ON NEAR–RINGS AND NEAR–RING MODULES

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INTRODUCTION

In recent years there has arisen an interest in algebraic systems with binary compositions of addition and multiplication satisfying all the ring axioms, save possibly, one of the distributive laws and commutivity of addition. Such systems are generally called near-rings. This concept arises very naturally if we define addition and multiplication on the set R of identity preserving mappings of an additive (not necessarily abelian) group G into itself. The product $x \cdot y$ of two mappings is given by the rule $g(x \cdot y) = (gx)y$, for all $g \varepsilon G$, and the sum $x + y$ by $g(x + y) = gx + gy$, for all $g \varepsilon G$. The system $(R,+,\cdot)$ is a near-ring, and more often than not is called the near-ring associated with the group G.

The study of endomorphisms of a group has been limited to the class of abelian groups. In a certain sense Fitting [11][1] was the first mathematician to investigate near-rings in his study of the normal endomorphisms of a non-abelian group.

Returning to the near-ring R, one can consider the sub-near-ring R' generated by an arbitrary subset T of R. Of particular importance is the sub-near-ring

[1] Numbers in the brackets refer to references in the bibliography.

generated by a set of endomorphisms of the group G. Such near-rings, which became known as distributively generated near-rings, were studied in some detail by Fröhlich [12, 13, 14, 15], Deskins [9], Laxton [22, 23, 24], and others.

We can also consider a second algebraic system consisting of the group G, the near-ring R, and the mapping $\eta: GxR \longrightarrow G$ given by $(g,r)\eta = gr$ where $g \varepsilon G$, $r \varepsilon R$. It is easily seen that $g(r_1+r_2) = gr_1 + gr_2$ and $(gr_1)r_2 = g(r_1r_2)$ where $g \varepsilon G$ and r_1, $r_2 \varepsilon R$. This is one of the motivating examples of the algebraic system known as a near-ring module. Of course, if η is restricted to GxR' where R' is the near-ring generated by a set T of endomorphisms of the group G, then we have another example of a near-ring module.

The abstract theory of near-ring modules has been discussed to some extent by Betsch [1, 2], Fröhlich [12, 13, 14, 15], Laxton [22, 23, 24], Roth [25], and others. It should be mentioned that Fröhlich and Laxton were concerned with the rather special situation of modules over distributively generated near-rings.

It is our purpose here to consider certain aspects of the abstract theory of near-ring modules. Firstly, we will develop the fundamental results and notions that will be needed for futher investigations. We mention, for example, the concepts of submodule and factor module, the analogous isomorphism theorems, the Jordan-Hölder theorem, and the chain conditions. Secondly, we will introduce the concept of a radical for near-ring modules which, in particular, generalizes many of the well known theorems of the so-called Jacobson radical for a ring. Thirdly, we will study the properties of the radical to be defined and apply the resulting theory to the particular case of a near-ring. Finally, after investigating a special class of near-ring modules which we will term strictly semi-simple, we will study the theory of near-vector spaces. Throughout this dissertation we will give numerous examples of the notions which are encountered.

CHAPTER I

ON THE FUNDAMENTAL PROPERTIES OF NEAR-RINGS AND NEAR-RING MODULES

The theory of near-rings and near-ring modules has been discussed in some detail. Betsch [1] and Blackett [5] studied some of the basic properties of near-rings. Fröhlich [12, 13, 14], Laxton [22, 23, 24] and Deskins [9] considered a special class of near-rings known as distributively generated near-rings. Fröhlich and Laxton investigated modules over this class of near-rings. In [25] Roth showed that many of the results from group theory can be carried over to the case of near-ring modules.

It is the purpose of this chapter to provide the basic results that will be of value for our later pursuits. After introducing the concepts of near-ring and near-ring module, we give some of the elementary properties of these algebraic systems. We show that many of the important concepts and results from ring theory and group theory can be adjusted accordingly to the case of near-rings and near-ring modules. For example, we will establish the three basic isomorphism theorems for near-ring modules. Also, the Schreier theorem and the Jordan-Hölder theorem for near-ring modules are given. Throughout this chapter we will give examples of the concepts introduced.

Definitions and Notation

Definition (1.1). A near-ring is an algebraic system consisting of a set R with two binary compositions called addition, denoted by +, and multiplication, denoted by $\cdot$, such that

(1.1.1) R is a group under addition.

(1.1.2) R is a semi-group under multiplication.

(1.1.3) $a(b+c) = ab + ac$ for all a, b, and c contained in R.

(1.1.4) $0 \cdot a = 0$ where 0 is the additive identity of R and a is an element of R.

Further, if R contains an element 1 such that $r \cdot 1 = 1 \cdot r = r$ for all $r \in R$, then the element 1 is called an identity for R. Throughout this thesis we will assume that if a near-ring has an identity, then $1 \neq 0$.

It follows from this definition that any ring is a near-ring. However, we do not assume the additive group of a near-ring is abelian. The most natural example of a near-ring is the set of all mappings of an additive group (not necessarily abelian) G into G that map the additive identity of G onto itself. Let this set of mappings be denoted S(G). If addition is defined pointwise i.e. $g(f+h) = gf + gh$ where $g \in G$, $f, h \in S(G)$ and if multiplication is defined by $g(f \cdot h) = (gf)h$ where $g \in G$, $f, h \in S(G)$, then the system $(S(G),+,\cdot)$ becomes a near-ring. The near-ring $S(G)$ is called the near-ring associated with the group G.

We now give some easy consequences of the definition above.

<u>Proposition (1.2)</u>. Let R be a near-ring. Then $r\cdot 0 = 0$ and $r(-s) = -rs$ where $r, s \in R$.

<u>Proof</u>: Let r and s be elements of R. Then $r\cdot 0 = r(0+0) = r\cdot 0 + r\cdot 0$ so that $r\cdot 0 = 0$. We also see that $0 = r(s+(-s)) = rs + r(-s)$ and so $r(-s) = -rs$. This establishes the proposition.

Let R be a near-ring. A subset R' of R is called a sub-near-ring of R if it is a subgroup of the additive group of R and if it is closed under multiplication. As in the case of rings, the intersection of an arbitrary number of sub-near-rings of R is a sub-near-ring. If A is a subset of R, then the intersection of all sub-near-rings that contain A is called the sub-near-ring generated by A.

The endomorphisms of an additive group G form a subset of the near-ring $S(G)$ given above. As might be expected, we are interested in sub-near-rings generated by subsets of the set of endomorphisms. Before pursuing these sub-near-rings of $S(G)$ in further detail we give

<u>Definition (1.3)</u>. An element r of an arbitrary near-ring R is called right distributive if, and only if, $(r_1 + r_2)r = r_1r + r_2r$ for all elements r_1, r_2 of R.

We now show an element s of S(G) is right distributive if, and only if, s is an endomorphism of G. Suppose s is an endomorphism and let $s_1, s_2 \varepsilon S(G)$. If $g \varepsilon G$, then

$$g(s_1 + s_2)\cdot s = (gs_1 + gs_2)s = (gs_1)s + (gs_2)s = g(s_1s + s_2s)$$

so that s is right distributive. Conversely, assume s is right distributive and let g_1, g_2 be non-zero elements of G. Then there exists elements $s_1, s_2 \varepsilon S(G)$ such that $g_1s_1 = g_1$ and $g_1s_2 = g_2$. Hence,

$$(g_1 + g_2)s = g_1(s_1 + s_2)s = (g_1s_1)s + (g_1s_2)s = g_1s + g_2s$$

so that s is an endomorphism of G. From this result, it is clear that S(G) is not a ring.

Let E be a multiplicative semi-group of endomorphisms of G and let E(G) denote the subgroup of the additive group of S(G) that is generated by E. Then E(G) is the near-ring generated by E. For if $x, y \varepsilon E(G)$, then we have representations

$x = m_1r_1 + \cdots + m_\ell r_\ell$, $y = n_1s_1 + \cdots + n_ks_k$ where $r_i, s_i \varepsilon E$ and m_i, n_i are integers taking on values 0, 1 or -1. Since the elements of E are right distributive

$$x\cdot y = (m_1r_1 + \cdots + m_\ell r_\ell)(n_1s_1 + \cdots + n_ks_k)$$
$$= n_1(m_1r_1s_1 + \cdots + m_\ell r_\ell s_1) + \cdots + n_k(m_1r_1s_k + \cdots + m_\ell r_\ell s_k)$$

so that $x\cdot y \varepsilon E(G)$. This shows that E(G) is a sub-near-ring of S(G) and it is easy to see that it is the sub-near-ring generated by E.

The preceeding example helps motivate the following

Definition (1.4). [12] A near-ring R is said to be distributively generated, denoted d.g. near-ring, if, and only if, R contains a multiplicative group E of right distributive elements that generates the additive group of R.

Fröhlich [12, 13, 14] was the first mathematician to study d.g. near-rings in detail. Deskins [9] and Laxton [22, 23, 24] also have investigated d.g. near-rings.

We now turn our attention to a more general algebraic system which we formulate in the next

Definition (1.5). By a near-ring module M over a near-ring R is meant a system consisting of an additive group M, a near-ring R, and a mapping $\eta:(m,r)\in M\times R \longrightarrow m\cdot r = (m,r)\eta \in M$ such that

(1.5.1) $m(r_1 + r_2) = mr_1 + mr_2, \quad r_1, r_2 \in R, \; m \in M$

(1.5.2) $m(r_1 \cdot r_2) = (mr_1)r_2, \quad r_1, r_2 \in R, \quad m \in M.$

If M is a near-ring module over R, then we denote this by M_R and M is said to be an R-module. Moreover, if R contains an identity 1 and $x\cdot 1 = x$ for all $x \in M$, then M_R is called unitary.

Unless otherwise stated, we will assume that the mapping η is not the zero mapping.

Fröhlich [13] considered near-ring modules M_R where R is a d.g. near-ring, and he called them R-groups. If E is the multiplicative semi-group that generates the additive group of R, then Fröhlich required the additional axiom $(m_1 + m_2)r = m_1r + m_2r$ where $m_1, m_2 \in M$, $r \in E$. Laxton [22, 23] also investigated near-ring modules over a d.g. near-ring. Whenever we speak of a near-ring module over a d.g. near-ring, we will mean it in the sense of Fröhlich [13]. In [1] Betsch used the concept of R-group for an arbitrary near-ring module M_R, whereas Blackett [5] called it a right R-space.

Any near-ring R can be considered as a near-ring module over itself. In particular, the near-rings S(G) and E(G) determined by an additive group G are near-ring modules over themselves. Moreover, G can be regarded as an S(G)-module as well as an E(G)-module.

It is interesting to note that every additive group G is the additive group of some near-ring. Let $g \in G$ and define $\varphi_g \in S(G)$ as follows

$h\varphi_g = \begin{cases} g \text{ if } h \neq 0 \\ 0 \text{ if } h = 0 \end{cases}$. Let $K = \{\varphi_g | g \in G\}$ and we observe that $\varphi_{g_1} \cdot \varphi_{g_2} = \varphi_{g_2}$ and $\varphi_{g_1} + \varphi_{g_2} = \varphi_{g_1+g_2}$ for all $g_1, g_2 \in G$. Moreover, it is clear that the mapping $T: g \in G \longrightarrow \varphi_g \in K$ is a group isomorphism of G onto the additive group of the sub-near-ring K of S(G).

The near-ring K is called the near-ring of constant mappings of the group G. We also notice that G is a K-module and G_K is not a unitary near-ring module.

Since the proof of the following Proposition is similar to that of Proposition (1.2) we will omit it.

Proposition (1.6). Let M_R be an R-module with additive identity 0_M. Then

(1.6.1) $0_M \cdot 0 = 0_M$.

(1.6.2) $m \cdot 0 = 0_M$ for all $m \in M$.

(1.6.3) $0_M \cdot r = 0_M$ for all $r \in R$.

(1.6.4) $m(-r) = -mr$ for all $r \in R$, $m \in M$.

If there is no confusion, from now on we will denote the additive identity of M_R by 0.

We next give some notation that will be used throughout this dissertation.

Definition (1.7). Let T be a mapping of the additive group G into the additive group G'. If A is a non-empty subset of G and B a non-empty subset of G', then

(1.7.1) $AT = \{aT \mid a \in A\}$ is called the image of A in G'

(1.7.2) $Ke(T) = \{g \in G \mid gT = 0 \in G'\}$ is called the kernel of T.

(1.7.3) $T^{-1}(B) = \{g \in G \mid gT \in B\}$ is called the pre-image of B in G.

We now generalize the concept of ring homomorphism.

Definition (1.8). A mapping T of a near-ring R into a near-ring R' is called a near-ring homomorphism if, and only if, $(x+y)T = xT + yT$ and $(x \cdot y)T = (xT) \cdot (yT)$ where $x, y \in R$.

Thus a near-ring homomorphism is a homomorphism of the additive group of R into the additive group of R' that preserves multiplication. If T is a one-to-one mapping, then T is called a near-ring isomorphism. Also, the near-rings R and R' are said to be isomorphic, denoted by $R \cong R'$, if there is an isomorphism of R onto R'.

As in the case of rings, we have

Proposition (1.9). If T is a near-ring homomorphism of R into R', then RT is a sub-near-ring of R'.

Proof: Since T is a group homomorphism of the additive group of R into the additive group of R', it follows that RT is a subgroup of the additive group of R'. If $x, y \in R$, then $xT \cdot yT = (x \cdot y)T \in RT$ and so RT is a sub-near-ring of R'.

As in the case of ring modules, we introduce the notion of near-ring module homomorphism.

Definition (1.10). A mapping T of an R-module M_R into an R-module M_R' is called an R-homomorphism if $(x+y)T = xT + yT$ and $(xT)r = (xr)T$ where $x, y \in T$, $r \in R$.

If T is one-to-one, then T is called an R-isomorphism. Moreover, the R-modules M_R and M_R' are said to be R-isomorphic, denoted by $M \underset{(R)}{\cong} M'$, if there is an R-isomorphism of M onto M'.

Let T be an R-homomorphism of the near-ring module M_R into the near-ring module M'_R. We are interested in MT. Before we determine the structure of MT, we give

<u>Definition (1.11)</u>. A subset A of a near-ring module M_R is called an R-subgroup if, and only if,

(1.11.1) A is a subgroup of the additive group M.

(1.11.2) $A \cdot R = \{a \cdot r \mid a \in A,\ r \in R\} \subseteq A$.

Now consider MT. Since T is a group homomorphism, MT is a subgroup of the additive group M'. If $m \in M$, $r \in R$, then $(mT)r = (mr)T \in MT$. Hence, we have proved

<u>Proposition (1.12)</u>. MT is an R-subgroup of the near-ring module M'_R.

The concept of an R-subgroup A of M_R has appeared in the literature under several different names. For example, Fröhlich [13] uses the notion of right R-module whenever R is a d.g. near-ring. If $M = R$, then Blackett [5] also referred to A as a right R-module. Roth [25] called A a submodule of M_R. We remark that the concept of submodule of M_R is reserved for the next section (see defn (1.14)). Roth called a submodule, in our sense of the word, an ideal of M_R.

We will say that a submodule (resp. R-subgroup) A of M_R is said to be proper if $A \neq M_R$ and $A \neq \{0\}$.

We conclude this section with

<u>Proposition (1.13)</u>. Let M_R be an R-module and T a group isomorphism of the additive group M onto the additive group A. Then

(1.13.1) A can be regarded as an R-module

(1.13.2) T is an R-isomorphism

(1.13.3) T^{-1} is an R-isomorphism.

<u>Proof</u>: If a is an element of A and r is an element of R, then we define $a \cdot r = (mr)T$ where $mT = a$. Since T is a group isomorphism, it is easy to verify that the mapping $\eta: A \times R \longrightarrow A$ given by the above is single-valued and A is an R-module. If $m \in M$, $r \in R$, then $(mT)r = a \cdot r = (mr)T$ where $mT = a$. Similarly, T^{-1} is an R-isomorphism, and the proof is complete.

Fundamental Theorem of R-Homomorphisms

In the present section we develop the analogue concepts of submodule and factor module of a ring module. Since we do not assume that the additive group of a near-ring module is abelian, these concepts will be given in terms of normal subgroups. A necessary and sufficient condition for a subset A of a near-ring module M_R to be a submodule is presented.

Finally, we state and prove the fundamental theorem.

We start with

Definition (1.14). A subset A of an R-module M_R is called a submodule if, and only if,

(1.14.1) A is a normal subgroup of the additive group M.

(1.14.2) $(m+a)r - mr \in A$ where $m \in M_R$, $a \in A$, and $r \in R$.

Let A be a submodule of the R-module M_R. As stated in the last section, Roth [25] referred to A as an ideal. Fröhlich [13] called A a normal right R-module. Throughout the literature A is called a right ideal in the special case when M = R.

Once again, assume A is a submodule of the R-module M_R. Let a be an element of A and r an element of R. Then since A is a submodule $a \cdot r = (0+a) \cdot r - 0 \cdot r \in A$. Hence, we have proved

Proposition (1.15). A is an R-subgroup of M_R.

We now present an example which shows that the converse of Proposition (1.15) is not true in general.

Let G be an additive group of order greater than two. Then the near-ring K of constant mappings of G is an S(G)-subgroup of S(G) but K is not a submodule of the near-ring module S(G). Let $\varphi_g \in K$, $r \in S(G)$. Then for every $h \in G$ $h(\varphi_g \cdot r) = gr$ if $h \neq 0$ and $h(\varphi_g \cdot r) = 0$ if $h = 0$. This shows $\varphi_g \cdot r = \varphi_{gr} \in K$ and so K is an S(G)-subgroup. Assume K is a submodule and let g be an element of order greater than two. We define $r \in S(G)$ as follows: $gr = g$ and $hr = 0$ for all $h \in G$ with $h \neq g$. Since K is a submodule, there exists an element $g' \in G$ such that $(r + \varphi_g)\varphi_g - r\varphi_g = \varphi_{g'}$. We observe that $g\varphi_{g'} = 0$ and so $g' = 0$. If $h \neq g$ is a non-zero element of G, then $0 = h[(r+\varphi_g)\varphi_g - r\varphi_g] =$ $= (hr + h\varphi_g)\varphi_g - (hr)\varphi_g = g$. This shows K is not a submodule of S(G).

Lemma (1.16). [25] Let A be a submodule of the near-ring module M_R and η the natural group homomorphism of M onto the factor group $M' = M/A$. Then $M\eta = M'$ can be regarded as an R-module and η an R-homomorphism.

Proof: Let $m \in M$, and define $(m + A)r = mr + A$. Assume $m_1 + A = m_2 + A$ where $m_1, m_2 \in M$. Then there is an element $a \in A$ such that $m_1 = m_2 + a$. Since A is a submodule, $m_1 r - m_2 r = (m_2 + a)r - m_2 r \in A$ for all $r \in R$. This shows that the above definition is well defined, and it is now evident that the factor group M' can be regarded as an R-module and η an R-homomorphism of M_R onto M'_R.

The near-ring module M' is often called a factor module and η is said to be the natural R-homomorphism of M_R onto M'_R.

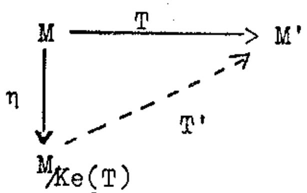
Theorem (1.17). [25] A subset A of an R-module M_R is a submodule if, and only if, A is the kernel of an R-homomorphism.

Proof: Assume A is a submodule. Then $A = Ke(\eta)$ where η is the natural R-homomorphism of M_R onto the factor-module M/A.

Conversely, let $A = Ke(T)$ where T is an R-homomorphism of M_R into the near-ring module M'_R. Since T is a group isomorphism, Ke(T) is a normal subgroup. Let $m \varepsilon M$, $a \varepsilon A$ and $r \varepsilon R$. Then $[(m+a)r - mr]T = (m+a)T \cdot r - (mT)r = (mT)r - (mT)r = 0 \varepsilon M'_R$. Hence, $A = Ke(T)$ is a submodule of M_R.

Finally, we state the fundamental theorem of R-homomorphisms.

Theorem (1.18). [25] Let M_R and M'_R be R-modules and T an R-homomorphism of M_R onto M'_R with kernel Ke(T). If η is the natural R-homomorphism of M_R onto the factor-module $^M/Ke(T)$, then T induces an R-isomorphism T' of $^M/Ke(T)$ onto M'_R such that the diagram is commutative.



Proof: It is well known from group theory that T induces a group isomorphism T' of the additive group $^M/Ke(T)$ onto the additive group M' such that $T = \eta T'$.

If $m \in M$, $r \in R$, then $[m + Ke(T)]T' \cdot r = (mT)r = (mr)T =$
$= [mr + Ke(T)]T' = [m + Ke(T)]r \cdot T'$ and so T' is an R-isomorphism.

First Isomorphism Theorem for Near-Ring Modules

In this section we will prove the first isomorphism theorem for near-ring modules. As in the case of operator groups, this theorem is of fundamental importance in the theory of near-rings and near-ring modules. We state this theorem as follows

__Theorem (1.19)__. Let T be an R-homomorphism of the near-ring module M_R onto the near-ring module M'_R with kernel Ke(T). Then

(1.19.1) If H is a submodule of M_R that contains Ke(T), then $M/H \underset{(R)}{\cong} M'/HT$.

(1.19.2) The mapping $f: H \longrightarrow HT$ is a one-to-one onto mapping between the R-subgroups H of M_R that contain Ke(T) and the R-subgroups of M'_R.

Before proving Theorem (1.19) we first give two lemmas that will be useful for later considerations and will enable us to simplify the proof of the theorem.

__Lemma (1.20)__. Let M_R and M'_R be two R-modules and T an R-homomorphism of M_R onto M'_R with kernel Ke(T). Then

(1.20.1) If H is an R-subgroup of M_R, then HT is an R-subgroup of M'_R.

(1.20.2) If H' is an R-subgroup of M', then $T^{-1}(H')$ is an R-subgroup of M_R that contains Ke(T).

(1.20.3) If H is an R-subgroup of M_R that contains Ke(T), then $T^{-1}(HT) = H$.

<u>Proof</u>: Let H be an R-subgroup of M_R. Since T induces an R-homomorphism of H_R into M_R, HT is an R-subgroup of M'_R by Proposition (1.12).

Let H' be an R-subgroup of M'_R. Since T is a group homomorphism, $T^{-1}(H')$ is a subgroup of the additive group M that contains Ke(T). If $h \in T^{-1}(H')$, $r \in R$, then there is an element $h' \in H'$ such that $h' = hT$ and so $h'r = (hT)r = (hr)T \in H'$. This shows $T^{-1}(H')$ is an R-subgroup of M_R.

Let H be an R-subgroup of M_R that contains Ke(T). Then by (1.20.1) and (1.20.2) it suffices to show $T^{-1}(HT) \subseteq H$. If $m \in T^{-1}(HT)$, then $mT = hT$ for some for some $h \in H$. Hence, $m - h \in Ke(T)$ and so $m - h + h = m \in H$.

Therefore, the proof of the lemma is complete.

<u>Lemma (1.21)</u>. Let M_R and M'_R be R-modules and T an R-homomorphism of M_R onto M'_R with kernel Ke(T). Then

(1.21.1) If A is a submodule of M_R, then AT is a submodule of M'_R.

(1.21.2) If A' is a submodule of M'_R, then $T^{-1}(A')$ is a submodule of M_R.

(1.21.3) If A is a submodule of M_R that contains Ke(T), then $T^{-1}(AT) = A$.

Proof: Let A be a submodule of M_R. Since T is an onto mapping, it follows that AT is a normal subgroup of the additive group M'. If $m' \varepsilon M'$, $a' \varepsilon AT$ and $r \varepsilon R$, then there exists elements $m \varepsilon M$, $a \varepsilon A$ such that $mT = m'$, $a' = aT$ and so $(m' + a')r - m'r = (mT + aT)r - (mT)r = (m + a)r \cdot T - (mr)T = [(m + a)r - mr]T$. Since A is a submodule of M_R $(m + a)r - mr \varepsilon A$ and so $(m' + a')r - m'r \varepsilon AT$. Therefore, (1.21.1) follows.

Let A' be a submodule of M'_R. Since T is a group homomorphism of the additive group M onto the additive group M', it follows that $T^{-1}(A')$ is a normal subgroup of M and $Ke(T) \subseteq T^{-1}(A')$. Let $m \varepsilon M$, $a \varepsilon T^{-1}(A')$ and $r \varepsilon R$. Then since A' is a submodule $[(m + a)r - mr]T = (mT + aT)r - (mT)r \varepsilon A'$ and so $(m + a)r - mr \varepsilon T^{-1}(A')$. From this (1.21.2) is proved.

Since every submodule of M_R is an R-subgroup (1.21.3) follows from Lemma (1.20).

At this point, we are ready to prove Theorem (1.19).

Proof: Let H be a submodule of M_R that contains Ke(T). From Lemma (1.21) HT is a submodule of M' and since T is a group homomorphism of M onto M', it is well known that T induces a group isomorphism T' of the factor group $M/_H$ onto the factor group $M'/_{HT}$ where $(m+H)T' = mT + HT$. It is sufficient to show T' is an R-isomorphism. If $m \varepsilon M$, $r \varepsilon R$, then $(m+H)T' \cdot r = (mT+HT)r = (mT)r + HT = (mr)T + HT = (mr+H)T' = [(m+H)\cdot r]T'$ so that the first part of the theorem is proved.

The second part of the theorem is an immediate consequence of Lemma (1.20).

The next result follows easily from the preceeding theorem.

Corollary (1.22). Let H be a submodule of the near-ring module M_R. If L is a submodule of M_R that contains H, then

(1.22.1) $L/_H$ is a submodule of the factor module $M/_H$.

(1.22.2) $M/_L \underset{(R)}{\cong} {}^{M/H}\big/_{L/H}$.

On Subgroups Generated by Subsets of a Near-Ring Module

It is evident from the definition of R-subgroup (resp. submodule) of a near-ring module M_R that the intersection of an arbitrary collection of R-subgroups (resp. submodules) is again an R-subgroup (resp. submodule). If A is a subset of M_R, then the intersection of all R-subgroups (resp. submodules) of M_R that contain A is called the R-subgroup (resp. submodule) of M_R generated by the set A. As in the case of ring modules, the concept of finitely generated is very important. Therefore, we introduce

Definition (1.23). Let M_R be an R-module. Then

(1.23.1) M_R is said to be finitely generated as an R-subgroup of itself if, and only if, there exists a finite subset A of M_R such that the R-subgroup of M_R generated by A is M_R.

(1.23.2) M_R is called a finitely generated near-ring module if, and only if, there is a finite subset A of M_R such that the submodule of M_R generated by A is M_R.

In this section we present two propositions that will be useful throughout the rest of this chapter. The first of which is

Proposition (1.24). Let M_R be an R-module, M' an R-subgroup of M_R, and H a submodule of M_R. Then the subgroup L of the additive group M that is generated by $M' \cup H$ is an R-subgroup and $L = H + M' = M' + H$.

Proof: Since H is a normal subgroup of M, it follows that $L = M' + H = H + M'$. Let $m' \in M'$, $h \in H$ and $r \in R$. Then since M' is a submodule, $(m' + h)r - m'r \in H$ and so $(m' + h)r = (m' + h)r - m'r + m'r \in L$. Therefore, L is an R-subgroup.

Proposition (1.25). Let M_R be an R-module and $\{M_\lambda \mid \lambda \in \Omega\}$ a collection of submodules of M_R. If L is the subgroup of the additive group M generated by $\bigcup_{\lambda \varepsilon \Omega} M_\lambda = M'$, then L is a submodule of M_R and $L = \sum_{\lambda \varepsilon \Omega} M_\lambda$ where $\sum_{\lambda \varepsilon \Omega} M_\lambda$ is the set of all finite sums of elements from M'.

Proof: It is well known that L is a normal subgroup of the additive group M and $L = \sum_{\lambda \varepsilon \Omega} M_\lambda$. Let $m \varepsilon M$, $r \varepsilon R$ and $b = b_{\lambda_1} + \cdots + b_{\lambda_n} \varepsilon L$ where $b_{\lambda_j} = M_{\lambda_j}$. Then $(m+b)r - mr =$

$$= \left[\left(m + \sum_{j=1}^{n-1} b_{\lambda_j}\right) + b_{\lambda_n}\right]r - \left(m + \sum_{j=1}^{n-1} b_{\lambda_j}\right)r + \left(m + \sum_{j=1}^{n-1} b_{\lambda_j}\right)r - mr.$$

Since M_{λ_n} is a submodule it is clear that

$$\left[\left(m + \sum_{j=1}^{n-1} b_{\lambda_j}\right) + b_{\lambda_n}\right]r - \left(m + \sum_{j=1}^{n-1} b_{\lambda_j}\right)r \; \varepsilon \; M_{\lambda_n}.$$ Proceeding in this way, we finally prove that $(m+b)r - mr$ is an element of L and so L is a submodule of M_R.

Zassenhaus Theorem for Near-Ring Modules

Before proving this famous theorem for the case of near-ring modules, we will establish three lemmas and a theorem which are important for future considerations.

We start with

Lemma (1.26). [25] Let T be an R-homomorphism of M_R onto M'_R with kernel $Ke(T)$. If L is an R-subgroup of M_R, then $L + Ke(T) = T^{-1}(LT)$.

Proof: Let L be an R-subgroup of M_R. Then since $Ke(T)$ is a submodule of M_R, $L + Ke(T)$ and LT are R-subgroups by Propositions (1.24) and (1.12). It suffices to show $T^{-1}(LT) \subseteq L + Ke(T)$. Let x be an element of $T^{-1}(LT)$. Then there is an element $y \in L$ such that $xT = yT$ and so $x - y \in Ke(T)$. From this $x - y + y = x \in$ $\in L + Ke(T)$. Therefore, $L + Ke(T) = T^{-1}(LT)$.

Lemma (1.27). [25] Let T be an R-homomorphism of M_R into M'_R. If L is an R-subgroup of M'_R and H is a submodule of L_R, then HT is a submodule of the R-module LT.

Proof: Since T induces an R-homomorphism of L_R into M'_R, it follows that LT is a near-ring module over R. Referring to Lemma (1.21) we see that HT is a submodule of the R-module LT.

An important theorem of this section is the analogue to the second isomorphism theorem for operator groups which we formulate as

Theorem (1.28). [25] Let H be a submodule and L an R-subgroup of the near-ring module M_R. If η is the natural R-homomorphism of M_R onto the factor module

$M' = M/_{H}$, then $L \cap H$ is a submodule of L_R and

$$L/_{L\cap H} \underset{(R)}{\cong} H+L/_{H} \underset{(R)}{\cong} L\eta.$$

__Proof__: The mapping $\eta^*: y \in L \longrightarrow y + H \in H+L/_{H}$ is an R-homomorphism of L_R onto the R-module $L+H/_{H}$ with kernel $Ke(\eta^*) = L\cap H$. From Theorems (1.17) and (1.18) $L\cap H$ is a submodule of L_R and $L/_{L\cap H} \underset{(R)}{\cong} L+H/_{H}$.

If T is the restriction of η to L_R, then T is an R-homomorphism of L_R onto the R-module $LT = L\eta$ with $Ke(T) = L\cap H$. By Theorem (1.18) it follows that

$$L\eta \underset{(R)}{\cong} L/_{L\cap H} \underset{(R)}{\cong} L+H/_{H}.$$

The latter theorem in conjunction with the two lemmas presented in this section allows us to conclude

__Lemma (1.29)__. [25] Let T be an R-homomorphism of M_R into M'_R with kernel Ke(T). If L is an R-subgroup of M_R and H is a submodule of L_R, then

(1.29.1) $H + Ke(T)$ is a submodule of the R-module $L + Ke(T)$, $H + (Ke(T) \cap L)$ is a submodule of L_R, and HT is a submodule of the near-ring module LT.

(1.29.2) $L + Ke(T)/_{H + Ke(T)} \underset{(R)}{\cong} L/_{K + (L\cap Ke(T))} \underset{(R)}{\cong} LT/_{KT}.$

We are now ready to give the Zassenhaus theorem for near-ring modules.

<u>Theorem (1.30)</u>. [25] Let H and K be R-subgroups of M_R. If H' is a submodule of H_R and K' is a submodule of K_R, then $H' + (H \cap K')$ is a submodule of the R-module $H' + (H \cap K)$, $K' + (H' \cap K)$ is a submodule of $K' + H \cap K$, and

$$H' + (H \cap K) / H' + (K' \cap H) \underset{(R)}{\cong} K' + (H \cap K) / K' + (H' \cap K)$$

<u>Proof:</u> From Propostion (1.24) $H' + (H \cap K)$ is an R-subgroup of H_R and $K' + (H \cap K)$ is an R-subgroup of K_R. It is clear that $H' + (H \cap K') \subseteq H' + (K \cap H)$ and $K' + (H' \cap K) \subseteq K' + (H \cap K)$. We first show that $K \cap H'$ is a submodule of the R-module $H \cap K$. Let $x \in H \cap K$, $y \in H' \cap K$ and $r \in R$. Then since H' is a submodule of H_R we see that $-x + y + x \in H' \cap K$ and $(x + y)r - xr \in H' \cap H$. Similarly, $H \cap K'$ is a submodule of the R-module $H \cap K$. Because of Proposition (1.25) $(H' \cap K) + (K' \cap H)$ is a submodule of $H \cap K$ and $(K' \cap H) + H'$ is a submodule of the R-module $(H \cap K) + H'$. Appealing to Theorem (1.28) we conclude that

$$H' + (H \cap K) / H' + (H \cap K') \underset{(R)}{\cong} H \cap K / (H' \cap K) + (K' \cap H)$$

Analogous to the above, we see that $K' + (H' \cap K)$ is a submodule of the R-module $K' + (H \cap K)$ and

$$K' + (H \cap K) / K' + (H' \cap K) \underset{(R)}{\cong} H \cap K / (H' \cap K) + (H \cap K')$$

Therefore, the Zassenhaus theorem follows.

Jordan-Hölder Theorem for Near-Ring Modules

As an important application of the Zassenhaus theorem we will obtain the Schreier theorem and Jordan-Hölder theorem for near-ring modules. In order to reach our objective we introduce

Definition (1.31). Let M_R be an R-module. By a normal series for M_R is meant a sequence

Σ: $M = M_o \supset M_1 \supset \ldots \supset M_n = \{0\}$ where M_{i+1} is a submodule of the R-module M_i.

The modules M_i/M_{i+1} are called the factors of the normal series Σ, and each M_i is called a term of Σ. If H is a submodule of M_R, then the sequence $M \supset H \supset \{0\}$ is a normal series.

Two normal series Σ_1 and Σ_2 are said to be equivalent if these is a one-to-one correspondence between the factors of the two series such that the paired factors are R-isomorphic. We will say that one normal series is a refinement of a second if its terms are also terms of the second. We can now state Schreier's theorem for near-ring modules.

Theorem (1.32). [25] Any two normal series for M_R have equivalent refinements.

Proof: Let Σ_1: $M = M_o \supset M_1 \supset M_2 \supset \ldots \supset M_n = \{0\}$ and Σ_2: $M_\ell = H_o \supset H_1 \supset \ldots \supset H_\ell = \{0\}$ be two normal series for M_R. Let

(1.32.1) $M_{i,j} = M_{i+1} + (M_i \cap H_j)$ for $i = 0,1,\ldots,n-1$; $j = 0,1,\ldots,\ell$.

(1.32.2) $H_{j,i} = H_{j+1} + (M_i \cap H_j)$ for $j = 0,1,\ldots,\ell-1$; $i = 0,1,\ldots,n$.

From Proposition (1.24) and Theorem (1.30) we note $M_{i,j+1}$ is a submodule of the R-module $M_{i,j}$, $H_{j,i+1}$ is a submodule of the R-module $H_{j,i}$, and

$$M_{i,j}/M_{i,j+1} \underset{(R)}{\cong} H_{j,i}/H_{j,i+1} \text{ for } i=0,1,\ldots,n-1;$$

$j=0,1,\ldots,\ell-1$. If

Σ_1': $M_{0,0} \supset M_{0,1} \supset \cdots \supset M_{0,\ell} \supset M_{1,\ell} \supset \cdots \supset M_{n-1,\ell} = \{0\}$ and

Σ_2': $H_{0,0} \supset H_{0,1} \supset \cdots \supset H_{0,n} \supset H_{1,0} \supset \cdots \supset H_{1,n} \supset$

$\supset H_{\ell-1,n} = \{0\}$, then it is clear that Σ_1' is a refinement of Σ_1 and Σ_2' is a refinement of Σ_2. From the above we see that Σ_1' is equivalent to Σ_2'.

Definition (1.33). A near-ring module M_R is said to be simple if, and only if, M_R contains no proper submodules.

In [25] Roth called a simple near-ring module M_R ideally simple and reserved the term simple for the case when M_R contained no proper R-subgroups. Laxton [24] referred to simple near-ring modules as irreducible R-groups where R is a d.g. near-ring with identity.

There exist many simple near-ring modules. For example, if G is an additive group and S(G) is the near-ring of mappings associated with G, then the S(G)-module G is simple. We show G contains no proper S(G)-subgroups. Let H be a non-zero S(G)-subgroup of G and let $g \in G$. If $h \in H$, $h \neq 0$, then there is an element $s \in S(G)$ such that $g = hs \in H$. This shows that G has no proper S(G)-subgroups and so the S(G)-module G can not have a proper submodule. $|G| > 2$

We will say that a proper submodule (resp. R-subgroup) A of a near-ring module M_R is maximal if for any other submodule (resp. R-subgroup) A' with A contained in A', then either $A = A'$ or $A' = M_R$.

As a consequence of Lemma (1.21) we have immediately

<u>Proposition (1.34)</u>. A proper submodule A of a near-ring-module M_R is maximal if, and only if, the factor module M/A is simple.

A normal series Σ for a near-ring module M_R is said to be strictly decreasing if each term is properly included in the preceeding. As the analogue of Jordan-Hölder series for operator groups we give

<u>Definition (1.35)</u>. A stictly decreasing normal series Σ for M_R is called a Jordan-Hölder series if, and only if, the factors of Σ are simple R-modules.

From Proposition (1.34) we conclude

<u>Proposition (1.36)</u>. A strictly decreasing normal

series $\Sigma: M = M_0 \supset M_1 \supset \cdots \supset M_n = \{0\}$ for M_R is a Jordan-Hölder series if, and only if, M_{i+1} is a maximal submodule of the R-module M_i.

If a near-ring module M_R has a Jordan-Hölder series Σ, then the factors of Σ are uniquely determined by M_R. This is the content of the Jordan-Hölder theorem that follows.

Theorem (1.37). [25] Any two Jordan-Hölder series for a near-ring module M_R are equivalent.

Proof: This follows from Theorem (1.32).

According to Theorem (1.37) the number of factors of a Jordan-Hölder series for M_R is invariant. Hence, if a near-ring module has a Jordan-Hölder series , then the number of factors of Σ is often called length of M_R.

Corollary (1.38). Let M_R be an R-module and H a submodule. If M_R has a Jordan-Hölder series Σ, then any strictly decreasing normal series can be refined to a Jordan-Hölder series and H_R has a Jordan-Hölder series.

Proof: Assume Σ_1 is a strictly decreasing normal series. By Theorem (1.32) Σ_1 and Σ have equivalent refinements. Let Σ' be a refinement of Σ_1 without duplicate terms. Then it is easy to see that

Σ' is equivalent to Σ and so it must be a Jordan-Hölder series.

Since H is a submodule of M_R, it follows that $M \supset H \supset \{0\}$ is a strictly decreasing normal series for M_R. By the first part of this result it is now clear that H_R has a Jordan-Hölder series.

CHAPTER II

ON THE THEORY OF IDEALS FOR NEAR-RINGS

Ideal theory for general near-rings was first discussed by Blackett [5]. In [12] Fröhlich considered the theory of ideals for d.g. near-rings. He developed many of the properties of ideals for this special class of near-rings.

After defining the concept of two-sided ideal of a near-ring R, we give a necessary and sufficient condition in order that a subset B of R be a two-sided ideal.

In the second section we present the fundamental theorem of near-ring homomorphisms and the first isomorphism theorem for near-rings. Another concept, right annihilator, is defined which plays an important role in the theory of near-rings. Several elementary results are obtained using this notion.

Ideals and Factor Near-Rings

The concepts of ideal of a near-ring and factor near-ring generalize the similar notions for a ring. Since we do not assume the additive group of a near-ring is abelian, these concepts will be given in terms of normal subgroups.

<u>Definition (2.1)</u>. Let R be a near-ring and B a subset of R. Then

(2.1.1) B is called a left ideal if, and only if, B is a normal subgroup of the additive group of R and $R \cdot B \subseteq B$.

(2.1.2) B is called a right ideal if, and only if, B is a submodule of R_R

(2.1.3) B is said to be a two-sided ideal of R if, and only if, B is a right ideal and a left ideal.

In the last section we will present several examples of two-sided ideals.

First, we show that the factor group determined by a two-sided ideal of a near-ring is also a near-ring.

<u>Lemma (2.2)</u>. Let B be a two-sided ideal of R. Then the factor group R/B can be made into a near-ring that is a homomorphic image of R.

<u>Proof</u>: If $x + B$ and $y + B$ are elements of R/B, then we define $(x + B)\cdot(y + B) = xy + B$. We will show this multiplication is single-valued. Assume $x + B = x' + B$ and $y + B = y' + B$ where x, y, y', and x' are elements of R. Then there exists elements $b_1, b_2 \in B$ such that $x + b_1 = x'$ and $y + b_2 = y'$. Hence, $x'\cdot y' = (x+b_1)\cdot(y+b_1) =$ $= (x + b_1)y - xy + xy + (x + b_1)b_2 - xb_2 + xb_2$. Since B is a two-sided ideal of R the elements $(x + b_1)y - xy$, $(x + b_1)b_2 - xb_2$, and xb_2 are contained in B and so our

multiplication is single-valued. It is easy to see that the system $(R/B, +, \cdot)$ is a near-ring. If η is the natural group homomorphism of the additive group of R onto the additive group of R/B, then $(xy)\eta =$ $= xy + B = (x+B)\cdot(y+B) = x\eta \cdot y\eta$ where $x, y \in R$. This shows η is a near-ring homomorphism, and the proof is complete.

The near-ring R/B is called a factor near-ring and η the natural near-ring homomorphism of R onto R/B.

<u>Theorem (2.3)</u>. [5] A subset B of a near-ring R is a two-sided ideal of R if, and only if, B is the kernel of a near-ring homomorphism.

<u>Proof</u>: Assume B is a two sided ideal. Then by Lemma (2.2) B is the kernel of the natural near-ring homomorphism of R onto R/B.

Conversely, let $B = Ke(T)$ where T is a near-ring homomorphism of R into the near-ring R'. In particular, B is the kernel of a group homomorphism so that B is a normal subgroup of the additive group of R. If $r_1, r_2 \in R$, $x \in K(T)$, then

(2.3.1) $(r_1x)T = (r_1T)(x\cdot T) = (r_1T)\cdot 0 = 0 \in R'$

(2.3.2) $[(r_1+x)r_2 - r_1r_2]T = (r_1T + xT)(r_2T) - (r_1r_2)T$
$= (r_1r_2)T - (r_1r_2)T = 0 \in R'$.

From (2.3.1) $B = Ke(T)$ is a left ideal and (2.3.2) shows

B is a right ideal. Therefore, B is a two-sided ideal of R.

Blackett [5] defined a two-sided ideal of a near-ring R to be the kernel of a near-ring homomorphism. From Theorem (2.3) our definition of two-sided ideals is equivalent to Blackett's.

The intersection of an arbitrary collection of two-sided ideals of a near-ring is also a two-sided ideal. We now show the subgroup generated by an arbitrary collection of two-sided ideals of a near-ring is a two-sided ideal.

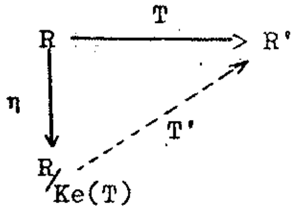
Lemma (2.4). Let $\{B_\lambda \mid \lambda \varepsilon \Omega\}$ be a collection of two-sided ideals of R. If $B = \sum_{\lambda \varepsilon \Omega} B_\lambda$, then B is a two-sided ideal of R.

Proof: By Proposition (1.25) it suffices to show $rb \varepsilon B$ where $r \varepsilon R$, $b \varepsilon B$. If $b = b_{\lambda_1} + \cdots + b_{\lambda_n}$ where $b_{\lambda_j} \varepsilon B_{\lambda_j}$, then $rb = rb_{\lambda_1} + \cdots + rb_{\lambda_n} \varepsilon B$ since B_{λ_j} is a two-sided ideal. This shows B is a two-sided ideal of R.

On Near-Ring Homomorphisms

We begin this section with the fundamental theorem of near-ring homomorphisms.

<u>Theorem (2.5)</u>. [25] Let T be a near-ring homomorphism of R onto R' with kernel Ke(T). If η is the natural near-ring homomorphism of R onto $R/_{Ke(T)}$, then T induces a near-ring isomorphism T' of $R/_{Ke(T)}$ onto R' such that the diagram is commutative.



<u>Proof</u>: Since T is a group homomorphism T induces a group isomorphism T' of the additive group $R/_{Ke(T)}$ onto the additive group R'. Moreover, the mapping T' is given by $T: r + Ke(T) \in R/_{Ke(T)} \longrightarrow rT \in R'$. It remains to show T' is a near-ring isomorphism. If $r_1 + Ke(T)$, $r_2 + Ke(T)$ are elements of $R/_{Ke(T)}$, then $(r_1 + Ke(T))(r_2 + Ke(T))\cdot T' = (r_1 r_2 + Ke(T))T' = (r_1 r_2)T =$ $= (r_1 T)(r_2 T) = (r_1 + Ke(T))\cdot T'(r_1 + Ke(T))T'$. From this the proof is concluded.

We now give the first isomorphism theorem for near-rings.

<u>Theorem (2.6)</u>. [3] Let T be a near-ring homomorphism of R onto R' with kernel Ke(T). Then

(2.6.1) If B is a two-sided ideal of R that contains Ke(T), then $R/_B \cong R'/_{BT}$

(2.6.2) The mapping $f: B \longrightarrow BT$ is a one-to-one onto correspondence between the collection of two-sided ideals B of R that contain Ke(T) and the collection of two-sided ideals of R'.

Before proving Theorem (2.6) we present the important

Lemma (2.7). Let T be a near-ring homomorphism of R onto R' with kernel Ke(T). Then
(2.7.1) If B is a two-sided ideal of R, then BT is a two-sided ideal of R.
(2.7.2) If B' is a two-sided ideal of R', then $T^{-1}(B')$ is a two-sided ideal of R that contains Ke(T).
(2.7.3) If B is a two-sided ideal of R that contains Ke(T), then $T^{-1}(BT) = B$.

Proof: Let B be a two-sided ideal of R. Then since T is a group homomorphism of R onto R' it follows that BT is a normal subgroup of the additive group of R'. If $r_1, r_2 \varepsilon R$, $b \varepsilon B$, then since B is a two-sided ideal of R, $(r_1 + b)r_2 - r_1r_2 \varepsilon B$ and so $(r_1T + bT)r_2T - (r_1T)(r_2T) = [(r_1 + b)r_2 - r_1r_2]T \varepsilon BT$. From this and the fact that T is an onto mapping we conclude that BT is a right ideal of R'. Next, we note that $(r_1T)(bT) = (r_1b)T$ and so BT is a left ideal. Hence, (2.7.1) is proved.

Similarly, if B' is a two-sided ideal of R', then

$T^{-1}(B')$ is a two-sided ideal of R that contains Ke(T) so that (2.7.2) follows.

Finally, assume B is a two-sided ideal of R that contains Ke(T). From (2.7.1) and (2.7.2) it suffices to show $T^{-1}(BT) \subseteq B$. If $x \in T^{-1}(BT)$, then there is an element $b \in B$ such that $xT = bT$ and so $x - b \in Ke(T) \subseteq B$. Therefore, $x = x - b + b \in B$ and so (2.7.3) is established.

We can now prove the theorem.

<u>Proof</u>: Let B be a two-sided ideal of R that contains Ke(T). By Lemma (2.7) BT is a two-sided ideal of R'. Since T is a group homomorphism T induces a group isomorphism T' of the additive group R/B onto the additive group R/BT. If $r_1, r_2 \in R$, then

$$(r_1 + B)(r_2 + B)T' = (r_1r_2 + B)T' = (r_1r_2)T + BT =$$
$$= (r_1T)(r_2T) + BT = (r_1T + BT)(r_2T + BT) = (r_1+B)T' \cdot (r_2+B)T'$$

and so T' is a near-ring isomorphism. Hence, the proof of (2.6.1) is complete.

Appealing to Lemma (2.7) it is clear that (2.6.2) follows.

As an easy application of Theorem (2.6) we obtain

<u>Corollary (2.8)</u>. Let B be a two-sided ideal of R and η the natural near-ring homomorphism of R onto R/B. If A is a two-sided ideal of R that contains B, then $A\eta = A/B$ is a two-sided ideal of R/B and

$$R/A \cong {}^{R/B}\Big/_{A/B} \ .$$

Right Annihilating Sets

Definition (2.9). Let A and B be two subsets of the near-ring module M_R. By the right annihilator of A in B is meant $\{r \in R \mid Br \subseteq A\} = \left[\frac{A}{B}\right]$

In the present section we give four easy results concerning right annihilators. We begin with

Proposition (2.10). If B is a submodule of M_R, then

$$\left[\frac{B}{M}\right] = \left[\frac{0}{M/B}\right] \ .$$

Proof: Let B be a submodule of the near-ring module M_R. If $r \in \left[\frac{0}{M/B}\right]$, then $mr + B = (m + B)r \subseteq B$ for all $m \in M$. Hence, $mr \in B$ and so $r \in \left[\frac{B}{M}\right]$. Conversely, assume r is an element of $\left[\frac{B}{M}\right]$. Then $mr \in B$ for all $m \in M$. Therefore, $(m + B)r = mr + B \subseteq B$ and so we see that $r \in \left[\frac{0}{M/B}\right]$. From this we can conclude that $\left[\frac{B}{M}\right] = \left[\frac{0}{M/B}\right]$.

Proposition (2.11). Let M_R be an R-module and m an element of M. Then the mapping $f_m \colon r \in R \longrightarrow mr \in M_R$ is an R-homomorphism and $\left[\frac{0}{M}\right] = \bigcap_{m \in M} Ke(f_m)$ is a two-sided ideal.

Proof: Let r_1, r_2 be elements of R. Then using the definition of near-ring module we see that $(r_1 + r_2)f_m =$ $= m(r_1 + r_2) = mr_1 + mr_2 = r_1 f_m + r_2 f_m$ and $(r_1 \cdot r_2)f_m =$

$= m(r_1r_2) = (mr_1)r_2 = (r_1f_m)r_2$. This shows f_m is an R-homomorphism of R_R into M_R.

It is clear that $\left[\frac{0}{M}\right] \subseteq \bigcap_{m \in M} Ke(f_m) = \bigcap_{m \in M} \left[\frac{0}{m}\right]$. If $r \in \bigcap \left[\frac{0}{M}\right]$, then $m \cdot r = 0$ for all $m \in M$ and so $M \cdot r = 0$. This shows $\left[\frac{0}{M}\right] = \bigcap_{m \in M} Ke(f_m) = \bigcap_{m \in M} \left[\frac{0}{m}\right]$ and $\left[\frac{0}{M}\right]$ is a right ideal. If $r \in \left[\frac{0}{M}\right]$, $r' \in R$, then $M(r'r) = (Mr')r \subseteq \subseteq M \cdot r = 0$ so that $\left[\frac{0}{M}\right]$ is a two-sided ideal.

<u>Proposition (2.12)</u>. Let B be a two-sided ideal of R Then

(2.12.1) If M is an R/B-module, then M can be regarded as an R-module.

(2.12.2) If M is an R-module and $B \subseteq \left[\frac{0}{M}\right] \subseteq R$, then M can be regarded as an R-module.

<u>Proof</u>: Assume M is an R/B-module. Let $m \in M$, $r \in R$ and define $mr = m(r + B) \in M$. With this definition M becomes an R-module and (2.12.1) follows.

Let $M = M_R$ be an R-module such that $B \subseteq \left[\frac{0}{M}\right]$. Let $m \in M$, $r \in R$ and define $m(r + B) = mr$. If $r_1, r_2 \in R$ and $r_1 + B = r_2 + B$, then there is an element $b \in B$ such that $r_1 = r_2 + b$. Since $b \in \left[\frac{0}{M}\right]$, it is clear that $mr_1 = mr_2$ and so the above definition is single-valued. It is now apparent that M can be regarded as an R/B-module.

<u>Proposition (2.14)</u>. Let M_R and A_R be R-modules. If $M \underset{(R)}{\cong} A$, then $\left[\frac{O}{M}\right] = \left[\frac{O}{A}\right]$.

<u>Proof</u>: Let φ denote the R-isomorphism of M_R onto A_R. If $r \in \left[\frac{O}{M}\right]$, $a \in A$, then $a \cdot r = (m\varphi)r = (mr)\varphi$ for some $m \in M$. By Proposition (2.12) $\left[\frac{O}{M}\right] \subseteq \left[\frac{O}{A}\right]$. Similarly, $\left[\frac{O}{A}\right] \subseteq \left[\frac{O}{M}\right]$ and so

$$\left[\frac{O}{M}\right] = \left[\frac{O}{A}\right].$$

CHAPTER III

CHAIN CONDITIONS

Betsch [1], Blackett [5], and Deskins [8] have discussed near-rings R that satisfy the descending chain condition on R-subgroups. In [22] Laxton considered d.g. near-rings that satisfy the descending chain condition on right ideals, and he investigated d.g. near-rings R that satisfy the descending chain condition on R-subgroups in [23, 24]. Roth [25] defined a near-ring module M_R to be Artinian if for every sequence $M_1 \supset M_2 \supset \cdots$ of R-subgroups such that H_1 is a submodule of M_R and H_{i+1} is a submodule of H_i, then there exists a positive integer n such that $H_i = H_n$ for all $i \geq n$. He called M_R Noetherian if H is any term of a normal series and $H_1 \subset H_2 \subset \cdots$ is an ascending sequence of submodules of H_R then there exists a natural number n such that $H_i = H_n$ for all $i \geq n$. Then he investigated the structure of Artinian and Noetherian near-ring modules.

It is the purpose of the present chapter to provide a foundation for near-ring modules M_R that satisfy the chain conditions on submodules and R-subgroups. Later, we will apply the results of this chapter to special classes of near-rings and near-ring modules.

We define the concept of minimum condition on submodules (resp. R-subgroups) of M_R, and we relate this concept to the descending chain condition. Another notion is given, maximum condition on submodules, which is equivalent to the ascending chain condition on submodules. In the last section we construct several examples of near-ring modules that satisfy the various chain conditions.

Descending Chain Condition

Definition (3.1). A near-ring module M_R is said to satisfy the descending chain condition, denoted d.c.c., on submodules (resp. R-subgroups) if for every decreasing sequence $M_1 \supset M_2 \supset \cdots$ of submodules (resp. R-subgroups) of M_R there is a positive integer n such that $M_i = M_n$ for all $i \geq n$.

A near-ring R is said to satisfy the d.c.c. on right ideals (resp. R-subgroups) if R_R satisfies the d.c.c. on submodules (resp. R-subgroups).

Every ring module M_R that satisfies the d.c.c. on submodules is a near-ring module that satisfies d.c.c. on submodules and R-subgroups. A simple near-ring module satisfies the d.c.c. on submodules. It is evident that any finite near-ring module satisfies both the d.c.c. on submodules and the d.c.c. on R-subgroups.

Let G be an additive group and S(G) the near-ring of mappings associated with G. We proved in the first chapter that G contained no proper S(G)-subgroups and so G considered as an S(G)-module satisfies the d.c.c. on S(G)-subgroups.

Since every submodule of a near-ring module M_R is an R-subgroup, it follows that the d.c.c. on R-subgroups implies the d.c.c. on submodules. The converse of this statement is not true in general. This is the content of Proposition (3.11).

Definition (3.2). A near-ring module M_R is said to saitisfy the minimum condition on submodules (resp. R-subgroups) if every non-empty collection of submodules (resp. R-subgroups) of M_R contains a minimal element relative to the partial ordering by set inclusion.

As might be expected, we present

Lemma (3.3). A near-ring module M_R satisfies d.c.c. on submodules (resp. R-subgroups) if, and only if, it satisfies the minimum condition on submodules (resp. R-subgroups).

Proof. Assume M_R satisfies the minimum condition on submodules. Let $M_1 \supset M_2 \supset \cdots$ be a descending sequence of submodules. Then since M_R satisfies the minimum condition on submodules the collection $\{M_i\}$ of submodules contains a minimal element. Therefore, there is a positive integer n such that $M_i = M_n$ for all $i \geq n$.

Conversely, assume M_R satisfies the d.c.c. on submodules. Let Φ be a non-empty collection of submodules of M_R. Let M_1 be an element of Φ. If M_1 is not minimal, then there is an element $M_2 \varepsilon \Phi$ such that $M_2 \subset M_1$. If M_2 is not minimal in Φ, then let $M_3 \varepsilon \Phi$ and $M_1 \supset M_2 \supset M_3$. Since M_R satisfies the d.c.c. on submodules we must arrive at a positive integer n such that M_n is minimal in Φ.

Similarly, M_R satisfies d.c.c. on R-subgroups if, and only if, it satisfies the minimum condition on R-subgroups.

Theorem (3.4). Let A be a submodule of the near-ring module M_R. If M_R satisfies d.c.c. on submodules (resp. R-subgroups), then the factor module M/A satisfies the d.c.c. on submodules (R-subgroups).

Proof: Let $M_1' \supset M_2' \supset \cdots$ be a decreasing sequence of submodules of M/A, and let η denote the natural R-homomorphism of M_R onto the factor module M/A. From Lemma (1.21)

$\eta^{-1}(M_1') \supset \eta^{-1}(M_2') \supset \cdots$ is a descending sequence of submodules of M_R. If M_R satisfies d.c.c. on submodules, then there is a positive integer n such that $\eta^{-1}(M_i') =$

$= \eta^{-1}(M'_n)$ for all $i \geq n$. Appealing to Lemma (1.21) the second time, it follows that $M'_i = M'_n$ for all $i \geq n$. This shows M/A satisfies the d.c.c. on submodules.

Now assume M_R satisfies d.c.c. on R-subgroups. Using Lemma (1.20) instead of Lemma (1.21) and proceeding in the same way as above we can conclude that M/A satisfies the d.c.c. on R-subgroups.

Ascending Chain Condition

Definition (3.5). A near-ring module M_R is said to satisfy the ascending chain condition, denoted a.c.c., on submodules if for every ascending sequence $M_1 \subseteq M_2 \subseteq \cdots$ of submodules there is a positive integer n such that $M_i = M_n$ for all $i \geq n$.

A near-ring R is said to satisfy a.c.c. on right ideals if R_R satisfies the a.c.c. on submodules.

Any finite near-ring satisfies the a.c.c. on right ideals. Let G be an additive group and S(G) its associated near-ring. As seen earlier, G considered as an S(G)-module has no proper S(G)-subgroups and so G satisfies a.c.c. on S(G)-subgroups.

In the last section of this chapter we will construct other examples of near-ring modules that satisfy the a.c.c. on submodules.

<u>Definition (3.6)</u>. A near-ring module M_R is said to satisfy the maximum condition on submodules if every non-empty collection of submodules contains a maximal element relative to the partial ordering by set inclusion.

The proof of the next result is similar to that of Lemma (3.3). For that reason it will not be given.

<u>Lemma (3.7)</u>. A near-ring module M_R satisfies a.c.c. on submodules if, and only if, it satisfies the maximum condition on submodules.

We now give the analogue of Theorem (3.4)

<u>Theorem (3.8)</u>. Let A be a submodule of M_R. If M_R satisfies a.c.c. on submodules, then the factor module M/A satisfies a.c.c. on submodules.

<u>Proof</u>: Assume that M_R satisfies a.c.c. on submodules. Let $M_1' \subseteq M_2' \subseteq \cdots$ be an ascending sequence of submodules of the factor module M/A. If η is the natural R-homomorphism of M_R onto $(M/A)_R$, then by repeated use of Lemma (1.20) we conclude that there is a positive integer n such that $\eta^{-1}(M_i') = \eta^{-1}(M_n')$ for all $i \geq n$ and so $M_i = M_n$ for all $i \geq n$. This shows M/A satisfies a.c.c. on submodules.

We conclude this section with a theorem that gives a sufficient condition for a near-ring module to satisfy both chain conditions on submodules.

Theorem (3.9). A near-ring module M_R that has a Jordan-Hölder series satisfies both the d.c.c. on submodules and the a.c.c. on submodules.

Proof: Let Σ be a Jordan-Hölder series for M_R with n-factors. From Theorem (1.37) every Jordan-Hölder series for M_R has exactly n-factors.

Suppose $M_1 \supset M_2 \supset \cdots$ is an infinite decreasing sequence of submodules where M_{i+1} is a proper subset of M_i. Then for $k > n$ the normal series $\Sigma' : M \supset M_1 \supset \cdots \supset \cdots \supset M_k \supset \{0\}$ is strictly decreasing and has more than n-factors. By Corollary (1.38) Σ' can be refined to a Jordan-Hölder series with more than n-factors. From this we conclude that M_R satisfies d.c.c. on submodules.

Assume $H_1 \subset H_2 \subset H_3 \subset \cdots$ is an infinite ascending sequence of submodules of M_R such that H_{i+1} is a proper subset of H_i. Then for $\ell > n$ the strictly normal series $M \supset H \supset H_{\ell+1} \supset \cdots \supset H_1 \supset \{0\}$ has more than n-factors. This shows M_R satisfies a.c.c. on submodules.

Construction of Examples

In this section we will construct several general examples that will be applicable throughout the remainder of this dissertation. In particular, we will be able to give examples of near-ring modules that satisfy the various chain conditions.

The most natural place to begin, of course, is with sub-near-rings of the near-ring of mappings associated with an additive (non-abelian) group.

Throughout this section G will denote an additive non-abelian group and $S = S(G)$ the near-ring associated with G. We will no doubt have to assume certain properties of G in order to reach our objective.

First, we will show that every subgroup of G determines a sub-near-ring of S(G).

Proposition (3.10). Let H be a subgroup of G. Then $S(H) = \{s \in S(G) \mid H \cdot s \subseteq H\}$ is a sub-near-ring of S(G).

Proof: If $h \in H$, $s_1, s_2 \in S(H)$, then $h(s_1 - s_2) = hs_1 - hs_2$ and $h(s_1 s_2) = (hs_1)s_2$. Since H is a subgroup, it follows that the elements $s_1 - s_2$ and $s_1 s_2$ are contained in S(H). From this we note that S(H) is a sub-near-ring of S(G).

The near-ring S(H) is called the near-ring determined by the subgroup H.

Let us assume G contains only a finite number of normal subgroups and does not satisfy the descending chain condition on subgroups (such groups are known to exist). Let $H_1 \supset H_2 \supset \cdots$ be an infinite descending sequence of subgroups of G and $S(H_i)$ the near-ring determined by H_i.

Proposition (3.11). If $R = \cap S(H_i)$, then the R-module G_R satisfies the d.c.c. on submodules and does not satisfy the d.c.c. on R-subgroups.

Proof: Since G contains only a finite number of normal subgroups, G_R must contain at most a finite number of submodules. Hence, G_R satisfies the d.c.c. on submodules. For each positive integer n the near-ring R is contained in $S(H_n)$ and so each H_n is an R-subgroup of G_R. Therefore, $H_1 \supset H_2 \supset \cdots$ is an infinite descending sequence of R-subgroups and so G_R does not satisfy the d.c.c. on R-subgroups.

We now consider d.g. sub-near-rings of S(G) where G is an arbitrary additive non-abelian group. Let E be a multiplicative semi-group of endomorphisms of G. A subgroup H of G is said to be E-invariant if, and only if, $H \cdot E \subseteq H$. As usual, E(G) will denote the d.g. near-ring generated by E.

Lemma (3.12). A subgroup H of G is E-invariant if, and only if, it is an E(G)-subgroup of G over the d.g. near-ring E(G).

Proof: First, assume H is an E(G)-subgroup. Since $E \subseteq E(G)$, H must be E-invariant.

Conversely, suppose H is an E-invariant subgroup.

Let s be any element of E(G). Then we have a representation $s = s_1 + \cdots + s_n$ where $s_i \in E$ or $-s_i \in E$. Since H is an E-invariant subgroup $h(\pm s_i) = \pm hs_i \in H$ where $h \in H$. From this we see that H is an E(G)-subgroup of G over the d.g. near-ring E(G). This completes the proof.

Let A denote the multiplicative group of inner-automorphisms of G. Throughout the remainder of this section we will assume $A \subseteq E$. Hence, we have

<u>Lemma (3.13)</u>. If H is an E(G)-subgroup, then H is a submodule of G over the d.g. near-ring E(G).

<u>Proof</u>: Let H be an E(G)-subgroup of G. In particular, H is an A-invariant subgroup and so it must be normal. Let s be an element of E(G). Then $s = s_1 + \cdots + s_n$ where $s_i \in E$ or $-s_i \in E$. If $h \in H$ and $g \in G$, then $(g+h)s - gs = (g+h)s_1 + \cdots + (g+h)s_n - gs_n - \cdots - gs_1$. If $s_n \in E$, then $(g+h)s_n - gs_n = gs_n + hs_n - gs_n \in H$. Also, if $-s_n \in E$, then $(g+h)s_n - gs_n = hs_n \in H$. Proceeding in this way we finally prove that $(g+h)s - gs$ is an element of H. Hence, we conclude that H is a submodule of the E(G)-module G.

According to Lemma (3.12) and Lemma (3.13) we can give the following list of examples.

<u>Example (3.14)</u>. If G satisfies the d.c.c. on normal

subgroups, then the E(G)-module G satisfies the d.c.c. on E(G)-subgroups.

<u>Example (3.15)</u>. If G satisfies the a.c.c. on normal subgroups, then the E(G)-module G satisfies the a.c.c. on submodules.

<u>Example (3.16)</u>. If G satisfies the d.c.c. on normal subgroups and does not satisfy the a.c.c. on normal subgroups, then the E(G)-module G satisfies the d.c.c. on submodules and does not satisfy the a.c.c. on submodules.

<u>Example (3.17)</u>. If G is a finite groups, then G considered as an E(G)-module satisfies both chain conditions on submodules.

CHAPTER IV

SEMI-SIMPLE NEAR-RING MODULES

According to Blackett [5] a near-ring R is semi-simple if R satisfies the d.c.c. on R-subgroups and contains no non-zero nilpotent R-subgroups. Betsch [1] introduced the analogue of the Jacobson radical for near-rings. For the class of near-rings R that satisfy the d.c.c. on R-subgroups, he proved that a necessay and sufficient condition for R to be semi-simple (in the sense of Blackett) is that the radical be zero. Using the same radical for d.g. near-rings R with identity, Laxton [22] called R semi-simple if the radical of R is zero. He showed that this definition did not conflict with Blackett's. Roth [25] defined a near-ring module to be semi-simple if every submodule is a direct summand. Then he developed some of the properties of semi-simple near-ring modules.

The present chapter is devoted to the essential properties of semi-simple near-ring modules. After introducing the concept of direct sums of submodules of a near-ring module and giving several elementary properties derived from this concept, we will present a definition of semi-simple near-ring modules that is equivalent to Roth's. The fundamental result of this chapter, which is to be used in the following chapters, is given in Theorem (4.12). The theorem gives four equivalent conditions for a near-ring module to be semi-simple. Another concept is defined, external direct sum, which is used to provide examples of semi-simple near-ring modules.

Direct Sums of Submodules

Definition (4.1). Let $\{M_\lambda \mid \lambda \in \Omega\}$ be a collection of submodules of the near-ring module M_R. Then M_R is said to be a direct sum of the submodules $\{M_\lambda \mid \lambda \in \Omega\}$ if, and only if, the additive group M is a direct sum of the normal subgroups $\{M_\lambda \mid \lambda \in \Omega\}$. If M_R is a direct sum of the submodules $\{M_\lambda \mid \lambda \in \Omega\}$, then we denote this fact by $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$.

Since our definition of direct sum of submodules is given in terms of group theoretic properties of a near-ring module, we can now formalize several well known results from group theory.

Proposition (4.2). Let $\{M_\lambda \mid \lambda \in \Omega\}$ be a collection of submodules of M_R. Then

(4.2.1) $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$ if, and only if, $M_R = \sum_{\lambda \in \Omega} M_\lambda$ and and the elements of any two distinct submodules M_λ permute.

(4.2.2) $M_R = \bigoplus_{\lambda\varepsilon\Omega} M_\lambda$ if, and only if $M_R = \sum_{\lambda\varepsilon\Omega} M_\lambda$ and $M_{\lambda_o} \cap \left(\sum_{\substack{\lambda\varepsilon\Omega \\ \lambda\neq\lambda_o}} M_\lambda \right) = \{0\}$ for each $\lambda_o \varepsilon \Omega$.

(4.2.3) $M_R = \bigoplus_{\lambda\varepsilon\Omega} M_\lambda$ if, and only if $M_R = \sum_{\lambda\varepsilon\Omega} M_\lambda$ and every element of M_R has a unique representation as a finite sum of elements chosen from the submodules M_λ.

Definition (4.3). Let $M_R = \bigoplus_{\lambda\varepsilon\Omega} M_\lambda$, M_λ is a submodule of M_R. If $m \varepsilon M$, $m = \sum_{j=1}^{n} m_{\lambda_j}$ where $m_{\lambda_j} \varepsilon M_{\lambda_j}$, then each m_{λ_j} is said to be the $\lambda_j{}^{th}$ component of the element m.

The next lemma will be very useful in our later considerations of near-ring modules M_R which can be written as direct sums of submodules. It will enable us to distribute the elements of R over the elements which are contained in distinct submodules that occur in the direct sum representation of M_R.

Lemma (4.4). Let $M_R = \bigoplus_{\lambda\varepsilon\Omega} M_\lambda$, M_λ a submodule of M_R. If $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$ where $m_{\lambda_j} \varepsilon M_{\lambda_j}$ and $r \varepsilon R$, then $mr = \left(\sum_{j=1}^{n} m_{\lambda_j}\right) \cdot r = m_{\lambda_1} r + \cdots + m_{\lambda_n} r$.

Proof: We will use induction on n. Assume the Lemma is true for all elements of M with $\leq n-1$ non-zero components. Let $m \varepsilon M$, $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$ where

$m_{\lambda_j} \varepsilon M_{\lambda_j}$ and is not zero. If $r \varepsilon R$, then $(m_{\lambda_1} + \cdots + m_{\lambda_{n-1}})r = m_{\lambda_1} r + \cdots + m_{\lambda_{n-1}} \cdot r$ by assumption. Appealing to Proposition (4.2) we see that $mr - (m_{\lambda_1} + \cdots + m_{\lambda_{n-1}})r - m_{\lambda_n} r \varepsilon M_{\lambda_n} \cap \left(\sum_{j=1}^{n-1} M_{\lambda_j} \right) = \{0\}$ and so $mr = \sum_{j=1}^{n} m_{\lambda_j} \cdot r$. This establishes the Lemma.

Definition (4.5). A submodule A of M_R is said to be a direct summand if, and only if, there is a submodule B such that $M_R = A \oplus B$.

It is evident that $M_R = M_R \oplus \{0\} = \{0\} \oplus M_R$. If these are the only two decompositions of the near-ring module M_R, then M_R is said to be indecomposable.

Proposition (4.6). Let A be a direct summand of the near-ring module M_R. Then every submodule of A_R is a submodule of M_R.

Proof: Since A is a direct summand of M_R, there is a submodule B of M_R such that $M_R = A \oplus B$. Let A' be a submodule of A_R. If $a' \varepsilon A$, $r \varepsilon R$, and $m = a + b$ where $a \varepsilon A$, $b \varepsilon B$, then appealing to Proposition (4.2) and Lemma (4.4) we have

(4.6.1) $-m + a' + m = -(a+b) + a' + (a+b) = -a + a' + a \varepsilon A'$

(4.6.2) $(m + a')r - mr = (a + a')r + br - ar - br =$

$= (a + a')r - ar \varepsilon A'$.

From these facts we conclude that A' is a submodule of M_R.

Definition and Fundamental Properties of Semi-Simple Near-Ring Modules

An important result in the structure theory of ring modules is that every finitely generated module over a ring contains a maximal submodule. We now give a similar result for near-ring modules.

Lemma (4.7). Let M_R be an R-module. Then

(4.7.1) If M_R is finitely generated as an R-subgroup of itself, then every proper R-subgroup of M_R is contained in a maximal R-subgroup.

(4.7.2) If M_R is finitely generated, then every proper submodule of M_R is contained in a maximal submodule.

Proof: Suppose M_R is finitely generated as an R-subgroup and let A be a proper R-subgroup. Let Φ denote the non-empty collection of all proper R-subgroups B of M_R that contain A. We partially order Φ by set inclusion. If Φ' is a chain of elements of Φ, then it follows that $A' = \cup\{B \varepsilon \Phi'\}$ is an R-subgroup that contains A. If $A' = M_R$ and $\{x_1, \ldots, x_n\}$ is a set of generators for M_R as an R-subgroup of itself, then there is an element $B \varepsilon \Phi'$ such that $\{x_1, \ldots, x_n\} \subseteq B$. From this we see that $B = M$ and so A' is a proper R-subgroup. By Zorn's lemma the collection Φ contains a maximal element so that (4.7.1) follows.

Assume M_R is finitely generated and let A be a proper submodule of M_R. By an analogous argument we can conclude that A is contained in a maximal submodule and so (4.72) is established.

If a near-ring R contains an identity, then it is evident that R is finitely generated as an R-subgroup of R_R and R is finitely generated as a submodule of R_R. Hence, we have the following

<u>Corollary (4.8)</u>. Every near-ring with identity contains a maximal right ideal and a maximal R-subgroup.

We next present a lemma and two theorems, which are of interest in themselves, that will enable us to establish the fundamental theorem of this particular chapter.

We proceed with

<u>Lemma (4.9)</u>. [25] Let M_R be an R-module such that every submodule is a direct summand. Then every submodule of M_R has the same property and M_R contains a simple submodule.

<u>Proof</u>: Let A be a submodule of M_R and A' a submodule of A_R. From Proposition (4.6) A' is a submodule of M_R so there is a submodule A'' of M_R such that $M_R = A' \oplus A''$. It is easily verified that $A = A' \oplus (A'' \cap A)$ and so every submodule of A_R is a direct summand.

Let B be a finitely generated submodule of M_R. If B = M, then M_R contains a maximal submodule B' by Lemma (4.7). Hence, there is a submodule B" such that $B'' \oplus B' = M$. From the Second Isomorphism Theorem for near-ring modules $M/_{B'} \underset{(R)}{\cong} B''$ and so B" is a simple submodule of M_R. Assume B is a proper submodule of M_R. From Lemma (4.7) B_R contains a maximal submodule C and C is a submodule of M_R. Let C' be a submodule of B_R such that $B = C' \oplus C$. It is now clear that C' is a simple submodule of M_R.

<u>Theorem (4.10)</u>. [25] Let $M_R = \sum_{\lambda \in \Omega} M_\lambda$, M_λ a simple submodule of M_R. Then there exists a subset $\Omega' \subseteq \Omega$ such that $M_R = \bigoplus_{\lambda' \in \Omega'} M_{\lambda'}$.

<u>Proof</u>: Let {H} denote the collection of all subsets H of Ω such that $\sum_{h \in H} M_h = \bigoplus_{h \in H} M_h$. The collection {H} is partially ordered by set inclusion. If $\{G_\ell \mid \ell \in L\}$ is any chain of subsets of {H}, then we will show $G = \bigcup_{\ell \in L} G_\ell$ is again in {H}. For this it suffices to show $\sum_{g \in G} M_g$ is direct. If $g_0 \in G$ and $m \in M_{g_0} \cap \left(\sum_{\substack{g \in G \\ g \neq g_0}} M_g \right)$, then $m \in M_{g_0} \cap (M_{g_1} + \dots + M_{g_n})$

for some set of indices $g_1, \dots, g_n \in G$. Since $\{G_\ell \mid \ell \in L\}$ is a chain, it follows that there exists an element $\ell_0 \in L$ such that the indices $g_0, g_1, \dots, g_n \in G_{\ell_0}$

But $\sum_{g \in G_{\ell_0}} M_g = \bigoplus_{g \in G_{\ell_0}} M_g$ and so m = 0. This shows

$\sum_{g\varepsilon G} M_g = \bigoplus_{g\varepsilon G} M_g$ and by Zorn's lemma Ω contains a maximal subset Ω' with the property that $\sum_{\lambda'\varepsilon\Omega'} M_{\lambda'} = \bigoplus_{\lambda'\varepsilon\Omega'} M_{\lambda'}$. We now show $M_R = \bigoplus_{\lambda'\varepsilon\Omega'} M_{\lambda'}$. Let M_{λ_o} be a simple submodule of M_R that occurs in $\sum_{\lambda\varepsilon\Omega} M_\lambda$. It is easy to see that either $M_{\lambda_o} \cap \left(\sum_{\lambda'\varepsilon\Omega'} M_{\lambda'}\right) = M_{\lambda_o}$ or $M_{\lambda_o} \cap \left(\sum_{\lambda'\varepsilon\Omega'} M_{\lambda'}\right) = 0$. Since Ω' is maximal we see that $M_{\lambda_o} \cap \left(\sum_{\lambda'\varepsilon\Omega'} M_{\lambda'}\right) = M_{\lambda_o}$ and so $M_{\lambda_o} \subseteq \sum_{\lambda'\varepsilon\Omega'} M_{\lambda'}$. From this we conclude $M_R = \bigoplus_{\lambda'\varepsilon\Omega'} M_{\lambda'}$.

__Theorem (4.11)__. [25] Let $M_R = \bigoplus_{\lambda\varepsilon\Omega} M_\lambda$, M_λ a simple submodule. Then every submodule of M_R is a direct summand.

__Proof__: Let B' be a submodule of M_R and Φ the collection of all submodules C of M_R such that $B' \cap C = \{0\}$. We partially order Φ by set inclusion and let Φ' be an ordered subset of Φ. Then it is clear that $C' = \cup\{C \mid C\varepsilon\Phi'\}$ is an element of Φ. By Zorn's lemma, the set Φ contains a maximal element B. In particular, $B + B' = B \oplus B'$. We now show $M = B \oplus B'$. If $\lambda \varepsilon \Omega$ and M_λ a simple submodule, then either $M_\lambda \cap (B \oplus B') = M_\lambda$ or $M_\lambda \cap (B \oplus B') = \{0\}$. If $M_\lambda \cap (B \oplus B') = \{0\}$, then $M_\lambda + B \varepsilon \Phi$ and $B \subset M_\lambda + B$. This contradicts the fact that B is maximal. Hence, $M_\lambda \subseteq B \oplus B'$ and so $M_R = B \oplus B'$.

We can now prove the fundamental theorem of this chapter.

<u>Theorem (4.12)</u>. [25] If M_R is a near-ring module, then the following statements are equivalent:

(4.12.1) Every submodule is a sum of simple submodules of M_R.

(4.12.2) M_R is a sum of simple submodules.

(4.12.3) M_R is a direct sum of simple submodules.

(4.12.4) Every submodule of M_R is a direct summand.

<u>Proof</u>: Appealing to Theorems (4.10) and (4.11) it suffices to show (4.12.4) implies (4.12.1). Let B be a submodule of M_R and B' the sum of all simple submodules of B_R. From Proposition (4.6) B' is a submodule of M_R and every simple submodule of B_R is a simple submodule of M_R. If B' is a proper submodule of B_R, then by Lemma (4.9) there is submodule $B'' \neq \{0\}$ such that $B' \oplus B'' = B_R$ and B'' contains a simple submodule of M_R. This contradicts the fact that $B' \cap B'' = \{0\}$ and so we conclude $B = B'$. Hence, we have proved that (4.12.4) implies (4.12.1).

<u>Definition (4.13)</u>. A near-ring module M_R is called semi-simple if, and only if, M_R is a direct sum of simple submodules.

We will say that a near-ring R is semi-simple if R_R is semi-simple.

In [25] Roth defined a near-ring module to be semi-simple if every submodule of M_R is a direct summand. From Theorem (4.12) we conclude that our definition is equivalent to his. It is clear that the class of all semi-simple R-modules includes all simple R-modules.

Corollary (4.14). Let M_R be a semi-simple R-module and H a submodule of M_R. Then H_R is semi-simple and the factor module M/H is semi-simple.

Proof: H_R is semi-simple by Theorem (4.12). There is a submodule H' of M_R such that $M_R = H \oplus H'$. From the second isomorphism theorem for near-ring modules we are allowed to conclude $M/H \underset{(R)}{\cong} H'$ so that the factor module M/H is semi-simple.

Corollary (4.15). Let M_R be a finitely generated semi-simple R-module. Then M_R satisfies d.c.c. on submodules and the a.c.c. on submodules.

Proof: Let $\{x_1, \ldots, x_n\}$ be a set of generators for M_R and let $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$ where M_λ is a simple submodule of M_R. Since each x_j is expressible as a finite sum of the form $\sum m_\lambda$ where $m_\lambda \in M_\lambda$, it follows that the index set Ω can be taken to be finite. If $M_R = \bigoplus_{i=1}^{n} M_i$, then $\sum: \bigoplus_{i=1}^{n} M_i \supset \bigoplus_{i=1}^{n-1} M_i \supset \cdots \supset M_1 \oplus M_2 \supset M_1 \supset \{0\}$ is a strictly decreasing normal series. From the second isomorphism theorem for near-ring modules

$$\bigoplus_{i=1}^{k} M_i \Big/ \bigoplus_{i=1}^{k-1} M_i \underset{(R)}{\cong} M_k$$ and so the factors of Σ are simple.

By Theorem (3.9) M_R satisfies d.c.c. on submodules and the a.c.c. on submodules.

External Direct Sum and Some Examples

As in the case of other algebraic systems, the notions of external direct product and external direct sum are useful for the construction of examples.

Let $\{M_\lambda \mid \lambda \in \Omega\}$ be a collection of R-modules and $M = \prod_{\lambda \in \Omega} M_\lambda$ denote the Cartesian product of the given collection. If (x_λ), $(y_\lambda) \in M$, and $r \in R$, then define $(x_\lambda) + (y_\lambda) = (x_\lambda + y_\lambda)$ and define $(x_\lambda)r = (x_\lambda \cdot r)$. With these definitions M becomes an R-module. The R-module M_R is called the module-product associated with the collection $\{M_\lambda \mid \lambda \in \Omega\}$.

<u>Lemma (4.16)</u>. Let M_R be the module-product associated with the collection $\{M_\lambda \mid \lambda \in \Omega\}$ of R-modules. Then

(4.16.1) For each $\lambda \in \Omega$ the set $M'_\lambda = \{(x_\lambda) \mid (x_\lambda) \in M_R, x_{\lambda_o} = 0 \text{ for } \lambda_o \neq \lambda\}$ is a submodule of M_R.

(4.16.2) For each λ, $M'_\lambda \underset{(R)}{\cong} M_\lambda$.

<u>Proof</u>: It is clear that M'_λ is a submodule of M_R and

so (4.16.1) follows.

For each $\lambda \varepsilon \Omega$, define the mapping π_λ as follows $\pi_\lambda : (x_\lambda) \varepsilon M'_\lambda \longrightarrow x_\lambda \varepsilon M_\lambda$. It is easy to verify that π_λ is an R-isomorphism of M'_λ onto M_λ.

The proof of the next lemma is straight forward and for that reason will be omitted.

Lemma (4.17). Let M_R be the product-module associated with the collection $\{M_\lambda \mid \lambda \varepsilon \Omega\}$ of R-modules. If M' is the set of all elements $(x_\lambda) \varepsilon M_R$ such that $x_\lambda = 0$ for all but a finite number of indices $\lambda \varepsilon \Omega$, then

(4.17.1) M' is a submodule of M_R

(4.17.2) $M'_\lambda \subseteq M'$ for each $\lambda \varepsilon \Omega$

(4.17.3) $M'_R = \bigoplus_{\lambda \varepsilon \Omega} M'_\lambda$

The near-ring module M'_R is called the external direct sum associated with the given collection $\{M_\lambda \mid \lambda \varepsilon \Omega\}$ of R-modules.

Now let $\{M_\lambda \mid \lambda \varepsilon \Omega\}$ be a collection of simple R-modules and M'_R the external direct sum associated with the collection $\{M_\lambda \mid \lambda \varepsilon \Omega\}$. Appealing to Lemmas (4.16) and (4.17) we have the following two examples.

Example (4.18). M'_R is a semi-simple R-module.

Example (4.19). If Ω is a finite index set, then M'_R satisfies d.c.c. on submodules and a.c.c. on submodules.

CHAPTER V

STRICTLY SEMI-SIMPLE NEAR-RING MODULES

The general algebraic theory of strictly semi-simple near-ring modules has not been discussed in detail. In this chapter we will develop some of the fundamental properties of this class of near-ring modules.

We introduce the concept of irreducible near-ring modules. If M_R is a direct sum of irreducible submodules and R contains an identity element, then we show in Theorem (5.6) that M_R is unitary.

Another concept is defined, regular submodule of a near-ring module, which plays an important role in our study of strictly semi-simple near-ring modules. As might be expected, every strictly semi-simple near-ring module is semi-simple.

In Theorem (5.17) we prove that a semi-simple near-ring module is strictly semi-simple if, and only if, every maximal submodule is regular. However, there exists semi-simple near-ring modules which are not strictly semi-simple.

Irreducible Near-Ring Modules

Definition (5.1). A near-ring module M_R is called irreducible if, and only if, M_R contains no proper R-subproups.

Roth [25] called an irreducible near-ring module M_R simple while Betsch [1] used the term minimal R-group.

Since a submodule of a near-ring module M_R is an R-subgroup, we note that every irreducible R-module is simple. In Chapter VII we will present an example to show the converse is not true in general.

Let G be an additive group and S(G) its associated near-ring. In Chapter I we showed that G contained no proper S(G)-subgroups and so G is an irreducible S(G)-module. If E is a multiplicative semi-group of endomorphisms of G and G contains no proper E-invariant subgroups, then G is an irreducible E(G)-module by Lemma (3.12).

The remainder of this section is devoted to the study of some of the elementary properties of irreducible near-ring modules.

We start with the useful

Lemma (5.2). Let M_R be an irreducible R-module. If $m \in M$, $m \cdot R = \{mr \mid r \in R\} \neq \{0\}$, then $M_R = m \cdot R$.

Proof: Let m be an element of M_R such that $m \cdot R \neq \{0\}$. Since M_R is irreducible it suffices to show $m \cdot R$ is an R-subgroup of M_R. If $r_1, r_2 \in R$, then $mr_1 - mr_2 = m(r_1 - r_2) \in m \cdot R$ and $(mr_1)r_2 = m(r_1 r_2) \in m \cdot R$.

From this we conclude that $m \cdot R$ is an R-subgroup of M_R.

Corollary (5.3). Let M_R be a unitary R-module. Then M_R is irreducible if, and only if, $m \cdot R = M_R$ for every non-zero element $m \in M$.

Proof: Since M_R is unitary the near-ring R contains an identity 1 such that $x \cdot 1 = x$ for all $x \in M_R$. First, we assume M_R is irreducible. If m is any non-zero element, then $m \cdot R \neq \{0\}$. From Lemma (5.2) it follows that $m \cdot R = M_R$.

Conversely, suppose $m \cdot R = M_R$ for all $m \in M_R$ with $m \neq 0$. Let A be any R-subgroup of M_R such that $A \neq \{0\}$. If $a \in A$ with $a \neq 0$, then $a \cdot R$ is an R-subgroup. Hence, we have the following $M_R = a \cdot R \subseteq A \subseteq M_R$ and so $M_R = A$.

Corollary (5.4). Every irreducible R-module M_R is R-isomorphic to a factor-module of R_R. Moreover, if R is a ring then M_R is a ring module.

Proof: Let m be a non-zero element of M_R such that $m \cdot R = M_R$. From Lemma (5.2) it is clear that the mapping $f_m: r \in R \longrightarrow mr \in M_R$ is an R-homomorphism of R_R onto M_R. Hence, $R/_{Ke(f)} \underset{(R)}{\cong} M_R$ by the fundamental theorem of R-homomorphisms.

Assume R is a ring. Since $R/_{Ke(f)} \underset{(R)}{\cong} M_R$, it follows that M is an additive abelian group. If $m_1, m_2 \in M$, $r \in R$, then there exists elements $r_1, r_2 \in R$

such that $mr_1 = m_1$, $mr_2 = m_2$ and so

$(m_1 + m_2)r = (mr_1 + mr_2)r = [m(r_1 + r_2)]r = m(r_1 r + r_2 r)$

$= (mr_1)r + (mr_2)r = m_1 r + m_2 r$. This shows that M_R is a ring module.

<u>Corollary (5.5)</u>. Let R be a near-ring with identity 1. If M_R is irreducible, then M_R is unitary.

<u>Proof</u>: Let m be a non-zero element of M_R such that $m \cdot R \neq \{0\}$. From Lemma (5.2) we see that $m \cdot R = M_R$. If $m' \in M_R$, then there is an element $r \in R$ such that $mr = m'$ and so $m' \cdot 1 = (mr) \cdot 1 = m(r \cdot 1) = mr = m'$. Hence, M_R is a unitary near-ring module.

We conclude this section with two theorems that will find direct application in our investigation of near-vector spaces which we will study in Chapter X.

<u>Theorem (5.6)</u>. Let $M_R = \bigoplus_{\lambda \varepsilon \Omega} M_\lambda$, M_λ an irreducible submodule of M_R. If R contains an identity 1, then M_R is a unitary near-ring module.

<u>Proof</u>: Let m be an element of M_R such that $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$ where $m_{\lambda_j} \varepsilon M_{\lambda_j}$. Appealing to Lemma (4.4) and Corollary (5.5) we see that $m \cdot 1 = (m_{\lambda_1} + \cdots + m_{\lambda_n}) \cdot 1 = m_{\lambda_1} \cdot 1 + \cdots + m_{\lambda_n} \cdot 1$. Therefore, M_R is a unitary near-ring module.

<u>Theorem (5.7)</u>. Let $M_R = \bigoplus_{\lambda \varepsilon \Omega} M_\lambda$, M_λ an irreducible

submodule of M_R. If R is a ring, then M_R is a ring module.

Proof: Assume R is a ring. From Corollary (5.4) and Lemma (4.2) it follows that the additive group M is abelian and the R-module M_λ is a ring module for each $\lambda \in \Omega$. Let $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$, $m' = m_{\lambda'_1} + \cdots + m_{\lambda'_k}$ be elements of M_R and let r be an element of R. If the indices λ_j and λ'_i are all distinct, then from Lemma (4.4) we have $(m + m')r =$ $[(m_{\lambda_1} + \cdots + m_{\lambda_n}) + (m_{\lambda'_1} + \cdots + m_{\lambda'_k})]r =$

$= m_{\lambda_1} \cdot r + \cdots + m_{\lambda_n} \cdot r + m_{\lambda'_1} \cdot r + \cdots + m_{\lambda'_k} \cdot r = mr + m'r.$

If $\lambda_j = \lambda'_i$ for some pair of integers j, i, then from Corollary (5.4) it follows that $(m_{\lambda_j} + m_{\lambda'_i})r =$ $= m_{\lambda_j} \cdot r + m_{\lambda'_i} \cdot r$. From this it is now evident that M_R is a ring module.

Regular Submodules of a Near-Ring Module

Definition (5.8). A submodule A of a near-ring module M_R is said to be regular if, and only if, A is maximal as an R-subgroup.

Thus a regular submodule of a near-ring module is a maximal submodule and a maximal R-subgroup of M_R. If M_R happens to be a ring module, then all maximal submodules are regular.

In [22] Laxton defined a right ideal of a d.g. near-ring R with identity which is maximal as an R-subgroup of R to be a modular right ideal. Hence, a regular right ideal of a d.g. near-ring with identity is modular in the sense of Laxton.

If M_R is an irreducible R-module, then the zero submodule is the unique regular submodule of M_R.

Lemma (5.9). A submodule A of a near-ring module M_R is regular if, and only if, the factor module M/A is irreducible.

Proof: Assume M/A is irreducible. From Lemma (1.20) it is easy to see that A is maximal as an R-subgroup of M_R and so A is regular.

Conversely, suppose A is a regular submodule of M_R. Since A is maximal as an R-subgroup it follows from Lemma (1.20) that the factor module M/A is irreducible.

As an immediate consequence of Lemma (5.9) we have the following

Corollary (5.10). A submodule A of a near-ring module M_R is regular if, and only if, A is the kernel of a non-zero R-homomorphism of M_R onto an irreducible R-module.

Let G be an additive group and S(G) the near-ring associated with G. In the previous section we noted that G considered as an S(G)-module is irreducible. From Corollary (5.10) it follows that for each $g \in G$, with $g \neq 0$, the right ideal $\left[\frac{0}{g}\right]$ is regular.

Theorem (5.11). A submodule A of a near-ring module M_R is contained in a regular submodule B if, and only if, the factor module M/A contains a regular submodule.

Proof: Let η denote the natural R-homomorphism of M_R onto $(M/A)_R$. Assume A is contained in a regular submodule B. According to the first isomorphism theorem for R-modules and Lemma (5.9) we have $M/B \underset{(R)}{\cong} \frac{M/A}{A\eta}$ and $A\eta = B/A$ is a regular submodule of the factor-module M/A.

Conversely, suppose M/A contains a regular submodule M'. From the first isomorphism theorem for near-ring modules $\eta^{-1}(M') \supseteq A$ and $M/\eta^{-1}(M') \underset{(R)}{\cong} \frac{M/A}{M'}$. Hence, $\eta^{-1}(M')$ is a regular submodule of M_R by Lemma (5.9).

Definition and Elementary Properties of Strictly Semi-Simple Near-Ring Modules

Definition (5.12). A near-ring module M_R is called strictly semi-simple if, and only if, M_R is a direct sum of irreducible submodules.

A near-ring R is said to be strictly semi-simple if R_R is strictly semi-simple.

It is clear that every irreducible R-module is strictly semi-simple. More generally, let $\{M_\lambda \mid \lambda \varepsilon \Omega\}$ be a collection of irreducible R-modules. If M'_R is the external direct sum associated with the given collection, then M'_R is a strictly semi-simple R-module.

Fröhlich [14] showed that the d.g. near-ring R generated by the multiplicative group of inner-automorphism of a finite, simple and non-abelian group is strictly semi-simple. Blackett [5] considered near-rings R that satisfy d.c.c. on R-subgroups which contained no non-zero nilpotent R-subgroups. Blackett proved that such near-rings could be written as a finite direct sum of irreducible right ideals.

<u>Theorem (5.13)</u>. A strictly semi-simple near-ring module is semi-simple.

<u>Proof</u>: Let M_R be a strictly semi-simple R-module. Since every irreducible R-module is simple it follows that M_R is a direct sum of simple submodules. Hence, M_R is a semi-simple.

We will give an example in Chapter VII (see Example (7.21)) that shows the converse of Theorem (5.13) is generally not true.

The contents of the next theorem give the essential difference between semi-simple near-modules and strictly semi-simple near-ring modules.

Theorem (5.14). Let $M_R = \bigoplus_{\lambda \varepsilon \Omega} M_\lambda$ where M_λ is an irreducible submodule of M_R. If $A \neq \{0\}$ is a simple submodule of M_R, then there exists $\lambda_o \varepsilon \Omega$ such that $A \underset{(R)}{\cong} M_{\lambda_o}$.

Proof: Let A be a non-zero simple submodule of M_R. If a is a non-zero element of A, then there is at least one index $\lambda_o \varepsilon \Omega$ such that the λ_o-component of a is non-zero. Let π denote the R-homomorphism of M_R onto the R-module M_{λ_o} that maps $m \varepsilon M$ onto its λ_o-component in M_{λ_o}. It is clear that π induces a non-zero R-homomorphism π_{λ_o} of A_R onto the R-module M_{λ_o}. From the fundamental theorem for R-homomorphisms it follows that $A/Ke(\pi_{\lambda_o}) \underset{(R)}{\cong} M_{\lambda_o}$. Since A is simple $Ke(\pi_{\lambda_o}) = 0$ so that $A \underset{(R)}{\cong} M_{\lambda_o}$. This completes the proof of the theorem.

From Theorem (5.14) we get the next two corollaries.

Corollary (5.15). Every simple submodule of a strictly semi-simple near-ring module is irreducible.

Corollary (5.16). A semi-simple near-ring M_R module is strictly semi-simple if, and only if, every simple submodule of M_R is irreducible.

Corollary (5.16) enables us to establish

<u>Theorem (5.17)</u>. A semi-simple near-ring module M_R is strictly semi-simple if, and only if, every maximal submodule of M_R is regular.

<u>Proof</u>: Assume every maximal submodule of M_R is regular. From Corollary (5.16) we note that it suffices to show every non-zero simple submodule is irreducible. If A is a non-zero simple submodule of M_R, then appealing to Theorem (4.12) and the second isomorphism theorem for near-ring modules it follows that there exists a submodule B of M_R such that $M/B \underset{(R)}{\cong} A$. Hence, B is a maximal submodule of M_R so that it is regular. Therefore, A_R is irreducible by Lemma (5.9).

Conversely, suppose M_R is strictly semi-simple. Let B be a maximal submodule of M_R. From Theorem (4.12) and the second isomorphism theorem for near-ring modules there is a submodule B' such that $M/B \underset{(R)}{\cong} B'$ and so B' is a simple submodule. Hence, B' is irreducible by Corollary (5.16) and so B is regular by Lemma (5.9).

The result to follow is a consequence of Theorem

(4.12) and Corollary (5.16).

Theorem (5.18). If $M_R = \sum_{\lambda \in \Omega} M_\lambda$, where M_λ is an irreducible submodule, then

(5.18.1) M_R is strictly semi-simple.

(5.18.2) Every submodule is a direct summand.

(5.18.3) Every submodule is strictly semi-simple.

Corollary (5.19). Every factor module of a strictly semi-simple module is strictly semi-simple.

Proof: Let M_R be strictly semi-simple R-module and A a submodule of M_R. From Theorem (5.18) and the second theorem isomorphsim theorem for near-ring modules there is a submodule B such that $M/A \underset{(R)}{\cong} B$. Hence, the factor-module M/A is strictly semi-simple by Theorem (5.18).

Corollary (5.20). A strictly semi-simple near-ring module contains at least one regular submodule.

Proof: Let M_R be a strictly semi-simple R-module, and let M' be an irreducible submodule of M_R. From Theorem (5.18) and the second isomorphism theorem for near-ring modules there is a submodule A of M_R such that $M/A \underset{(R)}{\cong} M'$. Hence, A is regular by Lemma (5.9).

Corollary (5.21). Let M_R be a unitary near-ring module such that every R-subgroup of M_R is a submodule. If R is a strictly semi-simple near-ring, then M_R is strictly semi-simple.

<u>Proof</u>: Let $R = \sum_{\lambda \varepsilon \Omega} A_\lambda$, where A_λ is an irreducible right ideal of R. If $m \varepsilon M$, then for each $\lambda \varepsilon \Omega$, the mapping f_λ: $a_\lambda \varepsilon A_\lambda \longrightarrow ma_\lambda \varepsilon m \cdot A_\lambda$ is either an R-isomorphism or the zero mapping. Since M_R is unitary, the near-ring R contains an identity so that every element of M_R is contained in an R-subgroup of the form $m \cdot A_\lambda$. From this it follows that $M_R = \sum_{\lambda \varepsilon \Omega} \sum_{m \varepsilon M} m \cdot A_\lambda$ and so M_R is strictly semi-simple by Theorem (5.18).

Let G be an additive group and E a multiplicative semi-group of endomorphisms of G that contains the set of inner-automorphisms of G. If G is considered as an E(G)-module, then every E(G)-subgroup of G is a sub-module by Lemma (3.12). From Corollary (5.21) we can state

<u>Corollary (5.22)</u>. If the d.g. near-ring E(G) is strictly semi-simple, then the E(G)-module G is strictly semi-simple.

CHAPTER VI

A RADICAL AND ITS THEORY FOR NEAR-RING MODULES

A radical for general near-ring modules has not been discussed in the literature. However, Betsch [1], Laxton [23, 24], and others have studied certain radicals for near-rings which were generally the analogues of the Jacobson radical for rings. Betsch showed that many of the results of the Jacobson radical generalize to the case of near-rings. Laxton restricted his work to d.g. near-rings with identity.

In this chapter we introduce a radical for near-ring modules which, when restricted to near-rings with identity, coincides with the radicals defined by Betsch and Laxton. The first part is devoted to the elementary properties of the radical which will be useful in what follows. Let M_R be a near-ring module and J(M) its radical. We show that $J(M/J(M)) = 0$ and J(M) is the smallest submodule A of M_R with the property that $J(M/A) = 0$.

In the second section, we introduce the concepts of small and strictly small submodule of a near-ring module. Assuming the radical J(M) of M_R to be small (resp. strictly small), we prove that J(M) is the intersection of all maximal submodules (resp. the intersection of all maximal R-subgroups). We also give a necessary and sufficient condition for J(M) to be the sum of all small (resp. strictly small) submodules of M_R.

In the third part, we apply our results to near-ring modules that satisfy the d.c.c. on submodules. We give two important theorems. First, we show that if J(M) = 0, then M_R is strictly semi-simple. Assuming J(M) is small, then we prove every maximal submodule is regular. The latter theorem will be useful in the following two chapters.

Definition and Properties of the Radical

Definition (6.1). Let M_R be a near-ring module over R, and let Φ denote the collection of all regular submodules of M_R. If Φ is not empty, then by the radical of M_R is meant $J(M) = \cap\{B \mid B \in \Phi\}$. In the case Φ is empty, the radical of M_R is defined to be J(M) = M and M_R is called a radical module.

In particular, J(R) is called the radical of near-ring R. Moreover, if J(R) = R, then R is said to be a radical near-ring. Any near-ring R that satisfies the d.c.c. on R-subgroups is not a radical near-ring. This result is a consequence of a theorem which will be given in the next chapter.

Because the intersection of an arbitrary collection of submodules of a near-ring module is a submodule we can state

<u>Proposition (6.2).</u> The radical of a near-ring module M_R is a submodule.

From Corollary (5.20) we note that a strictly semi-simple near-ring module is not a radical module. We now consider the radical of a strictly semi-simple near-ring module.

<u>Theorem (6.3).</u> If M_R is a strictly semi-simple near-ring module, then $J(M) = 0$.

<u>Proof:</u> Let $M_R = \bigoplus_{\lambda \varepsilon \Omega} M_\lambda$ where M_λ is an irreducible submodule of M_R. If $J(M) \neq 0$, then there is an element $x \varepsilon J(M)$ such that $x \neq 0$. From this it follows that there is an index $\lambda_o \varepsilon \Omega$ such that the λ_o-component m_{λ_o} of x is non-zero. Let $B = \bigoplus_{\substack{\lambda \varepsilon \Omega \\ \lambda \neq \lambda_o}} M_\lambda$

and observe that B is a regular submodule of M_R. Since $x \varepsilon J(M) \subseteq B$ it is clear that $m_{\lambda_o} \varepsilon B \cap M_{\lambda_o} = \{0\}$, a contradiction. Hence, the proof is complete.

We now give two important theorems concerning the radical of a near-ring module the first of which is

<u>Theorem (6.4).</u> Let M_R be an R-module. Then

$J(M/_{J(M)}) = 0$.

Proof: If M_R is a radical module, then $J(M) = M$ and so $J(M/_{J(M)}) = 0$. Assume M_R is not a radical-module and let η denote the natural R-homomorphism of M_R onto the factor module $M/_{J(M)}$. From Theorem (5.11) the factor module $M/_{J(M)}$ is not a radical-module and the module and $J(M/_{J(M)}) = \cap\{B\eta \mid B$ a regular submodule of $M_R\}$. If B is a regular submodule of M_R and $\bar{x} \varepsilon J(M/_{J(M)})$, then $\bar{x} \varepsilon B\eta$. Let x be a representative of the coset $\bar{x}$. Then x is an element of B and so $x \varepsilon J(M)$. This shows $\bar{x} = 0$ in the factor module $M/_{J(M)}$.

Betsch [1] and Laxton [24] proved a similar result for the particular case of near-rings.

Let M_R be a near-ring module over the near-ring R. In the following theorem we show J(M) is the smallest submodule A of M_R such that the radical of $M/_A$ is the zero submodule.

Theorem (6.5). Let M_R be an R-module and A a proper submodule of M_R such that $J(M/_A) = 0$. Then M_R is not a radical module and $J(M) \subseteq A$.

Proof: M_R contains a regular submodule by Theorem (5.11). Hence, M_R is not a radical module. Let η denote the natural R-homomorphism of M_R onto the factor module $M/_A$. From Theorem (5.11) $J(M/_A) = \cap\{B\eta \mid B$ a regular submodule of M_R that contains $A\}$. Therefore,

$0 = J(M/A) = J(M)+A/A$ so that $J(M) \subseteq A$.

Small Submodules and the Radical

Let M_R be a near-ring module over the near-ring R and A a submodule. Then

<u>Definition (6.6)</u>. A is said to be small (resp. strictly small) if, and only if, for every submodule (resp. R-subgroup) B such that $M_R = A + B$ then $M_R = B$.

Since every submodule of M_R is an R-subgroup, it is easily seen that every strictly small submodule is small. According to Theorem (6.3) the radical of a strictly semi-simple near-ring module is small and strictly small. It is well known that the radical of a ring with identity is small.

In the last section of the following chapter, we will give examples of near-rings whose radical is small and strictly small. We will also present a class of near-rings whose radical is neither small nor strictly small.

As a trivial consequence of the above definition, we give

<u>Proposition (6.7)</u>. Let A and A' be submodules of the near-ring module M_R. Then

(6.7.1) If A and A' are small, then $A + A'$ is small

(6.7.2) If A and A' are strictly small, then $A + A'$ is strictly small.

<u>Proof</u>: Suppose A and A' are small and B is a submodule of M_R such that $M_R = B + (A + A')$. Then $M_R = B + (A + A') = (B + A) + A' = B + A = B$ and so $A + A'$ is a small submodule. This proves the first part of the theorem. We can establish (6.7.2) in a similar manner.

For the remainder of this chapter we assume that all near-ring modules M_R are not radical-modules.

The first theorem of this section illustrates the importance of small and strictly small submodules of a near-ring module.

<u>Theorem (6.8)</u>. Let M_R be an R-module. Then

(6.8.1) If J(M) is small, then $J(M) = \cap\{B \mid B$ a maximal submodule of $M_R\}$.

(6.8.2) If J(M) is strictly small, then $J(M) = \cap\{B' \mid B'$ a maximal R-subgroup of $M_R\}$.

<u>Proof</u>: Assume J(M) is small and let $A = \cap\{B \mid B$ a maximal submodule of $M_R\}$. It suffices to show $J(M) \subseteq A$. If $J(M) \not\subseteq A$, then there is a maximal submodule B of M_R such that $J(M) \not\subseteq B$. From this it follows that $J(M) + B = M_R$. Since J(M) is small $M_R = B$. This shows $J(M) \subseteq A$ and (6.8.1) follows.

Suppose $J(M)$ is strictly small and let $A' = \cap\{B' \mid B'$ a maximal R-subgroup of $M_R\}$. It suffices to show $J(M) \subseteq A'$. If $J(M) \not\subseteq A'$, then there is a maximal R-subgroup B' such that $J(M) \not\subseteq B'$. Therefore, $M_R = J(M) + B'$. Since $J(M)$ is strictly small $M = B'$, a contradiction, which establishes (6.8.2).

From Theorem (6.8) we are allowed to conclude

Corollary (6.9). If the radical $J(M)$ of a near-ring module M_R is strictly small, then

$$J(M) = \cap\{B \mid B \text{ a maximal submodule of } M_R\}$$
$$= \cap\{B' \mid B' \text{ a maximal R-subgroup } M_R\}.$$

We now consider near-ring modules for which the converse of Theorem (6.8) also holds. The first result is

Theorem (6.10). Let M_R be a finitely generated R-module. $J(M)$ is small if, and only if, $J(M) = \cap\{B \mid B$ a maximal submodule of $M_R\}$.

Proof: Let $A = \cap\{B \mid B$ a maximal submodule of $M_R\}$. If $J(M)$ is small, then $J(M) = A$ by Theorem (6.8).

Assume $J(M) = A$. If $J(M)$ is not small, then there exists a proper submodule A' of M_R such that $M_R = J(M) + A'$. From Lemma (4.7) there is a maximal submodule B such that $A' \subseteq B$. Since $J(M) = A$, it follows $A' + J(M) \subseteq B$ and so $M_R = B$, a contradiction. Hence, $J(M)$ is a small submodule of M_R.

<u>Theorem (6.11)</u>. Let M_R be an R-module that is finitely generated as an R subgroup. $J(M)$ is strictly small if, and only if, $J(M) = \cap\{B \mid B$ a maximal R-subgroup of $M_R\}$.

<u>Proof</u>: Let $A = \cap\{B \mid B$ a maximal R-subgroup of $M_R\}$. If $J(M)$ is strictly small, then $J(M) = A$ by Theorem (6.8).

Assume $J(M) = A$. If $J(M)$ is not strictly small, then there is a proper R-subgroup A' such that $M_R = J(M) + A'$. From Lemma (4.7) there is a maximal R-subgroup B such that $A' \subseteq B$. Since $J(M) = A$, it follows that $M_R = J(M) + A' \subseteq B$, a contradiction. Hence, $J(M)$ is strictly small.

In Theorem (6.15) we will give a necessary and sufficient conditon for the radical of a near-ring module M_R to be the sum of all small submodules of M_R. As a preparation for this theorem, we first prove

<u>Lemma (6.12)</u>. Let A be a small submodule of M_R. Then $A \subseteq J(M)$.

<u>Proof</u>: If $A \not\subseteq J(M)$, then there is a regular submodule B such that $A \not\subseteq B$. From this it is clear that $M_R = A + B$. Since A is small, $M_R = B$, a contradiction.

Hence, A is a subset of J(M).

Corollary (6.13). Let M_R be an R-module. If J(M) is small, then J(M) is the sum of all small submodules.
Proof: Assume J(M) is small. Let A denote the sum of all small submodules. From Lemma (6.12) it is easily seen that $A \subseteq J(M) \subseteq A$ and so $J(M) = A$.

Corollary (6.14). Let M_R be an R-module. If J(M) is strictly small, then J(M) is the sum of all strictly small submodules of M_R.
Proof: Assume J(M) is strictly small. Let A denote the sum of all small submodules and A' the sum of all strictly small submodules. Since every strictly small submodule is small, it follows that $J(M) \subseteq A' \subseteq A = J(M)$ by Corollary (6.13). Hence, we conclude that $A = A' = J(M)$.

We use Lemma (6.12) to prove

Theorem (6.15). Let M_R be an R-module. J(M) is the sum of all small submodules if, and only if, every submodule of M_R that is generated by a finite subset of J(M) is small.
Proof: Let A denote the sum of all small submodules of M_R. If every submodule B of M_R that is generated by a finite subset of J(M) is small, then $J(M) \subseteq A$. From Lemma (6.12) we can conclude $A \subseteq J(M) \subseteq A$ and so

$J(M) = A$.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of $J(M)$ and let B denote the submodule of M_R that is generated by $\{x_1, \ldots, x_n\}$. Since $J(M) = A$, there exists small submodules $B_1, \ldots, B_\ell$ such that $B \subseteq \sum_{i=1}^{\ell} B_i$. If B is not small, then there exists a proper submodule C of M_R such that $M_R = B + C$. From this it is clear that $M_R = \sum_{i=1}^{\ell} B_i + C$. Appealing to Proposition (6.7) it follows that $M_R = C$, a contradiction. Therefore, B is a small submodule.

<u>Corollary (6.16)</u>. Let M_R be an R-module. $J(M)$ is the sum of all strictly small submodules if, and only if, every submodule of M_R that is generated by a finite subset of $J(M)$ is strictly small.

<u>Proof</u>: Let A denote the sum of all small submodules of M_R and A' the sum of all strictly small submodules. Since every strictly small submodule of M_R is small, it follows that $A' \subseteq A$. If every submodule of M_R that is generated by a finite subset of $J(M)$ is strictly small, then $J(M) \subseteq A'$. From Theorem (6.15) $J(M) \subseteq A' \subseteq A = J(M)$ so that $J(M) = A'$.

Conversely, let B be a submodule of M_R that is generated by a finite subset of $J(M)$. Since $J(M) = A'$, there exists strictly small submodules $B_1, \ldots, B_\ell$ of M_R such that $B \subseteq \sum_{i=1}^{\ell} B_i$. If B is not strictly small, then there is a proper R-subgroup C of M_R such that $M_R = B + C$. It is evident that $M_R = \sum_{i=1}^{\ell} B_i + C$. From Proposition (6.7) $M_R = C$, a contradiction. Hence, B is strictly small.

As an immediate consequence of Corollary (6.13) Corollary (6.14), Theorem (6.15), and Corollary (6.16) we get

Corollary (6.17). Let M_R be an R-module. Then
(6.17.1) If J(M) is small, then every submodule of M_R that is generated by a finite subset from J(M) is small.
(6.17.2) If J(M) is strictly small, then every submodule of M_R that is generated by a finite subset from J(M) is strictly small.

Chain Conditions on Submodules and the Radical

A well known theorem from ring theory states that if a ring R satisfies the descending chain condition on right ideals and has zero radical, then R can be written as a finite direct sum of simple right ideals. In Theorem (6.19) we present the analogue theorem for near-ring modules.

As a preparation, we first prove a lemma, which is also of interest in other areas of the theory of near-ring modules.

<u>Lemma (6.18)</u>. Let M_R be an R-module and $M_1, \ldots, M_n$ a set of submodules of M_R that satisfy

(6.18.1) $\bigcap_{i=1}^{n} M_i = \{0\}$.

(6.18.2) $C_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} (M_j \cap M) \neq \{0\}$.

(6.18.3) $M_R = M_i + C_i$ for all $i = 1, 2, \ldots, n$.

Then $M_R = \bigoplus_{i=1}^{n} C_i$.

<u>Proof</u>: We will use induction on n to establish the lemma. Assume that the result is true for arbitrary R-modules A_R and for all sets of at most n-1 submodules satisfying the three conditions.

First, we note that $M_R = M_n \oplus C_n$. Let $M_i' = M_i \cap M_n$ for $i = 1, 2, \ldots, n-1$. It is easy to verify that M_i' is a submodule of the R-module M_n and the three conditions hold for the set of submodules $M_1', \ldots, M_n'$. By assumption

$M_n = \bigoplus_{i=1}^{n-1} C_i$ and so $M_R = \bigoplus_{i=1}^{n} C_i$.

<u>Theorem (6.19)</u>. Let M_R be an R-module that satisfies d.c.c. on submodules and $J(M) = 0$. Then M_R is strictly semi-simple.

<u>Proof</u>: Let Φ denote the collection of all finite

intersections of regular submodules of M_R. Let C be a minimal element of Φ. If B is an arbitrary regular submodule, then $B \cap C = C$ since C is a minimal element of Φ. Hence, $C \subseteq J(M) = 0$ and so $C = 0$. We delete all superfluous elements in the intersection C and arrive at a set $B_1, \ldots, B_n$ of regular submodules of M_R such that $\bigcap_{i=1}^{n} B_i = \{0\}$ and $C_i = \bigcap_{\substack{j=1 \\ i \neq j}}^{n} B_j \neq \{0\}$. Since each B_i is regular, it is apparent that $M_R = B_i \oplus C_i$ for $i = 1, 2, \ldots, n$. From (6.18) $M_R = \bigoplus_{i=1}^{n} C_i$. It remains to show C_i is irreducible. By the second isomorphism theorem $M/B_i \underset{(R)}{\cong} C_i$ and so C_i is an irreducible submodule of M_R.

From the contents of Theorems (6.3) and (6.19) we are allowed to conclude

<u>Corollary (6.20)</u>. Let M_R be an R-module that satisfies d.c.c. on submodules. $J(M) = 0$ if, and only if, M_R is strictly semi-simple.

<u>Corollary (6.21)</u>. Let M_R be an R-module that satisfies the d.c.c. on submodules and $J(M) = 0$. Then M_R is finitely generated.

<u>Proof</u>: From Theorem (6.19) $M_R = \bigoplus_{i=1}^{n} C_i$, where C_i is a

non-zero irreducible submodule of M_R. By Lemma (5.2) there exist elements $x_i \in C_i$ such that $x_i \cdot R = C_i$. It is now easy to see that the set $\{x_1, \ldots, x_n\}$ is a set of generators for M_R.

According to Theorem (6.19), Corollary (4.15) and Corollary (6.21) we have

Corollary (6.22). Let M_R be an R-module that satisfies d.c.c. on submodules and $J(M) = 0$. Then M_R satisfies a.c.c. on submodules.

As mentioned in the introduction of the present chapter, it is important to consider the class of near-ring modules M_R satisfies the property that every maximal submodule is regular. In Chapter V we proved that every strictly semi-simple near-ring module is contained in this class. We are now in a position to enlarge our class. If M_R satisfies the d.c.c. on submodules and $J(M)$ is small, then we will prove in Theorem (6.25) that every maximal submodule of M_R is regular.

Before establishing our theorem, we first give two important lemmas.

Lemma (6.23). Let M_R be an R-module that satisfies d.c.c. on submodules. If $J(M) = 0$, then every maximal submodule is regular.

__Proof:__ Assume $J(M) = 0$. Then M_R is strictly semi-simple by Theorem (6.19). If B is a maximal submodule of M_R, then B is regular by Theorem (5.17).

__Lemma (6.24).__ Let M_R be an R-module that satisfies the d.c.c. on submodules. If B is a maximal submodule of M_R that contains J(M), then B is regular.

__Proof:__ Let η denote the natural R-homomorphism of M_R onto the factor module $M/J(M) = \overline{M}$. Theorems (3.4) and (6.4) show that $\overline{M}_R$ satisfies the d.c.c. on submodules and $J(\overline{M}) = 0$. From Theorem (6.19) $\overline{M}$ is a strictly semi-simple R-module. If B is a maximal submodule of M_R that contains J(M), then $M/B \underset{(R)}{\cong} \overline{M}/B\eta$ by the first isomorphism theorem for near-ring modules so that $B\eta$ is a maximal submodule of $\overline{M}_R$. By Theorem (5.17) $B\eta$ is regular and so B is regular.

We now use the preceeding lemmas to prove

__Theorem (6.25).__ Let M_R satisfy the d.c.c. on submodules. If J(M) is small, then every maximal submodule is regular.

__Proof:__ Let J(M) be a small submodule. From Theorem (6.8) $J(M) = \cap\{B \mid B \text{ a maximal submodule of } M_R\}$. If B is a maximal submodule, then $J(M) \subseteq B$ and so B is regular by Lemma (6.24). Therefore, every maximal submodule is regular.

Corollary (6.26). Let M_R be finitely generated and satisfy d.c.c. on submodules. J(M) is small if, and only if, every maximal submodule is regular.

Proof: If every maximal submodule is regular, then $J(M) = \cap\{B \mid B$ a maximal submodule of $M_R\}$. Hence, J(M) is small by Theorem (6.10).

The converse is a consequence of Theorem (6.25).

Finally, we consider some important consequences of the chain conditions on submodules of a near-ring module M_R. We prove first the following decomposition theorem.

Theorem (6.27). Let M_R satisfy the d.c.c. on submodules. Then M_R can be written as a finite direct sum of indecomposable submodules.

Proof: Let Φ denote the collection of all submodules of M_R that are direct summands. Since M_R satisfies d.c.c. on submodules, the collection Φ contains a minimal element A_1. If $M_R = A_1 \oplus A_1'$ where A_1' is a submodule of M_R, then submodules of the R-module A_1' are submodules of M_R by Proposition (4.6). Hence, A_1' satisfies the d.c.c. on submodules Let Φ' be the collection of all submodules of the R-module A_1' that are direct summands. If A_2 is a minimal element of Φ', then $M_R = A_1 \oplus A_2 \oplus A_2'$ where A_2' is a submodule of M_R. Proceeding in this way we will arrive at a positive integer n such that $A'_n = 0$. Therefore,

$M_R = \bigoplus_{i=1}^{n} A_i$ and A_i is an indecomposable R-module.

The famous Krull-Schmidt Theorem holds for near-ring modules. The proof is analogous to that presented in Jacobson [9] for operator groups. Therefore, we will not prove the important

Theorem (6.28). Let M_R satisfy both chain conditions on submodules and let

(6.28.1) $M_R = \bigoplus_{i=1}^{n} M_i$

(6.28.2) $M_R = \bigoplus_{j=1}^{\ell} M'_j$

be two decompositions of M_R into indecomposable submodules. Then $\ell = n$ and for a suitable ordering of M_i we have $M_i \underset{(R)}{\cong} M'_i$

Theorem (6.29). Let M_R satisfy the d.c.c. on submodules and let

$M_R = \bigoplus_{i=1}^{n} M_i$ be a decomposition of M_R into indecomposable submodules. If $J(M) = 0$, then each M_i is irreducible.

Proof: If $J(M) = 0$, then M_R is strictly semi-simple by Theorem (6.19). From Corollary (6.22) and the Krull-Schmidt Theorem for near-ring modules M_i is irreducible.

CHAPTER VII

ON THE RADICAL OF A NEAR-RING WITH IDENTITY

In this chapter we will apply many of our results from Chapter VI. We show that many of the properties of the Jacobson radical for rings can be generalized for near-rings.

We introduce the concept of quasi-regular module R-subgroup of a near-ring R. In Theorem (7.8) we show that the radical contains all quasi-regular R-subgroups. Another important result is given in Theorem (7.10) which tells us that J(R) is strictly small if, and only if, it is a quasi-regular two-sided ideal.

In the third section we will study the radical of a d.g. near-ring R that satisfies the d.c.c. on R-subgroups. The main result of this section is Theorem (7.15) which states that the radical, J(R), of R is nilpotent if, and only if, J(R) is strictly small. However, J(R) need not be nilpotent. In the last section we give an example of finite d.g. near-ring whose radical is not nilpotent.

Throughout this chapter it will be assumed that all near-rings R contain an identity element.

The Radical and Primitive Ideals

Definition (7.1). [1] A near-ring R is said to be primitive if, and only if, there is an irreducible R-module M_R such that $\left[\frac{0}{M}\right] = 0$.

A two-sided ideal of B of a near-ring R is called a primitive ideal if the factor near-ring $R/_B$ is primitive.

A natural example of a primitive near-ring is the near-ring associated with an additive group.

In the next lemma we show that the primitive ideals of a near-ring R are completely determined by irreducible R-modules.

Lemma (7.2). [1] A two sided ideal B of R is primitive if, and only if, there exists an irreducible R-module M_R such that $B = \left[\frac{0}{M}\right]$.

Proof: Assume there is an irreducible R-module M_R such that $B = \left[\frac{0}{M}\right]$. From Proposition (2.12) M can be regarded as an $R/_B$-module and so B is a primitive ideal of R.

Conversely, suppose B is a primitive two-sided ideal of R. Let M be an irreducible $R/_B$-module such that $\left[\frac{0}{M}\right] = 0 \in R/_B$. From Proposition (2.12) we conclude that M can be regarded as an irreducible R-module with $B = \left[\frac{0}{M}\right] \subseteq R$.

In the present section we will show that the radical of a near-ring R can be given in terms of irreducible R-modules, primitive ideals, and strictly semi-simple R-modules.

We first give

Lemma (7.3). [1] Let M_R be an irreducible R-module. If m is any non-zero element of M, then $\left[\frac{0}{m}\right]$ is a regular right ideal of R.

Proof: Since R contains an identity M_R is unitary by Corollary (5.5). Let m be any non-zero element of M. The mapping $f_m: r\varepsilon R \longrightarrow mr\varepsilon M_R$ is an R-homomorphism of R_R onto M_R. From the fundamental theorem of R-homomorphisms $R/\left[\frac{0}{m}\right] \underset{(R)}{\cong} M$ and so $\left[\frac{0}{m}\right]$ is a regular right ideal by Lemma (5.9).

Theorem (7.4). [1] Let R be a near-ring. Then $J(R) = \cap\{\left[\frac{0}{M}\right] \mid M$ an irreducible R-module$\}$

Proof: Let M_R be an arbitrary irreducible R-module. From Lemma (7.3) and Proposition (2.11) $J(R) \subseteq \bigcap_{m\varepsilon M} \left[\frac{0}{m}\right] = \left[\frac{0}{M}\right]$. Hence, $J(R) \subseteq \cap\{\left[\frac{0}{M}\right] \mid M$ an irreducible R-module$\}$.

Let $x \varepsilon \cap\{\left[\frac{0}{M}\right] \mid M$ an irreducible R-module$\} = B$. If A is a regular right ideal of R, then R/A is an irreducible R-module by Lemma (5.9). From Proposition (2.10) $x \varepsilon \left[\frac{0}{R/A}\right] = \left[\frac{A}{R}\right]$ and so $x\cdot 1 \varepsilon A$. This shows $B \subseteq J(R)$ and so $J(R) = B = \cap\{\left[\frac{0}{M}\right] \mid M$ an irreducible R-module$\}$.

Appealing to Lemma (7.2) and Theorem (7.4), we are allowed to conclude

<u>Corollary (7.5)</u>. [1] Let R be a near-ing. Then

(7.5.1) $J(R) = \cap\{B \mid B$ a primitive ideal of $R\}$.

(7.5.2) $J(R)$ is a two sided ideal of R.

(7.5.3) If R is a primitive near-ring, then $J(R) = 0$.

We now characterize the radical of a near-ring R by stricly semi-simple R-modules.

<u>Theorem (7.6)</u>. Let R be a near-ring and $B = \cap\{\left[\frac{0}{M}\right] \mid$ M a strictly semi-simple R-module$\}$. Then $J(R) = B$.

<u>Proof</u>: Since every irreducible R-module is strictly semi-simple $B \subseteq J(R)$ by Theorem (7.4). Let $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$, M_λ an irreducible submodule. To complete the proof it suffices to show $\left[\frac{0}{M}\right] = \bigcap_{\lambda \in \Omega}\left[\frac{0}{M_\lambda}\right]$. It is easy to see that $\left[\frac{0}{M}\right] \supseteq \bigcap_{\lambda \in \Omega}\left[\frac{0}{M_\lambda}\right]$. Let $r \in \bigcap_{\lambda \in \Omega}\left[\frac{0}{M_\lambda}\right]$ and $m = m_{\lambda_1} + \cdots + m_{\lambda_n}$ where $m_{\lambda_j} \in M_{\lambda_j}$. Then by Lemma (4.4), $mr = (m_{\lambda_1} + \cdots + m_{\lambda_n})r = m_{\lambda_1} \cdot r + \cdots + m_{\lambda_n} \cdot r = 0$ and so $r \in \left[\frac{0}{M}\right]$. This shows $\left[\frac{0}{M}\right] = \bigcap_{\lambda \in \Omega}\left[\frac{0}{M_\lambda}\right]$ and so the result follows.

Quasi-Regular R-Subgroups

Definition (7.7). An element r of a near-ring R is said to be quasi-regular if, and only if, there exists an element $r_o \in R$ such that $(1 - r)r_o = 1$.

A non-empty subset B of a near-ring R is called quasi-regular if every element of B is quasi-regular.

In ring theory, it is well known that the Jacobson radical of a ring contains all quasi-regular right ideals and is itself a quasi-regular two-sided ideal. This is not the case in the theory of near-rings. We now show that the radical of a near-ring R contains all quasi-regular R-subgroups. However, we will prove in Theorem (7.20) that the radical of a near-ring is not a quasi-regular two sided ideal in general.

Theorem (7.8). Let R be a near-ring and B a quasi-regular R-subgroup. Then $B \subseteq J(R)$.

Proof: If $B \not\subseteq J(R)$, then there exists a regular right ideal B' of R such that $B \not\subseteq B'$. Since B' is maximal as an R-subgroup, it follows that $R = B' + B$. If $1 = b' + b$ where $b' \in B'$, $b \in B$, then $1 - b = b' \in B'$. Now B is quasi-regular so that there is an element $b_o \in R$ such that $1 = (1 - b)b_o = b' \cdot b \in B'$ and so $1 \in B'$. From this it is easily seen that $B \subseteq J(R)$.

Lemma (7.9). Let R be a near-ring and let $A = \cap\{B | B$ a maximal R-subgroup}. Then A is a quasi-regular R-subgroup and A contains all quasi-regular right ideals of R.

__Proof:__ Let a be an element of A. We assert that $(1-a)\cdot R = R$. For if $(1-a)\cdot R$ is a proper R-subgroup, then $(1-a)\cdot R$ is contained in a maximal R-subgroup B by Lemma (4.7). Hence, $1 = (1-a) + a \in B$ and so $1 \in B$. This shows $(1-a)\cdot R = R$ so that there is an element $r \in R$ such that $(1-a)r = 1$. From this we conclude that A is a quasi-regular R-subgroup.

Let A' be a quasi-regular right ideal of R. If A' is not contained in A, then there is a maximal R-subgroup B such that $R = B + A'$. Let $1 = b + a'$ where $a' \in A'$, $b \in B$. Then there is an element $r \in R$ such that $1 = (1-a')r = br \in B$. Therefore, we can conclude that $A' \subseteq A$.

Finally, we give a necessary and sufficient condition for the radical of a near-ring to be quasi-regular.

__Theorem (7.10).__ Let R be a near-ring. J(R) is strictly small if, and only if, J(R) is a quasi-regular ideal.

__Proof:__ Since R contains an identity, it is finitely generated as an R-subgroup of R_R. Assume J(R) is a quasi-regular ideal. From Lemma (7.9) and Theorem (6.11) we conclude that J(R) is strictly small.

Conversely, suppose J(R) is strictly small. From Theorem (6.11) and Lemma (7.9) it follows that J(R) is a quasi-regular two-sided ideal.

Radical of Distributively Generated Near-Rings That Satisfy the Descending Chain Condition on R-subgroups.

Let R be an arbitrary d.g. near-ring. If A and B are non-empty subsets of R, then let $A \circ B$ denote the subgroup of the additive group of R that is generated by elements of the form $a \cdot b$ where $a \in A$, $b \in B$. In particular, if A and B are R-subgroups of R, then Fröhlich [12] showed $A \circ B$ is an R-subgroup.

For later use in this section we give

Lemma (7.11). Let A, B be R-subgroups of the d.g. near-ring R. If $r \in R$, then $r(A \circ B) = (rA) \circ B$.

Proof: First, we note that $r(A \circ B)$ and $r \cdot A$ are R-subgroups. If $x \in r(A \circ B)$, then $x = r(\sum_{i=1}^{n} a_i b_i) = \sum_{i=1}^{n} (ra_i) b_i$ where $a_i \in A$, $b_i \in B$. Hence, it follows that $x \in (rA) \circ B$. Similarly, $(rA) \circ B \subseteq r(A \circ B)$ and so $(rA) \circ B = r(A \circ B)$.

We now give Fröhlich's definition of nilpotent subset of a d.g. near-ring.

Definition (7.12). [12] Let R be an arbitrary d.g. near-ring. A non-empty subset B of R is said to be nilpotent if, and only if, there is a positive integer n such that $b_1 \cdots b_n = 0$ for all finite products of n-elements from B.

Laxton [23] (see thm 1.5) showed that the radical of a d.g. near-ring R contains all nilpotent R-subgroups. In particular, if R satisfies d.c.c. on R-subgroups, then Laxton proved $J(R) = 0$ if, and only if, R contains no non-zero nilpotent R-subgroups (see thm 2.3).

The contents of the following lemma will be useful throughout the remainder of this chapter.

Lemma (7.13). Let R be a d.g. near-ring (with identity) that satisfies d.c.c. on R-subgroups. If A is a quasi-regular R-subgroup, then A is nilpotent.

Proof: Let A be a non-zero quasi-regular R-subgroup of R. For each positive integer n let A_n denote the R-subgroup that is generated by finite products of n-elements from A. Hence, $A = A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ is a decreasing sequence of R-subgroups of R and so there is a positive integer k such that $A_k = A_{k+1} = \cdots$. Let $B = A_k$ and assume $B \neq 0$. We will obtain a contradiction to this assumption. Since $B \supseteq B \circ B \supseteq A_k \cdot A_k = \{a_k' \cdot a_k'' \mid a_k', a_k'' \in A_k\}$, it follows that $B \supseteq B \circ B \supseteq A_{2k} = A_k = B$ and so $B = B \circ B$. From this we observe that there exists a minimal R-subgroup B' that is contained in B and $B' \circ B \neq 0$. Therefore, there is an element $b' \varepsilon B'$ such that $b' \cdot B \neq 0$. From this it follows that $b' \cdot B$ is a non-zero R-subgroup that is contained in B'. By Lemma (7.11) $(b'B) \circ B = b'(B \circ B) = b'B \neq 0$. Since B' is minimal, we see that $b' \cdot B = B'$ so that there is an element $b \varepsilon B$ such that $b' \cdot b = b'$. But, A is a quasi-regular R-subgroup and so there exists an element $y \varepsilon R$ such that $(1-b)y = 1$ and so $0 = b'(1-b) \cdot y = b'$. This is a contradiction, and we are allowed to conclude that $B = 0$. In particular, A is a nilpotent R-subgroup.

From Lemma (7.9), Lemma (7.13), and Theorem (7.10) we obtain the following two results.

<u>Theorem (7.14)</u>. Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If $A = \cap\{B \mid B$ a maximal R-subgroup$\}$, then A is nilpotent.

<u>Theorem (7.15)</u>. Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. J(R) is nilpotent if, and only if, J(R) is strictly small.

Appealing to Theorems (7.14) and (7.15) we are allowed to give

<u>Corollary (7.16)</u>. Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups and let $A = \cap\{B \mid B$ a

maximal R-subgroup}. If J(R) is strictly small, then A is a two-sided ideal of R.

Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. Laxton [23] (see thm. 3.5) showed that the radical J(R) is nilpotent if, and only if, every maximal right ideal is regular. From this result, Theorem (6.25) and Theorem (6.10) we have

Theorem (7.17). J(R) is nilpotent if, and only if, J(R) is small.

Combining the results of Theroems (7.15) and (7.17) we can conclude

Theorem (7.18). If J(R) is nilpotent, then J(R) is small and strictly small.

Remarks and Examples

Laxton [24] constructed a finite d.g. near-ring with identity whose radical is non-nilpotent. We now give his example.

Let G be an additive group that is finite, non-abelian, and simple. Let A(G) denote the d.g. near-ring with identity that is generated by the inner-automorphisms of G. If $x \in A(G)$, then let φ_x: $y \in A(G) \longrightarrow x \cdot y \in A(G)$. Since the near-ring A(G) satisfies the left distributive law, the set $\{\varphi_x | x \in A(G)\}$ is a multiplicative semi-group of endomorphisms of the additive group of A(G).

Example (7.19). Let R be the d.g. near-ring with identity that is generated by $\{\varphi_x | x \in A(G)\}$, then J(R) is non-nilpotent.

From Example (7.19), Theorems (7.15), (7.17), and (7.10) we are allowed to conclude

Theorem (7.20). There exist finite d.g. near-rings R with identity that satisfy the following properties.

(7.20.1) J(R) is neither small nor strictly small.

(7.20.2) R contains at least one maximal right ideal B that is not regular.

(7.20.3) J(R) is not a quasi-regular two sided ideal.

Let R be a finite d.g. near-ring with identity and B a maximal right ideal of R that is not regular. Then the R-module R/B is simple and not irreducible. Let Ω be a non-empty set. For each $\lambda \in \Omega$ let $M_\lambda = R/B$. Hence, we have

Example (7.21). The external direct sum M_R' that is determined by $\{M_\lambda | \lambda \in \Omega\}$ is a semi-simple near-ring module that is not strictly semi-simple.

In [23] Laxton gave examples of finite d.g. near-rings R with identity whose radical is nilpotent. If the additive group of a finite d.g. near-ring R with identity is solvable, then J(R) is nilpotent. The following example is due to Laxton.

Let G be an additive p-group that is non-abelian. Let R denote the finite d.g. near-ring with identity 1 that is generated by the multiplicative semi-group of endomorphisms of G. It is clear that the additive order of the identity 1 is a power of p and so the additive order of R is a power of p. Hence, the additive group of R is solvable and so J(R) is nilpotent.

CHAPTER VIII

ON A DECOMPOSITION FOR DISTRIBUTIVELY GENERATED NEAR-RINGS WITH IDENTITY

Let R be a d.g. near-ring with identity element 1 that satisfies the descending chain condition on R-subgroups. From Theorem (6.27) $R = A_1 \oplus \cdots \oplus A_n$, where A_i is a non-zero indecomposable right ideal of R. We assume that the radical, J(R), of the d.g. near-ring R is nilpotent.

The purpose of this chapter is to determine the properties of the right ideals A_i.

We introduce the concept of minimal non-nilpotent R-subgroup and show that every minimal non-nilpotent R-subgroup is generated by an idempotent element. Moreover, a right ideal A is minimal non-nilpotent if, and only if, A is indecomposable. Also, if A is a minimal non-nilpotent right ideal of R, then $A \cap J(R)$ is the unique regular right ideal of R that is contained in A. Hence, each A_i, given by the above decomposition, is a minimal non-nilpotent right ideal of R that contains $A_i \cap J(R)$. Moreover, $A_i \cap J(R)$ is the unique regular right ideal of R that is contained in A_i.

In the last section we show that the decomposition given above induces a decomposition of the d.g. near-ring $R/J(R)$ into non-zero irreducible right ideals.

Orthogonal Idempotents

<u>Definition (8.1)</u>. An element $e \varepsilon R$ is called an idempotent if, and only if, $e^2 = e \neq 0$. A set of elements $e_1, \ldots, e_n$ of R is called a set of orthogonal idempotents if, and only if,

$$e_i \cdot e_j = \begin{cases} e_i \text{ for } i = j \\ 0 \text{ for } i \neq j. \end{cases}$$

We now show that a decomposition $R = \bigoplus_{i=1}^{n} A_i$, where A_i is a non-zero indecomposable right ideal of R, determines a set of orthogonal idempotents.

<u>Theorem (8.2)</u>. Let $R = \bigoplus_{i=1}^{n} A_i$ be a decomposition of R into non-zero indecomposable right ideals. If $1 = \sum_{i=1}^{n} e_i$ where $e_i \varepsilon A_i$, then $e_i \cdot R = A_i$ and $e_1, \ldots, e_n$ is a set of orthogonal idempotents.

<u>Proof</u>: Let $1 = e_1 + \cdots + e_n$ where $e_i \varepsilon A_i$. If $a_i \varepsilon A_i$, then $a_i = e_1 a_i + \cdots + e_n a_i$ by Lemma (4.4), and so $e_i \cdot a_i = a_i$. In particular,

$$e_i \cdot e_j = \begin{cases} e_i & i = j \\ 0 & i \neq j. \end{cases}$$

Hence, it suffices to show $e_i R = A_i$. Since A_i is a right ideal, $A_i = e_i A_i \subseteq e_i R \subseteq A_i$ and so $e_i R = A_i$. This completes the proof of the theorem.

Minimal Non-Nilpotent R-Subgroups of R

Definition (8.3). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. A non-zero R-subgroup A of R is called minimal non-nilpotent if, and only if, A is non-nilpotent and every proper R-subgroup of A_R is nilpotent.

Definition (8.4). Let R be a d.g. near-ring (with identity) that satisfies d.c.c. on R-subgroups. A non-empty subset B of R is called nil if, and only if, every element of B is nilpotent.

Laxton [23] (see Theorem 2.7) proved that every nil R-subgroup of R is nilpotent. He also showed that every nilpotent R-subgroup of R is contained in J(R) (see Theorem 1.5). Hence, we have

Corollary (8.5). An R-subgroup B of R is nil if, and only if, it is nilpotent.

Blackett [5] (see thm 1) proved that if R is a near-ring that satisfies d.c.c. on R-subgroups and contains no non-zero nilpotent R-subgroups, then every minimal R-subgroup is generated by an idempotent. In the following theorem we will prove a similar result for d.g. near-rings R with identity that have a nilpotent radical.

Theorem (8.6). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If A is a minimal non-nilpotent R-subgroup, then there exists an idempotent $e \in A$ such that $e \cdot A = e \cdot R = A$.

Proof: Let A be a minimal non-nilpotent R-subgroup. From Corollary (8.5) there exists an element $a_1 \in A$ such that a_1 is non-nilpotent. Since $a_1^2 \neq 0$, we see that $a_1 \cdot A = A$ and so there is an element $a_2 \in A$ such that $a_1 \cdot a_2 = a_1$. If a_2 is nilpotent, then $0 = a_1 a_2^n = (a_1 a_2) a_2^{n-1} = a_1 a_2^{n-1} = \dots = a_1 \cdot a_2 = a_1$ for some positive integer n. Hence, a_2 is non-nilpotent and $a_2 \cdot A = A$. There is an element $a_3 \in A$ such that $a_2 \cdot a_3 = a_2$. Proceeding in this way we obtain a sequence of elements $a_1, \dots, a_n, \dots$ from A with the following properties

1. a_i is non-nilpotent
2. $a_i \cdot a_{i+1} = a_i$
3. $a_i \cdot A = A$.

We next show $\left[\frac{0}{a_i}\right] \supseteq \left[\frac{0}{a_{i+1}}\right]$. If $r \in \left[\frac{0}{a_{i+1}}\right]$, then $a_i r = (a_i a_{i+1}) r = a_i (a_{i+1} r) = a_i \cdot 0 = 0$ and so $r \in \left[\frac{0}{a_i}\right]$. By the d.c.c. on R-subgroups there is a positive integer n such that $\left[\frac{0}{a_n}\right] = \left[\frac{0}{a_{n+1}}\right] = \dots$. Then by $a_n \cdot a_{n+1} = a_n$, it follows that $a_n(a_{n+1} - 1) = 0$. This shows $a_{n+1} - 1 \in \left[\frac{0}{a_n}\right] = \left[\frac{0}{a_{n+1}}\right]$ and so $a_{n+1}^2 = a_{n+1}$.

Hence, a_{n+1} is an idempotent element and it is clear that $a_{n+1}A = A = a_{n+1}R$.

<u>Corollary (8.7).</u> Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If A is a minimal non-nilpotent right ideal, then A is a direct summand.

<u>Proof</u>: Let A be a minimal non-nilpotent right ideal. By Theorem (8.6) there is an idempotent $e \in A$ such that $e \cdot R = A$. If $A' = \left[\frac{0}{e}\right]$, then we show that $R = A \oplus A'$. Let r be any element of R. Then $r = er + (-er + r)$ and $e(-er + r) = e(-er) + er = -e^2 r + er = 0$. From this we conclude that $R = A \oplus A'$.

<u>Corollary (8.8).</u> Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups and A a non-nilpotent right ideal. A is indecomposable if, and only if, A is minimal non-nilpotent.

<u>Proof</u>: Assume A is minimal non-nilpotent. If $A = A_1 \oplus A_2$ where A_1, A_2 are right ideals of R that are non-zero, then A_1, A_2 are nilpotent. As remarked earlier in this chapter every nilpotent R-subgroup is contained in J(R) and so $A_1 \oplus A_2 \subseteq J(R)$. From this we conclude A is indecomposable.

Conversely, suppose A is indecomposable. Let A' be a minimal non-nilpotent right ideal of R that is contained in A. From Corollary (8.7) there is a right ideal B of R such that $R = A' \oplus B$, and it follows that $A = A' \oplus (A \cap B)$. Hence, we see that $A = A'$.

Lemma (8.9). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If A is a minimal non-nilpotent right ideal of R, then $A \cap J(R)$ is the unique regular right ideal of R that is contained in A.

Proof: Let A be a minimal non-nilpotent right ideal of R. If A' is a proper R-subgroup of A_R, then A' is nilpotent and so $A' \subseteq A \cap J(R)$. Since J(R) is nilpotent and $A \cap J(R) \subseteq J(R)$, it follows that the right ideal $A \cap J(R)$ is nilpotent. Hence, $A \cap J(R)$ is the unique regular right ideal of R that is contained in A.

Corollary (8.10). Let R be a d.g. near-ring (with identity) that satisfies d.c.c. on R-subgroups. If A is a minimal non-nilpotent right ideal of R and e is an idempotent element such that eR = A, then $eJ(R) = A \cap J(R)$.

Proof: Since J(R) is a two-sided ideal of R, $eJ(R) \subseteq J(R)$ and so $eJ(R) \subseteq A \cap J(R)$. If $er \in A \cap J(R)$, then $er = e^2 r \in eJ(R)$. Hence, $eJ(R) = A \cap J(R)$.

Corollary (8.11). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If J(R) = 0 and A is a minimal non-nilpotent right ideal of R, then A_R is irreducible.

Proof: This result is an easy consequence of Lemma (8.9).

Corollary (8.12). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. If M_R is an irreducible R-module, then there exists a minimal non-nilpotent right ideal A of R such that

$$A/_{A\cap J(R)} \underset{(R)}{\cong} M.$$

Proof: Let M_R be an irreducible R-module. Since R contains an identity, M_R is unitary by Corollary (5.4). Let $R = \bigoplus_{i=1}^{n} A_i$ be a decomposition of R into indecomposable right ideals with $A_i \neq 0$. If $1 = e_1 + \cdots + e_n$, $e_i \varepsilon A_i$, $m \varepsilon M_R$, then $m \cdot e_i \neq 0$ for at least one index $i = 1, \ldots, n$. Hence, the mapping $f: a_i \varepsilon A_i \longrightarrow ma_i \varepsilon M_R$ is an R-homomorphism of the R-module A_i onto M_R. From Corollary (8.8), Lemma (8.9) and the fundamental theorem of R-homomorphisms

$$A_i/_{A_i \cap J(R)} \underset{(R)}{\cong} M_R$$ and A_i is minimal non-nilpotent.

If we summarize the results of this section, we have the fundamental

Theorem (8.13). Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups. Let $R = \bigoplus_{i=1}^{n} A_i$ be a decomposition of R into indecomposable right ideals.

Then

(8.13.1) The right ideals $A_1, \ldots, A_n$ are minimal non-nilpotent.

(8.13.2) The right ideals $A_1, \ldots, A_n$ are generated by orthogonal idempotents $e_1, \ldots, e_n$ where $e_i \in A_i$ and $1 = \sum_{i=1}^{n} e_i$.

(8.13.3). $A_i \cap J(R) = e_i J(R)$ is the unique regular right ideal of R that is contained in A_i.

A Decomposition Theorem for the Radical Factor Near-Ring $R/_{J(R)}$

Let R be a d.g. near-ring that satisfies d.c.c. on R-subgroups, and let η denote the natural near-ring homomorphism of R onto $R/_{J(R)} = \overline{R}$. Let $R = \bigoplus_{i=1}^{n} A_i$ be a decomposition of R into non-zero indecomposable right ideals. From Lemma (2.7) we know $A_i\eta$ is a right ideal of the near-ring $\overline{R}$. We now show that the decomposition $R = \bigoplus_{i=1}^{n} A_i$ determines a decomposition of $\overline{R}$ into irreducible right ideals.

<u>Theorem (8.14)</u>. $\overline{R} = \bigoplus_{i=1}^{n} A_i\eta$ is a decomposition of the d.g. near-ring $\overline{R}$ into non-zero irreducible right ideals.

<u>Proof</u>: Let $1 = e_1 + \cdots + e_n$ where $e_i \in A_i$. Then from

Theorem (8.2), $e_1, \ldots, e_n$ is a set of orthogonal idempotents. Since J(R) is nilpotent, it is easy to see that $A_i\eta = (e_iR)\eta = \bar{e}_i\bar{R}$ is non-zero in $\bar{R}$. From Lemma (4.4) $r = e_1r + \ldots + e_nr$ for all $r \in R$ and so we note that $\bar{r} = \bar{e}_1\bar{r} + \ldots + \bar{e}_n\bar{r}$. This shows

$\bar{R} = \sum_{i=1}^{n} \bar{e}_i\bar{R}$ and $\bar{e}_1, \ldots, \bar{e}_n$ is a set of orthogonal idempotents for $\bar{R}$. If $\bar{x} \in \bar{e}_i\bar{R} \cap \sum_{\substack{j=1 \\ i\neq j}}^{n} \bar{e}_j\bar{R}$, then

$$\bar{e}_i\bar{x} = \bar{x} = \bar{e}_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \bar{e}_j\bar{r}_j\right) =$$

$= \sum_{\substack{j=1 \\ i\neq j}}^{n} (\overline{e_ie_j})\bar{r}_j$ for some $\bar{r}_j \in \bar{R}$. Hence, $\bar{x} = 0$ and so

$\bar{R} = \bigoplus_{i=1}^{n} \bar{e}_i\bar{R} = \bigoplus_{i=1}^{n} A_i\eta$. We note that η can be considered as an R-homomorphism of R_R onto $\bar{R}_R$. From the second isomorphism theorem for R-homomorphisms

$$A_i\eta \underset{(R)}{\cong} {}^{A_i+J(R)}\!\big/_{A_i} \underset{(R)}{\cong} {}^{A_i}\!\big/_{A_i \cap J(R)} \; .$$

From Lemma (8.9) $A_i\eta$ is an irreducible R-module and so it is easy to see that $A_i\eta$ is an irreducible right ideal of $\bar{R}$.

CHAPTER IX

DIVISION NEAR-RINGS AND MATRICES OVER NEAR-RINGS

In the present chapter we introduce the basic concepts of division near-ring and matrices over an arbitrary near-ring. Zassenhaus [26] studied finite division near-rings and constructed many examples of of division near-rings that were not fields.

The first section is devoted to the study of division near-rings. In Theorem (9.2) we give a necessary and sufficient condition for a near-ring R with right identity to be a division near-ring.

In the last section of this chapter we show that matrices over an arbitrary near-ring with identity do not satisfy the usual properties of matrices over a ring.

The main objective of this chapter is to provide the basic theory of division near-rings that is needed to study near-vector spaces in the following chapter. Since matrices over a division ring play a fundamental role in the theory of vector spaces, it seems natural to investigate matrices over near-rings.

Division Near-Rings

Definition (9.1). A near-ring R that contains more than one element is said to be a division near-ring if, and only if, the set R' of non-zero elements is a multiplicative group.

Every division ring is an example of a division near-ring.

Zassenhaus [26] called a division near-ring a complete near-field. Zassenhaus showed that the additive group of a finite division near-ring is abelian.

Let R be an arbitrary near-ring with identity. As in the case of rings, $C(R) = \{a \in R \mid ar = ra$ for all $r \in R\}$ is called the multiplicative center of R. It is easy to see that C(R) is a multiplicative semi-group that contains the additive and multiplicative identity of R.

Let $q = p^{\ell}$ be a power of a prime p and let n be an integer all of whose prime factors divide q-1, where we also require $n \not\equiv 0 \pmod 4$ if $q \equiv 3 \pmod 4$. Then Zassenhaus [26] constructed a division near-ring R, that is not a ring, with q^n elements from a finite Galois field $GF(q^n)$ in the following way:

1. The elements of R are the same as the elements of $GF(q^n)$

2. Addition in R is the same as in $GF(q^n)$

3. A product $x \circ y$ in R can be defined in terms of the multiplication in $GF(q^n)$, in the following way:

Let z be a fixed primitive root of $GF(q^n)$; then if $x = z^{\ell n+j}$, an integer i is uniquely determined modulo n by $q^i \equiv 1 + j(q-1)$ [mod n(q-1)]. We define the product $x \circ y$ by the rule $x \circ y = y \cdot x^{q^i}$. Moreover, he proved that C(R) is just the Galois field of q-elements.

In this section we consider several easy consequences of the definition of division near-ring. First, we give

Theorem (9.2). Let R be a near-ring that contains a right identity element $e \neq 0$. R is a division near-ring if, and only if, R contains no proper R-subgroups.

Proof: Assume R is a division near-ring and let 1 denote the identity for the multiplicative group of non-zero elements R'. Let B be a non-zero R-subgroup and let b be a non-zero element of B. Since R is a division near-ring, there is an element $b' \in R$ such that $b \cdot b' = 1$ and so $1 \in B$. Hence, $1 \cdot r = r \in B$ for all $r \in R$. This shows $B = R$.

Conversely, suppose R contains no proper R-subgroups. We first show that e is a left identity for R. Since $e \notin \left[\frac{0}{e}\right]$, it follows that $\left[\frac{0}{e}\right] = 0$. Let r be any non-zero element of R. Then $e(er - r) =$ $= e^2 r + e(-r) = er - er = 0$ so that $er - r = 0$. This shows e is a left identity for R. Since $re = r \neq 0$, it is clear that $rR = R$ and so there is an element $r' \varepsilon R$ such that $rr' = e$. Similarly, there is an $r'' \varepsilon R$ such that $r' \cdot r'' = e$. From this we have $r'r = (r'r)(r'r'') = r'(rr')r'' = r'r'' = e$ and so R is a division near-ring.

As an easy consequence of Theorem (9.2) we have

<u>Corollary (9.3)</u>. A near-ring with identity $1 \neq 0$ is a division near-ring if, and only if, R contains no proper R-subgroups.

In Theorem (9.2) the assumption that R contains a right identity is essential. We construct an example of a near-ring K that contains no proper K-subgroups but is not a division near-ring.

<u>Example (9.4).</u> Let G be an additive group and K the near-ring of constant mappings of G into G. Then K contains no proper K-subgroups and K is not a division near-ring. No near-ring in this case!

<u>Proof</u>: Let K' denote the set of non-zero elements of K. If $\varphi_{g_1}, \varphi_{g_2} \varepsilon K'$, then we have seen earlier that $\varphi_{g_1} \cdot \varphi_{g_2} = \varphi_{g_2}$ and so K does not have a right identity. Let B be a non-zero K-subgroup of the near-ring K. Let φ_g be any non-zero element of B. If $\varphi_h \varepsilon K$, then $\varphi_h = \varphi_g \cdot \varphi_h \varepsilon B$ and so $B = K$. It is clear that the

near-ring K is not a division near-ring.

In particular, if G is finite, then K is not a division near-ring. However, the near-ring K has no non-zero divisors of zero. Therefore, the analogue of the famous Wedderburn Theorem for rings does not hold for finite near-rings.

Theorem (9.5). Let R be a primitive near-ring with identity $1 \neq 0$. R is a division ring if, and only if, R contains no proper right ideals.

Proof: If R is a division near-ring, then R contains no proper right ideals by Theorem (9.2).

Assume R contains no proper right ideals. Let M_R be an irreducible R-module. From Corollary (5.4) M_R is unitary. If $m \in M_R$, $m \neq 0$, then the mapping $f: r \in R \longrightarrow mr \in M_R$ is a non-zero R-homomorphism of R_R onto M_R. From the fundamental theorem of R-homomorphisms $R/\left[\frac{0}{m}\right] \underset{(R)}{\cong} M_R$ Since R contains no right ideals, $R \underset{(R)}{\cong} M_R$ and so R contains no proper R-subgroups.

Hence, R is a division near-ring by Theorem (9.2).

Matrices over Near-Rings

Definition (9.6). Let R be a near-ring. By an nxn matrix over R is meant an nxn rectangular array

$$(a) = (\alpha_{ij}) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

of n rows and n columns with elements α_{ij} contained in R.

Let $\mathcal{M}(R)$ denote the set of all nxn matrices over R. Two matrices (a) and (b) are regarded as equal if, and only if, $\alpha_{ij} = \beta_{ij}$ for every i,j.

We define addition in $\mathcal{M}(R)$ by the formula

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha_{11}+\beta_{11} & \alpha_{12}+\beta_{12} & \cdots & \alpha_{1n}+\beta_{1n} \\ \alpha_{21}+\beta_{21} & \alpha_{22}+\beta_{22} & \cdots & \alpha_{2n}+\beta_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1}+\beta_{n1} & \alpha_{n2}+\beta_{n2} & \cdots & \alpha_{nn}+\beta_{nn} \end{bmatrix} \in \mathcal{M}(R).$$

Thus to obtain the sum of two elements from $\mathcal{M}(R)$, we add the elements α_{ij} and β_{ij} in the same position. It is easy to verify that $\mathcal{M}(R)$ is an additive group with this addition. The zero matrix is the matrix whose elements are $0 \in R$ and the additive inverse of (α_{ij}) is the matrix $(-\alpha_{ij})$.

Multiplication of matrices is defined by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \Delta \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{bmatrix}$$

where $\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$.

In the remainder of this section we assume that $\mathcal{M}(R)$ is the set of all nxn matrices over R and the positive integer n is greater than one.

The next theorem tells us that generally, $(\mathcal{M}(R),\Delta)$ is a groupoid and not a semi-group.

<u>Theorem (9.7)</u>. Let R be a near-ring with identity and let the additive group of R be abelian. $(\mathcal{M}(R),\Delta)$ is a semi-group if, and only if, the near-ring R is a ring.

<u>Proof</u>: Assume R is a ring. Then it is well known that $(\mathcal{M}(R),\Delta)$ is a semi-group.

Conversely, suppose $(\mathcal{M}(R),\Delta)$ is a semi-group. Since the additive group of R is abelian, it suffices to show R satisfies the right distributive law. Let r_1, r_2, and r be any three elements of R Let

$$(a) = \begin{bmatrix} r_1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad (b) = \begin{bmatrix} 1 & r_2 & 0 & \cdots & 0 \\ r_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ and}$$

$$(c) = \begin{bmatrix} r & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ then}$$

$$(a)\Delta((b)\Delta(c)) = \begin{bmatrix} (r_1r+r_1r_2+r_2r) & 0 & 0 & \cdots & 0 \\ r_2r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} =$$

$$= ((a)\Delta(b))\Delta(c) = \begin{bmatrix} (r_1+r_2)r+r_1r_2 & 0 & 0 & \cdots & 0 \\ r_2r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and so $(r_1+r_2)r + r_1r_2 = r_1r + r_1r_2 + r_2r$. This shows that $(r_1 + r_2)r = r_1r + r_2r$.

As an immediate consequence of Theorem (9.7) we have

<u>Corollary (9.8)</u>. Let R be a near-ring with identity and let the additive group of R be abelian. If R is not a ring, then the system $(\mathcal{M}(R),\Delta)$ is a groupoid and not a semi-group.

Let G be an additive abelian group and S(G) the near-ring of mappings associated with G. It is clear that S(G) is not a ring and the additive group of S(G) is abelian. From Corollary (9.8) we conclude that $\mathcal{M}(S(G),\Delta)$ is a groupoid and not a semi-group.

<u>Theorem (9.9)</u>. Let R be a near-ring that contains an identity. The system $(\mathcal{M}(R),+,\Delta)$ satisfies the left distributive law if, and only if, the additive group of R is abelian.

<u>Proof</u>: Assume the additive group of R is abelian. Let (α_{ij}), (β_{ij}), and (γ_{ij}) be elements of $\mathcal{M}(R)$, and let $(\alpha_{ij})\Delta(\beta_{ij}+\gamma_{ij}) = (\Gamma_{ij})$ where $\Gamma_{ij} =$

$$= \sum_{k=1}^{n} \alpha_{ik}(\beta_{kj}+\gamma_{kj}) = \sum_{k=1}^{n} (\alpha_{ik}\beta_{kj} + \alpha_{ik}\gamma_{kj}).$$

If $(\alpha_{ij})\Delta(\beta_{ij}) = (\Gamma'_{ij})$ and $(\alpha_{ij})\Delta(\gamma_{ij}) = (\Gamma''_{ij})$, then

$$\Gamma'_{ij} + \Gamma''_{ij} = \sum_{k=1}^{n}\alpha_{ik}\beta_{kj} + \sum_{k=1}^{n}\alpha_{ik}\gamma_{kj} = \sum_{k=1}^{n}\alpha_{ik}(\beta_{kj} + \gamma_{kj})$$

since the additive group of R is abelian. From this it follows that $(\Gamma_{ij}) = (\Gamma'_{ij}) + (\Gamma''_{ij})$ and so the system $(\mathcal{M}(R),+,\Delta)$ satisfies the left distributive law.

Conversely, suppose the system $(\mathcal{M}(R),+,\Delta)$

satisfies the left distributive law. Let r_1 and r_2 be any elements of R, and, let

$$(\alpha_{ij}) = \begin{bmatrix} r_1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad (\beta_{ij}) = \begin{bmatrix} 1 & r_2 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$(\gamma_{ij}) = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{then}$$

$$(\alpha_{ij})\Delta[(\beta_{ij}) + (\gamma_{ij})] =$$

$$= \begin{bmatrix} (r_1+r_1+1) & (r_1r_2+r_1+r_2+1) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} =$$

$$= (\alpha_{ij}) \Delta (\beta_{ij}) + (\alpha_{ij}) \Delta (\gamma_{ij}) =$$

$$= \begin{bmatrix} (r_1+r_1+1) & (r_1r_2+r_2+r_1+1) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and so $r_1r_2 + r_2 + r_1+1 = r_1r_2 + r_1 + r_2 + 1$. From this $r_2 + r_1 = r_1 + r_2$ and we conclude that the additive group of R is abelian.

Corollary (9.10). Let R be a near-ring with identity. If the additive group of R is not abelian, then the system $(\mathcal{M}(R),+,\Delta)$ does not satisfy the left distributive law.

Let G be an additive group that is not abelian and S(G) the near-ring of mappings associated with G. From Corollary (9.10) the system $(\mathcal{M}(S(G)),+,\Delta)$ is not a near-ring.

From Theorem (9.7) and Theorem (9.9) we are allowed to conclude

Corollary (9.11). Let R be a near-ring with identity. Then the system $(\mathcal{M}(R),+,\Delta)$ is a near-ring if, and only if, it is a ring.

As the final result of this section we give

Proposition (9.12). Let R be a near-ring with identity. Then the additive group $\mathcal{M}(R)$ can be regarded as a unitary R-module.

Proof: If $(\alpha_{ij}) \in \mathcal{M}(R)$ and $r \in R$, then define $(\alpha_{ij})\cdot r = (\alpha_{ij}\cdot r) \in \mathcal{M}(R)$. With this definition the system $(\mathcal{M}(R),+)$ becomes a unitary near-ring module over R.

CHAPTER X

NEAR-VECTOR SPACES

The theory of near-vector spaces over a division near-ring has not been previously investigated. In this chapter we will apply many of the results encountered earlier in this thesis. For example, the results on strictly semi-simple near-ring modules will play a major role in our study of near-vector spaces.

We will give examples of near-vector spaces over division near-rings that are not vector spaces. Hence, our work generalizes many of the results of vector spaces over division rings. We introduce the concept of basis for a near-vector space and show that every near-vector space has a basis. Another concept is defined, the dimension of a near-vector space over a division near-ring, and we will prove that two near-vector spaces over the same division near-ring R are R-isomorphic if, and only if, they have the same dimension. We call an R-homomorphism from one near-vector space M_R into a second near-vector space A_R a linear mapping. As in the case of vector spaces over division rings, linear mappings play an important role in the theory of near-vector spaces. However, the image of a linear mapping need not be a subspace. This is the content of Theorem (10.34).

Let M_R and A_R be two near-vector spaces over the division near-ring R and let $Hom_R(M,A)$ denote the set of all linear mappings of M_R into A_R. If we define addition in $Hom_R(M,A)$ pointwise, i.e., $m(f+g) = mf + mg$ where $m \in M$, $f, g \in Hom_R(M,A)$, then the system $(Hom_R(M,A),+)$ is generally not a group. For this reason, some of the important results of the theory of vector spaces over division rings do not carry over to near-vector spaces. Assume M_R is finite dimensional over the division near-ring R and let $A_R = R_R$. In Theorem (10.53) we show that $Hom_R(M,R)$ is an additive group if, and only if, R is a division ring. Hence, we define the notion of the dual space of a finite dimensional near-vector space in terms of external direct sums.

Finally, suppose M_R is a finite dimensional near-vector space over R, and let S(M) denote the near-ring of mappings associated with the additive group of M. In Theorem (10.62) we prove that $Hom_R(M,M)$ is a subring of S(M) if, and only if, R is a division ring. However, since $Hom_R(M,M)$ is a multiplicative sub-semi-group of the multiplicative semi-group of the near-ring S(M) we can consider the d.g. near-ring S' that is generated by $Hom_R(M,M)$. If we assume that the additive group of R is abelian and the d.g.

near-ring S' satisfies the descending chain condition on right ideals, then we show in Theorem (10.73), that S' is isomorphic to a ring of linear mappings on a finite dimensional vector space over a division ring.

Definition and Elementary Properties of Near-Vector Spaces

Definition (10.1). A near-vector space is a strictly semi-simple R-module where R is a division near-ring.

From Theorem (5.6) we note that a near-vector space M_R is a unitary R-module.

It is evident that every vector space over a division ring is a near-vector space. If R is a division near-ring and M_R is an irreducible R-module, then M_R is a near-vector space. In particular, R_R is a near-vector space.

In Chapter IX we presented examples of division near-rings that were not division rings. Let R be a division near-ring that is not a division ring. Let $R = R_\lambda$, $\lambda \in \Omega$ where Ω is some index set. Then the external direct sum R' determined by the collection $\{R_\lambda \mid \lambda \in \Omega\}$ is a near-vector space that is not a vector space over a division ring.

Throughout this chapter we will assume that R is a division near-ring that is not a division ring in general.

Proposition (10.2). Every irreducible R-module is R-isomorphic to R_R.

Proof: Let M_R be an irreducible R-module. If $m \in M$, $m \neq 0$, then the mapping $f: r \in R \longrightarrow m \cdot r \in M_R$ is a non-zero R-homomorphism of R_R onto M_R. From Theorem (9.2) and the fundamental theorem of R-homomorphisms for near-ring modules we conclude that $R \underset{(R)}{\cong} M$.

Because of Proposition (10.2) we are allowed to conclude

Corollary (10.3). If M_R is a unitary R-module and m is any non-zero element of M_R, then $m \cdot R$ is an irreducible R-module.

Hence, we can construct another example of a near-vector space. Let M_R be a unitrary R-module. Then the external sum M_R' determined by the collection $\{mR \mid m \in M, m \neq 0\}$ is a near-vector space over R.

Another easy consequence of Proposition (10.2) is the following

Corollary (10.4). Let M_R be a near-vector space over R. The additive group of M is abelian if, and only if, the additive group of R is abelian.

Appealing to Theorem (5.7) and Corollary (10.4) we have

Corollary (10.5). If M_R is a near-vector space and R is a division ring, then M_R is a vector space over R.

The last part of this section is devoted to two proposition that will be useful in the remainder of this chapter.

Proposition (10.6). Let M_R be a unitary R-module such that $(m_1 + m_2)r = m_1r + m_2r$ for all $m_1, m_2 \varepsilon M_R, r \varepsilon R$. Then R is a division ring.

Proof: Let r_1, r_2, and r be elements of R. If m is any non-zero element of M_R, then $[m(r_1+r_2)]r = (mr_1+mr_2)r = m(r_1r) + m(r_2r) = m(r_1r+r_2r)$ and so $(r_1+r_2)r - (r_1r+r_2r) \varepsilon \left[\frac{0}{m}\right]$. Since R is a division near-ring $\left[\frac{0}{m}\right] = 0$ by Theorem (9.2). Hence, $(r_1+r_2)r = r_1r + r_2r$ so that the right distributive law holds in R. From this it follows that $(-1)\cdot r = -r$ for all $r \varepsilon R$. Therefore, $-r_1-r_2 = (-1)(r_1+r_2) = -(r_1+r_2) = -r_2-r_1$ and so $r_1 + r_2 = r_2 + r_1$. This shows that R is a division ring.

Proposition (10.7). Let M_R be a unitary R-module and let $m \varepsilon M$, $r \varepsilon R$. $mr = 0$ if, and only if, $m = 0$ or $r = 0$.

Proof: If $m = 0$ or $r = 0$, then it is clear that $m\cdot r = 0$.

Suppose $m\cdot r = 0$. If $r \neq 0$, then there is an element $r' \varepsilon R$ such that $r\cdot r' = 1$ and so $0 = mr =$

$= (mr)r' = m(rr') = m \cdot 1 = m$. Hence, the proof is complete.

Basis and Dimension of a Near-Vector Space

Before embarking on the concept of basis for a near-vector space, we first introduce the notions of linear combination and spanning set. We begin with

Definition (10.8). Let M_R be a near-vector space over the division near-ring R, and let $\{m_1, \ldots, m_n\}$ be a finite set of elements from M_R. An element $m \varepsilon M_R$ is said to be a linear combination of the elements $\{m_1, \ldots, m_n\}$ if, and only if, there exists elements $r_1, \ldots, r_n$ of R such that $m = \sum_{i=1}^{n} m_i r_i$.

Definition (10.9). A non-empty subset X of a near-vector space M_R is called a spanning set for M_R if, and only if,

(10.9.1) every element of X is contained in an irreducible submodule.

(10.9.2) every element of M_R can be written as a linear combination of a finite set of elements from X.

Definition (10.10). A non-empty subset X of a near-vector space M_R is called a basis if and only if,

(10.10.1) X is a spanning set for M_R.

(10.10.2) the representation of elements of M_R as a linear combination of elements from X is unique.

We now show that every near-vector space has a basis.

Theorem (10.11). If M_R is a near-vector space, then M_R has a basis.

Proof: Let $M_R = \bigoplus_{\lambda \in \Omega} M_\lambda$ where M_λ is a non-zero irreducible submodule. For each $\lambda \in \Omega$, let m_λ be a non-zero element. We show $\{m_\lambda \mid \lambda \in \Omega\}$ is a basis for M_R. Since each m_λ is non-zero it follows that $m_\lambda R = M_\lambda$ and so $M_R = \bigoplus_{\lambda \in \Omega} m_\lambda R$. From this it is clear that $\{m_\lambda \mid \lambda \in \Omega\}$ is a basis for M_R.

If M_R is an irreducible R-module, then every non-zero element is a basis for M_R. Let $R_i = R$ for $i = 1, 2, \ldots, n$. Then the set $\{(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)\}$ is a basis for the external direct sum determined by the collection $\{R_i \mid i = 1,2,\ldots,n\}$.

As in the case of vector spaces over division rings, the concept of linear independence plays an important part in the study of near-vector spaces. We introduce this notion in the following

Definition (10.12). Let M_R be a near-vector space over the division near-ring R. A finite subset

$X = \{m_1, \ldots, m_n\}$ of M_R is said to be linearly independent if, and only if,

(10.12.1) every element from X is contained in an irreducible submodule.

(10.12.2) If $m_1r_1 + \cdots + m_nr_n = 0$ for some set $r_1, \ldots, r_n$ of elements from R, then $r_1 = \cdots = r_n = 0$ in R.

A non-empty subset X of a near-vector space M_R is called a linearly independent set if every finite subset of X is linearly independent. If X is not linearly independent, then X is said to be linearly dependent.

<u>Proposition (10.13)</u>. Let M_R be a near-vector space over R and X a non-empty subset of M_R. If $0 \in X$, then X is a linearly dependent set.

<u>Proof</u>: From Proposition (10.7) $0 \cdot r = 0$ for all $r \in R$. Let $\{x_1, \ldots, x_n\}$ be any finite subset of X. Then $0 \cdot r + x_1 \cdot 0 + \cdots + x_n \cdot 0 = 0$ and so the set $\{0, x_1, \ldots, x_n\}$ is linearly dependent. Hence, the set X is linearly dependent.

<u>Lemma (10.14)</u>. Let X be a linearly independent subset of the near-vector space M_R. Then $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$.

<u>Proof</u>: First note that since X is a linearly independent set each $x \in X$ is contained in a non-zero irreducible submodule M_x. From Proposition (10.13) $x \neq 0$ and so $x \cdot R = M_x$. Hence, $\sum_{x \in X} x \cdot R$ is a submodule

of M_R. If $m \in x \cdot R \cap \left(\sum_{\substack{y \in X \\ y \neq x}} yR \right)$,

then there exists elements $y_1, \ldots, y_n$ from X and elements $r, r_1, \ldots, r_n$ from R such that $m = xr = \sum_{i=1}^{n} y_i r_i$. From this it follows that $y_1 r_1 + \cdots + y_n r_n + x(-r) = 0$ and so $r_1 = r_2 = \cdots = r_n = -r = 0$. Hence, $r = 0$ and it is now clear that $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$.

A significant result in the theory of vector spaces is that a linearly independent set can be extended to a basis. We now prove the analogous result for near-vector spaces.

<u>Theorem (10.15)</u>. Let X be a linearly independent subset of the near-vector space M_R. Then X can be extended to a basis for M_R.

<u>Proof</u>: From Lemma (10.14) $\sum_{x \in X} x \cdot R = \bigoplus_{x \in X} x \cdot R$ is a submodule of M_R. Since M_R is strictly semi-simple, it follows from Theorem (5.18) that there is a submodule M'_R such that $M_R = (\bigoplus_{x \in X} x \cdot R) \oplus M'$ and M'_R is a near-vector space over R. According to Theorem (10.11), M'_R has a basis Y. From this it is easy to see that the set $X' = X \cup Y$ is a basis for M_R. Hence, the set X can be extended to a basis.

We establish the important

<u>Theorem (10.16)</u>. Let X be a non-empty subset of the

near-vector space M_R. Then the following statements are equivalent

(10.16.1) X is a basis for M_R.

(10.16.2) X is a spanning linearly independent set.

(10.16.3) X is a maximal linearly independent set.

(10.16.4) X is a minimal spanning set.

by means of the following four lemmas.

<u>Lemma (10.17)</u>. Let X be a basis for the near-vector space M_R. Then X is a spanning set that is linearly independent.

<u>Proof</u>: Since X is a basis, it is a spanning set for M_R. Assume $m_1r_1 + \dots + m_nr_n = 0$ where $\{m_1, \dots, m_n\} \subseteq X$ and $r_1, \dots, r_n$ are elements from R. Since X is a basis and $0 = m_1 \cdot 0 + \dots + m_n \cdot 0$, we see that $r_1 = r_2 = \dots = r_n = 0 \varepsilon R$. This shows X is a linearly independent set.

<u>Lemma (10.18)</u>. Let X be a linearly independent set that spans the near-vector space M_R. Then X is a maximal linearly independent set.

<u>Proof</u>: Assume X' is a linearly independent subset of M_R and X is a proper subset of X'. Let x' be an element of X' such that $x' \not\varepsilon X$. Since X is a spanning set for M_R, there exists elements $x_1, \dots, x_n$ of X and elements $r_1, \dots, r_n$ of R such that $x' = \sum_{i=1}^{n} x_i r_i$.

From this we note that $x_1r_1 + \dots + x_nr_n + x'(-1) = 0$, so that the set $\{x_1, \dots, x_n, x'\} \subseteq X'$ is linearly dependent. Hence, X' must be a linearly dependent set.

<u>Lemma (10.19)</u>. Let X be a maximal linearly independent subset of the near-vector space M_R. Then X is a minimal spanning set.

<u>Proof</u>: From Theorem (10.15) X can be extended to a basis. Since every basis is a linearly independent set by Lemma (10.17), it follows that X must already be a basis. Hence, X is also a spanning set. It suffices to show X is a minimal spanning set. Assume Y is a proper subset of X that also spans M_R. If $x \in X$ and x is not an element of Y, then $x = \sum_{i=1}^{n} y_i r_i$ for some $y_1, \dots, y_n$ of Y and $r_1, \dots, r_n$ of R. This shows that the set $\{y_1, \dots, y_n, x\} \subseteq X$ is a linearly dependent set and so we have a contradiction. Therefore, X is a minimal spanning set.

<u>Lemma (10.20)</u>. Let M_R be a near-vector space over R and X a minimal spanning set for M_R. Then X is a basis.

<u>Proof</u>: Since X is a spanning set, $x \cdot R$ is an irreducible submodule for each $x \in X$ and $M_R = \sum_{x \in X} x \cdot R$. From Theorem (5.18) there is a subset $X' \subseteq X$ such that $M_R = \bigoplus_{x \in X'} x \cdot R$. It is clear that the set X' is a basis for M_R, and since X is minimal, we conclude that

$X = X'$.

Our next objective is to show that any two bases for a near-vector space have the same cardinal number. First, we prove

Lemma (10.21). Let M_R be a near-vector space. Let $M_R = \bigoplus_{i=1}^{n} M_i$ where M_i is a non-zero irreducible submodule. Then any other decomposition of M_R into non-zero irreducible submodules has exactly n-summands.

Proof: Let $\Sigma: M_1 \oplus \cdots \oplus M_n \supset M_2 \oplus \cdots \oplus M_n \supset \cdots \supset M_1 \supset \{0\}$ and note that Σ is a Jordan-Hölder series for M_R. Since a second decomposition of M_R into non-zero irreducible submodules would also determine a Jordan-Hölder series, it follows that such a decomposition has exactly n-summands by the Jordan-Hölder Theorem.

From Lemma (10.21) we note that if M_R has a finite basis, then any two bases have the same number of elements.

Let M_R be a near-vector space. If X is a non-empty subset of M_R, then we denote the cardinal number of X by $|X|$.

Theorem (10.22). Let X_1 and X_2 be two sets of bases for the near-vector space M_R. Then $|X_1| = |X_2|$.

Proof: It suffices to consider the case when X_1 is infinite. For each $x \in X_1$, let $X_2(x) = \{y_1, \ldots, y_n \in X_2 |$

$x = \sum_{i=1}^{n} y_j r_i$, $r_i \varepsilon R$ and $r_i \neq 0\}$. Since X_2 is a basis, it is clear that $X_2(x)$ is well defined. If $X = \bigcup_{x \varepsilon X_1} X_2(x)$, then note X is a spanning set for M_R and $X \subseteq X_2$. From Theorem (10.16) $X = X_2$. Since X_1 is infinite and each $X_2(x)$ is finite, it follows that $|X_2| \leq |X_1|$. Similarly, $|X_1| \leq |X_2|$ and so $|X_1| = |X_2|$.

We are now ready to give

Definition (10.23). If M_R is a near-vector space over R, then the cardinality of any basis is called the dimension of M_R and is denoted by dim M.

If the cardinality of any basis for M_R is finite, then M_R is called a finite dimensional near-vector space. However, if the cardinality of any basis for M_R is infinite, then M_R is said to be an infinite dimensional near-vector space. Since the zero submodule $\{0\}$ of a near-vector space does not have a basis, we will agree that dim $\{0\} = 0$.

Subspaces of a Near-Vector Space

Let M_R be a near-vector space over a division near-ring R. If M' is a submodule of M_R, then M' is called a subspace of M_R. From Theorem (5.18) every subspace of M_R is a near-vector space.

Definition (10.24). Let M' be a subspace of M_R. M' is said to be complemented by a subspace M" if, and only if, $M_R = M' \oplus M''$.

If M' is complemented by M", then M" is called the complementary subspace of M'.

Appealing to Theorem (5.18) and the second isomorphism theorem for near-ring modules we obtain the following

Theorem (10.25). Every subspace of a near-vector space is complemented uniquely up to R-isomorphism.

We next present the interesting

Theorem (10.26). Let $M_R = M' \oplus M''$ where M' and M" are proper subspaces of the near-vector space M_R. Then dim M = dim M' + dim M".

Proof: Let X' be a basis for M' and X" be a basis for M". It is clear that the set $X = X' \cup X''$ is a basis for M_R. By Theorem (10.16) X' is a linearly independent subset of M' and so Proposition (10.13) tells us that $0 \notin X'$. From this $X' \cap X''$ is the null set and so dim M = dim M' + dim M".

Corollary (10.27). Let M_R be an n-dimensional near-vector space and M' a subspace of M_R. M' = M if, and only if, dim M = dim M'.

Proof: Assume dim M' = dim M and M' is a proper subspace of M_R. From Theorem (10.25) there is a non-zero subspace M" such that $M_R = M' \oplus M"$. Hence, dim M = dim M' + dim M" by Theorem (10.26). This shows M' = M.

If M' = M, then it is easy to see that dim M' = = dim M.

Linear Mappings

We come now to the part of our study that makes the study of near-vector spaces interesting.

<u>Definition (10.28)</u>. An R-homomorphism T of a near-vector space M_R into a near-vector space A_R is called a linear mapping.

If T is an R-isomorphism, then T is called a linear isomorphism.

The next lemma tells us that the properties of a linear mapping T are determined by any basis for M_R.

<u>Lemma (10.29)</u>. Let T be a linear mapping of M_R into A_R. Then T is uniquely determined on any basis for M_R.

<u>Proof</u>: Let X be a basis for M_R and T' a linear mapping of M_R into A_R such that $xT = xT'$ for all $x \in X$. If $m \in M_R$, then $m = \sum_{i=1}^{n} x_i r_i$ where $x_i \in X$, $r_i \in R$. Hence,

$$mT = \left(\sum_{i=1}^{n} x_i r_i\right)T = \sum_{i=1}^{n} (x_i T) r_i = \sum_{i=1}^{n} (x_i T') r_i =$$

$= \left(\sum_{i=1}^{n} x_i r_i\right)T'$ and so $T = T'$ on M_R.

Let M_R and A_R be two near-vector spaces over the division near-ring R. Let T be a linear isomorphism of M_R onto A_R. Since T is an onto mapping and T maps irreducible subspaces of M_R onto irreducible subspaces of A_R by Lemma (1.21), it follows that dim M = dim A. Hence, we can give

Theorem (10.30). If two near-vector spaces M_R and A_R are R-isomorphic, then dim M = dim A.

We now prove the converse to Theorem (10.30).

Theorem (10.31). Let M_R and A_R be two near-vector spaces such that dim A = dim M. Then $A \underset{(R)}{\cong} M$.

Proof: Let X be a basis for A_R and Y a basis for M_R. Then $|X| = |Y|$ and so there is a one-to-one mapping T of X onto Y. Let $a \in A$, $a = \sum_{i=1}^{n} x_i r_i$ where $x_i \in X$, $r_i \in R$. Then define the mapping T' from A into M as follows: $aT' = \sum_{i=1}^{n} (x_i T) r_i$. Since a basis representation is unique and $xT \in Y$ for all $x \in X$, we see that T' is a single-valued onto mapping.

If $a' = x_1' r_1' + \cdots + x_k' r_k'$ where $x_j' \in X$, $r_j' \in R$, then

$(a + a')T' = \left(\sum_{i=1}^{n} x_i r_i + \sum_{j=1}^{k} x_j' r_j'\right)T' =$

$= \sum_{i=1}^{n}(x_iT)r_i + \sum_{j=1}^{k}(x_jT)r'_j$ and so T' is a group homomorphism of the additive group A onto the additive group M.

It remains to show $(aT)r = (ar)T$ where $r \in R$ and T is a one-to-one mapping.

Since Y is a basis for M_R, $xT \neq 0$ for all $x \in X$. Moreover, xT is contained in a non-zero irreducible submodule of M_R. From Lemma (4.4) it follows that

$$(aT')r = \left[\sum_{i=1}^{n}(x_iT)r_i\right]r = \sum_{i=1}^{n}(x_iT)r_ir = \left[\sum_{i=1}^{n}x_i(r_ir)\right]T' =$$

$= (ar)T'$. We now assume $a \neq 0$ where a is the element given above. Hence, there is at least one index $i = 1, 2, \ldots, n$ such that $x_ir_i \neq 0$. If $aT' = 0$, then $aT' = \sum_{i=1}^{n}(x_iT)r_i = 0$. Since Y is a basis, $r_1 = \cdots = r_n = 0$ by Theorem (10.16). This shows T' is a one-to-one mapping and so $A \underset{(R)}{\cong} M$.

We proceed now to apply several results of the last few sections to yield some information about linear mappings.

From now on we assume all near-vector spaces are finite dimensional.

Theorem (10.32). Let T be a linear mapping of the finite dimensional near-vector space M_R into the near-vector finite dimensional space A_R. Then MT is a near-vector space over R and dim M = dim Ke(T) + dim (MT).

Proof: From the fundamental theorem of R-homomophisms we conclude that $M/_{Ke(T)} \underset{(R)}{\cong}$ MT and so MT is a near-vector space over R. If Ke(T) = 0, then dim M = dim MT by Thoerem (10.29). Assume Ke(T) $\neq$ 0 and since Ke(T) is a subspace of M_R, it follows from Theorem (10.25) that there is a subspace M' of M_R such that M' $\oplus$ Ke(T) = M_R. As an easy consequence of Theorem (10.26), Theorem (10.29), and the second isomorphism theorem for near-ring modules we have dim M = dim M' + dim Ke(T) = dim (MT) + dim (Ke(T)).

Corollary (10.33). Let T be a linear mapping of M_R onto A_R. T is a one-to-one mapping if, and only if, dim M = dim A.

Proof: If T is one-to-one, then T is a linear isomorphism of M_R onto A_R. From Theorem (10.30) dim M = dim A.

Assume dim M = dim A and Ke(T) $\neq$ 0. From Theorem (10.33) dim M = dim A + dim Ke(T) and this is a contradicition. Therefore, T is a one-to-one linear mapping.

Let T be a linear mapping of the finite dimensional near-vector space M_R into the finite dimensional near-vector space A_R. In general, MT is not a subspace of A_R.

<u>Theorem (10.34)</u>. There exists a finite dimensional near-vector space M_R, a non-zero R-subgroup M_R' of M_R that is not a subspace of M_R and a linear mapping of M_R onto M_R'.

We now proceed to construct such a linear mapping.

Let R be a finite division near-ring that is not a division ring, and let the additive order of the identity 1 be $n > 2$. (Zassenhaus [26] showed that such division near-rings exist.) Let $S = S(R)$ denote the near-ring of mappings associated with the additive group of R.

<u>Lemma (10.35)</u>. For each $a \in R$, let δ_a denote the mapping δ_a: $r \in R \longrightarrow ra \in R$. Then $\{\delta_a \mid a \in R\} = S'$ is a sub-near-ring of S that is isomorphic to R.

<u>Proof</u>: If $a, b \in R$, then $r(\delta_{a+b}) = r(a+b) = ra+ab =$ $= r\delta_a + r\delta_b$ and $r(\delta_a \cdot \delta_b) = (r\delta_a)\delta_b = (ra)\delta_b = r(ab) = r\delta_{ab}$ for all $r \in R$. Hence, the mapping η: $a \in R \longrightarrow \delta_a \in S'$ is a near-ring homomorphism of the near-ring R into the near-ring S and $R\eta = S'$. From Theorem (9.2) $R \cong S'$.

The first step in our construction is to show that the near-ring module S over the division near-ring S' is a finite dimension near-vector space over

S'. To reach our objective, we present five easy lemmas. The first of which is the following

Lemma (10.36). Let A be a non-zero irreducible right ideal of the near-ring S. Then $A \underset{(S)}{\cong} R$

Proof: We already know that R_S is an irreducible S-module. Since A is non-zero, there exists elements $a \varepsilon A$, $r \varepsilon R$ such that $ra \neq 0$ in R. Let η denote the mapping defined by $\eta: a \varepsilon A \longrightarrow ra \varepsilon R$. It is clear that η is a non-zero S-homomorphism of A_S onto R_S and so $A \underset{(S)}{\cong} R$.

Lemma (10.37). If B is a non-zero S'-subgroup of the S'-module R, then B = R.

Proof: Let B be a non-zero S'-subgroup of $R_{S'}$. If b is a non-zero element of B, then there is an element $b' \varepsilon R$ such that $b \cdot b' = 1$ and so $1 = bb' = b\delta_{b'} \varepsilon B$. Let r be any element of R. Then $r = 1 \cdot r = 1 \cdot \delta_r \varepsilon B$ and so B = R.

Lemma (10.38). Let $s \varepsilon S$ and a a non-zero element of R such that $s \cdot \delta_a = 0$. Then s = 0.

Proof: If r is a non-zero element of R, then $0 = r(s\delta_a) = (rs)\delta_a = (rs) \cdot a$. Since R is a division near-ring, there is a non-zero element $a' \varepsilon R$ such that $a \cdot a' = 1$ so that $0 = 0 \cdot a' = rs$. Hence, rs = 0 for all $r \varepsilon R$. This shows s = 0.

<u>Lemma (10.39).</u> Let A be a non-zero irreducible right ideal of the near-ring S. Then $A_{S'}$ is a non-zero irreducible S'-module.

<u>Proof:</u> From Lemma (10.38) $A \cdot S' = \{as' \mid a \varepsilon A, s' \varepsilon S'\} \neq 0$. Hence, the mapping $\eta : a \varepsilon A \longrightarrow r \cdot a \varepsilon R$ given in the proof of Lemma (10.36) is a non-zero S'-isomorphism of $A_{S'}$ into $R_{S'}$. From Lemma (10.37) $A_{S'}$ is a non-zero irreducible S'-module.

<u>Lemma (10.40).</u> $S = \bigoplus_{i=1}^{n} A_i$ where A_i is a non-zero irreducible right ideal of the near-ring S.

<u>Proof:</u> S is a primitive near-ring and so $J(S) = 0$ by Theorem (7.4). Since S is finite it satisfies the descending chain condition on right ideals. Appealing to Theorem (6.19) there exists non-zero irreducible right ideals $A_1, \ldots, A_n$ of the near-ring S such that $S = \bigoplus_{i=1}^{n} A_i$.

Because of Lemma (10.39) and (10.40) we can conclude

<u>Theorem (10.41).</u> The near-ring S is an n-dimensional near-vector space over the division near-ring S'.

Let K denote the near-ring of constant mappings of the additive group of R We have seen, in the first chapter of this thesis, that K is a non-zero S-subgroup of the near-ring S. From which it follows

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that K is a non-zero S'-subgroup of $S_{S'}$.

We conclude our construction with three lemmas. We start with

Lemma (10.42). K is not a subspace of the near-vector space $S_{S'}$.

Proof: Assume K is a subspace. Then $(\delta_1 + \varphi_1)\varphi_1 - \delta_1\varphi_1 = \varphi_r \in K$ for some $r \in R$. Since the additive order of the identity 1 of R is $n > 2$, it follows that
$r = 1\varphi_r = 1(\delta_1 + \varphi_1)\varphi_1 - 1(\delta_1\varphi_1) = (1+1)\varphi_1 - 1\varphi_1 = 1 - 1 = 0$.
Hence, $(\delta_1 + \varphi_1)\varphi_1 = \delta_1\varphi_1$. If $(n-1)\cdot 1 = c \in R$, then $(n-1)\cdot 1(\delta_1 + \varphi_1)\varphi_1 = (n\cdot 1)\varphi_1 = 0$ and $(n-1)1\cdot(\delta_1\varphi_1) = 1$. This shows $1 = 0$ in the near-ring R. Therefore, K is not a subspace of the near-vector space $S_{S'}$.

Lemma (10.43). $K_{S'}$ is an irreducible S'-subgroup of the near-vector space $S_{S'}$.

Proof: Let B be a non-zero S'-subgroup of $K_{S'}$, and let φ_r be a non-zero element of B. Let φ_c be any non-zero element of K. If a is the multiplicative inverse of the non-zero element $r \in R$, then $r'(\varphi_r \cdot \delta_{ac}) = r(ac) = c$ for all non-zero elements r' of R. From this it follows that $\varphi_c = \varphi_r\delta_{(ac)} \in K$ and so B = K.

Lemma (10.44). Let φ_r be a non-zero element of K. Then the mapping $T: s \in S \longrightarrow \varphi_r \cdot s = \varphi_{r\cdot s} \in K$ is a non-zero S'-homomorphism of $S_{S'}$ onto $K_{S'}$ and $ST = K_{S'}$

is not a subspace of the near-vector space $S_{S'}$.

Proof: Let s_1, $s_2 \varepsilon S$, and $\delta_a \varepsilon S'$. Then
$(s_1 + s_2)T = \varphi_r(s_1 + s_2) = \varphi_{r(s_1+s_2)} = \varphi_{rs_1+rs_2} =$
$= \varphi_r \cdot s_1 + \varphi_r \cdot s_2 = s_1T + s_2T$ and $(s_1\delta_a)T = \varphi_r \circ s_1\delta_a =$
$= (\varphi_r s_1)\delta_a = (s_1T)\delta_a$. From Lemma (10.43) it is now clear that T is an S'-homomorphism of $S_{S'}$ onto $K_{S'}$ and $ST = K_{S'}$ is not a subspace of the near-vector space $S_{S'}$.

Therefore, the proof of Theorem (10.34) is now complete.

Let T be a linear mapping of a finite dimensional near-vector space M_R into itself. We seek conditions under which MT is a subspace of M_R. First, we give

Definition (10.45). T is called a normal linear mapping if, and only if, MT is a subspace of M_R.

Let S = S(M) denote the near-ring of mappings associated with the additive group M. For each pair of elements $m \varepsilon M_R$, $r \varepsilon R$ let $s_{m \cdot r}$ denote the element of S that maps $x \varepsilon M_R$ onto $(m+x)r - mr \varepsilon M_R$. Further, let $S(M,R) = \{s_{m \cdot r} \mid m \varepsilon M_R, r \varepsilon R\}$. We now give a sufficient condition for T to be normal.

Theorem (10.46). If T commutes with the elements S(M,R), then T is a normal linear mapping.

Proof: Assume T commutes with the elements of S(M,R).

Since M_R is unitary, it follows that T commutes with the inner-automorphisms of the additive group M and so MT is a normal subgroup. As T commutes with the elements of S(M,R), we see that $(m + xT)r - mr = [(m+x)r - mr]T \in MT$ and so MT is a subspace of M_R. This shows that T is a normal linear mapping.

Because of the next result we see that the number of normal linear mapping of a finite dimensional near-vector space into itself is quite large.

<u>Theorem (10.47)</u>. For each subspace A of a finite dimensional near-vector space M_R there exists a normal linear mapping T such that MT = A and T commutes with the elements of S(M,R).

<u>Proof</u>: From Theorem (10.25) there is a subspace B of M_R such that $M_R = A \oplus B$. If $m = a + b$ where $a \in A$, $b \in B$, then let T be the mapping given by $mT = a$. It is clear that T is a single-valued linear mapping of M_R onto A_R and so MT = A. Since A is a direct summand, T commutes with the inner-automorphisms of the additive group M. Let $m = a + b$, $m' = a' + b'$, and $r \in R$ where $a, a' \in A$, $b, b' \in B$. We are to show $s_{m \cdot r}T = T \cdot s_{mr}$. Appealing to Proposition (4.2) and Lemma (4.4), it follows that $m'(S_{m \cdot r}T) = [(m+m')r - mr]T =$ $= [(a+a')r + (b+b')r - ar - br]T = (aT+a'T)r + (bT+b'T)r -$

$(aT)r - (bT)r = [(a+a')r - ar]T$ and $(m'T)S_{m \cdot r} =$
$= (m+m'T)r - mr = (a+b+a')r - (a+b)r =$
$= (a+a')r + br - ar - br = (a+a')r - ar$. This shows
T commutes with the elements of S(M,R).

We combine the results Theorem (10.46) and Theorem (10.47) to obtain

Corollary (10.48). A non-empty subset A of a near-vector space M_R is a subspace if, and only if, there exists a linear mapping T of M_R into itself that commutes with the elements of S(M,R) and $MT = A$.

The last result of this section is the following

Theorem (10.49). Let T be a normal linear mapping of M_R into M_R such that $T^2 = T \neq 0$. Then $M_R = MT \oplus Ke(T)$.
Proof: Since T is a normal linear mapping, MT is a subspace of M_R. If $x \in MT \cap Ke(T)$, then there is an element $m \in M_R$ such that $mT = x$ and so $0 = xT = mT^2 =$
$= mT = x$. Hence, it suffices to show $M_R = MT + Ke(T)$.
Let m be any element of M_R. Then $m = mT - mT + m$ and $(-mT+m)T = (-mT)T + mT = -mT^2 + mT = 0$. This show $M_R = MT + Ke(T)$ and so $M_R = MT \oplus Ke(T)$.

Dual of a Finite Dimensional Near-Vector Space

Let M_R and A_R be two finite dimensional near-vector spaces over R, and let Map (M,A) denote the set of mappings of the additive group M into the additive group A. If addition in Map (M,A) is defined pointwise, i.e., $m(f+g) = mf + mg$ where $m \varepsilon M$, $f,g \varepsilon$ Map (M,A), then Map(M,A) becomes an additive group. Let $Hom_R(M,A)$ denote the subset of Map (M,A) of all linear mappings of M_R into A_R. In particular if A = R and R is a division ring, then $Hom_R(M,R)$ is just the dual vector space of M_R. However, when M_R is only a near-vector space over the division near-ring R, $Hom_R(M,R)$ is not necessarily a near-vector space. This is the essential content of Theorem (10.53). In Definition (10.55) we present the concept of the dual of a near-vector M_R space which is R-isomorphic to $Hom_R(M,R)$, whenever, M_R is a vector space over the division ring R.

Throughout the remainder of this chapter, we will assume all near-vector spaces are finite dimensional.

<u>Theorem (10.50)</u>. If dim M = dim A = 1, then $Hom_R(M,A)$ can be considered as a one-dimensional near-vector space over R.

<u>Proof</u>: Assume dim M = dim A = 1. Let m be a non-zero element of M_R. Since dim M = 1, we see that $mR = M_R$. If a is a non-zero element of A_R, then the mapping

f_a: $mr \varepsilon mR \longrightarrow a \cdot r \varepsilon A_R$ is a non-zero linear mapping of M_R onto A_R. First, we note that f_a is single-valued. For if $mr_1 = mr_2$, then $r_1 - r_2 \varepsilon \left[\frac{0}{m}\right]$ and by Theorem (9.2) $r_1 = r_2$. From this we see that f_a is single-valued. It is clear that f_a is a linear mapping.

Let f be any non-zero element of $Hom_R(M,A)$. If $mf = a$, then $(mr)f = (mf)r = ar$ and so $f_a = f$. If f_o denotes the zero mapping from M_R into A_R, then $Hom_R(M,A) = \{f_a \mid a \varepsilon A\}$.

Let f_{a_1} and f_{a_2} be elements of $Hom_R(M,A)$. Then define $f_{a_1} \oplus f_{a_2} = f_{a_1+a_2}$. With this definition of addition, $Hom_R(M,A)$ becomes a group. The mapping η: $a \varepsilon A_R \longrightarrow f_a \varepsilon Hom_R(M,A)$ is a group isomorphism of the additive group A onto $Hom_R(M,A)$. From Proposition (1.13) $Hom_R(M,A)$ can be regarded as a one dimensional near-vector space over R and $f_a \cdot r = f_{ar}$.

From the proof of Theorem (10.50) we have

<u>Corollary (10.51)</u>. Let dim M = dim A = 1. Then $Hom_R(M,A) = \{f_a \mid a \varepsilon A\}$ where f_a: $mr \varepsilon M_R \longrightarrow ar \varepsilon A_R$, m a non-zero element of M_R.

<u>Corollary (10.52)</u>. Let dim M = dim A = 1. $Hom_R(M,A)$ is a subgroup of the additive group Map (M,A) if, and only if, R is a division ring.

Proof: Assume R is a division ring. Because of Corollary (10.5), M_R and A_R are vector spaces over R. Hence, it is well known that $Hom_R(M,A)$ is a subgroup of the additive group Map (M,A).

Suppose $Hom_R(M,A)$ is a subgroup of the additive group Map (M,A). From Corollary (10.51) $Hom_R(M,A) = \{f_a \mid a \varepsilon A\}$ where $f_a: mr \varepsilon M_R \longrightarrow a \cdot r \varepsilon A_R$, m a non-zero element of M_R. It is clear from the proof of Theorem (10.50) that each f_a is uniquely determined by what it does to the element $m \varepsilon M_R$. Let a_1 and a_2 be elements of A_R. Then $m(f_{a_1} + f_{a_2}) = mf_{a_1} + mf_{a_2} = a_1 + a_2$ and $mf_{a_1+a_2} = a_1 + a_2$. This shows $f_{a_1} + f_{a_1} = f_{a_1+a_2}$. If $r \varepsilon R$, then $(a_1 + a_2)r = (mr)f_{a_1+a_2} = (mr)f_{a_1} + (mr)f_{a_2} = a_1 r + a_2 r$. Because of Proposition (10.6) R satisfies the right distributive law. Hence, it suffices to show the additive group of R is abelian. Since R satisfies the right distributive law, it is evident that $(-1)r = -r$ for all $r \varepsilon R$. Let r_1 and r_2 be elements of R. Then $-r_1-r_2 = (-1)(r_1+r_2) = -(r_1+r_2) = -r_2-r_1$ and so $r_2 + r_1 = r_1 + r_2$. We have now shown that R is a division ring.

Theorem (10.53). Let M_R be an n-dimensional near-vector space over the division near-ring R. $Hom_R(M,R)$ is a subgroup of the additive group Map (M,R) if, and only if, R is a division ring.

Proof: Assume R is a division ring. From Corollary (10.5) M_R is a vector space over R and so $Hom_R(M,R)$ is just the (algebraic) dual of M_R. Hence, $Hom_R(M,R)$ is a subgroup of the additive group Map (M,R).

Suppose $Hom_R(M,R)$ is an additive subgroup of the additive group Map (M,R). Let $M_R = \bigoplus_{i=1}^{n} M_i$ where M_i is a one dimensional subspace of M_R. From Corollary (10.52) it suffices to show $Hom_R(M_1,R)$ is a subgroup of the additive group Map (M_1,R). Let π_1 denote the linear mapping of M_R onto $(M_1)_R$ that maps each element of M_R onto its component in M_1. If $f,g \in Hom_R(M_1,R)$, then $\pi_1 f$, $\pi_1 g \in Hom_R(M,R)$ and so $\pi_1 f - \pi_1 g \in Hom_R(M,R)$. From this it is easy to see that $f - g \in Hom_R(M_1,R)$ and so $Hom_R(M_1,R)$ is a subgroup of the additive group Map (M_1,R). Hence, R is a division ring.

According to Corollary (10.5) and Theorem (10.53) we have

Corollary (10.54). Let M_R be an n-dimensional near-vector space over the division near-ring R. $Hom_R(M,R)$ is a subgroup of the additive group Map (M,R) if, and only if, M_R is an n-dimensional vector space over the division ring R.

Let M_R be an n-dimensional near-vector space over R that is not a vector space. Let $M_R = \bigoplus_{i=1}^{n} M_i$, where

M_i is a one dimensional subspace of M_R. From Theorem (10.50) the additive group $(\mathrm{Hom}_R(M_i, R, \oplus)$ is a one dimensional near-vector space over R. Let M'_R denote the external direct sum determined by the collection $\{\mathrm{Hom}_R(M_i,R) \mid i = 1,2,\ldots,n\}$ of one dimensional near-vector spaces. Because of Theorem (10.53) we give

<u>Definition (10.55)</u>. M'_R is called the dual near-vector space of M_R.

For simplicity of notation we will write

$M'_R = \bigoplus_{i=1}^{n} \mathrm{Hom}_R(M_i,R)$, and we note the elements of M'_R are sequences of the form $(f_{m_1}, \ldots, f_{m_n})$ where $f_{m_i} \in \mathrm{Hom}_R(M_i,R)$. Since M'_R is an n-dimensional near-vector space over R, it follows from Theorem (10.31) that $M \underset{(R)}{\cong} M'$.

Our final objective of this section is to develop the concept of dual basis for M'_R. First, we present

<u>Proposition (10.56)</u>. Let M_R be an n-dimensional near-vector space over R. If m is a non-zero element of M_R, then there exists an element $f \in \mathrm{Hom}_R(M,R)$ such that $mf=1$.

<u>Proof</u>: Let $m_1, \ldots, m_n$ be a basis for M_R. Then $M_R = \bigoplus_{i=1}^{n} m_i R$ where $m_i R$ is a one dimensional subspace. If m is a non-zero element of M_R, then there are elements $r_1, \ldots, r_n$ not all zero in R such that

$m = m_1r_1 + \cdots + m_nr_n$. Assume r_1 is not zero, and let π_1 denote the linear mapping that maps elements of M_R onto their components in m_1R. Hence, $m\pi_1 = m_1r_1$ and since $m_1r_1 \neq 0$, it follows that $(m_1r_1)R = m_1R$. R is a division near-ring and so there is an element $r \in R$ such that $rr_1 = 1$. Therefore, the mapping $f_{m_1r_1} \in \mathrm{Hom}_R(m_1r_1R,R)$ maps m_1r_1 onto 1. From this we see that $m(\pi_1 f_{m_1r_1}) = (m_1r_1)f_{m_1r_1} = 1$ and $\pi_1 f_{m_1r_1} \in \mathrm{Hom}_R(M,R)$. This completes the proof of the proposition.

Let $m_1, \ldots, m_n$ be a basis for M_R. Then appealing to the proof of the last proposition we can conclude

<u>Corollary (10.57)</u>. There exists elements $f_{m_i} \in \mathrm{Hom}_R(m_iR,R)$ such that $m_if_{m_i} = 1$.

We now give the important

<u>Theorem (10.58)</u>. Let $M_R' = \bigoplus_{i=1}^{n} \mathrm{Hom}_R(m_iR,R)$ be the dual near-vector space of M_R. Then the set $\{(f_{m_1},0,\ldots,0), (0,f_{m_2},\ldots,0), \ldots, (0,\ldots,f_{m_n})\}$ is a basis for M_R'.

<u>Proof</u>: For simplicity, let $f_{m_i} = (0,\ldots,0,f_{m_i},0,\ldots,0)$. By Theorem (10.16) it suffices to show that $f_{m_1}, \ldots, f_{m_n}$ is a linearly independent subset of M_R'. If $\sum_{i=1}^{n} f_{m_i} \cdot r_i = 0$ where $r_i \in R$, then $f_{m_1}r_1 + \cdots + f_{m_n}r_n = 0$

and so $f_{m_i r_i} = 0$ for $i = 1,2,\ldots,n$. Appealing to Theorem (10.50) and Proposition (10.7) $r_1=r_2=\cdots=r_n=0$ so that the set $\{f_{m_1}, \ldots, f_{m_n}\}$ is a basis for M_R'.

Definition (10.59). The basis $\{f_{m_1}, \ldots, f_{m_n}\}$ for the near-vector space M_R' is called the dual basis of $\{m_1, \ldots, m_n\}$.

Distributively Generated Near-Rings of Linear Mappings of a Finite Dimensional Near-Vector Space

Let M_R be an n-dimensional near-vector space over the division near-ring R and let S = S(M) be the near-ring of mappings associated with the additive group M. Further, let $Hom_R(M,M)$ denote the subset of linear mappings of M_R into itself. Since the product of two elements from $Hom_R(M,M)$ is a linear mapping of M_R into itself, it follows that the multiplicative semi-group $Hom_R(M,M)$ generates a d.g. sub-near-ring S' of S(M). In particular, if R is a division ring, then S' is just the ring of linear mappings of M_R.

The purpose of this section is to investigate the d.g. near-ring S'. The main result given in Theorem (10.73) tells us that if the additive group of R is abelian and S' satisfies the descending chain condition on right ideals, then S' is isomorphic to the ring of linear mappings on a finite dimensional vector space over a division ring.

From the proof of Theorem (10.30) we can conclude

Theorem (10.60). Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be two sets of bases for the n-dimensional near-vector space M_R. Then there exists a linear isomorphism T of M_R onto itself such that $x_iT = y_i$ for $i = 1, 2, \ldots, n$.

Corollary (10.61). Let M_R be an n-dimensional near-vector space over R. For each $\alpha \varepsilon R$ there exists an element $\bar{\alpha} \ \varepsilon \ \mathrm{Hom}_R(M,M)$ that is determined by α. Moreover, if α is not zero, then $\bar{\alpha}$ is a linear isomorphism.

Proof: Let α be an element in R. If $\alpha = 0$, then we let $\bar{\alpha}$ denote the zero mapping of $\mathrm{Hom}_R(M,M)$. Assume $\alpha \neq 0$ and let $x_1, \ldots, x_n$ be a basis for M_R. Then $M_R = \bigoplus_{i=1}^{n} x_iR$ where x_iR is a one dimensional subspace of M_R. From Propositions (10.7) and (10.13) $x_i\alpha \neq 0$. Hence, $(x_i\alpha)R = x_iR$ and so the set $\{x_1\alpha, \ldots, x_n\alpha\}$ is a basis for M_R. By Theorem (10.60) there exists a linear isomorphism $\bar{\alpha} \ \varepsilon \ \mathrm{Hom}_R(M,M)$ such that $x_i\bar{\alpha} = x_i\alpha$. This completes the proof.

Theorem (10.62). Let M_R be an n-dimensional near-vector space. $\mathrm{Hom}_R(M,M)$ is a subring of the near-ring S(M) if, and only if, R is a division ring.

Proof: Assume R is a division ring. Because of

Corollary (10.5) M_R is a vector space over R. Hence, $Hom_R(M,M)$ is just the ring of linear mappings of M_R into M_R. In particular, it is a subring of S(M).

Suppose $Hom_R(M,M)$ is a subring of the near-ring S(M). Let $x_1, \ldots, x_n$ be a basis for M_R, and let α and β be non-zero elements of the division near-ring R. Then $\bar{\alpha} + \bar{\beta}$ and $\bar{\alpha}\cdot\bar{\beta}$ are elements of $Hom_R(M,M)$. We now show that $\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$ and $\overline{\alpha\cdot\beta} = \bar{\alpha}\cdot\bar{\beta}$. From Lemma (10.29) it suffices to show $x_j(\overline{\alpha+\beta}) = x_j\bar{\alpha} + x_j\bar{\beta}$ and $x_j(\overline{\alpha\beta}) = (x_j\bar{\alpha})\bar{\beta}$. Since $x_j(\overline{\alpha+\beta}) = x_j(\alpha+\beta) = x_j\alpha + x_j\beta = x_j\bar{\alpha} + x_j\bar{\beta}$ and $x_j(\overline{\alpha\beta}) =$ $= x_j(\alpha\cdot\beta) = (x_j\bar{\alpha})\bar{\beta}$, it follows that $\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$ and $\overline{\alpha\cdot\beta} = \bar{\alpha}\cdot\bar{\beta}$. Hence, the mapping $\eta : \alpha \in R \longrightarrow \bar{\alpha} \in Hom_R(M,M)$ is a near-ring homomorphism. Because of Theorem (9.2) we can conclude that η is a near-ring isomorphism and so R must be a division ring.

Let M_R be an n-dimensional near-vector space over the division near-ring R. Let S' denote the d.g. sub-near-ring of S(M) that is generated by the multiplicative semi-group $Hom_R(M,M)$ of right distributive elements. Then we have the interesting

<u>Theorem (10.63).</u> If x and y are non-zero elements of M_R, then there exists an element $f \in S'$ such that $xf = y$.

<u>Proof</u>: We prove the theorem by induction on n. Assume the result is true for all near-vector spaces

over R of dimension $\leq n-1$. Let $x_1, \ldots, x_n$ be a basis for M_R. Then $M_R = x_1R \oplus x_2R \oplus \ldots \oplus x_nR$ where x_iR is a one dimensional subspace of M_R. Let

$x = \sum_{i=1}^{n} x_i r_i$ and $y = \sum_{i=1}^{n} x_i r_i'$ where $r_i, r_i' \in R$. The

subspace $M' = \bigoplus_{i=2}^{n} x_iR$ is an n-1 dimensional near-vector space over R and so there is an element $f \in \mathrm{Hom}_R(M',M')$ such that $\left(\sum_{i=2}^{n} x_i r_i\right) f = \sum_{i=2}^{n} x_i r_i'$.

Similarly, there exists an element $g \in \mathrm{Hom}_R(x_1R, x_1R)$ such that $(x_1r_1)g = x_1r_1'$.

It is clear that $M_R = x_1R \oplus M'$. Let π_1 denote the element of $\mathrm{Hom}_R(M,M)$ that π_1 maps an element of M_R onto its component in x_1R. In the same manner, let π_2 denote the element of $\mathrm{Hom}_R(M,M)$ that maps an element of M_R onto its component in M'. It is easy to see that $\mathrm{Ke}(\pi_1) = M'$ and $\mathrm{Ke}(\pi_2) = x_1R$. If $T = \pi_1 \cdot g + \pi_2 \cdot f$, then T is an element of S'. Hence,

$$xT = \left(\sum_{i=1}^{n} x_i r_i\right) T = \left(\sum_{i=1}^{n} x_i r_i\right)\pi_1 g + \left(\sum_{i=1}^{n} x_i r_i\right)\pi_r f =$$

$$= (x_1r_1)g + \left(\sum_{i=2}^{n} x_i r_i\right) f = x_1r_1' + \sum_{i=2}^{n} x_i r_i' = y.$$

From this the theorem follows.

As an immediate consequence of Theorem (10.63) we obtain the next two corollaries.

Corollary (10.64). The S'-module $M_{S'}$ is irreducible.

Corollary (10.65). The near-ring S' is a primitive near-ring.

Let M_R be a n-dimensional near-vector space over the division near-ring R. In the particular case when R is a division ring, $\mathrm{Hom}_R(M,M)$ is just the ring of linear mappings of the vector space M_R and $\mathrm{Hom}_R(M,M)$ is anti-isomorphic to a ring of matrices $\mathcal{M}(R)$ with coefficients in R. We will now show that this is not true when R is a division near-ring that is not a ring.

Let $X = \{x_1, \ldots, x_n\}$ be a fixed basis for M_R. If $T \in \mathrm{Hom}_R(M,M)$ and $x_jT = \sum_{i=1}^{n} x_i\alpha_{ij}$ where $\alpha_{ij} \in R$, then the matrix (α_{ij}) is called the matrix associated with T relative to the basis X.

Let f be another element of $\mathrm{Hom}_R(M,M)$, and let $x_jf = \sum_{i=1}^{n} x_i\beta_{ij}$ where $\beta_{ij} \in R$. Then by Lemma (4.4) we have $x_j(T\cdot f) = (x_jT)f = \left(\sum_{i=1}^{n} x_if\right)\alpha_{ij} =$

$$(x_1f)\alpha_{1j} + \cdots + (x_nf)\alpha_{nj} = \left(\sum_{\ell=1}^{n} x_\ell\beta_{\ell 1}\right)\alpha_{1j} + \cdots + \left(\sum_{\ell=1}^{n} x_\ell\beta_{\ell n}\right)\alpha_{nj}$$

$$= x_1(\beta_{11}\alpha_{1j}) + \cdots + x_n(\beta_{n1}\alpha_{1j}) + \cdots + x_1(\beta_{1n}\alpha_{nj}) + \cdots$$

$$+ x_n(\beta_{nn}\alpha_{nj}) = x_1(\beta_{11}\alpha_{1j} + \cdots + \beta_{1n}\alpha_{nj}) + \cdots +$$

$+ x_n(\beta_{n1}\alpha_{1j} + \cdots + \beta_{nn}\alpha_{nj}) = \sum_{i=1}^{n} x_i c_{ij}$ where

$c_{ij} = \sum_{k=1}^{n} \beta_{ik}\alpha_{kj}$. Hence, $(c_{ij}) = (\beta_{ij})\Delta(\alpha_{ij}) \in \mathcal{M}(R)$.

Appealing to Lemma (10.29) we can conclude

Lemma (10.66). The mapping η: $T \in (\mathrm{Hom}_R(M,M),\cdot) \longrightarrow (\alpha_{ij}) \in (\mathcal{M}(R),\Delta)$ is a semi-group anti-isomorphism of $\mathrm{Hom}_R(M,M)$ onto a sub-semi-group of the groupoid $\mathcal{M}(R)$.

According to Theorem (9.7), we have the following two theorems.

Theorem (10.67). If R is not a division ring, then η is not an onto mapping.

Theorem (10.68). Let the additive group of R be abelian. η is an onto mapping if, and only if, R is a division ring.

We now assume the additive group of the division near-ring R is abelian. As pointed out in the last chapter, Zassenhaus [26] showed that the additive group of a finite division near-ring is abelian. Let M_R be an n-dimensional near-vector space over R. From Corollary (10.4) we note that the additive group M is abelian and so the additive group of the near-ring S(M) is also abelian. Fröhlich [13] proved that a d.g. near-ring with identity whose additive group is abelian is a ring. In particular, the d.g. sub-near-ring S' of S(M) generated by the multiplicative semi-group $Hom_R(M,M)$ is a ring.

According to Corollary (10.64) we have

Proposition (10.69). The S'-module $M_{S'}$ is a ring module and S' is a primitive ring.

The next three theorems are well known results from ring theory and vector-space theory. For proofs of these theorems, the reader is referred to Jacobson [19] or Kasch [20].

Theorem (10.70). A primitive ring that satisfies the descending chain condition on right ideals is isomorphic to a ring of linear mappings on a finite dimensional vector space over a division ring.

Theorem (10.71). The ring of linear mappings of a finite dimensional vector space over a division ring satisfies the descending chain condition on right ideals.

Theorem (10.72). The ring of linear mappings of an n-dimensional vector space over a division ring is anti-isomorphic to a ring of nxn matrices over a division ring.

If we conbine the results of Proposition (10.69), Theorem (10.70), Theorem (10.71) and Theorem (10.72), then we are allowed to give the following two theorems.

Theorem (10.73). The primitive ring S' satisfies the descending chain consition on right ideals, if, and only if, S' is isomorphic to a ring of linear mappings on a finite dimensional vector space over a division ring.

Theorem (10.74). If S' satisfies the descending chain condition on right ideals, then S' is anti-isomorphic to a ring of matrices over a division ring.

Theorem (10.75). If the division near-ring R is finite, then S' is isomorphic to a ring of linear mappings of a finite dimensional vector space over a division ring.

Proof: Assume R is finite. Then the n-dimensional near-vector space M_R is finite by Proposition (10.2). Hence, the ring S' is finite and so it satisfies the descending chain condition on right ideals. From Theorem (10.73) the desired result follows.

Sets of Linear Mappings of a Finite Dimensional Near-Vector Space

Let M_R be an n-dimensional near-vector space over R. As in the previous section, let $Hom_R(M,M)$ denote the subset of S(M) that consists of all linear mappings of M_R into itself.

In this section we investigate certain subsets of $Hom_R(M,M)$ in terms of subspaces of M_R. First, we give

Definition (10.76). Let Ω be a non-empty subset of $Hom_R(M,M)$. Then

(10.76.1) A subspace M' of M_R is called Ω-invariant if, and only if, $M'\Omega \subseteq M'$.

(10.76.2) Ω is said to be decomposable if, and only if, there exists proper subspaces $M_1, \ldots, M_S$ such that $M_R = \bigoplus_{i=1}^{s} M_i$ and each M_i is Ω-invariant.

From the definition, it is easy to see that the intersection and sum of Ω-invariant subspaces of M_R are Ω-invariant.

An element $f \in Hom_R(M,M)$ such that $f^2 = f \neq 0$ is called a projection. We now construct a simple example of a decomposable subset $\Omega \subseteq Hom_R(M,M)$ in which normal projections play an important role.

Let M_1 be a proper subspace of M_R. Then because of Theorem (10.25) there is a proper subspace M_2 such that $M_R = M_1 \oplus M_2$. Let $m = m_1 + m_2$ where $m_1 \in M$, $m_2 \in M_2$. Then for the integers i = 1,2 define the mapping f_i by $mf_i = m_i$. It is easy to see that f_i is a normal projection of M_R onto M_i for

$i = 1,2$. Also, $1 = f_1 + f_2$ and $f_1 \cdot f_2 = f_2 \cdot f_1 = 0$. If $\Omega = \{f_1, f_2\}$, then since $Ke(f_1) = M_2$ and $Ke(f_2) = M_1$, it is evident that Ω is a decomposable subset of $Hom_R(M\ M)$.

Our construction helps motivate the following

Theorem (10.77). Let M_R be an n-dimensional near-vector space and Ω a non-empty subset of $Hom_R(M,M)$. Ω is decomposable if, and only if, there exist normal projections $f_1, \ldots, f_k$ such that $1 = \sum_{i=1}^{k} f_i$, $f_i \cdot f_j = 0$ for $i \neq j$, and $Af_i = f_iA$ for all $A \varepsilon \Omega$ and all f_i.

Proof: Assume Ω is decomposable. Then there exists proper Ω-invariant subspaces $M_1, \ldots, M_k$ such that $M_R = \bigoplus_{i=1}^{k} M_i$. Let f_i denote the normal projection that maps M_R onto M_i. Then it follows that $1 = \sum_{i=1}^{k} f_i$. Let A be an element of Ω and $x = x_1 + \cdots + x_k$, where $x_i \varepsilon M_i$. Since M_i is A invariant we have $x(Af_i) = (xA)f_i = x_iA = (x_if_i)A = (xf_i)A = x(f_iA)$ and so $f_iA = Af_i$. It is evident that $f_i \cdot f_j = 0$ for $i \neq j$.

Conversely, suppose there exists normal projections $f_1, \ldots, f_k$ such that $1 = \sum_{i=1}^{k} f_i$, $f_i \cdot f_j = 0$ for $i \neq j$, and $Af_i = f_iA$ for all $A \varepsilon \Omega$ and all f_i. Since f_i is normal, it follows that $M_i = Mf_i$ is a subspace of M_R. If $A \varepsilon \Omega$, then $M_iA = (Mf_i)A = M(f_iA) = M(Af_i) = (MA)f_i \subseteq M_i$ and so M_i is Ω-

invariant. Since the elements $f_1, \ldots, f_k$ form a set of orthogonal idempotents of S(M) and

$$1 = \sum_{i=1}^{k} M_i, \text{ it is easily verified that } M_R = \bigoplus_{i=1}^{n} M_i.$$

Hence, Ω is indecomposable.

<u>Theorem (10.79)</u>. Let M_R be an n-dimensional near-vector space over R and A a linear mapping of M_R into M_R. A subspace M' is A-invariant if, and only if, $f\cdot A\cdot f = f\cdot A$ for every normal projection $f \in Hom_R(M,M)$ such that $Mf = M'$.

<u>Proof</u>: Assume the subspace M' is A-invariant and let f be a normal projection such that $Mf = M'$. Because of Theorem (10.49) we have $M_R = M' \oplus Ke(f)$. If $m = m' + m''$ where $m' \in M'$, $m'' \in Ke(f)$, then $m(fAf) = (mf)(Af) = m'(Af) = (m'A)f = m'A$ and $m(fA) = (mf)A = m'A$. From this it follows that $f\cdot A = fAf$.

Conversely, suppose $f\cdot Af = f\cdot A$ for every normal projection $f \in Hom_R(M,M)$ such that $Mf = M'$. Since M' is a subspace by Theorem (10.25), there exists a subspace M" of M_R such that $M_R = M' \oplus M''$. Let f denote the normal projection of M_R that maps an element in M onto its component in M'. If $x \in M'$, then $xf = x$ and so $xA = (xf)A = x(fA) = x(fAf) = (xfA)f \in M'$. This shows M' is A-invariant.

We conclude this section with

<u>Theorem (10.78)</u>. Let M_R be an n-dimensional near-vector space. If $f \varepsilon \mathrm{Hom}_R(M,M)$ and M' is an f-invariant subpace of M_R, then f induces an element $\overline{f} \varepsilon \mathrm{Hom}_R(M/M', M/M')$.

<u>Proof</u>: Let $f \varepsilon \mathrm{Hom}_R(M,M)$ and M' an f-invariant subspace. Define the mapping $\overline{f}$ as follows: $\overline{x}\,\overline{f} = \overline{xf}$ where x is a representative of the coset $\overline{x} \varepsilon M/M'$. If $\overline{x} = \overline{y}$, then $x - y \varepsilon M'$. Since M' is f-invariant $xf - yf = (x-y)f \varepsilon M'$ and so the mapping $\overline{f}$ is single-valued. It is easily verified that $\overline{f}$ is a linear mapping of the factor space M/M' into itself.

CHAPTER XI

SUMMARY

The purpose of our investigation was to consider certain aspects of the abstract theory of near-ring modules. One of our primary objectives was to define an appropriate radical for near-ring modules and develop its theory. After having done this, we then used our results to study, among other things, the properties of this radical for near-rings. Finally, we investigated near-vector spaces and showed that many of the properties of vector spaces over a division ring generalized to the case of near-vector spaces.

In the first four chapters we developed the fundamental results and concepts that were needed for further investigation. Thus, for example, we considered the three isomorphism theorems for near-ring modules, the Jordan-Hölder Theorem, the chain conditions on submodules of a near-ring module, and the notion of semi-simple near-ring module.

In the fifth chapter, our interest turned to strictly semi-simple near-ring modules. We showed that every strictly semi-simple near-ring module was semi-simple, and every simple submodule of a strictly semi-simple module was irreducible. Also, we proved that a semi-simple module was strictly semi-simple if, and only if, every maximal submodule was regular. However, we showed the existence of semi-simple modules that were not strictly semi-simple.

The sixth chapter was devoted to the study of an appropriate radical for near-ring modules. We defined the radical, denoted by J(M), of a module M to be the intersection of all regular submodules of M. It was proved that J(M) = 0 whenever M was strictly semi-simple. We next showed that J(M) was the smallest submodule A such that $J(M/A) = 0$. If M satisfied the descending chain condition on submodules and J(M) = 0, then M was strictly semi-simple. The notion of small submodule was introduced, and we proved that if M satisfied the descending chain condition on submodules and J(M) was small, then the set of maximal submodules of M coincided with the set of regular submodules.

Throughout the seventh chapter we used the results of the previous chapter. It was proved that the radical, J(R), of a near-ring R was a quasi-regular ideal if, and only if, J(R) was strictly small. Turning our attention to distributively generated near-rings R which satisfied the descending chain condition on R-subgroups, we proved that J(R) was nilpotent if, and only if, it was strictly small. However, we established the existence of finite near-rings whose radical was non-nilpotent.

The objective of the eighth chapter was to study a decomposition $R = \bigoplus_{i=1}^{n} A_i$ where R was a distributively generated near-ring that satisfied the descending chain condition on R-subgroups and A_i a non-zero indecomposable right ideal. We introduced the concept of minimal non-nilpotent R-subgroup A and proved that A was generated by an idempotent. Assuming J(R) was nilpotent, we showed each A_i was a minimal non-nilpotent right ideal. Moreover, $A_i \cap J(R)$ was the unique regular right ideal of R contained in A_i.

In the ninth chapter we studied division near-rings and matrices over arbitrary near-rings.

In the tenth chapter a strictly semi-simple module over a division near-ring was called a near-vector space. Examples of near-vector spaces that were not vector spaces were constructed. After we defined basis of a near-vector space in an appropriate way, we showed that many of the well known theorems from the theory of vector spaces generalized to the case of near-vector spaces.

The theory of near-ring modules is relatively new and so it seems that further research in this area would be fruitful. For example, topological near-ring modules have not been investigated. In particular, compact near-ring modules might be of interest. Further study of distributively generated near-rings might find applications in the theory of non-abelian groups. Finally, we mention that necessary and sufficient conditions for the radical of a general near-ring R (with descending chain condition on R-subgroups) to be nilpotent would be most useful.

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VITA

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