 \ldots, u_s$ and $a = g^{\mathbf{A}}(b_1, \ldots, b_r)$. Then by Lemma 2.2, $f = g$, $r = s$ and $t_i = u_i$ for $i \in \underline{r}$. Since $(t_i, a_i) \in e$, $(t_i, b_i) \in e$ and $t_i$ is shorter than $u$, we have $a_i = b_i$. Thus $a = b$.

From its construction, we see that $e$ is the subuniverse of $\mathbf{T}(X) \times \mathbf{A}$ that is generated by $\boldsymbol{a}$. Hence $e$ is a function and a subuniverse, and thus a homomorphism. □

For this $e$, we denote $e(t)$ also by $t^{\mathbf{A}}(\boldsymbol{a})$. Let $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in X$. We write $t(x_1, \ldots, x_n)$ to indicate that $t \in T(\{x_1, \ldots, x_n\})$ and $t^{\mathbf{A}}(a_1, \ldots, a_n)$ for $t^{\mathbf{A}}(\{(x_1, a_1), \ldots, (x_n, a_n)\})$. For $u, v \in T(X)$, we write $\mathbf{A} \models u \approx v$ if $u^{\mathbf{A}}(\boldsymbol{a}) = v^{\mathbf{A}}(\boldsymbol{a})$ for all $\boldsymbol{a} \in A^X$. For a class $K$ of similar algebras, we say that $u$ and $v$ are $K$-equivalent if $\mathbf{A} \models u \approx v$ for all $\mathbf{A}$ from $K$. In this case, we write $u \sim_K v$ and $K \models u \approx v$.

## 3. Class operators

Let $\mathcal{F}$ be an algebraic language, and let $K$ be a class of algebras of type $\mathcal{F}$. We use the operators $I, H$ as in [BS81]. By $P\,K$ we denote the class of all algebras that can be written as $\prod_{i \in I} \mathbf{A}_i$ for some set $I$ and some family $(\mathbf{A}_i)_{i \in I}$ of algebras from $K$. Deviating from the notation from [BS81], we also allow the empty product $\prod_{i \in \varnothing} \mathbf{A}_i$ to be a member of $PK$. Hence for every class of similar algebras $K$, the one-element algebra with universe $\{\varnothing\}$ of type $\mathcal{F}$ belongs to $P\,K$. When we restrict $I$ to be finite, then the class is $P_{\text{fin}}\,K$. By $P_{\text{u}}\,K$ we denote the class of all ultraproducts of members of $K$.

## 4. The free algebra

Let $K$ be a class of algebras of type $\mathcal{F}$, and let $X$ be a set such that $\mathcal{F}_0 \cup X \neq \varnothing$. The *free algebra* $\mathbf{F}(X, K)$ *constructed from* $X$ *and* $K$ is defined as follows: We let

$$\Theta_K := \bigcap \{\theta \in \operatorname{Con}(\mathbf{T}(X)) \mid \mathbf{T}(X)/\theta \text{ lies in } IS\,K\},$$

and we define $\mathbf{F}(X, K) := \mathbf{T}(X)/\Theta_K$. We observe that by its definition, $\mathbf{F}(X, K) = \mathbf{T}(X)/\Theta_K$ is isomorphic to a subdirect product of algebras in $IS\,K$, and hence $\mathbf{F}(X, K)$ is an element of $ISP\,K$. This algebra is denoted by $\mathbf{F}_K(\overline{X})$ in [BS81, Definition II.10.9] and is called the *$K$-free algebra* over $\overline{X} = \{x/\Theta_K \mid x \in X\}$.

**Lemma 4.1.** *Let $X$ be a set, let $s, t \in T(X)$. Then $(s,t) \in \Theta_K$ if and only if $s$ and $t$ are $K$-equivalent.*

*Proof.* For the "only if"-direction, we assume that $(s,t) \in \Theta_K$. Let $\mathbf{A}$ be an algebra in $K$, and let $\boldsymbol{a} \in A^X$. Then by Theorem 2.3, the mapping $e$ defined by

$$e(u) := u^{\mathbf{A}}(\boldsymbol{a})$$

is a homomorphism from $\mathbf{T}(X)$ to $\mathbf{A}$. The image of $e$ is the universe of an algebra that lies in $S\{\mathbf{A}\}$, and thus for $\theta := \ker e$, we have $\mathbf{T}(X)/\theta \in IS\,K$. Hence $\Theta_K \subseteq \theta$ and therefore $(s,t) \in \ker e$, which implies $s^{\mathbf{A}}(\boldsymbol{a}) = t^{\mathbf{A}}(\boldsymbol{a})$. Thus $s \sim_K t$.

For the "if"-direction, we assume that $s \sim_K t$. Let $\theta \in \mathrm{Con}(\mathbf{T}(X))$ be such that $\mathbf{T}(X)/\theta$ lies in $IS\,K$. Let $x_1, \ldots, x_m$ be the variables occurring in $s$ and $t$. By the definition of $\sim_K$, we have $K \models s(x_1, \ldots, x_m) \approx t(x_1, \ldots, x_m)$. Since $\mathbf{T}(X)/\theta \in IS\,K$, we have $\mathbf{T}(X)/\theta \models s(x_1, \ldots, x_m) \approx t(x_1, \ldots, x_m)$, and therefore

$$\begin{aligned} s/\theta &= s^{\mathbf{T}(X)}(x_1, \ldots, x_m)/\theta \\ &= s^{\mathbf{T}(X)/\theta}(x_1/\theta, \ldots, x_m/\theta) \\ &= t^{\mathbf{T}(X)/\theta}(x_1/\theta, \ldots, x_m/\theta) \\ &= s^{\mathbf{T}(X)}(x_1, \ldots, x_m)/\theta \\ &= t/\theta. \end{aligned}$$

Thus $(s,t) \in \theta$. Hence $(s,t) \in \Theta_K$. □

From this property, it is easy to see that $\mathbf{F}(X, K)$ is free for $K$ over $\{x/\Theta_K \mid x \in X\}$ in the sense of [MMT87, Definition 4.107].

## 5. Birkhoff's HSP-Theorem

**Theorem 5.1.** *Let $K$ be a class of similar algebras, and let $Y$ be a countably infinite set. Then $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(Y)}(K)) \subseteq HSP\,K$.*

*Proof.* Let $\mathbf{B}$ in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(Y)}(K))$, and let $G$ be a nonempty subset of $B$ that generates $\mathbf{B}$. Let $X$ be a set of the same cardinality as $G$, and let $\boldsymbol{b} = (b_x)_{x \in X}$ be such that $\{b_x \mid x \in X\} = G$. Let $\mathbf{F} := \mathbf{F}(X, K) = \mathbf{T}(X)/\Theta_K$ be the free algebra constructed from $X$ and $K$, and let

$$\varphi := \{(t/\Theta_K, t^{\mathbf{B}}(\boldsymbol{b})) \mid t \in T(X)\}.$$

We first show that $\varphi$ is a function from $F$ to $B$. To this end, we suppose that $(s, t) \in \Theta_K$. By the "only if"-direction of Lemma 4.1, we have $s \sim_K t$. Let $x_1, \ldots, x_m$ be the variables occurring in $s$ and $t$. By the definition of $\sim_K$, we have $K \models s(x_1, \ldots, x_m) \approx t(x_1, \ldots, x_m)$. Since $\mathbf{B} \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(Y)}(K))$, we then also have $\mathbf{B} \models s(x_1, \ldots, x_m) \approx t(x_1, \ldots, x_m)$, and therefore $s^{\mathbf{B}}(\boldsymbol{b}) = t^{\mathbf{B}}(\boldsymbol{b})$. Hence $\varphi$ is a function.

Since $\varphi$ is the image of the homomorphism $\psi : \mathbf{T}(X) \to \mathbf{T}(X)/\Theta_K \times \mathbf{B}$, $\psi(t) = (t/\Theta_K, t^{\mathbf{B}}(\boldsymbol{b}))$, $\varphi$ is a subuniverse of $\mathbf{T}(X)/\Theta_K \times \mathbf{B}$. As a function that is a subuniverse, $\varphi$ is a homomorphism, and therefore $\mathbf{B} \in H\{\mathbf{F}\}$. Thus $\mathbf{B}$ lies in $HSP\,K$. □

We call a class $K$ of similar algebras a *variety* if there is a set $Y$ and a set $\Phi$ of identities in the variables $Y$ such that $K = \mathrm{Mod}(\Phi)$.

**Corollary 5.2.** *Let $K$ be a class of similar algebras. Then the variety generated by $K$ is $HSP\,K$.*

*Proof.* Let $V(K)$ be the smallest variety containing $K$. Then there are a set $Y$ and a set $\Phi$ of identities in the variables $Y$ such that $V(K) = \mathrm{Mod}(\Phi)$. Let $X$ be a countably infinite set. By renaming the variables in each of the identities in $\Phi$, we obtain identities $\Phi'$ in $X$ such that $\mathrm{Mod}(\Phi) = \mathrm{Mod}(\Phi')$. Then we have $\Phi' \subseteq \mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Phi')) = \mathrm{Th}_{\mathrm{Id}(X)}(V(K)) \subseteq \mathrm{Th}_{\mathrm{Id}(X)}(K)$, and therefore every algebra in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(X)}(K))$ lies in $\mathrm{Mod}(\Phi') = V(K)$. Since $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(X)}(K))$ is a variety containing $K$, and $V(K)$ is the smallest variety containing $K$, $V(K)$ and $\mathrm{Mod}(\mathrm{Th}_{\mathrm{Id}(X)}(K))$ contain the same algebras. Thus by Theorem 5.1, every algebra in $V(K)$ lies in $HSP\,K$.

Since validity of an identity is preserved by forming products, subalgebras and homomorphic images, every algebra in $HSP\,K$ satisfies $\Phi$, and therefore every algebra in $HSP\,K$ lies in $\mathrm{Mod}(\Phi) = V(K)$. □

## 6. Quasi-identities

A *quasi-identity* over $X$ is a formula $(\bigwedge_{i\in\underline{r}} s_i \approx t_i) \to u \approx v$ with $r \in \mathbb{N}_0$ and $s_1, \ldots, s_r, t_1, \ldots, t_r, u, v \in T(X)$. $\mathrm{QId}(X)$ is the set of all quasi-identities over $X$. A *quasivariety* is a class of similar algebras that is axiomatized by a set of quasi-identities.

In [BS81], the following result is proved.

**Theorem 6.1** (cf. [BS81, Theorem V.2.25])**.** *Let $K$ be a class of similar algebras, let $X$ be a countably infinite set, and let $\mathbf{A}$ be an algebra in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{QId}(X)}(K))$. Then $\mathbf{A}$ lies in $ISP_{\mathrm{u}}P_{\mathrm{fin}}K$.*

*Proof.* We specialize the proof in [BS81]. Let $\mathbf{A}^*$ be an expansion of $\mathbf{A}$ where for each $a \in A$, we add a nullary operation symbol $\hat{a}$ interpreted by $\hat{a}^{\mathbf{A}} := a$. We let $\mathcal{F}$ be the language of $\mathbf{A}$, and $\mathcal{F}^*$ be the language of $\mathbf{A}^*$. Let $T^*(X)$ be the terms over $X$ in the language $\mathcal{F}^*$, and $T^*(\varnothing)$ be the set of terms using no variables. Each term $t'$ in $T^*(\varnothing)$ can be written as $t(a_1, \ldots, a_n)$, where $t \in T(\{x_1, \ldots, x_n\})$ is a term of language $\mathcal{F}$, $a_1, \ldots, a_n \in A$ and $t(a_1, \ldots, a_n)$ is understood as an abbreviation of $t^{\mathbf{T}^*(\{x_1,\ldots,x_n\})}(\hat{a_1}, \ldots, \hat{a_n})$.

We define the set $D$ of formulae in the language $\mathcal{F}^*$ by

$$D := \{s \approx t \mid s, t \in T^*(\varnothing), s^{\mathbf{A}^*} = t^{\mathbf{A}^*}\} \cup \{s \not\approx t \mid s, t \in T^*(\varnothing), s^{\mathbf{A}^*} \neq t^{\mathbf{A}^*}\},$$

and we let

$$F := \{\varphi \mid \varphi \text{ is a finite subset of } D\}.$$

For $\varphi \in F$, we define $\varphi\!\uparrow := \{\psi \in F \mid \varphi \subseteq \psi\}$. Let

$$\mathcal{A} := \{\varphi\!\uparrow \mid \varphi \in F\}.$$

This $\mathcal{A}$ is a filter on the set $F$ because $(\varphi_1\!\uparrow) \cap (\varphi_2\!\uparrow) = (\varphi_1 \cup \varphi_2)\!\uparrow$. Hence there exists an ultrafilter $\mathcal{U}$ on the set $F$ with $\mathcal{A} \subseteq \mathcal{U}$. Altogether, $\mathcal{U}$ is an ultrafilter on the set $F$ such that for every $\varphi \in F$, we have

$$\varphi\!\uparrow = \{\psi \in F \mid \varphi \subseteq \psi\} \in \mathcal{U}.$$

Now for every $\varphi \in F$, we construct an $\mathcal{F}^*$-algebra $\mathbf{B}^*_\varphi$ such that $\mathbf{B}^*_\varphi \models \varphi$ and the $\mathcal{F}$-reduct of $\mathbf{B}^*_\varphi$ lies in $P_{\mathrm{fin}}\,K$. Since $\varphi \in F$, there are $k, m, n \in \mathbb{N}_0$, a finite subset $\{x_1, \ldots, x_n\}$ of $X$, $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^n$, and for each $i \in \underline{k}$ and $j \in \underline{m}$ there are

$\mathcal{F}$-terms $s_i, t_i, u_j, v_j \in T(\{x_1, \ldots, x_n\})$ such that

$$\varphi = \{s_1(\boldsymbol{a}) \approx t_1(\boldsymbol{a}), \ldots, s_k(\boldsymbol{a}) \approx t_k(\boldsymbol{a})\} \cup \{u_1(\boldsymbol{a}) \not\approx v_1(\boldsymbol{a}), \ldots, u_m(\boldsymbol{a}) \not\approx v_m(\boldsymbol{a})\}$$

and for $i, j \in \underline{n}$ with $i \neq j$, we have $a_i \neq a_j$. Here, $s_1(\boldsymbol{a})$ is a shorthand for $s_1^{\mathbf{T}^*(\{x_1,\ldots,x_n\})}(\hat{a_1}, \ldots, \hat{a_n})$.

For each $i \in \underline{m}$, we will now construct an algebra $\mathbf{B}_i$ in $K$ with certain properties. We fix $i \in \underline{m}$. Then we have

$$\mathbf{A} \models \exists \boldsymbol{x} : s_1(\boldsymbol{x}) \approx t_1(\boldsymbol{x}) \wedge \ldots \wedge s_k(\boldsymbol{x}) \approx t_k(\boldsymbol{x}) \wedge u_i(\boldsymbol{x}) \not\approx v_i(\boldsymbol{x}).$$

We show that there is $\mathbf{B}_i$ in $K$ such that

$$\mathbf{B}_i \models \exists \boldsymbol{x} : s_1(\boldsymbol{x}) \approx t_1(\boldsymbol{x}) \wedge \ldots \wedge s_k(\boldsymbol{x}) \approx t_k(\boldsymbol{x}) \wedge u_i(\boldsymbol{x}) \not\approx v_i(\boldsymbol{x}). \tag{6.1}$$

Suppose that there is no such $\mathbf{B}_i$. Then

$$K \models \forall \boldsymbol{x} : (s_1(\boldsymbol{x}) \approx t_1(\boldsymbol{x}) \wedge \ldots \wedge s_k(\boldsymbol{x}) \approx t_k(\boldsymbol{x})) \to u_i(\boldsymbol{x}) \approx v_i(\boldsymbol{x}). \tag{6.2}$$

Since $\mathbf{A} \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{QId}(X)}(K))$, also $\mathbf{A}$ satisfies this quasi-identity. We know that we have $s_1^{\mathbf{A}}(\boldsymbol{a}) \approx t_1^{\mathbf{A}}(\boldsymbol{a})$, $\ldots$, $s_k^{\mathbf{A}}(\boldsymbol{a}) = t_k^{\mathbf{A}}(\boldsymbol{a})$ and $u_i^{\mathbf{A}}(\boldsymbol{a}) \neq v_i^{\mathbf{A}}(\boldsymbol{a})$. This contradicts the fact that $\mathbf{A}$ satisfies the quasi-identity in (6.2). Hence there is $\mathbf{B}_i$ in $K$ with (6.1). Let $\boldsymbol{b} := (b_1, \ldots, b_n) \in B_i^n$ be such that

$$s_1^{\mathbf{B}_i}(\boldsymbol{b}) = t_1^{\mathbf{B}_i}(\boldsymbol{b}), \ldots, s_k^{\mathbf{B}_i}(\boldsymbol{b}) = t_k^{\mathbf{B}_i}(\boldsymbol{b}) \text{ and } u_i^{\mathbf{B}_i}(\boldsymbol{b}) \neq v_i^{\mathbf{B}_i}(\boldsymbol{b}).$$

We will now form an $\mathcal{F}^*$-expansion $\mathbf{B}_i^*$ of $\mathbf{B}_i$. For each $j \in \underline{n}$, set $\hat{a_j}^{\mathbf{B}_i^*} := b_j$, and for $a \in A \setminus \{a_1, \ldots, a_n\}$, set $\hat{a}^{\mathbf{B}_i^*}$ to some element of $B_i$.

We set $\mathbf{B}_\varphi^* := \prod_{i \in \underline{m}} \mathbf{B}_i^*$ and note that in the case $m = 0$, $\mathbf{B}_\varphi^*$ is a one element algebra. Then $\mathbf{B}_\varphi^* \models \varphi$, and the $\mathcal{F}$-reduct of $\mathbf{B}_\varphi^*$ lies in $P_{\mathrm{fin}}\, K$.

Next, we show that $\mathbf{C} := \prod_{\varphi \in F} \mathbf{B}_\varphi^* / \mathcal{U}$ satisfies $D$. To this end, let $\delta \in D$. Now for all $\varphi$ with $\delta \in \varphi$, we have $\mathbf{B}_\varphi^* \models \delta$ because $\mathbf{B}_\varphi^* \models \varphi$. Hence $\{\varphi \in F \mid \mathbf{B}_\varphi^* \models \delta\} \supseteq \{\delta\}{\uparrow}$, and thus $\{\varphi \in F \mid \mathbf{B}_\varphi^* \models \delta\} \in U$. By Łoś's Theorem [BS81, Theorem V.2.9], we therefore have $\mathbf{C} \models \delta$. Thus $\mathbf{C} \models D$.

Now we define a mapping $h : A \to C$ by $h(a) := \hat{a}^{\mathbf{C}}$. We claim that this mapping is an embedding of $\mathbf{A}$ into the $\mathcal{F}$-reduct of $\mathbf{C}$. First, if $f$ is an $n$-ary operation symbol of $\mathbf{A}$, $a_1, \ldots, a_n \in A$ and $b = f^{\mathbf{A}}(a_1, \ldots, a_n)$, the identity

$$\hat{b} \approx f(\hat{a_1}, \ldots, \hat{a_n})$$

is an element of $D$. Therefore, since $\mathbf{C} \models D$, $h(b) = \hat{b}^{\mathbf{C}} = f^{\mathbf{C}}(\hat{a_1}^{\mathbf{C}}, \ldots, \hat{a_n}^{\mathbf{C}}) = f^{\mathbf{C}}(h(a_1), \ldots, h(a_n))$ and thus $h$ is a homomorphism.

Second, if $a_1, a_2$ are elements of $A$ such that $a_1 \neq a_2$, then $\hat{a_1} \not\approx \hat{a_2}$ is an element of $D$. Thus, since $\mathbf{C} \models D$, we have $h(a_1) = \hat{a_1}^{\mathbf{C}} \neq \hat{a_2}^{\mathbf{C}} = h(a_2)$, and therefore $h$ is injective.

Therefore, since $\mathbf{C} \in P_{\mathrm{u}} P_{\mathrm{fin}}\, K$, we have $\mathbf{A} \in ISP_{\mathrm{u}} P_{\mathrm{fin}}\, K$. □

Similarly to Corollary 5.2, we obtain:

**Corollary 6.2.** *Let $K$ be a class of similar algebras. Then the quasi-variety generated by $K$ is $ISP_{\mathrm{u}} P_{\mathrm{fin}}\, K$.*

For a class $K$ and $m \in \mathbb{N}$, we write $P_m$ for the class of direct products of exactly $m$ algebras from $K$.

**Theorem 6.3.** *Let $K$ be a class of similar algebras of finite type, let $X$ be a countably infinite set, and let $\mathbf{A}$ be a finite algebra in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{QId}(X)}(K))$ with $n$ elements. Then $\mathbf{A} \in ISP_{\binom{n}{2}} K$.*

*Proof.* Let $\mathbf{A}^*$, $\mathcal{F}$ and $\mathcal{F}^*$ be as in the proof of Theorem 6.1. Let $a_1, \ldots, a_n$ be the elements of $A$. Let $T$ (the operation tables) be defined by

$$T := \{(f, (i_0, \ldots, i_m)) \mid m \in \mathbb{N}_0, i_0, \ldots, i_m \in \underline{n},\ f \text{ is an } m\text{-ary function symbol in } \mathcal{F},\ a_{i_0} = f^{\mathbf{A}}(a_{i_1}, \ldots, a_{i_m})\},$$

and let

$$D^+ := \{\hat{a_{i_0}} \approx f(\hat{a_{i_1}}, \ldots, \hat{a}_{i_m}) \mid (f, (i_0, \ldots, i_m)) \in T\},$$

and

$$D := D^+ \cup \{\hat{a}_i \not\approx \hat{a}_j \mid i, j \in \underline{n}, i < j, a_i \neq a_j\}.$$

We fix $i, j \in \underline{n}$ with $i < j$. Let

$$\varphi(x_1, \ldots, x_n) := \bigwedge \{x_{i_0} = f^{\mathbf{A}}(x_{i_1}, \ldots, x_{i_m}) \mid (f, \boldsymbol{i}) \in T\}.$$

Then

$$\mathbf{A} \models \exists \boldsymbol{x} \,:\, \varphi(x_1, \ldots, x_n) \wedge x_i \not\approx x_j.$$

We want to show that there is $\mathbf{B}_{i,j}$ in $K$ such that

$$\mathbf{B}_{i,j} \models \exists \boldsymbol{x} \,:\, \varphi(x_1, \ldots, x_n) \wedge x_i \not\approx x_j.$$

Suppose that there is no such $\mathbf{B}_{i,j}$. Then

$$K \models \forall \boldsymbol{x} \, : \, \varphi(x_1, \ldots, x_n) \to x_i \approx x_j.$$

Since $\mathbf{A} \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{QId}(X)}(K))$, we have

$$\mathbf{A} \models \forall \boldsymbol{x} \, : \, \varphi(x_1, \ldots, x_n) \to x_i \approx x_j. \tag{6.3}$$

Setting $(x_1, \ldots, x_n) := (a_1, \ldots, a_n)$, we see that (6.3) does not hold. This contradiction shows that there is $\mathbf{B}_{i,j}$ in $K$ with

$$\mathbf{B}_{i,j} \models \exists \boldsymbol{x} \, : \, \varphi(x_1, \ldots, x_n) \wedge x_i \not\approx x_j.$$

Let $\boldsymbol{b}$ be an $n$-tuple witnessing the existence of these $x_k$'s, and for $k \in \underline{n}$, set $a_k^{\mathbf{B}^*_{i,j}} = b_k$. Now the mapping defined by $h(a_k) := b_k = \hat{a_k}^{\mathbf{B}^*_{i,j}}$ is a homomorphism from $\mathbf{A}^*$ to $\mathbf{B}^*_{i,j}$. To prove this, let $f$ be an $m$-ary operation symbol in $\mathcal{F}$, and let $i_1, \ldots, i_m \in \underline{n}$. We assume that $a_{i_0} = f^{\mathbf{A}}(a_{i_1}, \ldots, a_{i_m})$. Then

$$\mathbf{B}^*_{i,j} \models \hat{a_{i_0}} \approx f(\hat{a_{i_1}}, \ldots, \hat{a_{i_m}}),$$

and therefore

$$h(a_{i_0}) = \hat{a_{i_0}}^{\mathbf{B}^*_{i,j}} = f^{\mathbf{B}^*_{i,j}}(\hat{a_{i_1}}^{\mathbf{B}^*_{i,j}}, \ldots, \hat{a_{i_m}}^{\mathbf{B}^*_{i,j}}) = f^{\mathbf{B}^*_{i,j}}(h(a_{i_1}), \ldots, h(a_{i_m})),$$

which concludes the proof that $h$ is a homomorphism. In addition, for $i < j$, we have $h(a_i) \neq h(a_j)$.

Let $\mathbf{B}^* := \prod(\mathbf{B}^*_{i,j})_{i,j \in \underline{n},\, i<j}$. Then $\mathbf{B}^* \models D$, and $\mathbf{A}$ embeds into the $\mathcal{F}$-reduct $\mathbf{B}$ of $\mathbf{B}^*$. Hence $\mathbf{A}$ embeds into a direct product of exactly $\binom{n}{2}$ algebras in $K$. $\square$

**Corollary 6.4.** *Let $K$ be a finite set of similar finite algebras of finite type, let $X$ be a countably infinite set, and let $\mathbf{A}$ be a subdirectly irreducible algebra in* $\mathrm{Mod}(\mathrm{Th}_{\mathrm{QId}(X)}(K))$*. Then $\mathbf{A} \in I S\, K$.*

*Proof.* We know that there is a set $I$ and a family $(\mathbf{B}_i)_{i \in I}$ from $K$ such that $\mathbf{A}$ embeds into $\prod_{i \in I} \mathbf{B}_i$. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{A}$ embeds into some $\mathbf{B}_j$, which impies $\mathbf{A} \in I S\, K$. $\square$

## 7. Generalized Quasi-identities

A *generalized quasi-identity* over $Y$ is a formula $(\bigwedge_{i \in I} s_i \approx t_i) \to u \approx v$, where $I$ is a (possibly infinite) set and there exists a finite subset $X$ of $Y$ such that $u, v \in T(X)$, and for all $i \in I$, $s_i \in T(X)$ and $t_i \in T(X)$. $\mathrm{GQId}(Y)$ is the class of all generalized quasi-identities over $Y$. For a class $K$ of similar algebras, let $L\, K$

be the class of those algebras of the same signature that have the property that every finitely generated subalgebra embeds into some member of $K$.

**Theorem 7.1.** *Let $K$ be a class of similar algebras, and let $Y$ be a countably infinite set. If $\mathbf{A}$ lies in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{GQId}(Y)}(K))$, then $\mathbf{A}$ lies in $LSP\,K$. Hence $\mathrm{Mod}(\mathrm{Th}_{\mathrm{GQId}(Y)}(K)) = LSP\,K$.*

*Proof.* Let $\mathbf{A}'$ be a finitely generated subalgebra of $\mathbf{A}$, and let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$ be such that $\{a_1, \ldots, a_n\}$ generates $\mathbf{A}'$.

$$D^+ = \{(s,t) \mid s,t \in T(\{x_1, \ldots, x_n\}), s^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n)\}$$

and

$$D^- = \{(u,v) \mid u,v \in T(\{x_1, \ldots, x_n\}), s^{\mathbf{A}}(a_1, \ldots, a_n) \neq v^{\mathbf{A}}(a_1, \ldots, a_n)\}.$$

For every $(u,v) \in D^-$, we find an algebra $\mathbf{B}_{u,v}$ in $K$ such that $\mathbf{B}_{u,v}$ satisfies

$$\exists x_1, \ldots, x_n : \quad (\bigwedge_{(s,t)\in D^+} s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)) \wedge u(x_1, \ldots, x_n) \not\approx v(x_1, \ldots, x_n).$$

Then $\mathbf{C} := \prod_{(u,v)\in D^-} \mathbf{B}_{u,v}$ satisfies

$$\exists x_1, \ldots, x_n : \quad (\bigwedge_{(s,t)\in D^+} s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)) \wedge (\bigwedge_{(u,v)\in D^-} u(x_1, \ldots, x_n) \not\approx v(x_1, \ldots, x_n)).$$

If we choose $(c_1, \ldots, c_n)$ as witnesses for these $x_1, \ldots, x_n$, then $h : A' \to C$, $h(t^{\mathbf{A}}(a_1, \ldots, a_n)) = t^{\mathbf{C}}(c_1, \ldots, c_n)$ is well-defined because for $\boldsymbol{c} = (c_1, \ldots, c_n)$, we have

$$\bigwedge_{(s,t)\in D^+} s^{\mathbf{C}}(c_1, \ldots, c_n) = t^{\mathbf{C}}(c_1, \ldots, c_n)),$$

and injective because of

$$(\bigwedge_{(u,v)\in D^-} u^{\mathbf{C}}(c_1, \ldots, c_n) \neq v^{\mathbf{C}}(c_1, \ldots, c_n)).$$

Therefore $\mathbf{A}' \in ISP\,K$. Thus $\mathbf{A} \in LSP\,K$.

Now similar to Corollary 5.2, we can argue that $\mathrm{Mod}(\mathrm{Th}_{\mathrm{GQId}(Y)}(K))$ is equal to $LSP\,K$. □

A *fully generalized quasi-identity* over $X$ is a formula $(\bigwedge_{i\in I} s_i \approx t_i) \to u \approx v$, where $I$ is a (possibly infinite) set, $u, v \in T(X)$, and for all $i \in I$, $s_i \in T(X)$ and $t_i \in T(X)$. $\mathrm{FGQId}(X)$ is the class of all quasi-identities over $X$. Then we have

**Theorem 7.2.** *Let $K$ be a class of similar algebras, and let $X$ be a set such that $\mathbf{A}$ can be generated by $|X|$ elements. If $\mathbf{A}$ lies in $\mathrm{Mod}(\mathrm{Th}_{\mathrm{FGQId}(X)}(K))$, then $\mathbf{A}$ lies in $ISP\,K$. Thus, $\mathrm{Mod}(\mathrm{Th}_{\mathrm{FGQId}(X)}(K)) = ISP\,K$.*

*Proof.* Let $\boldsymbol{a} = (a_x)_{x\in X}$ be such that $\{a_x \mid x \in X\}$ generates $\mathbf{A}$.

$$D^+ = \{(s,t) \in T(X) \times T(X) \mid s^{\mathbf{A}}(\boldsymbol{a}) = t^{\mathbf{A}}(\boldsymbol{a})\}$$

and

$$D^- = \{(u,v) \in T(X) \times T(X) \mid u^{\mathbf{A}}(\boldsymbol{a}) \neq v^{\mathbf{A}}(\boldsymbol{a})\}.$$

For every $(u,v) \in D^-$, $(\bigwedge_{(s,t)\in T^+} s \approx t) \to u \approx v$ is a fully generalized quasi-identity that does not hold in $\mathbf{A}$. Thus we find an algebra $\mathbf{B}_{u,v}$ in $K$ and $\boldsymbol{b}_{u,v} \in B_{u,v}^X$ with

$$(\bigwedge_{(s,t)\in D^+} s(\boldsymbol{b}_{u,v}) \approx t(\boldsymbol{b}_{u,v}))\ \wedge\ u(\boldsymbol{b}_{u,v}) \not\approx v(\boldsymbol{b}_{u,v}).$$

Then $\mathbf{B} := \prod_{(u,v)\in D^-} \mathbf{B}_{u,v}$ and $\boldsymbol{b}$ with $\boldsymbol{b}_x := ((\boldsymbol{b}_{u,v})_x)_{(u,v)\in D^-}$ satisfies

$$(\bigwedge_{(s,t)\in D^+} s(\boldsymbol{b}) \approx t(\boldsymbol{b}))\ \wedge\ (\bigwedge_{(u,v)\in D^-} u(\boldsymbol{b}) \not\approx v(\boldsymbol{b})).$$

Now the mapping $h(t^{\mathbf{A}}(\boldsymbol{a})) = t^{\mathbf{B}}(\boldsymbol{b})$ is well-defined because of

$$\bigwedge_{(s,t)\in D^+} s^{\mathbf{B}}(\boldsymbol{b}) = t^{\mathbf{B}}(\boldsymbol{b}),$$

and injective because of

$$\bigwedge_{(u,v)\in D^-} u^{\mathbf{B}}(\boldsymbol{b}) \neq v^{\mathbf{B}}(\boldsymbol{b}).$$

Therefore $\mathbf{A} \in ISP\,K$.

Now similar to Corollary 5.2, we can argue that $\mathrm{Mod}(\mathrm{Th}_{\mathrm{FGQId}(Y)}(K))$ is equal to $ISP\,K$. □

## 8. Completeness of the equational calculus

In this section, we see an identity $s \approx t$ as a pair $(s,t)$.

**Theorem 8.1.** *Let $X$ be a set, and let $\Sigma \subseteq T(X) \times T(X)$. Then $\mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma))$ is the fully invariant congruence $\Theta_{\mathrm{FI}}(\Sigma)$ generated by $\Sigma$.*

*Proof.* For $\Theta_{\mathrm{FI}}(\Sigma) \subseteq \mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma))$, we observe that $\mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma))$ is a fully invariant congruence of $\mathbf{T}(X)$ that contains $\Sigma$ as a subset.

For $\Theta_{\mathrm{FI}}(\Sigma) \supseteq \mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma))$, let $\theta := \Theta_{\mathrm{FI}}(\Sigma)$. We first establish

$$\mathbf{T}(X)/\theta \models \Sigma. \tag{8.1}$$

To this end, let $(s(x_1,\ldots,x_n), t(x_1,\ldots,x_n)) \in \Sigma$, and let $t_1/\theta,\ldots,t_n/\theta \in T(X)/\theta$. Since $(s,t) \in \Sigma$, invariance under the endomorphism obtained from extending $\{(x_i,t_i) \mid i \in \underline{n}\}$ yields $(s^{\mathbf{T}(X)}(t_1,\ldots,t_n), t^{\mathbf{T}(X)}(t_1,\ldots,t_n)) \in \theta$. Thus $s^{\mathbf{T}(X)/\theta}(t_1/\theta,\ldots,t_n/\theta) = s^{\mathbf{T}(X)}(t_1,\ldots,t_n)/\theta = t^{\mathbf{T}(X)}(t_1,\ldots,t_n)/\theta = s^{\mathbf{T}(X)/\theta}(t_1/\theta,\ldots,t_n/\theta)$, completing the proof of (8.1).

Now let $(s,t) \in \mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma))$. Then by (8.1), $\mathbf{T}(X)/\theta \models s(x_1,\ldots,x_n) \approx t(x_1,\ldots,x_n)$, and thus $s/\theta = s^{\mathbf{T}(X)}(x_1,\ldots,x_n)/\theta = s^{\mathbf{T}(X)/\theta}(x_1/\theta,\ldots,x_n/\theta) = t^{\mathbf{T}(X)/\theta}(x_1/\theta,\ldots,x_n/\theta) = t^{\mathbf{T}(X)}(x_1,\ldots,x_n)/\theta = t/\theta$, which completes the proof of $\mathrm{Th}_{\mathrm{Id}(X)}(\mathrm{Mod}(\Sigma)) \subseteq \theta$. □

Now the calculus can be obtained by seeing fully invariant congruences as subalgebras of the expansion of $\mathbf{A} \times \mathbf{A}$ constructed in [BS81, Lemma II.14.4], and applying the subalgebra generation process of [BS81, Theorem II.3.2].

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