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The Groups of Order 2^n $(n \leq 6)$

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Preface

It is the hope of the authors that the following tables, giving for the first time the complete list of the 267 groups of order 64, will be of enduring value to those interested in finite groups. Theories change, but the groups remain.

No single presentation of a group or list of groups can be expected to yield all the information which a reader might desire. Here, each group is presented in three different ways: (1) by generators and defining relations; (2) by generating permutations; and (3) by its lattice of normal subgroups, together with the identification of every such subgroup and its factor group. In this lattice the characteristic subgroups are distinguished.

For each group, additional information is given. Here are included the order of the group of automorphisms and the number of elements of each possible order 2, 4, 8, 16, 32, and 64. Thus the groups containing exactly three elements of order 2, or the groups of exponent 4, or the groups in which every normal subgroup is characteristic, may readily be found. All the groups are divided into twenty-seven families, following Philip Hall's theory of isotopy.

Chapters 3 and 4 give the theoretical background for the construction of the tables. But these chapters are not necessary for the use of those tables; for that purpose Chapter 2 is adequate. Chapter 5 draws attention to a number of the more interesting individual groups.

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CHAPTER 1

Introduction*

The work reported in the ensuing chapters of this monograph was begun about 1935. I was then attempting to determine all the groups of order 64 by methods which I now see to have been absurdly cumbersome and inept. All that I had to go by was a paper by G. A. Miller† which I found to be neither clear nor accurate.‡

At that time, a mathematical friend called my attention to the fact that Professor Philip Hall of King's College, Cambridge, England, was also working on the groups of order 64. He advised me to get in touch with Professor Hall, and to request him to let me compare my results with his. Professor Hall kindly permitted me to do this. We found some slight discrepancies between the two lists, but these were rapidly cleared up. A month or two later, we settled the one-to-one correspondence between his groups and mine.

By this time, it was evident to me that Professor Hall's methods were much superior to the ones I had been using. We set to work to decide such questions as the following:

> The linear sequence of the families.
> The linear sequence of the genera within one stem or branch.
> The linear sequence of the groups within one genus.

These decisions involved a prolonged correspondence, but, by the summer of 1939, we were within a few months of being ready to send in our results (minus the diagrams) to *Acta Mathematica*, where we hoped to have them published.

Then World War II broke out, and since that time ill luck has dogged our footsteps. For five years Professor Hall and I were forced to lay groups aside and engage in far different occupations. In 1945, when the war was over, we attempted to start the work once more, but something always interfered. For example, I was twice incapacitated for over a year by illness.

About five years ago, Professor Philip Hall indicated that he wished to withdraw from the project. Professor Marshall Hall, Jr., was willing to take up the work at the point where Professor Philip Hall had left it, and this arrangement met with the latter's approval. Since that time, Professor Marshall Hall and I have collaborated in the preparation of the present monograph. We are, however, fully aware how much that work owes to the labors of Philip Hall. The only reason why his name does not appear on the title page as coauthor is that he requested us to omit it.

* Introduction by James K. Senior.

† G. A. Miller. "Determination of all the groups of order 64." *Am. J. of Math.*, vol. 52 (1930), pp. 617–634.

‡ Miller states that there are 294 groups of order 64. As a matter of fact, there are only 267 of them.

CHAPTER 2

Use of the Tables; Notation and Terminology

An individual group in these tables is given a designation such as 32 $\Gamma_3 c_2$. Here the 32 gives the order of the group, Γ_3 the *family** to which it belongs, and c_2 means that the group in question is the second group of *genus c* in that family. The groups are listed by families. The groups of lowest order in a family are called *stem groups*. If the stem groups of a family Γ are of order 2^r, then Γ is of *rank r*. The groups of order 2^{r+s} in a family Γ of rank r are said to form the sth *branch*. Thus, the family Γ_3 is of rank 4, and the group 32 $\Gamma_3 c_2$ is in its first branch. This group is numbered 28 in the list. The groups of each order are numbered from family to family, going from Γ_1, the family of Abelian groups, to Γ_{27}, the last family including groups of order 64.

The following table shows the number of groups of each order treated.

Order	*Number of Groups*
2	1
4	2
8	5
16	14
32	51
64	267

Before proceeding further, it is desirable to define *family* and *genus*. For a group G, let $Z(G)$ designate its center and G' its derived group, generated by all commutators $x^{-1}y^{-1}xy$, $x, y \in G$; here the notation is $[x, y] = x^{-1}y^{-1}xy$.

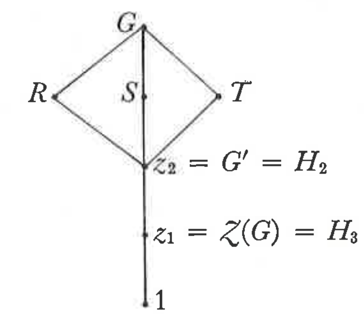
* This term and others will be defined below. They arise from the paper by Philip Hall, "Classification of prime power groups." *J. für die reine u. ang. Math.*, vol. 182 (1940), pp. 130–141.

DEFINITION. Two groups G_1 and G_2 belong to the same *family* Γ if (1) $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are isomorphic; (2) G_1' and G_2' are isomorphic; (3) It is possible to choose the isomorphisms (1) and (2) in such a way that whenever, under (1), the elements $a_1Z(G_1)$ and $b_1Z(G_1)$ of $G_1/Z(G_1)$ correspond respectively to the elements $a_2Z(G_2)$ and $b_2Z(G_2)$ of $G_2/Z(G_2)$, then, under (2), the element $[a_1, b_1]$ of G_1' corresponds to the element $[a_2, b_2]$ of G_2'.

With respect to property (3), note that, if z_1 and z_2 are elements of $Z(G)$, then $[x, y] = [xz_1, yz_2]$; whence a commutator $[x, y]$ may be regarded as a function with arguments in $G/Z(G)$ and values in G'.

DEFINITION. Two groups G_1 and G_2 of the same order are in the same *genus* if there is an isomorphism between the lattice of normal subgroups of G_1 and G_2 such that corresponding normal subgroups belong to the same family.

This definition tells when two groups of the same order are in the same genus. Furthermore, a group and its direct product with a group of order 2 are (by definition) in the same genus. Thus every genus in a branch appears again in the next branch. For example, the three stem groups of Γ_3 are in a single genus, all three having the following lattice of normal subgroups:



$G \in \Gamma_3$

$R \in \Gamma_2$

$S, T, z_1, z_2 \in \Gamma_1$

Defining relations are given for every group G of a family of rank r in terms of r elements $\alpha_1, \alpha_2, \ldots, \alpha_r$, using only these elements if G is a stem group, and using further elements $\beta_1, \ldots, \beta_m$ (which are a basis of the center $Z_1(G)$) when G is not a stem group. There is in G a chain of subgroups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_r = G$$

where $G_i = (G_{i-1}, \alpha_i)$ is the subgroup generated by G_{i-1} and the element α_i. For a stem group $G_0 = 1$, and, for other groups, G_0 is the center generated by the β's. A complete set of defining relations for G is given by the values of α_i^2, $i = 1, \ldots, r$, the commutators $[\alpha_i, \alpha_j]$, $i < j$, and the orders of the β's, together with the fact that the β's are in the center of G. The α's are so chosen that certain relations hold for every group of the family.

These relations are listed at the beginning of the family. Certain squares α_i^2 are given, as well as all commutators $[\alpha_i, \alpha_j]$, $i < j$ which are not the identity. For an individual group, further relations involving α's are given in columns headed "Defining Relations." The orders of the β's are obtained from the column subtitled Z_1 under the heading "Generic Invariants," these being the invariants of the center, which is an Abelian group. If there is more than one β, the β's are numbered so that β_1 is of highest order, β_2 of next highest order, and so on. It may happen that some of the β's do not occur in the defining relations involving the α's. When such is the case, these β's form an Abelian direct factor of G.

As an illustration, consider the group $64\ \Gamma_2 r_2$. Here Γ_2 is of rank 3, and every G of Γ_3 is generated by three elements α_1, α_2, α_3, and the β's. Hence, for $64\ \Gamma_2 r_2$,

$$\alpha_1^2 = 1, \qquad [\alpha_2, \alpha_3] = \alpha_1,$$
$$[\alpha_i, \alpha_j] = 1, \qquad i < j, \text{ otherwise}$$
$$\beta_1^2 = 1,\ \beta_2^2 = 1,$$
$$\alpha_1 = \beta_2,\ \alpha_2^2 = \beta_2,\ \alpha_3^2 = \beta_2.$$

The first relations are given at the beginning of Γ_2. Here the orders 8 and 2 for β_1 and β_2 respectively are determined by the generic invariants (3, 1) for Z_1 in genus r. The last relations come from the columns headed "Defining Relations." Since β_1 does not occur in these columns, β_1 generates an Abelian direct factor of G. Hence G is the direct product of the quaternion group and a cyclic group of order 8.

There are a number of invariants common to all groups of a family. These are listed in a series of tables on plates I, II, III, and IV.

The *rank* of a family Γ is r if the stem groups of Γ are of order 2^r. The *class* is the length of the lower central series. The families $\Gamma_1, \Gamma_2, \ldots, \Gamma_{27}$ are arranged in increasing order of the following invariants taken in turn:

The rank r.

The middle length b, where $|G':Z_1(G)| = 2^b$ for a stem group G.

The class c in decreasing order.

The symbol for the family in the second column of the table of family invariants has the form $^u_v X_i$, where, since G is a stem group, $2^u = |G:G'|$, $2^v = |Z_1(G)|$, and so $r = u + b + v$. Thus there is a correspondence between the letter $X = B, C, D, E, F$ and the values of b and c as follows

$$B \rightleftarrows b = 0, c = 2, \qquad D \rightleftarrows b = 2, c = 4,$$
$$C \rightleftarrows b = 1, c = 3, \qquad E \rightleftarrows b = 2, c = 3,$$
$$F \rightleftarrows b = 3, c = 5.$$

The group of inner automorphisms of a group of G is of course $G/Z_1(G)$; by definition it is a family invariant. The terms of the lower central series (except for G itself) are family invariants. H_k is the kth term in this series. H_k/H_{k+1} is an Abelian group.

In the column headed H_k (if $H_k \neq 1$) are given the invariants of H_k/H_{k+1}.

Under class numbers, the value j_k is the number of classes of conjugates in G (a stem group) having 2^k elements each, and $j_k{}^*$ is the number of inequivalent absolutely irreducible representations of degree 2^k. For a group $\bar{G}$ of the same family and sth branch (i.e., $|\bar{G}| = 2^s\,|G|$), $j_k(\bar{G}) = 2^s j_k(G)$, and $j_k{}^*(\bar{G}) = 2^s j_k{}^*(G)$.

A self-centralizer is a maximal Abelian subgroup. The table lists the number of these, there being s_k maximal Abelian subgroups of index 2^k.

On plate II the invariants of the Abelian groups $H_2 \cap Z_1$ and G/H_2Z_1, which are of course family invariants, are given.

Here, d is the minimum number of generators of a stem group G, and (by the Burnside basis theorem) $2^d = |G:\Phi(G)|$, where $\Phi(G)$ is the Frattini subgroup of G, the intersection of all the maximal subgroups.

The tensor product $T_0 = (G/G') \times Z_1(G)$, where G is a stem group, is isomorphic with the group of all automorphisms of G which induce the identity on $G/Z_p(G)$. The column headed T_0 gives the invariants of this group.

The group of autologisms of the family is represented by U. This is the group of those automorphisms of $G/Z_1(G)$ which induce automorphisms on $G' = H_2$. The column headed u gives the order of U. The group U itself is given in most instances. Here Σ_3, Σ_4, Σ_6 are the respective symmetric groups. In a number of cases U is one of the groups of the table. Aut (1^3) is the simple group of order 168, which is the group of automorphisms of the elementary group of order 8. $\Sigma_3 \wr \Sigma_2$ is the "wreath product" of Σ_3 by Σ_2, i.e., the direct product of a Σ_3 on 1, 2, 3 and another on 4, 5, 6 together with an element (1, 4)(2, 5)(3, 6). $(\Sigma_3 \times \Sigma_5)^+$ is the subgroup of even elements of the direct product of a Σ_3 and a Σ_5. Hol (4) is the holomorph of the cyclic group of order 16.

The group U_2 is induced on $G' = H_2$ by U.

Most of the rest of the table of family invariants is related to the diagrams for the families. These diagrams give the lattice of normal subgroups of $G/Z_1(G)$ and that of $G' = H_2(G)$, together with dotted horizontal lines which show the identification of groups in $G/Z_1(G)$ with those in $H_2(G)$, when G is a stem group. To simplify the diagrams, a box $\boxed{X_i^j}$ represents j subgroups equivalent under automorphism. The general procedure may be illustrated by reference to the diagram for Γ_5. Here G contains 15 subgroups of index 2 labeled X_1; the 15 below the G box indicates this. Each of the subgroups X_1 contains 3 subgroups of type X_2 and 4 of type X_3. There are 15 groups of type X_2 each contained in 3 of type X_1. There are 2 groups of type X_3 each contained in 3 of type X_1. A group of type X_2 contains 3 subgroups of type X_4, and a group of type X_3 contains 3 subgroups of type X_4. There are 15 groups X_4 each contained in 3 groups X_2 and in 4 groups X_3. The group $Z = Z_1$ is the center of G and is contained in all 15 of the X_4 subgroups. H_2, the derived group, is of order 2. In a stem group $Z = H_2$.

A heavily outlined box means that the corresponding group is characteristic.

Plate II, by reference to the diagrams, lists the self-centralizers (i.e., maximal Abelian subgroups) and the pairs of groups which centralize each other. Thus in Γ_5, an X_1 and an appropriate X_4 centralize one another; this happens 15 times. Similarly, in ten instances, two groups of type X_3 centralize one another.

The lower central series of G is

$$G \supset H_2 \supset H_3 \supset \ldots \supset H_c \supset 1,$$

and the upper central series is

$$1 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_{c-1} \subset G.$$

As is well known, any central series for G is of the form

$$G \supseteq B_2 \supseteq \ldots \supseteq B_t \supseteq 1,$$

where $B_i \supseteq H_i$ and $B_{t-j} \subseteq Z_{j+1}$. Under maximal central factors are listed those maximal central factors which may occur in some central series which are not trivially consequences of the upper and lower series.

Plate III gives the defining relations on the α's which are common to all groups of the family. Also, there are listed the congruences modulo Z_1 which hold for the α's. Plate IV gives, in relation to the family diagrams, the choice of the groups $G_1, \ldots, G_5$, where $G_i = (G_{i-1}, \alpha_i)$ is used in determining the relations on the α's.

The first signals of a 2-group G are the subgroups of index 2, and the factor groups modulo a normal subgroup of order 2. For every stem group G, the families of the first signals are determined. The second signals are similarly the normal subgroups of index 4 and the factor groups modulo normal subgroups of order 4.

$H_2{}^*$ is the centralizer of H_2 and its type is indicated; so also are the invariants of G/H^* and $H_2{}^*/Z_1$, these last being Abelian groups.

Further information is given for individual groups. Here $\Upsilon_2 = G/H_2$. The first subgroup signals and quotient signals are given. Under order structure is given the number of elements in G of order 2, 4, 8, 16, 32.

The order of the group of automorphisms $A(G)$ of G is the product t_1t_2. Here t_1 is the order of the subgroup of $A(G)$ inducing the identity on $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. Hence t_1 is always a power of 2 and is a multiple of the order of $G/Z_1(G)$, the group of inner automorphisms. Hence also t_2 is the order of the group of automorphisms of $G/\Phi(G)$ induced by automorphisms of G. The symbol t_3, when listed, gives supplementary information. It is the order of the group of automorphisms of $G/Z_1(G)$ induced by automorphisms of G. Hence t_3 always divides u, the order of U, which is the group of autologisms of Γ. When $H_2 = Z_1$, then $t_2 = t_3$.

Index of Terms and Symbols

A, B, C, D, E, F. These letters correspond to a division of the 27 families of groups according to a systematic ordering. (See p. 4).

Autologisms. The group U of those automorphisms of $G/Z_1(G)$ which induce automorphisms on $G' = H_2$ (see p. 5).

Branch. The sth branch of a family of rank r consists of the groups of order 2^{r+s} in the family.

Family. A collection of closely related groups (See the definition on p. 3).

First quotient signal. The list of factor groups modulo normal subgroups of order 2.

First subgroup signal. The list of subgroups of index 2.

Γ_i. The ith family, $i = 1, \ldots, 27$.

Genus. For definition of *genus* see p. 3.

H_i. The ith term of the descending (lower) central series. Here $H_1 = G$.

H_i^*. The centralizer of H_i.

I_j. The factor group G/Z_j.

$\Lambda(a)$. A family invariant when it exists. There is an Abelian subgroup of index 2 which is of type (a) modulo Z_1. For example, Γ_{14} is of type $\Lambda(2, 1)$.

Order structure. These columns list the number of elements of orders 2, 4, 8, 16, 32, 64 respectively in the group.

Rank. A family is of rank r if the stem groups are of order 2^r.

Self-centralizer. A maximal Abelian subgroup.

Stem. The groups of lowest order in a family.

t_1. Order of group of automorphisms of G which are the identity on $G/\Phi(G)$ (see p. 6).

t_2. Order of group of automorphisms of $G/\Phi(G)$ induced by automorphisms of G (see p. 6).

t_3. Order of group of automorphisms of $G/Z_1(G)$ induced by automorphisms of G (see p. 6).

Type. An Abelian 2-group is of type $(a, b, \ldots, e)$ if it has a basis of elements of orders $2^a, 2^b, \ldots, 2^e$.

U. The group of autologisms of a family (see p. 5).

U_2. The group induced on $G' = H_2$ by U.

Verbally characteristic. Same as fully invariant. A subgroup is verbally characteristic if it is mapped into itself by every endomorphism of the group.

Υ_i. The factor group G/H_i.

Z_i. The ith term of the ascending (upper) central series. Here Z_0 is the identity.

CHAPTER 3

The Families of 2-Groups of Rank ≤ 6

Let G be a stem group (i.e., $Z_1 = Z_1(G)$, the center of G, $\leq G'$) of order 2^r.

3.1 Let G have an Abelian subgroup A of index 2

Let $G = \{A, x\}$ and let $y \in A$. Then (since A is Abelian and normal in G) $yx^{-1}yx = x^{-1}yxy = x^{-1}(yx^{-1}yx)x$. $x^2 \in A$, and is therefore commutative with y. Hence $yx^{-1}yx$ is commutative with x, and therefore lies in Z_1. Or

$$x^{-1}yx \equiv y^{-1} \bmod Z_1.$$

Since $x^2 \in Z_1$, it follows that G/Z_1 is the split extension of the Abelian group A/Z_1 by an element of order 2 which transforms each element of A/Z_1 into its inverse.

Let A/Z_1 be of type λ with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$. Let $G^* = \{x^*, y_1^*, \ldots, y_s^*\}$, with the defining relations $(x^*)^2 = 1$, $(y_i^*)^{2^{\lambda_i+1}} = 1$, $x^*y_i^*x^* = (y_i^*)^{-1}$, $[y_i^*, y_j^*] = 1$ $(i, j = 1, 2, \ldots, s)$. Put $z_i^* = (y_i^*)^{2^{\lambda_i}}$. The center Z_1^* of G^* is then the elementary group $\{z_1^*, \ldots, z_s^*\}$ of order 2^s, and if $A^* = \{y_1^*, \ldots, y_s^*\}$, then A^*/Z_1^* is of type λ, A^* is Abelian of index 2 in G^*, and $G^*/Z_1^* \cong G/Z_1$. More precisely, if $y_1, \ldots, y_s$ are elements of A such that $y_1Z_1, \ldots, y_sZ_1$ form a basis of A/Z_1 with y_iZ_1 of order 2^{λ_i}, then there is an isomorphism of G^*/Z_1 formed by making the cosets of Z_1^* which contain x^*, y_i^* correspond to the cosets of Z_1 which contain x, y_i $(i = 1, \ldots, s)$.

Since A is Abelian, the mapping $y \rightarrow [y, x]$ is homomorphic (for $y \in A$), and maps A onto G', with Z_1 as kernel. Hence G' is also of type λ, with a basis consisting of the elements $[y_i, x] = t_i$

$(i = 1, \ldots, s)$, the order of t_i being 2^{λ_i}. Write $t_i^* = [y_i^*, x^*] = (y_i^*)^{-2}$. Then the above isomorphism of G/Z_1 with G^*/Z_1 induces an isomorphism of G' with $(G^*)'$, where t_i corresponds to t_i^* $(i = 1, \ldots, s)$. Thus G and G^* belong to the same family. Hence the following theorem:

THEOREM 1.1. *There is a 1-to-1 correspondence between the set of all partitions λ and the set of all families of 2-groups with an Abelian subgroup of index 2.*

Denote by $\mathfrak{A}_\lambda$ the family corresponding to λ. If λ has s parts, the rank of $\mathfrak{A}_\lambda$ is $1 + s + \sum_i \lambda_i$. This expression is ≤ 6 in the following cases:

$\mathfrak{A}_0 = \Gamma_1$, the family of Abelian 2-groups. Here zero stands for the empty partition.
$\mathfrak{A}_{(1)} = \Gamma_2$, the family of all non-Abelian 2-groups with more than one Abelian subgroup of index 2 (hence with a center of index 4), $\mathfrak{A}_{(2)} = \Gamma_3$; $\mathfrak{A}_{(1,2)} = \Gamma_4$; $\mathfrak{A}_{(3)} = \Gamma_8$; $\mathfrak{A}_{(2,1)} = \Gamma_{14}$; $\mathfrak{A}_{(4)} = \Gamma_{27}$.

Note that the class of $G \in \mathfrak{A}_\lambda$ is $\lambda_1 + 1$.

THEOREM 1.2. *A group G of order 2^n $(n \geq 3)$ and maximal class (i.e., $n - 1$) is a stem group of $\mathfrak{A}_{(n-2)}$, and has a cyclic subgroup of index 2.*

Proof by induction on *n*. When $n = 3$, there occur the octic and quaternion groups of order 8, which contain elements of order 4. These are the stem groups of $\Gamma_2 = \mathfrak{A}_{(1)}$. Suppose $n > 3$. Since G is of class $n - 1$, Z_1 must be of order 2 and G/Z_1 is a group of maximal class and order 2^{n-1}. By the induction hypothesis, G/Z_1 has a cyclic subgroup of index 2. Hence G has a subgroup $A = \{y, z\}$ of index 2, where z generates Z_1. If A were noncyclic, it would be the direct product of y of order 2^{n-2} and Z_1. But then $\{y^{2^{n-3}}\}$ (since it is characteristic in A) would be normal in G; since it is of order 2, it would belong to Z_1—a contradiction. Hence $A = \{y\}$ is cyclic, A/Z_1 is of type $(n - 2)$, and $G \in \mathfrak{A}_{(n-2)}$. G is necessarily a stem group since Z_1 is of order 2.

For $n > 2$, there are three stem groups in $\mathfrak{A}_{(n-2)}$ with generators x, y, and the defining relations:

$$y^{2^{n-1}} = 1,\ x^{-1}yx = y^{-1},\ x^2 = 1 \qquad \text{say } a_1$$
$$y^{2^{n-1}} = 1,\ x^{-1}yx = y^{-1},\ x^2 = y^{2^{n-2}} \qquad a_3$$
$$y^{2^{n-1}} = 1,\ x^{-1}yx = y^{-1+2^{n-2}},\ x^2 = 1 \text{ (or } y^{2^{n-2}}) \qquad a_2$$

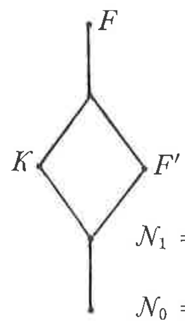
with subgroups of index 2 other than A these are given by $\mathfrak{A}_{(n-3)}$ a_1^2, a_3^2 and a_1a_3 (for $\mathfrak{A}_{(n-2)}$ a_1, a_3, and a_2 respectively).

3.2 Maximal Families

Notation. H is a given abstract group, and F/K is any factor group of a free group F which happens to be isomorphic with H:

$$F/K \cong H. \qquad (1)$$

Hence the lattice diagram



$$N_1 = K \cap F' \qquad (2)$$

$$N_0 = [K, F]$$

Note that

K/N_0 is a central factor of F, hence Abelian; (3)

$K/N_1 \cong F'K/F'$, a subgroup of the free Abelian group F/F'; hence it is itself a free Abelian group; (4)

whence

K/N_0 is the direct product of N_1/N_0 with a free Abelian group. (5)

DEFINITION. Call H *capable* if there exists a group G such that $G/\mathfrak{Z}_1(G) \cong H$.

Suppose H is capable.

Problem. Given H, to classify all such G groups into families.

Write $G \sim G_1$ to signify that G and G_1 belong to the same family. Recall that

if G_1 is a subgroup of G such that $G = G_1\mathfrak{Z}_1(G)$, then $G \sim G_1$; (6) and if $M \lhd G$ and $M \cap G' = 1$, then $G \sim G/M$. (7)

Assuming that $G/\mathfrak{Z}_1(G) \cong H$, (8) let $x_1, x_2, \ldots$, be a set of free generators of F; let α and β be fixed isomorphisms mapping F/K onto H and H onto $G/\mathfrak{Z}_1(G)$. Let $T = T(H)$ be the group of automorphisms of H. Then, for any $\theta \in T$, $\alpha\theta\beta = \gamma$ (say) will be an isomorphism of F/K onto $G/\mathfrak{Z}_1(G)$. Suppose γ maps Kx_i onto $\mathfrak{Z}_1(G)y_i$ ($i = 1, 2, \ldots$), the y's being chosen arbitrarily in their cosets. Then the map $x_i \to y_i$ extends uniquely to a homomorphism $\bar{\gamma}$ of F onto $G_1 = \{y_1, y_2, \ldots\}$ with kernel M (say), so that

$$F/M \cong G_1, \qquad (9)$$

since $G = G_1\mathfrak{Z}_1(G)$ by construction, then by (6),

$$G \sim G_1, \qquad (10)$$

and therefore

$$\mathfrak{Z}_1(G_1) = G_1 \cap \mathfrak{Z}_1(G),\ G_1/\mathfrak{Z}_1(G_1) \cong G/\mathfrak{Z}_1(G) \cong H. \qquad (11)$$

By definition of γ,

$M \leq K$ and $K = \mathfrak{Z}_1(G_1)^{\bar{\gamma}^{-1}}$ is the inverse image of the center of G_1 under $\bar{\gamma}$. (12)

Thus K/M is the center of F/M and therefore

$$M \geq N_0 = [K, F], \text{ and } K/N_0 \text{ is the center of } F/N_0. \qquad (13)$$

Let $M \cap N_1 = N$ so that $N_0 \leq N \leq N_1$. By (4), $M/N \cong N_1 M/N_1$, a subgroup of the free Abelian group K/N_1. So M/N is free Abelian, and hence

$$M/N_0 = M/N_0 \times L/N_0, \qquad (14)$$

for a suitable choice of L. All these subgroups L, M, N are normal in F since they lie between K and $N_0 = [K, F]$.

Since $M \leq K$, $M \cap F' = M \cap N_1 = N$, and so the subgroup M/N of F/N intersects the derived group F'/N of F/N in the unit subgroup. Hence $F/M \sim F/N$ and, in view of (6) and (9),

$$G \sim F/N. \qquad (15)$$

Also K/N is the center of F/N. The conclusion is that every family of groups G with $G/\mathfrak{Z}_1(G) \cong H$ has at least one representative among the factor groups F/N of F/N_0 for which $N_0 \leq N \leq N_1$, (16a) and K/N is the center of F/N. (16b)

The choice $N = N_0$ is always possible by (13). The family $M(H)$ to which F/N_0 belongs is called the *maximal family* associated with the given H. The remaining problem is to decide when $F/N \sim F/N^*$, where N and N^* are subgroups satisfying (16a and b).

Now by (16a), the central quotient groups of F/N and F/N^* effectively coincide; in $F/K \cong H$. Therefore $F/N \sim F/N^*$, if and only if there exists an automorphism φ of F/K which induces an isomorphism mapping the derived group F'/N of the former onto the derived group F'/N^* of the latter. *Induces* is to be understood in the obvious sense: if φ maps Ku and Kv onto Ku^* and Kv^* respectively, then $\bar{\varphi}$ maps $N_0[u, v]$ onto $N_0[u^*, v^*]$. Since $N_0 = [K, F]$ these cosets of N_0 are uniquely determined by the cosets Ku, Kv, Ku^*, and Kv^* of K.

Thus $\varphi \to \bar{\varphi}$ may be regarded as a homomorphism of $T(F/K) \cong T = T(H)$ into $T(F'/N_0)$. Obviously the automorphism induced by φ on the subgroup KF'/K of F/K and that induced by $\bar{\varphi}$ on the factor group F'/N_1 of F'/N_0 are related by the natural isomorphism of KF'/K with F'/N_1. However, $\bar{\varphi}$ leaves N_1/N_0 invariant; and the only really interesting items are the automorphisms of N_1/N_0 induced by $\bar{\varphi}$ for the various $\varphi \in T$. For it has been shown

(LEMMA 2.1) *that* $F/N \sim F/N^*$ if and only if there exists $\varphi \in T$ *such that* $\bar{\varphi}$ *maps* N *onto* N^* (*or more strictly* N/N_0 *onto* N^*/N_0).

Note that inner automorphisms of F/K can be realized by transforming with elements of F. Since N_1/N_0 is a central factor of F, $\bar{\varphi}$ acts trivially on N_1/N_0 whenever φ is an inner automorphism.

Schur proved that the group N_1/N_0 is independent of the choice of presentation (1) of H as a factor group of a free group, and depends only on the abstract group H itself. More precisely, he proved that F'/N_0 depends only on H. Both these results are embodied in the still more precise fact that *the maximal family*

$\mathfrak{m}(H)$ depends only on H. For if a different presentation of H

$$\text{(say) } F^*/K^* \cong H \qquad (17)$$

is taken and the above argument is applied, then, by (15), with the obvious notation,

$$F^*/N_0^* \sim F/N, \qquad (18)$$

for some N between N_0 and N_1. Similarly, by interchanging the roles of the two presentations,

$$F/N_0 \sim F^*/N^*, \qquad (19)$$

for some N^* between N_0^* and N_1^*. In order to obtain (18) and (19), it is necessary to use an isomorphism γ of F/K onto F^*/K^* and an isomorphism γ^* of F^*/K^* onto F/K. But it is always possible to choose $\gamma^* = \gamma^{-1}$. If this is done, the induced homomorphisms $\bar{\gamma}$ of F'/N_0 onto $(F^*)'/N_0^*$ (with kernel N/N_0) and $\bar{\gamma}^*$ of $(F^*)'/N_0^*$ onto F'/N_0 (with kernel N^*/N_0^*) immediately make $\bar{\gamma}\bar{\gamma}^*$ and $\bar{\gamma}^*\bar{\gamma}$ the identity maps of F'/N_0 and $(F^*)'/N_0$, respectively, so that $\bar{\gamma}^* = (\bar{\gamma})^{-1}$ and $N = N_0$, $N^* = N_0^*$. Summarizing:

THEOREM 2.2. *The outer automorphisms of H (strictly F/K) are represented in a natural way by automorphisms of the Schur multiplier N_1/N_0 of H. The subgroups N/N_0 such that $N_0 \leq N \leq N_1$ and such that $K/N = \mathfrak{Z}_1(F/N)$ are permuted among themselves in this representation; and $F/N \sim F/N^*$ if and only if N and N^* belong to the same transitive set.*

The distinct families of groups G with $G/\mathfrak{Z}_1(G) \cong H$ are in one-to-one correspondence with these transitive sets. In particular, the maximal family corresponds to the set consisting of N_0 alone.

The group of autologisms of the maximal family is $T = T(H)$, the group of automorphisms of H. For the family to which F/N belongs, the group of autologisms is the stabilizer of N/N_0 in T; i.e., it consists of all the automorphisms of $F/K \cong H$ which induce in N_1/N_0 an automorphism leaving N/N_0 invariant. An easy corollary of the foregoing general theory is the following theorem:

THEOREM 2.3. *Every family contains stem groups.*

For by (5), K/N_0 is the direct product of N_1/N_0 with a free Abelian group (say) L/N_0. Here $L \lhd F$; and $F/L \sim F/N_0$ so that $F/L \in \mathfrak{m}(H)$, the maximal family. Further F/L is a stem group. If $N_0 \leq N \leq N_1$, and if K/N is the center of F/N, then $F/LN \sim F/N$ and F/LN is again a stem group. But every family of groups G with $B/\mathfrak{Z}_1(G) \cong H$ contains members of the form F/N, with N as described.

Remark. In principle, the above theory gives a method which can be used systematically to classify the 2-groups of order ≤ 64 into families. It is most useful in those cases where $\mathfrak{m}(H)$ is small. When $\mathfrak{m}(H)$ is too large, other methods are easier.

3.3 Applications of the Multiplier Method

Not every group H can function as the group of inner automorphisms of some other group G. The following criterion is the simplest:

LEMMA 3.1. *Let $H = \{x_1, \ldots, x_r\}$, and suppose H contains an element $u \neq 1$ such that $u \in \{x_i\}$ for each $i = 1, \ldots, r$. Then $G/\mathcal{Z}_1(G) \cong H$ is impossible.*

For if $\mathcal{Z}_1(G)v$ is the coset corresponding to u, and $y_1, \ldots, y_r$ are arbitrary elements from the cosets of $\mathcal{Z}_1(G)$ corresponding to $x_1, \ldots, x_r$, then $v \equiv y_i^{m_i} \bmod \mathcal{Z}_1(G)$ for some integer m_i. Hence v commutes with each y_i. But $G = \{\mathcal{Z}_1(G), y_1, \ldots y_r\}$. Hence $v \in \mathcal{Z}_1(G)$, which contradicts $u \neq 1$.

COROLLARY 3.2. *A finite Abelian group is capable if and only if its two largest invariants coincide.*

Suppose for example that $H = \{x_1'\} \times \ldots \times \{x_r'\}$ with $\{x_i'\}$ of order h_i, where h_{i+1} divides $h_i (i = 1, \ldots r - 1)$. If $h_1/h_2 > 1$, write $x_1 = x_1'$ and $x_i = x_1'x_i'$ for $i > 1$, and let $u = x_1^{h_2}$. Then $H = \{x_1, \ldots, x_r\}$, $u \neq 1$ and $u = x_i^{h_2}$ for each i. Thus H is incapable.

But it is easy to see that the multiplier of H has as invariants 1 equal to h_2, 2 equal to h_3, ..., $r - 1$ equal to h_r, and that, if $h_1 = h_2$, there is a group of the maximal family $\mathfrak{M}(H)$ in the form $G = \{y_1, \ldots, y_r\}$ with the defining relations $[y_i, y_j]^{h_j} = 1$ $(i < j)$, together with those which express the fact that the $[y_i, y_j]$ lie in the center. This group has $\mathcal{Z}_1(G)$ generated by the $[y_i, y_j]$ and $y_k^{h_{k-1}}$; hence, effectively, $G/\mathcal{Z}_1(G) \cong H$.

Where H is an Abelian 2-group of Type λ, and the partition λ has parts

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0, \qquad (1)$$

then necessarily $\lambda_1 = \lambda_2$, if H is to be capable. Assuming this condition, the rank of the maximal family is

$$\sum_{i=1}^{r} \lambda_i + \sum_{i<j} \min(\lambda_i, \lambda_j) = \lambda_1 + 2\lambda_2 + \ldots + r\lambda_r.$$

When $r = 2$, H is of type (λ_1, λ_1) and the multiplier is cyclic of order 2^{λ_1}. Here (in the notation of sec. 2) $\mathcal{N} > \mathcal{N}_0$ implies that the center of $F/\mathcal{N}$ is larger than $K/\mathcal{N}$. Thus only the maximal family exists. This family is Γ_2 when $\lambda_1 = 1$, Γ_{12} when $\lambda_1 = 2$.

Equally simple is the case $\lambda = (1^3)$. Here the multiplier $\mathcal{N}_1/\mathcal{N}_0$ is an elementary group of order 8 and therefore isomorphic with H itself. Hence it is easy to see that the group of automorphisms $T = T(H)$ of order 168 is faithfully represented on $\mathcal{N}_1/\mathcal{N}_0$. Besides the maximal family Γ_9, there occurs only one other family, Γ_4, formed by taking $\mathcal{N}/\mathcal{N}_0$ of order 2 (all such $\mathcal{N}$'s being equivalent under T). It is impossible for $\mathcal{N}/\mathcal{N}_0$ to be of order 4, since that condition would make the center of $F/\mathcal{N}$ of index 4, not 8.

It is easy to see that 7 is the smallest rank of a family of 2-groups with $G/\mathcal{Z}_1(G)$ of type (1^5) or $(2^2 1)$. Hence *the groups G of class 2 and order dividing 64 which are not included in the families Γ_2, Γ_4, Γ_9, and Γ_{12} obtained above all have central quotient groups of type (1^4).* For if H is of type (1^4), the multiplier is of type (1^6), and since this type is rather large, these remaining class-2 groups are best divided into families by another method (see sec. 3.5.).

Of the two non-Abelian groups of order 8, the quaternion group is incapable by Lemma 3.1. More generally, of the three groups of order 2^n $(n \geq 4)$ and maximal class $n - 1$, only the dihedral group is capable, again by Lemma 3.1.

LEMMA 3.3. *Let $H = \{a, b\}$ with $a^{2^n} = b^2 = 1$, $bab = a^{-1}$ $(n \geq 1)$, be the dihedral group of order 2^{n+1}. Then the multiplier of H is of order 2. Only the maximal family $\mathfrak{M}(H)$ exists, and its stem consists of the groups of order 2^{n+2} and of maximal class.*

For let G be a stem group with $G/\mathcal{Z}_1 \cong H$ where $\mathcal{Z}_1 = \mathcal{Z}_1(G)$. Let $\mathcal{Z}_1 x$ correspond to a. Then $A = \{\mathcal{Z}_1, x\}$ is of index 2 in G and Abelian. By Theorem 1, if G' is of type λ, $G/\mathcal{Z}_1$ will be the split extension of an Abelian group of type λ by an element of order 2 which transforms each element of the Abelian group into its inverse. Hence $\lambda = (n)$, and G' is cyclic of order 2^n. But $G'/\mathcal{Z}_1 \cong H'$, which is cyclic of order 2^{n-1}. Hence $\mathcal{Z}_1$ is of order 2, and G (by Theorem 1) is of maximal class.

In particular, for $H = 8\,\Gamma_2\, a_1$, the only family is $\Gamma_3 = \mathfrak{M}(H)$; for $H = 16\,\Gamma_3\, a_1$, the only family is $\Gamma_8 = \mathfrak{M}(H)$.

It follows that the groups with $G/\mathcal{Z}_1$ of order 8 fall into the three families Γ_3, Γ_4, and Γ_9. Consider next those with $G/\mathcal{Z}_1$ of order 16. Of these, those of class 2 but not in Γ_{12} are (as already remarked) to be treated later. By Lemma 3.3, those of class 4 constitute the family Γ_8. It remains to deal with those of class 3. For these, $H \cong G/\mathcal{Z}_1$ is a group of the first branch of Γ_2. Now, in the notation of the tables, the groups a_2, b, c_2, and d of this branch are incapable by Lemma 3.1.

For $16\Gamma_2 a_2 = \{\alpha_2, \alpha_3, \alpha_2\beta_2\}$, take $u = \beta_1$.
For $16\Gamma_2 b = \{\alpha_2\beta, \alpha_3\beta, \beta\}$, take $u = \beta^2$.
For $16\Gamma_2 c_2 = \{\alpha_2\alpha_3, \alpha_3\}$, take $u = \beta_2$.
For $16\Gamma_2 d = \{\alpha_2\alpha_3, \alpha_3\}$, take $u = \beta^2$.

Thus there remain only the groups a_1 and c_1 to be discussed.

Now $16\Gamma_2 a_1$ is the direct product of an octic group with a group of order 2. Rather more generally, consider the case where H is the direct product of a group of order 2 with a dihedral group of order 2^{n+1} $(n > 1)$. Say $H = \{a, b, c\}$, where

$$a^{2^n} = b^2 = c^2 = 1, \; bab = a^{-1}, \; bc = cb, \; ac = ca. \qquad (3)$$

Let G be any group with $G/\mathcal{Z}_1 \cong H$, and suppose that to a, b, and c there correspond the cosets of $\mathcal{Z}_1$ containing α, β, and γ. Then γ^2 and $[\alpha, \gamma]$ lie in $\mathcal{Z}_1$. Hence $1 = [\alpha, \gamma^2] = [\alpha, \gamma]^2 = [\alpha^2, \gamma]$. Hence, γ commutes with α^2; and $\xi = [\alpha, \gamma]$ is of order 2. Similarly $\eta = [\beta, \gamma]$ is of order 2. Also $G_1 = \{\alpha, \beta\}$ has $G_1/\mathcal{Z}_1 \cap G_1$, which is dihedral and of order 2^{n+1}. Hence, by Lemma (3.3),

G_1' is cyclic, of order at most 2^n, and generated by $[\alpha, \beta]$. But $[\alpha, \beta]$ is of order $2^{n-1} \bmod \mathcal{Z}_1$. So, if $\zeta = [\alpha, \beta]^{2^{n-1}}$, $\gamma^2 = 1$. Thus $\mathcal{N} = \{\xi, \eta, \zeta\}$ is elementary and of order ≤ 8; it lies in $\mathcal{Z}_1$. Now $[\alpha, \beta] = \alpha^{-2} \bmod \mathcal{Z}_1$; therefore $[\alpha, \beta]$ commutes with γ; and G_1' is normal in G. But $G = \{\mathcal{Z}, \alpha, \beta, \gamma\}$; and so G' is generated by $[\alpha, \beta]$, ξ, and η, together with their conjugates in G. Since it has been shown that $\{[\alpha, \beta]\} = G_1'$ is normal in G, it follows that $G' = \{[\alpha, \beta], \xi, \eta\}$ and that $G' \cap \mathcal{Z}_1 = \{\xi, \eta, \zeta\}$. Hence the multiplier of H is elementary and of order ≤ 8.

To prove that this multiplier is actually of order 8, define a group $G = \{\alpha, \beta, \gamma, \xi, \eta\}$ by the relations:

$$\alpha^{2^{n+1}} = \beta^2 = \gamma^2 = \xi^2 = \eta^2 = 1, \; \beta\,\alpha\,\beta = \alpha^{-1}, \; [\alpha, \gamma] = \xi,$$
$$[\beta, \gamma] = \eta, \; \xi \in \mathcal{Z}_1, \; \eta \in \mathcal{Z}_1.$$

Thus G is the split extension of the direct product $\{\alpha, \beta\} \times \{\xi\} \times \{\eta\}$ of a dihedral group of order 2^{n+2} and two cyclic groups of order 2 by an element γ of order 2 which induces the automorphism $\alpha, \beta, \xi, \eta \rightarrow \alpha\,\xi, \beta\,\eta, \xi, \eta$. Note that $[\alpha^2, \gamma] = 1$, and that $\mathcal{Z}_1 = \mathcal{Z}_1(G)$ is $\{\xi, \eta, \zeta\}$ where $\zeta = \alpha^{2^n}$. So in fact $G/\mathcal{Z}_1 \cong H$. Since $G' = \{\alpha^2, \xi, \eta\}$, $\mathcal{Z}_1$ is contained in G'. The multiplier of H is of order ≥ 8, since $\mathcal{Z}_1$ is of order 8. Combining this result with those previously obtained proves that the group in question is a stem group of the maximal family $\mathfrak{M}(H)$. This family therefore has rank $n + 5$. It is next necessary to obtain representatives of all families with central quotient groups isomorphic with H in the form $G/\mathcal{N}$, where $\mathcal{N} \leq \mathcal{Z}_1$. To avoid extra notation, $\mathcal{Z}_1$ may be regarded as itself the multiplier of H. There remains to be considered how its subgroups $\mathcal{N}$ are affected by automorphisms of H.

Let θ map a, b, c into a, ab, c. The induced automorphism $\bar{\theta}$ maps ξ, η, ζ into $\xi, \xi\eta, \zeta$. Again, let φ map a, b, c into $a, b, a^{2^{n-1}}c$. Then $\bar{\varphi}$ maps ξ, η, ζ into $\xi, \eta\zeta, \zeta$; and if ψ maps a, b, c into ac, b, c, then $\bar{\psi}$ is the identity on $\mathcal{Z}_1$. These three automorphisms suffice, since they generate $T = T(H)$ modulo inner automorphisms of H.

In order that $G/\mathcal{N}$ shall have the center $\mathcal{Z}_1/\mathcal{N}$, it is obviously necessary that $\mathcal{N}$ shall not contain ζ, since, if it did, $\alpha^{2^{n-1}}\mathcal{N}$ would belong to the center of $G/\mathcal{N}$. Excluding $\mathcal{N} = \{\zeta\}$, the remaining six subgroups of order 2 in $\mathcal{Z}_1$ fall into three transitive sets under the influence of T. The first consists of $\{\xi\}$ alone, the second of $\{\xi\zeta\}$ alone; the remaining four are all equivalent. Thus there are three distinct families of rank $n + 4$, when $\mathcal{N} = \{\xi\}$, $\{\xi\zeta\}$ or $\{\eta\}$. When $n = 2$, these families are Γ_{14}, Γ_{16} and Γ_{15} respectively.

If $\mathcal{N}$ is to be of order 4, it must contain neither ζ nor ξ. For if $\xi \in \mathcal{N}$, $G/\mathcal{N}$ has an Abelian subgroup of index 2 (as, for example, in family Γ_{14}). But when $\mathcal{N}$ is of order 4, the derived group of $G/\mathcal{N}$ is cyclic, since $\mathcal{N}$ must not contain $\zeta = [\alpha, \beta]^{2^{n-1}}$, and G' is of type $(n, 1, 1)$. Thus if $\mathcal{N}$ is of order 4 and contains ξ, $G/\mathcal{N}$ has as a central factor group a dihedral group which is not isomorphic with H. But of the seven subgroups of order 4 in $\mathcal{Z}_1$, there are just two which contain neither ξ nor ζ, namely,

$\{\xi\zeta, \eta\}$ and $\{\xi\zeta, \eta\zeta\}$, and these two are equivalent under the influence of T. Thus there is only a single family of rank $n+3$. When $n = 2$ this family is Γ_6, and when $n = 3$ it is Γ_{19}—as may easily be verified. The following Theorem summarizes this situation:

THEOREM 3.4. *Let H be the direct product of a dihedral group of order 2^{n+1} with a cyclic group of order 2, where $n > 1$. Then the Schur multiplier of H is elementary and of order 8. The groups G with $G/\mathcal{Z}_1(G) \cong H$ fall into five distinct families; the maximal family of rank $n+5$, three families of rank $n+4$, and one family of rank $n+3$. Those among these families which have rank ≤ 6 are Γ_6, Γ_{14}, Γ_{15}, Γ_{16} (for $n = 2$), and Γ_{19} (for $n = 3$).*

Turning next to the case $H = 16\Gamma_2 c_1$, it is again just as easy to deal with a rather more general case, $H = \{a, b\}$, with the defining relations:

$$a^{2^n} = b^2 = 1,\ bab = a^{-1}c,\ c^2 = 1,\ ca = ac,\ cb = bc, \qquad (4)$$

where $n > 1$. When $n = 2$, $H = 16\Gamma_2 c_1$. When $n = 3$, $H = 32\Gamma_3 c_1$. When $n = 4$, $H = 64\Gamma_8 c_1$. In general, the H in question has the same order and belongs to the same family as the H of Theorem 3.4.

Let G be a group with $G/\mathcal{Z}_1 \cong H$, and let the cosets of $\mathcal{Z}_1$ in G which contain α, β, γ correspond as before to the elements a, b, c of H. Hence $G = \{\mathcal{Z}_1, \alpha, \beta\}$ and G' is generated by $[\alpha, \beta]$ together with its conjugates in G. Now H' is cyclic and of order 2^{n-1}, since it is generated by $[a, b] = a^{-2}c$. Hence $[\alpha, \beta]$ is of order 2^{n-1} mod $\mathcal{Z}_1$. If ζ is taken as $[\alpha, \beta]^{2^{n-1}}$, then $\zeta \in \mathcal{Z}_1$. As before, since γ^2 and $[\alpha, \gamma] = \eta$ lie in $\mathcal{Z}_1$, $\eta^2 = 1$ and α^2 commutes with γ. Write $\alpha_0 = \alpha$ and $\alpha_{i+1} = [\alpha_i, \beta]$. Since $\beta^2 \in \mathcal{Z}_1$, it follows that $1 = [\alpha_{i-1}, \beta^2] = [\alpha_{i-1}, \beta]^2 [\alpha_{i-1}, \beta, \beta]$, so that $\alpha_{i+1} = \alpha_i^{-2}$ for $i > 0$ or $\alpha_{i+1} = \alpha_1^{(-2)^i}$. But $\alpha_1 = [\alpha, \beta] \equiv \alpha_1^{(-2)^j}$ mod $\mathcal{Z}_1$, so that $\alpha_1^{2^{n-1}} \in \mathcal{Z}_1$. Therefore $\alpha_n \in \mathcal{Z}_1$, and so $\alpha_{n+1} = \alpha_n^{-2} = 1$. Hence $\alpha_n = \zeta$ and $\zeta^2 = 1$. Thus $\{\alpha_1\}$ is of order $\leq 2^n$, and is transformed into itself by β. But $[\alpha_1, \alpha] = [\alpha^{-2}\gamma, \alpha] = [\gamma, \alpha] = \eta$, since $\eta^2 = 1$. And since $\eta \in \mathcal{Z}_1$, $\{\alpha_1, \eta\}$ is transformed into itself by both α and β and is therefore normal in G. But G' is generated by α_1 and its conjugates in G. Hence $G' = \{\alpha_1, \eta\}$ and $G' \cap \mathcal{Z}_1 = \{\eta, \zeta\}$ is elementary and of order ≤ 4. To prove that the multiplier of H is of order 4, use the group $G = \{\alpha, \beta, \gamma\}$ with the defining relations $\alpha^{2^{n+1}} = \beta^2 = \gamma^2 = 1$, $[\alpha, \gamma] = [\beta, \gamma] = \eta$, $\eta^2 = 1$, $\eta \in \mathcal{Z}_1$, $\beta\alpha\beta = \alpha^{-1}\gamma$. Note that $G = \{\alpha, \beta\}$ and is the split extension (by β of order 2) of the group $\{\alpha, \gamma\}$ of class 2, with the automorphism $\alpha, \gamma \to \alpha^{-1}\gamma, \gamma\eta$ which is effectively of period 2. Since $n > 1$, the centralizer of β in $\{\alpha, \gamma\}$ is the group $\{\alpha^{2^n}, \eta\}$ of order 4, which is also in the center of $\{\alpha, \gamma\}$. Since β transforms $\{\alpha, \gamma\}$ by an outer automorphism, it follows that $\mathcal{Z}_1 = \mathcal{Z}_1(G)$ is precisely $\{\alpha^{2^n}, \eta\}$. Hence $G/\mathcal{Z}_1 \cong H$. Finally, G' contains η and $[\alpha, \beta] = \alpha^{-2}\gamma$, and hence also $\alpha^{2^n} = (\alpha^{-2}\gamma)^{2^{n-1}}$. Thus $\mathcal{Z}_1 \leq G'$ and G is a stem group. It follows that $G \in \mathfrak{M}(H)$, and the multiplier of H is the elementary group of order 4.

Any families other than $\mathfrak{M}(H)$ with H as central factor group

may be represented by groups G/N, where N is one of the three subgroups of order 2 in $\mathcal{Z}_1$. But N must not contain η, for if it did $N\gamma$ would lie in the center of G/N. If $n > 2$, N must also not contain $\alpha^{2^n} = \alpha_n$, for if it did it would make $\alpha_{n-1} = [\alpha, \beta]^{2^{n-2}} = \alpha^{-2^{n-1}}$ commute with both α and β mod N, so that $N\alpha_{n-1}$ would lie in the center of G/N. Finally if $n = 2$, N must not contain $\alpha^4\eta$, since if it did, $[\alpha, \gamma, \alpha] = 1$ and $[\alpha_1\gamma, \beta] = \alpha^4\eta$; hence again G/N would have too large a center. The only admissible choice of N is therefore $N = \{\alpha^4\eta\}$ if $n > 2$ and $N = \{\alpha^4\}$ if $n = 2$. Thus the maximal family has rank $n+4$; hence there is just one other family, of rank $n+3$. For $n = 2$, $\mathfrak{M}(H) = \Gamma_{17}$ and the second family, where this $H = 16\Gamma_2 c_1$ is Γ_7. For $n = 3$, the unique family of rank 6 is Γ_{21}. To summarize:

THEOREM 3.5. *Let H be the group of order 2^{n+2} $(n > 1)$ defined by equation (4). Then the Schur multiplier of H is elementary and of order 4. The groups G with $G' \supset \mathcal{Z}_1(G)$ fall into two distinct families; the maximal family has rank $n+4$, and there is one other family of rank $n+3$. Those among these families which have rank ≤ 6 are Γ_7 and Γ_{19} (for $n = 2$) and Γ_{21} (for $n = 3$).*

3.4 Groups G of Class 2 or 3 with Center of Order 2

These groups are stem groups. Suppose first that G is of class 2. Then $G' = \mathcal{Z}_1 = \mathcal{Z}_1(G)$ is of order 2. Hence $G/\mathcal{Z}_1$ is elementary and every element of $G - \mathcal{Z}_1$ has exactly two conjugates in G. The following theorem may be proved:

THEOREM 4.1. *If G is a 2-group of class 2 with center $\mathcal{Z}_1$ of order 2, then G is the central product of a certain number r of octic groups O with a certain number s of quaternion groups Q: symbolically $G \cong O^r Q^s$, so that $G/\mathcal{Z}_1$ is elementary and of order 2^{2n} where $n = r + s$. Also $O^r Q^s \cong O^\rho Q^\sigma$ if and only if $r \equiv \rho$ mod 2 (and hence $s \equiv \sigma$ mod 2). For given n, there are just two distinct such groups, forming the stem of a single family $\mathfrak{G}_n$.*

When $|G:\mathcal{Z}_1|$ has its smallest possible value 4, G is a non-Abelian group of order 8, hence, as is well known, G is either octic or quaternion. These two groups form the stem of the family $\mathfrak{G}_1 = \Gamma_2$. Suppose then that $|G:\mathcal{Z}_1| > 4$. Choose any two elements x_1, y_1 in G which do not commute and let their centralizers in G be X_1 and Y_1. Since X_1 and Y_1 are distinct and are both of index 2 in G, it follows that $G_1 = X_1 \cap Y_1$ is of index 4 in G. Also x_1 and y_1 are independent mod G_1. Hence $G = P_1 G_1$ where $P_1 = \{x_1, y_1\}$, and $P_1 \cap G_1 = \mathcal{Z}_1$. Since $[P_1, G_1] = 1$, the center of G_1 is again $\mathcal{Z}_1$, and consequently G is the central product of P_1 and G_1 (i.e., the direct product with amalgamated centers). P_1, being non-Abelian and of order 8, is either octic or quaternion. The first result now follows by induction.

It is easy to verify that the central product O^2 of two octic groups is isomorphic with the central product Q^2 of two quaternion groups, while O^2 and OQ are distinct. More generally, (for example, by counting the numbers of elements of order 4)

O^n and $O^{n-1}Q$ may be shown to be distinct. Thus the order of G has the form 2^{2n+1} and either $G \cong O^n$ or $G \cong O^{n-1}Q$. In either case, if x_i and y_i are chosen to generate the ith of these central factors $(i = 1, \ldots, n)$ and z generates $\mathcal{Z}_1$, then $x_i^2 \equiv y_i^2 \equiv 1$ mod $\mathcal{Z}_1$, $z^2 = 1$, $[x_i, y_i] = z$ $(i = 1, \ldots, n)$, and the x's and y's with distinct suffixes commute. Thus the two groups belong to the same family $\mathfrak{G}_n$: $O^n \sim O^{n-1}Q$. Note that $\mathfrak{G}_n$ is of rank $2n+1$, $\mathfrak{G}_2 = \Gamma_5$. Note, as a corollary, that there is no family of rank 6 of groups G where $G/\mathcal{Z}_1$ is elementary, Abelian, and of order 32.

Now let G be of class 3 with center $\mathcal{Z}_1$ of order 2. Then $\mathcal{Z}_2/\mathcal{Z}_1$ is elementary. Let W be the centralizer of $\mathcal{Z}_2$ in G. For $x \in G$ and $y \in \mathcal{Z}_2$, the function $[x, y]$, with values 1 or z, where $\mathcal{Z}_1 = \{z\}$ is distributive with respect to both arguments, and establishes a dual correspondence δ between $\mathcal{Z}_2/\mathcal{Z}_1$ (regarded as a linear space over the field of two elements) on the one hand and G/W on the other. Thus G/W is also elementary and $|G:W| = |\mathcal{Z}_2:\mathcal{Z}_1|$. By δ, any subgroup L of G lying between $\mathcal{Z}_1$ and $\mathcal{Z}_2$ is associated with its centralizer L^* in G, which lies between W and G. Moreover $|L:\mathcal{Z}_1| = |G:L^*|$.

Now let L be complementary in $\mathcal{Z}_2/\mathcal{Z}_1$ to the center $W \cap \mathcal{Z}_2$ of $\mathcal{Z}_2$, so that $L(W \cap \mathcal{Z}_2) = \mathcal{Z}_2$ and $L \cap W = \mathcal{Z}_1$. The centralizer $L^* \cap \mathcal{Z}_2$ of L in $\mathcal{Z}_2$ must then coincide with the center $W \cap \mathcal{Z}_2$ of $\mathcal{Z}_2$. Hence $L^* \cap \mathcal{Z}_2 = W \cap \mathcal{Z}_2$. But $|G:L^*| = |L:\mathcal{Z}_1| = |\mathcal{Z}_2:W \cap \mathcal{Z}_2| = |\mathcal{Z}_2:L^* \cap \mathcal{Z}_2|$. Thus $G = L^*\mathcal{Z}_2 = L^*L$ is the central product of L with L^*. Since the center of L is $\mathcal{Z}_1$ and L is of class 2, it follows from Theorem 4.1 that either $L = \mathcal{Z}_1$ (in which case $L^* = G$) or else $L \cong O^n$ or $O^{n-1}Q$ for some $n > 0$. On the other hand, L^* must be of class 3, for otherwise G would not be of that class. And since $[L, L^*] = 1$, $G = LL^*$, it follows that $\mathcal{Z}_2(L^*) = L^* \cap \mathcal{Z}_2(G)$, which is the center of $\mathcal{Z}_2 = \mathcal{Z}_2(G)$. Thus the second center of L^* is Abelian. For the same reason, the center of L^* is $\mathcal{Z}_1$.

THEOREM 4.2. *A group G of class 3 with center of order 2 is the central product of a subgroup L^* of class 3 with a center of order 2 and an Abelian second center containing a certain number n (possibly zero) of octic or quaternion groups.*

The only family of class-3 groups of rank as small as 4 is Γ_3. Applying Theorem 4.2, where L^* is in Γ_3 and $n = 1$, there is obtained a single family Γ_{18} of rank 6. All other families obtained by applying Theorem 4.2, with $n \geq 1$, are of rank > 6 and hence are no further considered here. *Hence it will be assumed that $\mathcal{Z}_2 = \mathcal{Z}_2(G)$ is Abelian.*

In this case $\mathcal{Z}_2 \leq W$, and if $|\mathcal{Z}_2:\mathcal{Z}_1| = |G:W| = 2^s$, the order of G is $2^{2s+1}|W:\mathcal{Z}_2|$. It may therefore also be supposed that $s = 1$ or 2.

Since G is of class 3, $\mathcal{Z}_1 < G' \leq \mathcal{Z}_2$. Suppose first that G' is of order 4 and let U be its centralizer in G. Since $|G':\mathcal{Z}_1|$ is then 2, U is of index 2 in G. Also $U' \leq \mathcal{Z}_1$ (a general truth about the centralizer of the derived group). Thus $G/\mathcal{Z}_1$ is a group of class 2 with a derived group $G'/\mathcal{Z}_1$ of order 2, and with an Abelian subgroup $U/\mathcal{Z}_1$ of index 2. These facts imply that $G/\mathcal{Z}_1 \in \Gamma_2$ and hence that $|G:\mathcal{Z}_2| = 4$. Consequently $|G| = 2^{s+3} = 16$ or 32. It follows that $G/\mathcal{Z}_1$ is either the octic group or else $16\Gamma_2 a_1$ or else $16\Gamma_2 c_1$, cases already dealt with in sec. 3. In these cases, G is a stem group of one of the families Γ_3, Γ_6, Γ_7, already found.

Thus it remains only to deal with the case $s = 2$, $G' = \mathcal{Z}_2$ of order 8. Assuming $|G| \leq 64$, $G/\mathcal{Z}_1$ is then a group of class 2; its order is ≤ 32 and its derived group is $G'/\mathcal{Z}_1$ of order 4. $G/\mathcal{Z}_1$ must therefore be a stem group of family Γ_4. Hence there is a uniquely determined subgroup A of index 2 in G such that $A' \leq \mathcal{Z}_1$. A itself cannot be Abelian (see sec. 1) for, since $\mathcal{Z}_1$ is of order 2, that would mean that G' was cyclic, whereas in fact $G'/\mathcal{Z}_1$ is an elementary group of order 4. Thus either (1) A is a stem group of Γ_5 or (2) $A \in \Gamma_2$.

Case (1). Since $|G':\mathcal{Z}_1| = |\mathcal{Z}_2:\mathcal{Z}_1| = 4$, it follows that $|G:W| = 4$, where (as previously) W is the centralizer of G' in G. Now by hypothesis $\mathcal{Z}_2 = G'$ is Abelian. Therefore, in case (1), $\mathcal{Z}_2$ must be its own centralizer in A. Hence $A \cap W = \mathcal{Z}_2$, $AW = G$. Also, since $A \in \Gamma_5$ and has $\mathcal{Z}_1$ as a center, $A/\mathcal{Z}_1$ is Abelian and elementary. Therefore $G/\mathcal{Z}_1$ is $32\Gamma_4 a_1$, since this is the only stem group of Γ_4 whose Abelian subgroup of index 2 is elementary.

Case (2). Inasmuch as $\mathcal{Z}_2$ is either of type (1^3) or (21), $\mathcal{Z}_2 = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\{\alpha_1\} = \mathcal{Z}_1$ and $\alpha_1^2 = \alpha_2^2 = 1$; either $\alpha_3^2 = 1$ ($\mathcal{Z}_2$ elementary) or $\alpha_3^2 = \alpha_1$ ($\mathcal{Z}_2$ of type (21)). Then α_2 and α_3 form a base of $\mathcal{Z}_2$ mod $\mathcal{Z}_1$. Since $G = AW$ and $\mathcal{Z}_2 = A \cap W$, the dual relation δ between $\mathcal{Z}_2/\mathcal{Z}_1$ and G/W induces an isomorphism between $\mathcal{Z}_2/\mathcal{Z}_1$ and $A/\mathcal{Z}_2$. The "linear spaces" are only of dimension 2. Hence α_5, α_6 in $A - \mathcal{Z}_2$ may be chosen so that

$$[\alpha_2,\alpha_5] = [\alpha_3,\alpha_6] = 1, \; [\alpha_2,\alpha_6] = [\alpha_3,\alpha_5] = \alpha_1. \qquad (1)$$

Let $\alpha_4 \in W - \mathcal{Z}_2$, so that $\alpha_4, \alpha_5, \alpha_6$ form a base of G mod $\mathcal{Z}_2$. Since $A/\mathcal{Z}_1$ is elementary,

$$\alpha_5^2 \equiv \alpha_6^2 \equiv 1 \bmod \mathcal{Z}_1. \qquad (2)$$

For $\xi \in A$, write $\xi^* = [\alpha_4, \xi]$. Then, since $G/\mathcal{Z}_1 \in \Gamma_4$, the mapping $\mathcal{Z}_2\xi \to \mathcal{Z}_1\xi^*$ is an isomorphism of $A/\mathcal{Z}_2$ onto $\mathcal{Z}_2/\mathcal{Z}_1$; and $[\xi^*, \xi] = 1$ or α_1. From (2) it follows that $1 = [\alpha_4, \alpha_5^2] = (\xi^*)^2[\xi^*, \xi]$ so that

$$(\xi^*)^2 = [\xi^*, \xi]. \qquad (3)$$

Also $[\xi, \alpha_4^2] = [\xi, \alpha_4]^2$, since $\alpha_4 \in W$, the centralizer of G'. Hence

$$(\xi^*)^2 = [\alpha_4^2, \xi]. \qquad (4)$$

When $\mathcal{Z}_2$ is elementary, $(\xi^*)^2 = 1$ for all $\xi \in A$. Hence by (4), α_4^2 belongs to the center $\mathcal{Z}_1$ of A; and by (3), $[\xi^*, \xi] = 1$ for every ξ. Since the relations (1) remain unaffected when α_2, α_3 are replaced (either or both) by $\alpha_1\alpha_2$, $\alpha_1\alpha_3$ respectively, the present case may be written

$$\alpha_4^2 \equiv 1 \bmod \mathcal{Z}_1 \quad \text{and} \quad [\alpha_4,\alpha_5] = \alpha_2, \; [\alpha_4,\alpha_6] = \alpha_3. \qquad (5)$$

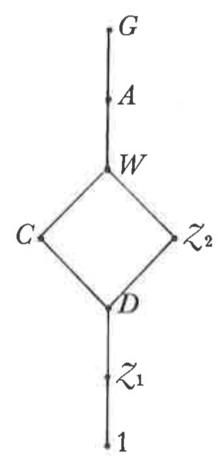
Relations (1), (2), and (5) define family Γ_{25}.

When $\mathcal{Z}_2$ is of type (21), $\alpha_4^2 \notin \mathcal{Z}_1$ since ξ may be chosen so as to make ξ^* of order 4. On the other hand, (4) gives $[\alpha_4^4, \xi] = 1$ for every ξ in A, since $(\xi^*)^2 \in \mathcal{Z}_1$. Hence $\alpha_4^2 \equiv \alpha_2 \bmod \mathcal{Z}_1$. Relations (1), (3), and (4) now give $(\alpha_5^*)^2 = 1 = [\alpha_5^*, \alpha_5]$, so that α_5^* is either α_2 or $\alpha_1\alpha_2$, and it may be assumed that $\alpha_5^* = \alpha_2$. α_2 may be replaced by $\alpha_1\alpha_2$ if necessary. On the other hand $(\alpha_6^*)^2 = [\alpha_2,\alpha_6] = \alpha_1 = [\alpha_6^*,\alpha_6]$. Hence $\alpha_6^* \not\equiv \alpha_3 \bmod \mathcal{Z}_1$. Nevertheless, $\alpha_6^* \not\equiv \alpha_2 \bmod \mathcal{Z}_1$ also, since $\alpha_2 = \alpha_5^*$, as has already been shown. Thus it may be assumed that $\alpha_6^* = \alpha_2\alpha_3$ (α_3 may be replaced by $\alpha_1\alpha_3$ if necessary). Thus in this case $\mathcal{Z}_2$ is of type (21) and the relations obtained are

$$\alpha_4^2 \equiv \alpha_2 \bmod \mathcal{Z}_1 \quad \text{and} \quad [\alpha_4,\alpha_5] = \alpha_2, \; [\alpha_4,\alpha_6] = \alpha_2\alpha_3, \qquad (6)$$

which together with (1) and (2) define the family Γ_{26}.

It remains only to discuss the case where $A \in \Gamma_2$. Naturally A' is again $\mathcal{Z}_1$, since this is the only normal subgroup of order 8. But $C \neq \mathcal{Z}_2$ since the centralizer W of $\mathcal{Z}_2$ is of index 4 in G. Since $\mathcal{Z}_2$ is Abelian and A is not, $C\mathcal{Z}_2$ must be of order 16. But $C\mathcal{Z}_2 = W$. Hence the following diagram where $D = C \cap \mathcal{Z}_2$ and all subgroups given are characteristic in G. Choosing $\alpha_1 \neq 1$ in $\mathcal{Z}_1$, α_2 in $D - \mathcal{Z}_1$, α_3 in $\mathcal{Z}_2 - D$, α_4 in $C - D$, α_5 in $A - W$, and α_6 in $G - A$, gives the following relations:



$$[\alpha_2,\alpha_6] = \alpha_1, \; [\alpha_4,\alpha_6] = \alpha_2, \qquad (7)$$

since $[\alpha_2,A] = 1$ and α_2 is not in $\mathcal{Z}_1$; whereas $[\alpha_4,A] = 1$ and α_4 is not in $\mathcal{Z}_2$. (It might be that $[\alpha_4,\alpha_6] = \alpha_1\alpha_2$, but then α_4 could be replaced by $\alpha_2\alpha_4$.) Note also that (as just remarked)

$$[\alpha_2,\alpha_i] = [\alpha_4,\alpha_i] = 1 \text{ for } i < 6. \qquad (8)$$

In view of the dual correlation of $\mathcal{Z}_2/\mathcal{Z}_1$ with G/W, in which $D/\mathcal{Z}_1$ corresponds to A/W, α_6 may be chosen so that $[\alpha_3,\alpha_6] = 1$. Then

$$[\alpha_3,\alpha_5] = \alpha_1; \quad [\alpha_3,\alpha_i] = 1 \text{ for } i \neq 5. \qquad (9)$$

Finally, $[\alpha_5,\alpha_6] \in \mathcal{Z}_2 - D$ since $[\alpha_4,\alpha_6]$ is in D. By adjusting the choice of α_5 (which does not affect (8) or (9)) it may be assumed (in view of (7)) that

$$[\alpha_5,\alpha_6] = \alpha_3. \qquad (10)$$

Now by (9) $\alpha_6^2 \in \mathcal{Z}_2$ and commutes with α_3. But by (7) it does not commute with α_2 or $\alpha_2\alpha_3$. Hence

$$\alpha_6^2 = \alpha_3 \bmod \mathcal{Z}_1. \qquad (11)$$

Next (by (9)) $\alpha_1 = [\alpha_3,\alpha_5] = [\alpha_6^2,\alpha_5] = \alpha_3^{-2}$ (by (11)) since $[\alpha_5,\alpha_6] = \alpha_3$ commutes with α_6. Hence

$$\alpha_3^2 = \alpha_1. \qquad (12)$$

Next $[\alpha_5^2,\alpha_6] = \alpha_3^{\alpha_5}\alpha_3 = \alpha_3^2\alpha_1 = 1$ by (12). $\alpha_5^2 \in \mathcal{Z}_2$, and the centralizer of α_6 in $\mathcal{Z}_2$ is $\{\alpha_3\}$. But α_5 does not commute with α_3, and therefore $\alpha_5^2 \not\equiv \alpha_3 \bmod \mathcal{Z}_1$. Hence

$$\alpha_5^2 \equiv 1 \bmod \mathcal{Z}_1. \qquad (13)$$

Again, (by (11)) $1 = [\alpha_4,\alpha_3] = [\alpha_4,\alpha_6^2] = [\alpha_4,\alpha_6]^2 = [\alpha_4,\alpha_6,\alpha_6] = \alpha_2^2\alpha_1$ by (7); so

$$\alpha_2^2 = \alpha_1. \qquad (14)$$

Finally, $[\alpha_4^2,\alpha_6] = \alpha_2^2$, since $[\alpha_2,\alpha_4] = 1$. Hence by (14), α_4^2 does not commute with α_6. But $\alpha_4^2 \in D$. Hence

$$\alpha_4^2 \equiv \alpha_2 \bmod \mathcal{Z}_1. \qquad (15)$$

The relations (7) to (15) define the family Γ_{24} and so $G/\mathcal{Z}_1 \cong 32\Gamma_4 b_1$. These statements may be summarized as follows:

THEOREM 4.3. *There are just three families of rank 6 for which the central quotient group is a stem group of Γ_4, to wit, the families Γ_{24}, Γ_{25}, and Γ_{26}.*

In view of the results obtained in secs. 3 and 4, the determination of the families of rank ≤ 6 of 2-groups of class 3 is now complete. The families in question are Γ_3, Γ_6, Γ_7, Γ_{14}–Γ_{18} inclusive and Γ_{24}, Γ_{25}, Γ_{26}.

3.5 Let $r = 6$ and Let G Be of class 2

Note first that, since G is a stem group, $G' = \mathcal{Z}_1$. Also, the two highest invariants of the Abelian group $G/\mathcal{Z}_1$ must be equal; for otherwise if x_1 is of highest order $2^{\lambda_1} \bmod \mathcal{Z}_1$, $x_1^{2\exp(\lambda_1-1)}$ must be in $\mathcal{Z}_1$—a contradiction. Therefore $|G:\mathcal{Z}_1| = 4$ implies $G \in \Gamma_2$, and $|G:\mathcal{Z}_1| = 32$ implies that $\mathcal{Z}_1$ is cyclic. Therefore r (by sec. 2) is odd. It follows that the only possibilities for the type of $G/\mathcal{Z}_1$ are the partitions (1^3), (2^2), and (1^4).

It is easy to see that for $G/\mathcal{Z}_1$ of type (1^3), the chief family is precisely of rank 6, namely, Γ_9. For $G/\mathcal{Z}_1$ of type (2^2), the chief family is again of rank 6, namely, Γ_{12} (already obtained in sec. 2). Indeed, for given partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 = \lambda_2$, the rank of the chief family is easily shown to be

$$\sum_{i=1}^{r} \lambda_i + \sum_{i<j} \min(\lambda_i, \lambda_j).$$

Therefore it remains to consider only the case $G/\mathcal{Z}_1$ of type (1^4), where $\mathcal{Z}_1$ is an elementary group $\{u, v\}$ of order 4.

The three groups $G_1 = G/\{u\}$, $G_2 = G/\{v\}$, and $G_3 = G/\{uv\}$

belong either to Γ_2 or else to Γ_5, since their derived groups are of order 2.

(1) Suppose two of these groups G_i belong to Γ_2. Without loss of generality, let G_1 and G_2 belong to Γ_2. Let $H_1/\{u\}$ and $H_2/\{v\}$ be the centers of G_1 and G_2. Then $|G:H_1| = |G:H_2| = 4$ and $H_i \geq \mathcal{Z}_1$. From $[H_1, G] \leq \{u\}$ and $[H_2, G] \leq \{v\}$, it follows that $H_1 \cap H_2 \leq \mathcal{Z}_1$. Since $G/\mathcal{Z}_1$ is of type (1, 4), it follows that $H_1H_2 = G$ and that, if $H_1 = \{\mathcal{Z}_1, x_1, x_2\}$ and $H_2 = \{\mathcal{Z}_1, x_3, x_4\}$, then $x_1, \ldots, x_4$ form a base of G mod $\mathcal{Z}_1$. Also $[H_1, H_2] \leq \{u\} \cap \{v\} = 1$. Thus $[x_1, x_3] = [x_1, x_4] = [x_2, x_3] = [x_2, x_4] = 1$. It follows that $[x_1, x_2] = u$, $[x_3, x_4] = v$ (since otherwise $[x_1, x_2] = 1$, and x_1 would belong to $\mathcal{Z}_1$). Thus $G \in \Gamma_{10}$.

(2) Suppose $G_1 \in \Gamma_2$, but neither G_2 nor G_3 is so contained. Let $H_1/\{u\}$ be the center of G_1. As before, $|G:H_1| = 4$ and it may be assumed that $H_1 = \{\mathcal{Z}_1, x_1, x_2\}$. Since G_2 and G_3 belong to Γ_5, every element of $G - H_1$ has 4 conjugates in G, whereas x_1 and x_2 have only two such conjugates. Let X_1, X_2 be the centralizers of x_1, x_2 in G. Then $|G:X_1| = 2$ and so $|G:X_1 \cap X_2| \leq 4$. It follows that $X_1 \cap X_2 = H_1$, since otherwise $X_1 \cap X_2$ would be an element of $G - H_1$, commuting with both x_1 and x_2, and hence having only two conjugates in G. Thus $[x_1, x_2] = 1$ and $X_1 \neq X_2$. Let $x_3 \in X_1 - H_1$ and $x_4 \in X_2 - X_1$. Then $[x_1, x_3] = [x_2, x_4] = 1$, and $[x_1, x_4] = [x_2, x_3] = u$, since $x_3 \notin X_2$ and $x_4 \notin X_1$. Clearly $G = \{\mathcal{Z}_1, x_1, x_2, x_3, x_4\}$ and $G' = \mathcal{Z}_1$, since it is generated by the six commutators $[x_i, x_j]$ $(i < j)$. Hence necessarily $[x_3, x_4] = v$ or uv (the choice is irrelevant). Thus $G \in \Gamma_{11}$.

(3) Suppose G_1, G_2, G_3 all belong to Γ_5. Then every element of $G - \mathcal{Z}_1$ has four conjugates in G. The centralizers of two of these elements either coincide or intersect in $\mathcal{Z}_1$, since such centralizers (of order 16) are Abelian. There are therefore just five of these centralizers $C_1, \ldots, C_5$, each containing, besides $\mathcal{Z}_1$, three of the fifteen remaining cosets of $\mathcal{Z}_1$. Also $G = C_iC_j$ for $i \neq j$, since $C_i \cap C_j = \mathcal{Z}_1$. Let $C_1 = \{\mathcal{Z}_1, x_1, x_2\}$ and $C_2 = \{\mathcal{Z}_1, x_3, x_4\}$. Then $[x_1, x_2] = [x_3, x_4] = 1$ and it may be assumed (by choosing x_3, x_4 in C_2 suitably) that $[x_1, x_3] = u$, $[x_1, x_4] = v$. Then $[x_2, x_3] = v$ or uv (since x_3 has four conjugates) and similarly $[x_2, x_4] = u$ or uv. But since x_2 has four conjugates, $[x_2, x_3]$ and $[x_2, x_4]$ cannot both be uv. The case $[x_2, x_3] = v$, $[x_2, x_4] = u$ is also impossible, since it makes x_1x_2 commute with x_3x_4 as well as with x_1 and x_2; so that x_1x_2 would then have only two conjugates. The remaining two cases are equivalent. They are interchanged by an exchange of x_1 and x_2, and the implied change $u \to v$, $v \to uv$. Thus there is here only a single family, Γ_{13}. Hence the following conclusion:

THEOREM 5.1. *The families of 2-groups of class 2 with rank 6 are* $\Gamma_9, \Gamma_{10}, \Gamma_{11}, \Gamma_{12}$, *and* Γ_{13}.

In view of Theorems 1.2 and 4.1, 4.2, 4.3, the groups G with $r = 6$ which do not belong to any one of the families $\Gamma_1 - \Gamma_{17}$ or Γ_{27} (already obtained) are either (1) of class 4 or else (2) of class 3 with centers of order 2.

3.6 Groups of Class 4 or 5

If G (of an order dividing 64) is of class 5, then by Theorem 1.2, $G \in \Gamma_{27}$. If G is of class 4 and order ≤ 32, then, by the same theorem, $G \in \Gamma_8$. Hence it may be assumed that $|G| = 64$ and that G is of class 4. Then the order of the center $\mathcal{Z}_1$ of G is ≤ 4. If $|\mathcal{Z}_1| = 4$, then $G/\mathcal{Z}_1$ is a class-3 group of order 16, and can only be dihedral. By Lemma 3.3, this happens only when G belongs to the first branch of Γ_8. Thus $\mathcal{Z}_1$ may be supposed to be of order 2. Then $G/\mathcal{Z}_1$ is of class 3 and order 32; it belongs therefore to one of the families $\Gamma_3, \Gamma_6, \Gamma_7$.

Applying the criterion of Lemma 3.1 to $G/\mathcal{Z}_1 = H$, it is evident that the following groups (in the notation of the tables) are incapable:

$$\begin{aligned} H = 32\,\Gamma_3\,a_2 &= \{\alpha_3, \alpha_4\,\alpha_3, \alpha_3\,\beta_2\}, & u &= \beta_1,\\ 32\,\Gamma_3\,a_3 &= \{\alpha_3, \alpha_4, \alpha_3\,\beta_2\}, & u &= \beta_1,\\ 32\,\Gamma_3\,b &= \{\alpha_3, \beta\alpha_4, \beta\}, & u &= \beta^2,\\ 32\,\Gamma_3\,c_2 &= \{\alpha_3, \alpha_4\}, & u &= \beta_1,\\ 32\,\Gamma_3\,d_1 &= \{\alpha_4\,\alpha_3, \alpha_4\}, & u &= \beta_2,\\ 32\,\Gamma_3\,e &= \{\alpha_4\,\alpha_3, \alpha_4\,\beta\}, & u &= \beta^2,\\ 32\,\Gamma_3\,f &= \{\alpha_3, \alpha_4\}, & u &= \beta^2. \end{aligned}$$

Thus only the groups a_1, c_1, and d_2 in this branch of Γ_3 are left as possibly capable.

$$\begin{aligned} H = 32\,\Gamma_6\,a_1 &= \{\alpha_4, \alpha_4\,\alpha_3, \alpha_5\}, & u &= \alpha_1,\\ 32\,\Gamma_6\,a_2 &= \{\alpha_4, \alpha_4\,\alpha_3, \alpha_5\}, & u &= \alpha_1, \end{aligned}$$

so neither of the two Γ_6 groups is capable.

$$\begin{aligned} H = 32\,\Gamma_7\,a_2 &= \{\alpha_5, \alpha_5\,\alpha_4\}, & u &= \alpha_1,\\ 32\,\Gamma_7\,a_3 &= \{\alpha_4, \alpha_5\}, & u &= \alpha_1. \end{aligned}$$

Hence only $32\,\Gamma_7\,a_1$ is possibly capable.

But the cases $H = 32\,\Gamma_3\,a_1$ and $32\,\Gamma_3\,c_1$ have already been settled by Theorems 3.4 and 3.5, where n is taken as 3. They yield one family each, namely, Γ_{19} and Γ_{21}. Thus it remains only to deal with

(1) $H = 32\,\Gamma_3\,d_2$ and (2) $H = 32\,\Gamma_7\,a_1$.

Case(1). Here it may be assumed that $H = \{a, b\}$, with the defining relations:

$$a^8 = b^4 = 1, \; b^{-1}ab = a^3. \tag{1}$$

Let $G = \{\mathcal{Z}_1, \alpha, \beta\}$ be any group with $G/\mathcal{Z}_1 \cong H$, as given by (1), so that $\mathcal{Z}_1\alpha$ and $\mathcal{Z}_1\beta$ correspond respectively to a and b. Then $\alpha_1 = [\alpha, \beta] = \alpha^2$ mod $\mathcal{Z}_1$. Hence α_1 commutes with α and consequently G' is generated by $\alpha_1, \alpha_2, \ldots,$ where $\alpha_{i+1} = [\alpha_i, \beta]$. Note that $G' \leq \{\alpha^2, \mathcal{Z}_1\}$; it is therefore Abelian, so that $\alpha_1, \alpha_2, \ldots,$ are commutative. Hence $\alpha_2 = [\alpha_1, \beta] = [\alpha^2, \beta] = \alpha_1{}^{\alpha}\alpha_1 = \alpha_1{}^2$, so that $\beta^{-1}\alpha_1\beta = \alpha_1{}^3$. Transforming with β gives $\alpha_3 = \alpha_2{}^2$, $\alpha_4 = \alpha_3{}^2$, and so on. Hence $G' = \{\alpha_1\}$ is cyclic. Since $\alpha_1 \equiv \alpha^2$ mod $\mathcal{Z}_1$ and $a^8 = 1$, it follows that $\alpha_1{}^4 = \alpha_3 \in \mathcal{Z}_1$. Hence $\alpha_4 = 1$. Thus G' is of order ≤ 8, and since $\alpha_1{}^2 = \alpha^4$ mod $\mathcal{Z}_1$, it follows that $G' \cap \mathcal{Z}_1 = \{\alpha_3\}$ is of order ≤ 2. The multiplier of H is

therefore of order ≤ 2. But the group $G = \{\alpha, \beta\}$ of order 64 defined by the relations,

$$\alpha^{16} = \beta^4 = 1, \; \beta^{-1}\alpha\beta = \alpha^3, \tag{2}$$

has as a center $\mathcal{Z}_1 = \{\alpha^8\}$ and $G/\mathcal{Z}_1 \cong H$. Thus there is obtained the following theorem:

THEOREM 6.1. *The Schur multiplier of* $H = 32\,\Gamma_3\,d_2$ *is of order* 2, *and the groups* G *with* $G/\mathcal{Z}_1 \cong H$ *form the single family* $\mathfrak{M}(H) = \Gamma_{20}$.

Case(2). $H = 32\,\Gamma_7\,a_1$. This case is the split extension of an elementary Abelian group of order 8 by a single element inducing an automorphism of order 4. Take $H = \{a, b\}$, with the defining relations,

$$a^4 = 1; \; [b, a] = c; \; [c, a] = d;$$
$$[d, a] = [d, b] = [d, c] = [c, b] = 1; \; b^2 = c^2 = d^2 = 1.$$

As in previous cases, let $G = \{\mathcal{Z}_1, \alpha, \beta\}$ have $G/\mathcal{Z}_1 \cong H$, with $\mathcal{Z}_1\alpha$ and $\mathcal{Z}_1\beta$ corresponding respectively to a and b. Define $[\beta, \alpha] = \gamma$, $[\gamma, \alpha] = \delta$, $[\delta, \alpha] = \epsilon$. Then ϵ is in $\mathcal{Z}_1$, as also are $\beta^2, \gamma^2, \delta^2$. Thus $1 = [\delta^2, \alpha] = \epsilon^2$. Further, the commutators of β, γ, δ in pairs lie in $\mathcal{Z}_1$ and so $1 = [\beta^2, \gamma] = [\beta, \gamma]^2$ and, similarly, the orders of $[\beta, \delta]$ and $[\gamma, \delta]$ are ≤ 2. Also, $1 = [\gamma^2, \alpha] = \delta[\delta, \gamma]\delta$ gives $\delta^2 = [\gamma, \delta]$. Similarly $\gamma^2 = [\beta, \gamma]$, so that $\delta^4 = \gamma^4 = 1$. But $\alpha^4 \in \mathcal{Z}_1$ and $[\beta, \alpha^2] = \gamma^2\,\delta$ and $[\delta, \alpha^2] = \delta^2 = 1$, so that $[\beta, \alpha^2]$ is commutative with α^2. Hence $1 = [\beta, \alpha^4] = [\beta, \alpha^2]^2 = \gamma^4\,\delta^2 = \delta^2$, since $\gamma^4 = 1$. Finally $[\beta, \alpha^2] = \gamma^2\,\delta$, and hence $1 = [\beta^2, \alpha^2] = [\beta, \alpha^2]^{\beta}[\beta, \alpha^2] = \gamma^2\,\delta[\delta, \beta]\gamma^2\,\delta$; or, since γ^2 is in $\mathcal{Z}_1$ and $\gamma^4 = 1$, $[\beta, \delta] = \delta^2 = 1$. Now G' is generated by $[\beta, \alpha] = \gamma$ and its conjugates in G. The foregoing relations therefore show that $G' = \{\gamma, \delta, \epsilon\}$, where ϵ and γ^2 are in $\mathcal{Z}_1$ and $[\gamma, \delta] = \delta^2 = \epsilon^2 = 1$. Hence G' is Abelian of type at most (2, 1, 1), because commutation with α maps γ, δ into δ, ϵ, and commutation with β maps γ, δ into $\gamma^2, 1$. The fact that $G = \{\mathcal{Z}_1, \alpha, \beta\}$ proves that $\{\gamma, \delta, \epsilon\}$ is normal in G. Since it contains γ it must coincide with G'.

It follows that the multiplier of H is at most elementary and of order 4, since $G' \cap \mathcal{Z}_1 = \{\gamma^2, \epsilon\}$. But it is possible to define a group $G = \{\alpha, \beta, \gamma, \delta, \epsilon\} = \{\alpha, \beta\}$ (of order 128) by the relations which express the fact that $\{\beta, \gamma, \delta, \epsilon\}$ is the direct product of the octic group $\beta^2 = \gamma^4 = (\beta\gamma)^2 = 1$, with the two cyclic groups $\{\delta\}$ and $\{\epsilon\}$ of order 2 extended by the element α (where $\alpha^4 = 1$), which transforms $\beta, \gamma, \delta, \epsilon$ into $\beta\gamma, \gamma\delta, \delta\epsilon, \epsilon$. This transformation effectively defines an automorphism of $\{\beta, \gamma, \delta, \epsilon\} = G_1$ of period 4. Moreover the centralizer of α in G_1 is $\{\gamma^2, \epsilon\}$, which is easily seen to be the center of G, since α^2 transforms G_1 by an outer automorphism. It then follows that $G/\mathcal{Z}_1 \cong H$. The multiplier of H is therefore elementary of order 4, and the group G of order 128 described above is a stem group of $\mathfrak{M}(H)$. Hence $\{\gamma^2, \epsilon\} = \mathcal{Z}_1(G)$ may be used as the Schur multiplier of H.

Any families of rank 6 which have H as a central factor group may be represented by factor groups G/N, where N is one of the three subgroups of order 2 in $\mathcal{Z}_1(G)$. Moreover, $N = \{\alpha\}$ is inadmissible, since it puts $\mathcal{N}\delta$ in the center of $G/\mathcal{N}$. In fact $G/\{\epsilon\}$ is in Γ_{17}. But $\mathcal{N} = \{\gamma^2\}$ gives a stem group of Γ_{22} with a derived group $G'/\{\gamma^2\}$ of type (1^3), whereas $\mathcal{N} = \{\gamma^2\,\epsilon\}$ gives a stem group of Γ_{23} with $G'/\{\gamma^2\,\epsilon\}$ of type (21). Thus these two families are seen to be distinct without the necessity of considering the effect of automorphisms of H on the multiplier. We may summarize as follows;

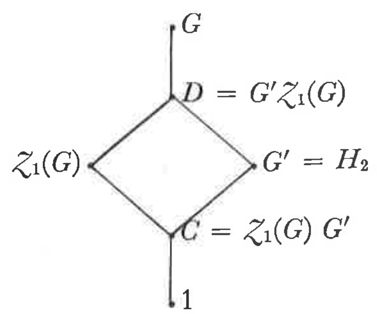
THEOREM 6.2. *The Schur multiplier of* $H = 32\,\Gamma_7 a_1$ *is elementary of order* 4; *and in the groups of the maximal family* $\mathfrak{M}(H)$ *of rank* 7, *the derived groups are Abelian of type* (2, 1, 1). *Thus the groups* G *with* $G/\mathcal{Z}_1 \cong H$ *fall into three distinct families, to wit,* $\mathfrak{M}(H)$ *and the two families* Γ_{22} *and* Γ_{23} *of rank* 6.

C H A P T E R 4

Construction of the Groups in a Family

Let Γ be a given family of 2-groups, say of rank r, and let G_0 be any stem group in Γ; hence, $|G_0| = 2^r$. Let G_0 be used as a "group of reference." If G is an arbitrary group in Γ, there will be exactly $u = u(\Gamma)$ isomorphisms θ mapping $G_0/\mathcal{Z}_1(G_0)$ onto $G/\mathcal{Z}_1(G)$ in such a way as to induce isomorphisms of G'_0 onto G'. Here u is the order of the group of "autologisms" of Γ.

Suppose G belongs to the sth branch of Γ;



and let $|C| = 2^c$, so that c depends only on Γ and not on G. Also $|\mathcal{Z}_1(G):c| = 2^s$, so that $|\mathcal{Z}_1(G)| = 2^{c+s}$.

Let $G_0 = (x_1, \ldots, x_a)$. Suppose any one of the u isomorphisms θ maps $x_i\,\mathcal{Z}_1(G_0)$ onto $y_i\,\mathcal{Z}_1(G)$, $i = 1, \ldots, a$. Then if $z_1, \ldots, z_h$ in $\mathcal{Z}_1(G)$ are chosen so that

$$\mathcal{Z}_1(G) = \{c, z_1, \ldots, z_h\},$$

then

$$G = \{y_1, \ldots, y_a, z_1, \ldots, z_h\}.$$

Here h has only to be chosen sufficiently large, say $h \geq s$. Since $z_1, \ldots, z_h$ are in the center of G, G is evidently the homomorphic image of the direct product $F \times A$, where F is a free group of rank a, $F = \{\bar{y}_1, \ldots, \bar{y}_a\}$, and $A = \{\bar{z}_1, \ldots, \bar{z}_h\}$ is a free Abelian group of rank h. Let $\mathcal{N}$ be the kernel of this homomorphism, so that $G \cong (F \times A)/\mathcal{N}$. By the given choice of $y_1, \ldots, y_a$, the kernel $\mathcal{N}$ intersects F' in one and the same subgroup M for every G of Γ. Hence $F'/M \cong G'_0 \cong G'$. It may be supposed for simplicity that $\bar{y}_1, \ldots, \bar{y}_a$ generate $\Upsilon = F'/M$.

It is easy to see that Υ (and hence also $H = A \times \Upsilon$) belongs to Γ. Also $\mathcal{Z}_1(H) = A \times \mathcal{Z}_1(\Upsilon) \cap \Upsilon' = \bar{C}$, where (say) $\bar{C} \cong C$, and $\mathcal{Z}_1(H)$ is thus dependent only on Γ and not on G. Also $\mathcal{Z}_1(\Upsilon) = B \times \bar{C}$, where B is a free Abelian group of rank a. The fact that $\mathcal{Z}_1(\Upsilon)$ has this structure depends on the general theory of the multiplicator.*

The general theorem covering this situation is as follows;

THEOREM. *Every group* G *of a family* Γ *is the homomorphic image of a direct product* $\Upsilon \times A = H$, *where* Υ *is a particular group depending only on* Γ, *and* A *is a free Abelian group with a sufficient number of generators. If* G *is in the* s*th branch of* Γ, A *may be taken as a free Abelian group with* s *generators. Here* $\mathcal{Z}_1(H) = A \times \mathcal{Z}_1(\Upsilon) = A \times B \times \bar{C}$, $\mathcal{Z}_1(\Upsilon) \cap \Upsilon' = \bar{C} \cong C$. *Also,* $G \cong H/\mathcal{N}$, *where the kernel* $\mathcal{N}$ *satisfies (i)* $\mathcal{N} \cap \bar{C} = 1$ *and (ii)* $A\bar{C}\mathcal{N} = \mathcal{Z}_1(H)$.

This theorem is next illustrated by application to the family Γ_3. Here $G/\mathcal{Z}$ is the octic group and G' is cyclic of order 4.

$$G = \mathcal{Z} + \mathcal{Z}x + \mathcal{Z}x^2 + \mathcal{Z}x^3 + \mathcal{Z}y + \mathcal{Z}xy + \mathcal{Z}x^2y + \mathcal{Z}x^3y$$

If $x \to g$ and $y \to h$ in the homomorphism $G \to G/\mathcal{Z}$, then

$$g^4 = 1,\ h^2 = 1,\ hgh = g^{-1}.$$

Hence, in G,

$$b_0 = x^4 \in \mathcal{Z},$$
$$b_1 = y^2 \in \mathcal{Z},$$
$$b_2 = xyxy \in \mathcal{Z}.$$

Also in G, $u = \{x, y\} = x^{-1}y^{-1}xy$ generates G', and $u^4 = 1$.

Next define a group $\Upsilon = \{x, y\}$ subject to the requirements, (1) $b_0 = x^4$, $b_1 = y^2$, and $b_2 = xyxy$ are in the center of Υ, and (2) $(x^{-1}y^{-1}xy)^4 = 1$. Here $u = x^{-1}y^{-1}xy = x^{-2}xy^{-2}x^{-1}xyxy = x^{-2}xb_1^{-1}x^{-1}b_2 = x^{-2}b_1^{-1}b_2u^2 = x^{-4}b_1^{-2}b_2^2 = b_0^{-1}b_1^{-2}b_2^2 = c \in \mathcal{Z}(\Upsilon)$. Hence $u^4 = 1 = c^2$. It follows that $\mathcal{Z}(\Upsilon) = \{b_1\} \times \{b_2\} \times \bar{C}$ where $\bar{C} = \{c\}$, and $\bar{C} = \Upsilon' \cap \mathcal{Z}(\Upsilon)$. Recalling that every G of Γ_3 is generated by the elements x and y and elements of $\mathcal{Z}$, it appears that G is a homomorphic image of the direct product $\Upsilon \times A$, where Υ is as given above and A is a free Abelian group.

Having found the group Υ for the family Γ, other groups of the same family may be found by taking kernels $\mathcal{N}$ with $\mathcal{N} \cap \bar{C} = 1$. Here $G \cong (\Upsilon \times A)/\mathcal{N}$, with $\mathcal{N} \subseteq \mathcal{Z}(\Upsilon \times A) = A \times B \times \bar{C}$. If $(A \times B \times \bar{C})/\mathcal{N}\bar{C} = 2^s$, then G is in the sth branch of Γ. An autologism γ of Γ, as applied to $\Upsilon \times A$ has the property that a kernel $\mathcal{N}$ and $\mathcal{N}\gamma$ yield isomorphic groups.

For Γ_3 the autologisms of Υ are generated by $\gamma{:}(x)\gamma = x$, $(y)\gamma = xy$ and $\delta{:}(x)\delta = x^{-1}$, $(y)\delta = y$.

$$\gamma{:}\begin{cases}(b_1)\ \gamma = b_2,\\ (b_2)\ \gamma = b_1^{-1}b_2^2,\\ (c)\ \ \gamma = c,\end{cases}\qquad \delta{:}\begin{cases}(b_1)\ \delta = b_1,\\ (b_2)\ \delta = b_1^2b_2^{-1},\\ (c)\ \ \delta = c,\end{cases}$$

and $\gamma^4 = 1$, $\delta^2 = 1$, $\delta\gamma = \gamma^{-1}\delta$. For a stem group, the group A is not needed and $\mathcal{N}\bar{C} = B \times \bar{C}$. The possible kernels $\mathcal{N}$ are

$$\mathcal{N}_1 = \{b_1, b_2\},$$
$$\mathcal{N}_2 = \{b_1, b_2c\},$$
$$\mathcal{N}_3 = \{b_1c, b_2\},$$
$$\mathcal{N}_4 = \{b_1c, b_2c\}.$$

The second and third of these are interchanged by the autologism γ, since

$$\{b_1, b_2c\}\ \gamma = \{b_2, b_1^{-1}b_2^2c\} = \{b_1c, b_2\}.$$

Thus there are three stem groups in Γ_3.

In the notation of the tables for Γ_3, $\alpha_3 = x$, $\alpha_4 = y$, $u = [\alpha_3, \alpha_4] = [x, y] = \alpha_2$, $a_2^2 = \alpha_1 = c$, $\alpha_1^2 = c^2 = 1$. For the three stem groups;

(1) $16\,\Gamma_3\,a_1{:}\alpha_3^2 = \alpha_2^{-1}$, $\alpha_4^2 = 1$,

or in terms of the above notation,

$$x^2 = u^{-1},\ y^2 = 1,$$

or (since $u = x^{-2}b_1^{-1}b_2$, $u^2 = c$, $y^2 = b_1$) it follows that $b_1^{-1}b_2 = 1$, $b_1 = 1$; hence the kernel is $\mathcal{N}_1 = \{b_1, b_2\}$. Similarly,

(2) $16\,\Gamma_3\,a_2{:}\alpha_3^2 = \alpha_2$, $\alpha_4^2 = 1$.

Whence $x^2 = u = x^{-2}b_1^{-1}b_2$, $y_2 = b_1 = 1$ or $b_1 = 1$, $b_1^{-1}b_2c = 1$. Here the kernel is $\mathcal{N}_2 = \{b_1, b_2c\}$.

(3) $16\,\Gamma_3\,a_3{:}\alpha_3^2 = \alpha_2^{-1}$, $\alpha_4^2 = \alpha_1$

or $b_1^{-1}b_2 = 1$, $b_2 = c$, and the kernel is $\mathcal{N}_4 = \{b_1c, b_2c\}$.

One way of constructing the groups in a branch is first to find the possible groups which are generated by the coset representatives of the center, and then if necessary to adjoin further central elements to these groups. In other words, first find the homomorphic images of the group Υ of the theorem, and then adjoin further central elements.

This process is illustrated by application to the first branch of Γ_3. Here the groups are of order 32; in an individual group G

* I. Schur. "Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen." *Crelle*, vol. 127 (1904), pp. 20–50, and *ibid.*, vol. 132 (1907), pp. 85–137.

the elements $x = \alpha_3$ and $y = \alpha_4$ generate a group of Γ_3, which is either a stem group of Γ_3 or the entire group of order 32. If x and y generate a stem group G_1 of order 16, then G/G_1 is of order 2. And if $G = G_1 + G_1 z$ then z^2 is in the center of G_1; whence $z^2 = 1$ or $z^2 = \alpha_1 = c$. With $z^2 = 1$, G is the direct product of a stem group and a group $\{z\}$ of order 2. This extension gives the first three groups of the branch in question. Their symbols are a_1, a_2, a_3. If $z^2 = \alpha_1$, and G_1 is any one of the three stem groups of order 16, the group of order 32 obtained in all three cases is the same. Its symbol is b.

There remain cases in which x and y generate the entire group. With the notation used above, there are twelve possible kernels N;

$N_1\ \{b_1^2, b_2\}$,	$N_5\ \{b_1, b_2^2\}$,	$N_9\ \{b_1 b_2, b_2^2\}$,
$N_2\ \{b_1^2 c, b_2\}$,	$N_6\ \{b_1 c, b_2^2\}$,	$N_{10}\ \{b_1 b_2 c, b_2^2\}$,
$N_3\ \{b_1^2, b_2 c\}$,	$N_7\ \{b_1, b_2^2 c\}$,	$N_{11}\ \{b_1 b_2 c, b_2^2 c\}$,
$N_4\ \{b_1^2 c, b_2 c\}$,	$N_8\ \{b_1 c, b_2^2 c\}$,	$N_{12}\ \{b_1 b_2 c, b_2^2 c\}$.

The autologism γ permutes these in the following way;

$$(N_1, N_5)(N_2, N_8, N_4, N_7)(N_3, N_6)(N_9)(N_{10})(N_{11})(N_{12}).$$

The group c_1 is given by N_5, c_2 by N_6, d_1 by N_9, d_2 by N_{10}, e by N_7, and f by N_{12}. Thus these are all the possible groups except that given by kernel N_{11}. But N_{11} gives group f just as does N_{12} Using N_{11} the table would read $\alpha_1 = \beta^2$, $\alpha_3^2 = \alpha_2^{-1}\beta^2$, $\alpha_4^2 = \beta$.

For N_{11}, $G_{11} = \{x, y\}$ and $x^8 = 1$, $y^8 = 1$, $y^{-1}xy = x^{-1}$, $y^4 = x^4$. If, in G_{11}, x_1 is taken as xy^2 then $x_1^8 = 1$, $y^{-1}x_1 y = x^3 y^2 = x^{-1}y^{-2} = x_1^{-1}$. It follows that $G_{11} = \{x, y\} = \{x_1, y\}$ is isomorphic with G_{12}.

CHAPTER 5

Notes on Various Groups of Orders 8, 16, 32, and 64

S_x is the symmetric group of degree x.
A_x is the alternating group of degree x.

Γ_2	8 a_1	The octic group. Dihedral. Sylow subgroup of S_4, S_5, A_6, and A_7.
	8 a_2	The quaternion group. Dicyclic. Hamiltonian.
	16 a_1	Generalized dihedral. Sylow subgroup of S_6 and S_7.
	16 a_2	Hamiltonian.
	32 a_1	Generalized dihedral.
	32 a_2	Hamiltonian.
	64 a_1	Generalized dihedral.
	64 a_2	Hamiltonian.
Γ_3	16 a_1	Dihedral.
	16 a_2	Every invariant subgroup characteristic. The Sylow subgroup in the group of automorphisms of the Abelian group of order p^2 and type (1^2), where $p \equiv 3(8)$.
	16 a_3	Dicyclic.
	32 a_1	Generalized dihedral.
	32 a_2	The Sylow subgroup in the group of automorphisms of the Abelian group of order p^3 and type (1^3), where $p \equiv 3(8)$.
	32 e	Every invariant subgroup characteristic. The Sylow subgroup in the group of automorphisms of the Abelian group of order p^2 and type (1^2) where $p \equiv 5(8)$.
	64 a_1	Generalized dihedral.
	64 p	Every invariant subgroup characteristic.
Γ_4	32 a_2	Generalized dihedral.
	32 d	This group has only one invariant subgroup of order 4, and it is noncyclic. For this group t_2 is odd and >1. In both of these respects, the group is unique among the groups of order 32.
	64 a_2	Generalized dihedral.
Γ_5	32 a_1	Generated by 8 Γ_2 a_1 expressed as a regular group, and its conjoint. Also generated by 8 Γ_2 a_2 expressed as a regular group, and its conjoint.
	32 a_2	The only non-Abelian group of order 32 which has (1) an automorphism of order 5 and (2) an insolvable group of automorphisms.
	64 a_1	Generated by 16 Γ_2 a_1 and its conjoint or by 16 Γ_2 a_2 and its conjoint.
	64 a_2	Has an insolvable group of automorphisms.
	64 b	Generated by 16 Γ_2 b and its conjoint. Has an insolvable group of automorphisms.
	64 c_1	Generated by 16 Γ_2 c_1 and its conjoint or by 16 Γ_2 c_2 and its conjoint.
	64 d	Generated by 16 Γ_2 d and its conjoint.

Γ_6	32 a_1	When expressed as a transitive group of degree 8, this group is the holomorph of the cyclic group of order 8.
Γ_7	32,	Genus a. The only genus whose order divides 32 where neither t_1 nor the maximal order of the operators is a generic invariant.
	64,	Genera a and b. In these genera, t_1 is not a generic invariant.
	64,	Genera a, b, and c. In these genera, the maximal order of the operators is not a generic invariant.
Γ_8	32 a_1	Dihedral.
	32 a_2	Every invariant subgroup characteristic. The Sylow subgroup in the group of automorphisms of the Abelian group of order p^2 and type (1^2), where $p \equiv 7(16)$.
	32 a_3	Dicyclic.
	64 a_1	Generalized dihedral.
	64 a_2	Sylow subgroup in the group of automorphisms of the Abelian group of order p^3 and type (1^3), where $p \equiv 7(16)$.
	64 e	Every invariant subgroup characteristic.
Γ_9	Stem of order 64	Every group in this stem contains at least one subgroup 32 Γ_2 h.
	{64 c, 64 e}	The only two groups of order 64 where t_2 is odd and >1.
	64 e	The only non-Abelian group of order 64 which contains only one invariant subgroup of order 8. Sylow subgroup of the simple group of order 29,120 discovered by M. Suzuki (*Proc. Nat. Acad. Sci.*, vol. 46 (1960), p. 868).
Γ_{10}	64 a_1	This group can be expressed as an intransitive group of degree 8.
Γ_{12}	Stem of order 64	Of all the stems whose orders divide 64, the stem of Γ_{12} is the only one which has a non-elementary center.
Γ_{13}	64 a_5	Has a solvable group of automorphisms the order of which is divisible by three distinct primes.
Γ_{14}	64 a_1	Generalized dihedral.
	Stem of order 64	In genera a and d, t_1 is not a generic invariant.
Γ_{15}	{64 f_1, 64 f_2}	Every invariant subgroup characteristic.

Γ_{16}	64	Genus *a*. In this genus, t_1 is not a generic invariant.
Γ_{17}	$\begin{cases} 64\ b_1 \\ 64\ b_2 \end{cases}$	Every invariant subgroup characteristic.
Γ_{21}	64 a_2	Every invariant subgroup characteristic.
Γ_{22}	64 a_1	This group can be expressed in two distinct ways as a transitive group of degree 8.
	$\begin{cases} 64\ a_1 \\ 64\ a_2 \end{cases}$	Every invariant subgroup characteristic.
Γ_{23}	64 a_1	This group can be expressed in one way as a transitive group of degree 8.
	Stem of order 64	(Genus *a*) The maximal order of the operators is not a generic invariant.
Γ_{24}	64 a_2	Every invariant subgroup characteristic.
Γ_{25}	64 a_1	Can be expressed in two distinct ways as a transitive group of degree 8. Its positive expression is the Sylow subgroup in A_8 and A_9. This same expression is the holomorph of the Abelian group of order 8 and type (2, 1, 1), as well as the Sylow subgroup in the holomorph of the Abelian group of order 8 and type (1^3).
Γ_{26}	64 a_1	This group can be expressed in one way as a transitive group of degree 8. As such, it is the holomorph of 8 $\Gamma_2 a_1$ and the Sylow subgroup in the group of automorphisms of 8 $\Gamma_2 a_2$.
Γ_{27}	64 a_1	Dihedral.
	64 a_2	Every invariant subgroup characteristic. The Sylow subgroup in the group of automorphisms of the Abelian group of order p^2 and of type (1^2), where $p \equiv 15(32)$.
	64 a_3	Dicyclic.

FAMILY		RANK	CLASS	GROUP OF INNER AUTOMORPHISMS	LOWER CENTRAL SERIES				CLASS NUMBERS									NUMBERS OF SELF-CENTRALIZERS		
					H_2	H_3	H_4	H_5	TOTAL	j_0	j_1	j_2	j_3	j_4	j_0^*	j_1^*	j_2^*	s_1	s_2	s_3
Γ_2	${}^2_1B$	3	2	(1^2)	(1)				5	2	3				4	1		3		
Γ_3	${}^2_1C$	4	3	$8\Gamma_2 a_1$	(2)	(1)			7	2	3	2			4	3		1		
Γ_4	${}^3_2B$	5	2	(1^3)	(1^2)				14	4	6	4			8	6		1	4	
Γ_5	${}^4_1B$	5	2	(1^4)	(1)				17	2	15				16	0	1		15	
Γ_6	${}^3_1C_1$	5	3	$16\Gamma_2 a_1$	(2)	(1)			11	2	3	6			8	2	1		3	
Γ_7	${}^3_1C_2$	5	3	$16\Gamma_2 c_1$	(1^2)	(1)			11	2	3	6			8	2	1		3	
Γ_8	${}^2_1D$	5	4	$16\Gamma_3 a_1$	(3)	(2)	(1)		11	2	7	0	2		4	7		1		
Γ_9	${}^3_3B$	6	2	(1^3)	(1^3)				22	8	0	14			8	14			7	
Γ_{10}	${}^4_2B_1$	6	2	(1^4)	(1^2)				25	4	12	9			16	8	1		9	
Γ_{11}	${}^4_2B_2$	6	2	(1^4)	(1^2)				22	4	6	12			16	4	2		7	
Γ_{12}	${}^4_2B_3$	6	2	(2^2)	(2)				22	4	6	12			16	4	2		7	
Γ_{13}	${}^4_2B_4$	6	2	(1^4)	(1^2)				19	4	0	15			16	0	3		5	
Γ_{14}	${}^3_2C_1$	6	3	$16\Gamma_2 a_1$	(21)	(1)			22	4	14	0	4		8	14		1		
Γ_{15}	${}^3_2C_2$	6	3	$16\Gamma_2 a_1$	(21)	(1)			19	4	4	9	2		8	10	1		3	
Γ_{16}	${}^3_2C_3$	6	3	$16\Gamma_2 a_1$	(21)	(1)			16	4	2	6	4		8	6	2		3	
Γ_{17}	${}^3_2C_4$	6	3	$16\Gamma_2 c_1$	(21)	(1^2)			19	4	4	9	2		8	10	1		1	
Γ_{18}	${}^4_1C$	6	3	$32\Gamma_2 a_1$	(2)	(1)			22	2	9	11			16	4	2		3	
Γ_{19}	${}^3_1D_1$	6	4	$32\Gamma_3 a_1$	(3)	(2)	(1)		16	2	5	5	4		8	6	2		3	
Γ_{20}	${}^3_1D_2$	6	4	$32\Gamma_3 d_2$	(3)	(2)	(1)		16	2	5	5	4		8	6	2		3	
Γ_{21}	${}^3_1D_3$	6	4	$32\Gamma_3 c_1$	(3)	(2)	(1)		16	2	5	5	4		8	6	2		1	
Γ_{22}	${}^3_1D_4$	6	4	$32\Gamma_7 a_1$	(1^3)	(1^2)	(1)		13	2	1	5	5		8	2	3		1	2
Γ_{23}	${}^3_1D_5$	6	4	$32\Gamma_7 a_1$	(21)	(1^2)	(1)		13	2	1	5	5		8	2	3		1	2
Γ_{24}	${}^3_1E_1$	6	3	$32\Gamma_4 b_1$	(21)	(1)			16	2	5	5	4		8	6	2		1	
Γ_{25}	${}^3_1E_2$	6	3	$32\Gamma_4 a_1$	(1^3)	(1)			16	2	3	8	3		8	6	2		1	6
Γ_{26}	${}^3_1E_3$	6	3	$32\Gamma_4 a_1$	(21)	(1)			16	2	3	8	3		8	6	2		1	2
Γ_{27}	${}^2_1F$	6	5	$32\Gamma_8 a_1$	(4)	(3)	(2)	(1)	19	2	15	0	0	2	4	15		1		

FAMILY	$H_2 \cap Z_1$	G/H_2Z_1	$\underline{d}$	T_0	u	U	U_2	CENTRALIZERS SELF	CENTRALIZERS PAIRS	MAXIMAL CENTRAL FACTORS
Γ_2	(1)	(1^2)	2	(1^2)	6	Σ_3	(0)	X_1^3		
Γ_3	(1)	(1^2)	2	(1^2)	8	$\Gamma_2 a_1$	(1)	X_1		Z_2/H_3
Γ_4	(1^2)	(1^3)	3	(1^8)	24	Σ_4	Σ_3	$X_3 X_4^4$		$(X_3/K_1)^3$
Γ_5	(1)	(1^4)	4	(1^4)	720	Σ_6	(0)	X_2^{15}	$(X_1,X_4)^{15}(X_3,X_3)^{10}$	
Γ_6	(1)	(1^3)	3	(1^3)	32	$\Gamma_4 a_1$	(1)	$Z_2 X_4^2$	$(X_1,X_7)(X_2,X_8)^2$	Z_2/H_3
Γ_7	(1)	(21)	2	(1^2)	32	$\Gamma_4 a_1$	(1)	$Z_2 X_3^2$	$(X_1,X_4)^2(X_2,X_5)$	Z_2/H_3
Γ_8	(1)	(1^2)	2	(1^2)	32	$\Gamma_6 a_1$	(1^2)	X_1		$Z_3/H_3, Z_2/H_4$
Γ_9	(1^3)	(1^3)	3	(1^9)	168	$Aut\ (1^3)$	$Aut\ (1^3)$	X_2^7		$(X_2/K_1)^7$
Γ_{10}	(1^2)	(1^4)	4	(1^8)	72	$\Sigma_3 \wr \Sigma_2$	(1)	X_4^9	$(X_1,X_7)^6(X_3,X_3)$	$(X_3/K_1)^2$
Γ_{11}	(1^2)	(1^4)	4	(1^8)	96	$Aut\ (2^2)$	(1)	$X_3 X_4^6$	$(X_1,X_7)^3$	X_3/K_1
Γ_{12}	(2)	(2^2)	2	(2^2)	96	$Aut\ (2^2)$	(1)	$X_2 X_3^6$	$(X_1,X_4)^3$	X_2/K_1
Γ_{13}	(1^2)	(1^4)	4	(1^8)	360	$(\Sigma_3 \times \Sigma_5)^+$	Σ_3	X_2^5		
Γ_{14}	(1^2)	(1^3)	3	(1^6)	64	$\Gamma_{25} a_1$	$8\Gamma_2 a_1$	X_1		$Z_2/K_1, (X_4/K_2)^2, (X_6/K_3)^2, X_7/H_3$
Γ_{15}	(1^2)	(1^3)	3	(1^6)	16	(1^4)	(1^2)	$Z_2 X_5^2$	$(X_1,X_{10})(X_2,X_8)$	$Z_2/K_1, (X_6/K_2)^2, X_8/K_3, X_{10}/H_3$
Γ_{16}	(1^2)	(1^3)	3	(1^6)	64	$\Gamma_{25} a_1$	$8\Gamma_2 a_1$	$Z_2 X_4^2$	(X_1,X_7)	$Z_2/K_1, (X_4/K_2)^2, X_7/H_3$
Γ_{17}	(1^2)	(21)	2	(1^4)	32	$\Gamma_4 a_1$	$8\Gamma_2 a_1$	Z_2	$(X_1,X_4)^2$	$Z_2/H_3, (X_4/K_1)^2$
Γ_{18}	(1)	(1^4)	4	(1^4)	192	$8\Gamma_2 a_1 \times \Sigma_4$	(1)	X_6^3	$(X_1,X_{13})(X_4,X_{15})^6(X_9,X_{10})^4$	Z_2/H_3
Γ_{19}	(1)	(1^3)	3	(1^3)	128		(1^2)	$Z_3 X_4^2$	$(X_1,X_7)(X_2,X_8)^2$	$Z_3/H_3, Z_2/H_4$
Γ_{20}	(1)	(21)	2	(1^2)	128		(1^2)	$Z_3 X_3^2$	$(X_1,X_6)^2(X_2,X_5)$	$Z_3/H_3, Z_2/H_4$
Γ_{21}	(1)	(21)	2	(1^2)	64	$\Gamma_2 a_1$	(1^2)	Z_3	$(X_1,X_5)(X_2,X_7)(X_3,X_8)$	$Z_3/H_3 . Z_2/H_4$
Γ_{22}	(1)	(21)	2	(1^2)	64	$\Gamma_{25} a_1$	$8\Gamma_2 a_1$	$X_4 X_5^2$	(X_2,Z_2)	$Z_3/H_3 . Z_2/H_4$
Γ_{23}	(1)	(21)	2	(1^2)	64	$\Gamma_{25} a_1$	$8\Gamma_2 a_1$	$X_4 X_5^2$	(X_2,Z_2)	$Z_3/H_3, Z_2/H_4$
Γ_{24}	(1)	(1^3)	3	(1^3)	64	$\Gamma_2 a_1$	(1^2)	X_8	$(X_1,X_{16})(X_3,X_{12})(X_5,X_{14})$	$X_6/K_1, X_8/K_2, X_{10}/K_3, Z_2/H_3$
Γ_{25}	(1)	(1^3)	3	(1^3)	192		Σ_4	$X_4 X_7^6$	$(X_2,X_8)^3$	$(X_5/K_1)^3, Z_2/H_3$
Γ_{26}	(1)	(1^3)	3	(1^3)	64	$\Gamma_4 a_1$	$8\Gamma_2 a_1$	$X_6 X_{12}^2$	$(X_2,X_{13})^2(X_3,X_{14})$	$(X_7/K_1)^2, X_8/K_2, Z_2/H_3$
Γ_{27}	(1)	(1^2)	2	(1^2)	128	$Hol\ (4)$	(21)	X_1		$Z_4/H_3, Z_3/H_4, Z_2/H_5$

III

FAMILY	LOCATION OF CLASSES		DEFINING RELATIONS AND CONGRUENCES mod Z_1 $\gamma_{mn} = \alpha_m^{-1}\alpha_n^{-1}\alpha_m\alpha_n$															
	J_1	J_2	γ_{23}	γ_{24}	γ_{34}	γ_{25}	γ_{35}	γ_{45}	γ_{26}	γ_{36}	γ_{46}	γ_{56}	α_1^2	α_2^2	α_3^2	α_4^2	α_5^2	α_6^2
Γ_2	G	G	α_1										1	$\equiv 1$	$\equiv 1$			
Γ_3	X_1	G	1	α_1	α_2								1	α_1	$\equiv \alpha_2^{-1}$	$\equiv 1$		
Γ_4	X_1	G	1	1	1	1	α_1	α_2					1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$	
Γ_5	G	G	1	1	α_1	α_1	1	1					1	$\equiv 1$	$\equiv 1$	$\equiv 1$	$\equiv 1$	
Γ_6	Z_2	G	1	1	α_1	α_1	1	α_2					1	α_1	$\equiv 1$	$\equiv \alpha_2$	$\equiv 1$	
Γ_7	Z_2	G	1	1	α_1	α_1	1	α_2					1	1	$\equiv 1$	$\equiv 1$	α_3	
Γ_8	X_1	X_1	1	1	1	α_1	α_2	α_3					1	α_1	α_2^{-1}	$\equiv \alpha_3^{-1}$	$\equiv 1$	
Γ_9	Z_1	G	1	1	1	1	1	α_1	1	1	α_2	α_3	1	1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{10}	X_3^2	G	1	1	α_1	1	1	1	1	1	1	α_2	1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{11}	X_3	G	1	1	1	1	1	α_1	1	α_1	1	α_2	1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{12}	X_2	G	1	1	1	1	1	α_1	1	α_1	1	α_2	1	α_1	$\equiv 1$	$\equiv 1$	α_3	α_4
Γ_{13}	Z_1	G	1	1	1	1	α_1	$\alpha_1\alpha_2$	1	$\alpha_1\alpha_2$	α_2	1	1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{14}	X_1	X_1	1	1	1	1	1	1	1	α_1	α_2	α_3	1	1	α_1	$\equiv 1$	$\equiv \alpha_3$	$\equiv 1$
Γ_{15}	X_8X_{10}	X_1X_2	1	1	1	1	1	α_2	1	α_1	1	α_3	1	1	α_1	$\equiv 1$	$\equiv \alpha_3$	$\equiv 1$
Γ_{16}	X_7	X_1	1	1	1	1	1	α_1	1	α_1	α_2	α_3	1	1	α_1	$\equiv 1$	$\equiv \alpha_3$	$\equiv 1$
Γ_{17}	X_4^2	X_1^2	1	1	1	1	α_1	α_2	1	$\alpha_1\alpha_2$	1	α_3	1	1	α_1	$\equiv 1$	$\equiv 1$	α_4
Γ_{18}	Z_2X_{13}	G	1	1	1	1	1	α_1	α_1	α_2	1	1	1	α_1	$\equiv \alpha_2$	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{19}	Z_2X_7	X_1	1	1	1	1	1	α_1	α_1	α_2	1	α_3	1	α_1	α_2^{-1}	$\equiv 1$	$\equiv \alpha_3^{-1}$	$\equiv 1$
Γ_{20}	Z_2X_5	X_2	1	1	1	1	1	α_1	α_1	α_2	1	α_3	1	α_1	α_2	$\equiv 1$	$\equiv \alpha_3$	α_4
Γ_{21}	X_5X_6	X_1	1	1	1	1	α_1	1	α_1	α_2	α_2	α_3	1	α_1	α_2^{-1}	$\equiv \alpha_2$	α_4	$\equiv 1$
Γ_{22}	Z_2	Z_3X_4	1	1	1	1	α_1	α_2	α_1	α_2	α_3	1	1	1	1	$\equiv 1$	$\equiv 1$	α_5
Γ_{23}	Z_2	Z_3X_4	1	1	1	1	α_1	α_2	α_1	α_2	$\alpha_2\alpha_3$	1	1	1	α_1	$\equiv \alpha_2$	$\equiv 1$	α_5
Γ_{24}	$X_{12}Z_2$	X_3	1	1	1	1	α_1	1	α_1	1	α_2	α_3	1	α_1	α_1	$\equiv \alpha_2$	$\equiv 1$	$\equiv \alpha_3$
Γ_{25}	Z_2	X_1X_4	1	1	1	1	α_1	α_2	α_1	1	α_3	1	1	1	1	$\equiv 1$	$\equiv 1$	$\equiv 1$
Γ_{26}	Z_2	X_3X_8	1	1	1	1	α_1	α_2	α_1	1	$\alpha_2\alpha_3$	1	1	1	α_1	$\equiv \alpha_2$	$\equiv 1$	$\equiv 1$
Γ_{27}	X_1	X_1	1	1	1	1	1	1	α_1	α_2	α_3	α_4	1	α_1	α_2^{-1}	α_3^{-1}	$\equiv \alpha_4^{-1}$	$\equiv 1$

IV

FAMILY	CORRELATION OF DEFINING RELATIONS AND DIAGRAMS					FIRST SIGNALS		CENTRALIZER OF H_2			SECOND SIGNALS	
	G_1	G_2	G_3	G_4	G_5	QUOTIENT GROUPS	SUBGROUPS	H_2^*	G/H_2^*	H_2^*/Z_1	QUOTIENT GROUPS	SUBGROUPS
Γ_2	H_2	anX_1				Γ_1	Γ_1^3	G		(1^2)		Γ_1
Γ_3	H_3	H_2	X_1			Γ_2	$\Gamma_1\Gamma_2^2$	X_1	(1)	(2)	Γ_1	Γ_1
Γ_4	aK_1	H_2	anX_3	X_1		Γ_2^3	$\Gamma_1\Gamma_2^6$	G		(1^3)	Γ_1	$\Gamma_1^3\Gamma_1^4$
Γ_5	H_2	anX_4	anX_2	anX_1		Γ_1	Γ_2^{15}	G		(1^4)		$\Gamma_1^{15}\Gamma_2^{20}$
Γ_6	H_3	H_2	Z_2	X_1		Γ_2	$\Gamma_2\Gamma_2^2\Gamma_3^4$	X_1	(1)	(21)	Γ_1	$\Gamma_1\Gamma_1^2\Gamma_2^4$
Γ_7	H_3	H_2	Z_2	X_2		Γ_2	$\Gamma_2^2\Gamma_2$	X_2	(1)	(1^3)	Γ_1	$\Gamma_1\Gamma_1^2$
Γ_8	H_4	H_3	H_2	X_1		Γ_3	$\Gamma_1\Gamma_3^2$	X_1	(1)	(3)	Γ_2	Γ_1
Γ_9	aK_2	aK_1	H_2	anX_2	anX_1	Γ_4^7	Γ_2^7	G		(1^3)	Γ_2^7	Γ_1^7
Γ_{10}	aK_1	H_2	anX_7	anX_3	anX_1	$\Gamma_2^2\Gamma_5$	$\Gamma_2^6\Gamma_4^9$	G		(1^4)	Γ_1	$\Gamma_2^2\Gamma_1^9\Gamma_2^{18}\Gamma_2^6$
Γ_{11}	K_1	H_2	anX_7	X_3	anX_1	$\Gamma_2\Gamma_5^2$	$\Gamma_2^3\Gamma_4^{12}$	G		(1^4)	Γ_1	$\Gamma_1\Gamma_1^6\Gamma_2^{12}\Gamma_2^{16}$
Γ_{12}	K_1	H_2	anX_4	X_2	anX_1	Γ_2	Γ_2^3	G		(2^2)	Γ_1	$\Gamma_1\Gamma_1^6$
Γ_{13}	aK_1	H_2	anX_4	anX_2	anX_1	Γ_5^3	Γ_4^{15}	G		(1^4)	Γ_1	$\Gamma_1^5\Gamma_2^{30}$
Γ_{14}	H_3	K_1	H_2	Z_2	X_1	$\Gamma_3^2\Gamma_4$	$\Gamma_1\Gamma_4^2\Gamma_3^4$	X_1	(1)	(21)	$\Gamma_2\Gamma_2^2$	$\Gamma_1\Gamma_1^2\Gamma_2^4$
Γ_{15}	H_3	K_1	H_2	Z_2	X_1	$\Gamma_3\Gamma_6\Gamma_4$	$\Gamma_2\Gamma_2\Gamma_4\Gamma_3^4$	X_1	(1)	(21)	$\Gamma_2\Gamma_2^2$	$\Gamma_1\Gamma_1^2\Gamma_2^2\Gamma_2^2$
Γ_{16}	H_3	K_1	H_2	Z_2	X_1	$\Gamma_6^2\Gamma_4$	$\Gamma_2\Gamma_4^2\Gamma_3^4$	X_1	(1)	(21)	$\Gamma_2\Gamma_2^2$	$\Gamma_1\Gamma_1^2\Gamma_2^4$
Γ_{17}	K_2	H_3	H_2	Z_2	X_2	$\Gamma_3^2\Gamma_7$	$\Gamma_2^2\Gamma_4$	Z_2	(1^2)	(1^2)	Γ_2	$\Gamma_1\Gamma_2^2$
Γ_{18}	H_3	H_2	X_{13}	anX_6	X_1	Γ_2	$\Gamma_2\Gamma_5^2\Gamma_6^6\Gamma_3^6$	X_1	(1)	(21^2)	Γ_1	$\Gamma_2\Gamma_2^3\Gamma_1^3\Gamma_2^{12}\Gamma_3^{12}\Gamma_3^4$
Γ_{19}	H_4	H_3	H_2	Z_3	X_1	Γ_3	$\Gamma_2\Gamma_3^2\Gamma_8^4$	X_1	(1)	(31)	Γ_2	$\Gamma_1\Gamma_1^2\Gamma_3^4$
Γ_{20}	H_4	H_3	H_2	Z_3	X_2	Γ_3	$\Gamma_3^2\Gamma_2$	X_2	(1)	(31)	Γ_2	$\Gamma_1\Gamma_1^2$
Γ_{21}	H_4	H_3	H_2	Z_3	X_1	Γ_3	$\Gamma_2\Gamma_3\Gamma_3$	Z_3	(1^2)	(21)	Γ_2	$\Gamma_1\Gamma_3^2$
Γ_{22}	H_4	H_3	H_2	X_4	X_2	Γ_7	$\Gamma_7^2\Gamma_4$	X_4	(2)	(1^3)	Γ_2	$\Gamma_2\Gamma_2\Gamma_1$
Γ_{23}	H_4	H_3	H_2	X_4	X_2	Γ_7	$\Gamma_7^2\Gamma_4$	X_4	(2)	(21)	Γ_2	$\Gamma_2\Gamma_2\Gamma_1$
Γ_{24}	H_3	K_2	H_2	X_8	X_3	Γ_4	$\Gamma_3\Gamma_6^2\Gamma_2\Gamma_7^2\Gamma_3$	X_8	(1^2)	(21)	$\Gamma_2\Gamma_2\Gamma_2$	$\Gamma_2\Gamma_2^2\Gamma_1\Gamma_2^2\Gamma_2$
Γ_{25}	H_3	aK_1	H_2	X_4	anX_2	Γ_4	$\Gamma_5\Gamma_4^3\Gamma_7^3$	X_4	(1^2)	(1^3)	Γ_2^3	$\Gamma_1\Gamma_2^3\Gamma_2^3$
Γ_{26}	H_3	K_2	H_2	X_6	X_3	Γ_4	$\Gamma_5\Gamma_3^2\Gamma_4\Gamma_6^2\Gamma_7$	X_6	(1^2)	(21)	$\Gamma_2^2\Gamma_2$	$\Gamma_1\Gamma_2^2\Gamma_2\Gamma_2^2\Gamma_2$
Γ_{27}	H_5	H_4	H_3	H_2	X_1	Γ_8	$\Gamma_1\Gamma_8^2$	X_1	(1)	(4)	Γ_3	Γ_1

GROUPS BY FAMILY

FAMILY $\Gamma_1 = \mathbf{A}$. Rank 0. Class 1. $u=1$.

Abelian. $H_2 = I_1 =$ Identity. $Y_2 = Z_1 = G$. $t_3 = t_2$.

The first quotient signal and the first subgroup signal are identical.

Each group constitutes a complete genus.

Order	Group Number	Group Symbol	First Signal Γ_1^m	Order Structure 1	2	4	8	16	32	64	Automorphisms t_1	t_2
1	1	(0)	---	1							2^0	1
2	1	(1)	(0)	1	1						2^0	1
4	1	(1^2)	$(1)^3$	1	3						2^0	$2\cdot3$
	2	(2)	(1)	1	1	2					2^1	1
8	1	(1^3)	$(1^2)^7$	1	7						2^0	$2^3\cdot3\cdot7$
	2	(21)	$(1^2)(2)^2$	1	3	4					2^2	2
	3	(3)	(2)	1	1	2	4				2^2	1
16	1	(1^4)	$(1^3)^{15}$	1	15						2^0	$2^6\cdot3^2\cdot5\cdot7$
	2	(21^2)	$(1^3)(21)^6$	1	7	8					2^3	$2^3\cdot3$
	3	(2^2)	$(21)^3$	1	3	12					2^4	$2\cdot3$
	4	(31)	$(21)(3)^2$	1	3	4	8				2^3	2
	5	(4)	(3)	1	1	2	4	8			2^3	1
32	1	(1^5)	$(1^4)^{31}$	1	31						2^0	$2^{10}\cdot3^2\cdot5\cdot7\cdot31$
	2	(21^3)	$(1^4)(21^2)^{14}$	1	15	16					2^4	$2^6\cdot3\cdot7$
	3	(2^21)	$(21^2)^3(2^2)^4$	1	7	24					2^6	$2^3\cdot3$
	4	(31^2)	$(21^2)(31)^6$	1	7	8	16				2^4	$2^3\cdot3$
	5	(32)	$(2^2)(31)^2$	1	3	12	16				2^6	2
	6	(41)	$(31)(4)^2$	1	3	4	8	16			2^4	2
	7	(5)	(4)	1	1	2	4	8	16		2^4	1
64	1	(1^6)	$(1^5)^{63}$	1	63						2^0	$2^{15}\cdot3^4\cdot5\cdot7^2\cdot31$
	2	(21^4)	$(1^5)(21^3)^{30}$	1	31	32					2^5	$2^{10}\cdot3^2\cdot5\cdot7$
	3	(2^21^2)	$(21^3)^3(2^21)^{12}$	1	15	48					2^8	$2^6\cdot3^2$
	4	(31^3)	$(21^3)(31^2)^{14}$	1	15	16	32				2^5	$2^6\cdot3\cdot7$
	5	(2^3)	$(2^21)^7$	1	7	56					2^9	$2^3\cdot3\cdot7$
	6	(321)	$(2^21)(31^2)^2(32)^4$	1	7	24	32				2^8	2^3
	7	(41^2)	$(31^2)(41)^6$	1	7	8	16	32			2^5	$2^3\cdot3$
	8	(3^2)	$(32)^3$	1	3	12	48				2^8	$2\cdot3$
	9	(42)	$(32)(41)^2$	1	3	12	16	32			2^7	2
	10	(51)	$(41)(5)^2$	1	3	4	8	16	32		2^5	2
	11	(6)	(5)	1	1	2	4	8	16	32	2^5	1

FAMILY $\Gamma_2 = {}^2_1\mathbf{B} = \Lambda_{(1)}$. Rank 3. Class 2. u = 6.

Commutator $\neq 1$: $[\alpha_2, \alpha_3] = \alpha_1$.

Square: $\alpha_1^2 = 1$.

Congruences (mod Z_1): $\alpha_2^2 \equiv \alpha_3^2 \equiv 1$.

$H_2 \sim (1)$. $I_1 \sim (1^2)$.

Y_2 is the unique Γ_1 group in the first quotient signal.

STEM. Order 8.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

Relation: $\alpha_2^2 = \alpha_1$.

Group		Defining Relation	First Quot. Sig.	First Subgroup Signal.	Order Structure		Automorphisms
Number	Symbol	α_3^2	Γ_1	Γ_1^3	2	4	$t_1 \cdot t_2$
4	a_1	1	(1^2)	$(1^2)^2(2)$	5	2	$2^2 \cdot 2$
5	a_2	α_1	(1^2)	$(2)^3$	1	6	$2^2 \cdot 6$

FAMILY Γ_2 (Continued).

FIRST BRANCH. Order 16.

Group		Defining Relations			Generic Invariants		First Quotient Signal		First Subgroup Signal		Order Structure			Automorphisms	
Number	Symbol	α_1	α_2^2	α_3^2	Y_2	Z_1	Γ_1	Γ_2^m	Γ_1^3	Γ_2^n	2	4	8	$t_1 \cdot t_2$	t_3
6	a_1	β_1	β_1	1	(1^3)	(1^2)	(1^3)	a_1^2	$(1^3)^2(21)$	a_1^4	11	4		$2^3 \cdot 8$	2
7	a_2	β_1	β_1	β_1			(1^3)	a_2^2	$(21)^3$	a_2^4	3	12		$2^3 \cdot 24$	6
8	b	β^2	1	1	(1^3)	(2)	(1^3)		$(21)^3$	$a_1^3a_2$	7	8		$2^3 \cdot 6$	6
9	c_1	β_1	1	β_2	(21)	(1^2)	(21)	a_1^2	$(1^3)(21)^2$		7	8		$2^4 \cdot 2$	2
10	c_2	β_1	β_1	β_2			(21)	a_1a_2	$(21)(21)^2$		3	12		$2^4 \cdot 2$	2
11	d	β^2	1	β	(21)	(2)	(21)		$(21)(3)^2$		3	4	8	$2^3 \cdot 2$	2

FAMILY Γ_2 (Continued).
SECOND BRANCH. Order 32.

Group Number	Group Symbol	Defining Relations α_1	α_2^2	α_3^2	Generic Invariants Y_2	Z_1	First Quotient Signal Γ_1	Γ_2^m	First Subgroup Signal Γ_1^3	Γ_2^n	Order Structure 2	4	8	16	Automorphisms $t_1 \cdot t_2$	t_3
8	a_1	β_1	β_1	1	(1^4)	(1^3)	(1^4)	a_1^6	$(1^4)^2(21^2)$	a_1^{12}	23	8			$2^4 \cdot 192$	2
9	a_2	β_1	β_1	β_1			(1^4)	a_2^6	$(21^2)^3$	a_2^{12}	7	24			$2^4 \cdot 576$	6
10	b	β_1^2	1	1	(1^4)	(21)	(1^4)	b^2	$(21^2)^3$	$a_1^3 a_2 b^8$	15	16			$2^4 \cdot 48$	6
11	c_1	β_1	1	β_2	(21^2)	(1^3)	(21^2)	$a_1^2 c_1^4$	$(1^4)(21^2)^2$	c_1^4	15	16			$2^6 \cdot 8$	2
12	c_2	β_1	β_1	β_2			(21^2)	$a_1 a_2 c_2^4$	$(21^2)(21^2)^2$	c_2^4	7	24			$2^6 \cdot 8$	2
13	d	β_1^2	1	β_1	(21^2)	(21)	(21^2)	d^2	$(21^2)(31)^2$	d^4	7	8	16		$2^4 \cdot 8$	2
14	e_1	β_2	β_2	1	(21^2)	(21)	(21^2)	$a_1 b$	$(21^2)^2(2^2)$	$a_1 c_1^2 c_2$	11	20			$2^6 \cdot 2$	2
15	e_2	β_2	β_2	β_2			(21^2)	$a_2 b$	$(2^2)^3$	$a_2 c_2^3$	3	28			$2^6 \cdot 6$	6
16	f	β_1^2	1	β_2	(21^2)	(21)	(21^2)	b^2	$(21^2)(2^2)^2$	$c_1^2 c_2^2$	7	24			$2^6 \cdot 4$	2
17	g	β^4	1	1	(21^2)	(3)	(21^2)		$(31)^3$	$b d^3$	7	8	16		$2^4 \cdot 6$	6
18	h	β_1	β_2	β_3	(2^2)	(1^3)	(2^2)	$c_1^3 c_2^3$	$(21^2)^3$		7	24			$2^6 \cdot 6$	6
19	i	β_1^2	β_2	β_1	(2^2)	(21)	(2^2)	d^2	$(2^2)(31)^2$		3	12	16		$2^6 \cdot 2$	2
20	j_1	β_2	1	β_1	(31)	(21)	(31)	$c_1 d$	$(21^2)(31)^2$		7	8	16		$2^5 \cdot 2$	2
21	j_2	β_2	β_2	β_1			(31)	$c_2 d$	$(2^2)(31)^2$		3	12	16		$2^5 \cdot 2$	2
22	k	β^4	1	β	(31)	(3)	(31)		$(31)(4)^2$		3	4	8	16	$2^4 \cdot 2$	2

FAMILY Γ_2 (Continued).

THIRD BRANCH. *Order 64.*

Group Number	Group Symbol	Defining Relations α_1	α_2^2	α_3^2	Generic Invariants Y_2	Z_1	First Quotient Signal Γ_1	Γ_2^m	First Subgroup Signal Γ_1^3	Γ_2^n	Order Structure 2	4	8	16	32	Automorphisms $t_1 \cdot t_2$	t_3
12	a_1	β_1	β_1	1	(1^5)	(1^4)	(1^5)	a_1^{14}	$(1^5)^2(21^3)$	a_1^{28}	47	16				$2^5 \cdot 21504$	2
13	a_2	β_1	β_1	β_1			(1^5)	a_2^{14}	$(21^3)^3$	a_2^{28}	15	48				$2^5 \cdot 64512$	6
14	b	β_1^2	1	1	(1^5)	(21^2)	(1^5)	b^6	$(21^3)^3$	$a_1^3a_2b^{24}$	31	32				$2^5 \cdot 2304$	6
15	c_1	β_1	1	β_2	(21^3)	(1^4)	(21^3)	$a_1^2c_1^{12}$	$(1^5)(21^3)^2$	c_1^{12}	31	32				$2^8 \cdot 192$	2
16	c_2	β_1	β_1	β_2			(21^3)	$a_1a_2c_2^{12}$	$(21^3)(21^3)^2$	c_2^{12}	15	48				$2^8 \cdot 192$	2
17	d	β_1^2	1	β_1	(21^3)	(21^2)	(21^3)	d^6	$(21^3)(31^2)^2$	d^{12}	15	16	32			$2^5 \cdot 192$	2
18	e_1	β_2	β_2	1	(21^3)	(21^2)	(21^3)	$a_1be_1^4$	$(21^3)^2(2^21)$	$a_1c_1^2c_2e_1^8$	23	40				$2^8 \cdot 16$	2
19	e_2	β_2	β_2	β_2			(21^3)	$a_2be_2^4$	$(2^21)^3$	$a_2c_2^3e_2^8$	7	56				$2^8 \cdot 48$	6
20	f	β_1^2	1	β_2	(21^3)	(21^2)	(21^3)	b^2f^4	$(21^3)(2^21)^2$	$c_1^2c_2^2f^8$	15	48				$2^8 \cdot 32$	2
21	g	β_1^4	1	1	(21^3)	(31)	(21^3)	g^2	$(31^2)^3$	bd^3g^8	15	16	32			$2^5 \cdot 48$	6
22	h	β_1	β_2	β_3	(2^21)	(1^4)	(2^21)	$c_1^3c_2^3h^8$	$(21^3)^3$	h^4	15	48				$2^9 \cdot 24$	6
23	i	β_1^2	β_2	β_1	(2^21)	(21^2)	(2^21)	d^2i^4	$(2^21)(31^2)^2$	i^4	7	24	32			$2^8 \cdot 8$	2
24	j_1	β_2	1	β_1	(31^2)	(21^2)	(31^2)	$c_1dj_1^4$	$(21^3)(31^2)^2$	j_1^4	15	16	32			$2^7 \cdot 8$	2
25	j_2	β_2	β_2	β_1			(31^2)	$c_2dj_2^4$	$(2^21)(31^2)^2$	j_2^4	7	24	32			$2^7 \cdot 8$	2
26	k	β_1^4	1	β_1	(31^2)	(31)	(31^2)	k^2	$(31^2)(41)^2$	k^4	7	8	16	32		$2^5 \cdot 8$	2
27	l	β_1^2	1	1	(21^3)	(2^2)	(21^3)	b^2	$(2^21)^3$	$be_1^6e_2^2f^3$	15	48				$2^8 \cdot 12$	6
28	m_1	β_2	1	β_3	(2^21)	(21^2)	(2^21)	$c_1e_1^4f$	$(21^3)(2^21)^2$	$c_1^2h^2$	15	48				$2^9 \cdot 4$	2
29	m_2	β_2	β_2	β_3			(2^21)	$c_2e_1^2e_2^2f$	$(2^21)(2^21)^2$	$c_2^2h^2$	7	56				$2^9 \cdot 4$	2
30	n	β_1^2	β_2	β_3	(2^21)	(21^2)	(2^21)	f^6	$(2^21)^3$	h^4	7	56				$2^9 \cdot 24$	6
31	o	β_1^2	1	β_1	(2^21)	(2^2)	(2^21)	d^2	$(2^21)(32)^2$	d^2i^2	7	24	32			$2^8 \cdot 4$	2
32	p	β_1^4	1	β_2	(2^21)	(31)	(2^21)	g^2	$(31^2)(32)^2$	dfi^2	7	24	32			$2^8 \cdot 2$	2
33	q	β_1^2	1	β_2	(31^2)	(2^2)	(31^2)	df	$(2^21)(32)^2$	$j_1^2j_2^2$	7	24	32			$2^7 \cdot 4$	2
34	r_1	β_2	β_2	1	(31^2)	(31)	(31^2)	e_1g	$(31^2)^2(32)$	$e_1j_1^2j_2$	11	20	32			$2^7 \cdot 2$	2
35	r_2	β_2	β_2	β_2			(31^2)	e_2g	$(32)^3$	$e_2j_2^3$	3	28	32			$2^7 \cdot 6$	6
36	s	β^8	1	1	(31^2)	(4)	(31^2)		$(41)^3$	gk^3	7	8	16	32		$2^5 \cdot 6$	6
37	t	β_2	β_3	β_1	(32)	(21^2)	(32)	$hij_1^2j_2^2$	$(2^21)(31^2)^2$		7	24	32			$2^8 \cdot 2$	2
38	u	β_1^2	β_1	β_2	(32)	(2^2)	(32)	i^2	$(32)(32)^2$		3	12	48			$2^8 \cdot 2$	2
39	v	β_1^4	β_2	β_1	(32)	(31)	(32)	k^2	$(32)(41)^2$		3	12	16	32		$2^7 \cdot 2$	2
40	w_1	β_2	1	β_1	(41)	(31)	(41)	j_1k	$(31^2)(41)^2$		7	8	16	32		$2^6 \cdot 2$	2
41	w_2	β_2	β_2	β_1			(41)	j_2k	$(32)(41)^2$		3	12	16	32		$2^6 \cdot 2$	2
42	x	β^8	1	β	(41)	(4)	(41)		$(41)(5)^2$		3	4	8	16	32	$2^5 \cdot 2$	2

FAMILY $\Gamma_3 = {}^{2}_{1}\mathbf{C} = \Lambda_{(2)}$. Rank 4. Class 3. u=8.

Commutators $\neq 1$: $[\alpha_2, \alpha_4] = \alpha_1$, $[\alpha_3, \alpha_4] = \alpha_2$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_3^2 \equiv \alpha_2$, $\alpha_4^2 \equiv 1$.

$H_2 \sim (2)$, $H_3 \sim (1)$. $I_1 \sim 8\Gamma_2 a_1$, $I_2 \sim (1^2)$.

$Y_3 = \widetilde{I}_1$ is the unique Γ_2 group in the first quotient signal.

STEM. Order 16.

$Z_1 = H_3$, $Z_2 = H_2$. $Y_2 = I_2$, $Y_3 = I_1$.

Group Number	Group Symbol	Defining Relations α_3^2	Defining Relations α_4^2	First Quot. Sig. Γ_2	First Subgroup Signal Γ_1	First Subgroup Signal Γ_2^2	Order Structure 2	4	8	Automorphisms $t_1 \cdot t_2$	t_3
12	a_1	α_2^{-1}	1	a_1	(3)	a_1^2	9	2	4	$2^4 \cdot 2$	8
13	a_2	α_2	1	a_1	(3)	a_1a_2	5	6	4	$2^4 \cdot 1$	4
14	a_3	α_2^{-1}	α_1	a_1	(3)	a_2^2	1	10	4	$2^4 \cdot 2$	8

FAMILY Γ_3 (Continued).

FIRST BRANCH. Order 32.

$Z_2 \sim (21)$.

Group Number	Group Symbol	Defining Relations α_1	α_3^2	α_4^2	Generic Invariants Y_2	Z_1	First Quotient Signal Γ_2	Γ_3^m	First Subgroup Signal Γ_1	Γ_2^2	Γ_3^n	Order Structure 2	4	8	Automorphisms $t_1 \cdot t_2$	t_3
23	a_1	β_1	α_2^{-1}	1	(1^3)	(1^2)	a_1	a_1^2	(31)	a_1^2	a_1^4	19	4	8	$2^5 \cdot 8$	8
24	a_2	β_1	α_2	1			a_1	a_2^2	(31)	a_1a_2	a_2^4	11	12	8	$2^5 \cdot 4$	4
25	a_3	β_1	α_2^{-1}	β_1			a_1	a_3^2	(31)	a_2^2	a_3^4	3	20	8	$2^5 \cdot 8$	8
26	b	β^2	α_2^{-1}	1	(1^3)	(2)	a_1		(31)	b^2	$a_1a_2^2a_3$	11	12	8	$2^5 \cdot 2$	8
27	c_1	β_1	$\alpha_2^{-1}\beta_2$	1	(21)	(1^2)	c_1	a_1a_2	(31)	a_1c_2		11	12	8	$2^6 \cdot 1$	4
28	c_2	β_1	$\alpha_2^{-1}\beta_2$	β_1			c_1	a_2a_3	(31)	a_2c_2		3	20	8	$2^6 \cdot 1$	4
29	d_1	β_1	α_2^{-1}	β_2	(21)	(1^2)	c_2	a_1a_3	(31)	c_2^2		3	20	8	$2^6 \cdot 2$	8
30	d_2	β_1	α_2	β_2			c_2	a_2^2	(31)	c_2^2		3	20	8	$2^6 \cdot 2$	8
31	e	β^2	$\alpha_2^{-1}\beta$	1	(21)	(2)	c_1		(2^2)	bd		7	16	8	$2^5 \cdot 1$	4
32	f	β^2	α_2^{-1}	β	(21)	(2)	c_2		(31)	d^2		3	4	24	$2^5 \cdot 2$	8

FAMILY Γ_3 (Continued).

SECOND BRANCH. Order 64.

Group Number	Group Symbol	Defining Relations α_1	Defining Relations α_3^2	Defining Relations α_4^2	Generic Invariants Y_2	Generic Invariants Z_1	First Quotient Signal Γ_2	First Quotient Signal Γ_3^m	First Subgroup Signal Γ_1	First Subgroup Signal Γ_2^2	First Subgroup Signal Γ_3^n	Order Structure 2	Order Structure 4	Order Structure 8	Order Structure 16	Automorphisms $t_1 \cdot t_2$	Automorphisms t_3	Z_2 $16\Gamma_1$
43	a_1	β_1	α_2^{-1}	1	(1^4)	(1^3)	a_1	a_1^6	(31^2)	a_1^2	a_1^{12}	39	8	16		$2^6 \cdot 192$	8	(21^2)
44	a_2	β_1	α_2	1			a_1	a_2^6	(31^2)	$a_1 a_2$	a_2^{12}	23	24	16		$2^6 \cdot 96$	4	(21^2)
45	a_3	β_1	α_2^{-1}	β_1			a_1	a_3^6	(31^2)	a_2^2	a_3^{12}	7	40	16		$2^6 \cdot 192$	8	(21^2)
46	b	β_1^2	α_2^{-1}	1	(1^4)	(21)	a_1	b^2	(31^2)	b^2	$a_1 a_2^2 a_3 b^8$	23	24	16		$2^6 \cdot 16$	8	(21^2)
47	c_1	β_1	$\alpha_2^{-1}\beta_2$	1	(21^2)	(1^3)	c_1	$a_1 a_2 c_1^4$	(31^2)	$a_1 c_2$	c_1^4	23	24	16		$2^8 \cdot 4$	4	(21^2)
48	c_2	β_1	$\alpha_2^{-1}\beta_2$	β_1			c_1	$a_2 a_3 c_2^4$	(31^2)	$a_2 c_2$	c_2^4	7	40	16		$2^8 \cdot 4$	4	(21^2)
49	d_1	β_1	α_2^{-1}	β_2	(21^2)	(1^3)	c_2	$a_1 a_3 d_1^4$	(31^2)	c_2^2	d_1^4	7	40	16		$2^8 \cdot 8$	8	(21^2)
50	d_2	β_1	α_2	β_2			c_2	$a_2^2 d_2^4$	(31^2)	c_2^2	d_2^4	7	40	16		$2^8 \cdot 8$	8	(21^2)
51	e	β_1^2	$\alpha_2^{-1}\beta_1$	1	(21^2)	(21)	c_1	e^2	$(2^2 1)$	bd	e^4	15	32	16		$2^6 \cdot 4$	4	(21^2)
52	f	β_1^2	α_2^{-1}	β_1	(21^2)	(21)	c_2	f^2	(31^2)	d^2	f^4	7	8	48		$2^6 \cdot 8$	8	(21^2)
53	g	β_1^2	$\alpha_2^{-1}\beta_2$	1	(21^2)	(21)	c_1	b^2	(31^2)	bf	$c_1^2 c_2^2$	15	32	16		$2^8 \cdot 2$	4	(21^2)
54	h	β_1^2	α_2^{-1}	β_2	(21^2)	(21)	c_2	b^2	(31^2)	f^2	$d_1^2 d_2^2$	7	40	16		$2^8 \cdot 4$	8	(21^2)
55	i_1	β_2	α_2^{-1}	1	(21^2)	(21)	e_1	$a_1 b$	(32)	e_1^2	$a_1 c_1^2 d_1$	19	28	16		$2^8 \cdot 2$	8	(2^2)
56	i_2	β_2	α_2	1			e_1	$a_2 b$	(32)	$e_1 e_2$	$a_2 c_1 c_2 d_2$	11	36	16		$2^8 \cdot 1$	4	(2^2)
57	i_3	β_2	α_2^{-1}	β_2			e_1	$a_3 b$	(32)	e_2^2	$a_3 c_2^2 d_1$	3	44	16		$2^8 \cdot 2$	8	(2^2)
58	j	β^4	α_2^{-1}	1	(21^2)	(3)	e_1		(32)	g^2	$b e^2 f$	11	20	32		$2^6 \cdot 2$	8	(31)
59	k	β_1	$\alpha_2^{-1}\beta_2$	β_3	(2^2)	(1^3)	h	$c_1^2 c_2^2 d_1 d_2$	(31^2)	c_2^2		7	40	16		$2^8 \cdot 2$	8	(21^2)
60	l	β_1^2	$\alpha_2^{-1}\beta_1$	β_2	(2^2)	(21)	h	e^2	$(2^2 1)$	df		7	40	16		$2^8 \cdot 1$	4	(21^2)
61	m	β_1^2	$\alpha_2^{-1}\beta_2$	β_1	(2^2)	(21)	h	f^2	(31^2)	d^2		7	8	48		$2^8 \cdot 2$	8	(21^2)
62	n_1	β_2	$\alpha_2^{-1}\beta_1$	1	(31)	(21)	j_1	$c_1 e$	(32)	$e_1 j_2$		11	20	32		$2^7 \cdot 1$	4	(2^2)
63	n_2	β_2	$\alpha_2^{-1}\beta_1$	β_2			j_1	$c_2 e$	(32)	$e_2 j_2$		3	28	32		$2^7 \cdot 1$	4	(2^2)
64	o_1	β_2	α_2^{-1}	β_1	(31)	(21)	j_2	$d_1 f$	(32)	j_2^2		3	12	48		$2^7 \cdot 2$	8	(2^2)
65	o_2	β_2	α_2	β_1			j_2	$d_2 f$	(32)	j_2^2		3	12	48		$2^7 \cdot 2$	8	(2^2)
66	p	β^4	$\alpha_2^{-1}\beta$	1	(31)	(3)	j_1		(41)	gk		7	8	16	32	$2^6 \cdot 1$	4	(31)
67	q	β^4	α_2^{-1}	β	(31)	(3)	j_2		(32)	k^2		3	12	16	32	$2^6 \cdot 2$	8	(31)

FAMILY $\Gamma_4 = {}^{3}_{2}\mathbf{B} = \Lambda_{(1^2)}$. Rank 5. Class 2. u=24.

Commutators $\neq 1$: $[\alpha_3, \alpha_5] = \alpha_1$, $[\alpha_4, \alpha_5] = \alpha_2$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$.

Congruences (mod Z_1): $\alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv 1$.

$H_2 \sim (1^2)$. $I_1 \sim (1^3)$.

STEM. Order 32.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

n is the number of self-conjugate subgroups of order 4.

Group		Defining Relations			First Quot Signal	First Subgroup Signal		Order Structure		Automorphisms	Self-centralizers		n
Number	Symbol	α_3^2	α_4^2	α_5^2	Γ_2^3	Γ_1	Γ_2^6	2	4	$t_1 \cdot t_2$	$16\Gamma_1$	$(8\Gamma_1)^4$	
33	a_1	1	1	1	a_1^3	(1^4)	$a_1^3 c_1^3$	19	12	$2^6 \cdot 6$	(1^4)	$(1^3)(21)^3$	7
34	a_2	α_1	α_2	1	a_1^3	(2^2)	a_1^6	19	12	$2^6 \cdot 24$	(2^2)	$(1^3)^4$	7
35	a_3	α_1	α_2	α_1	$a_1 a_2^2$	(2^2)	$a_2^2 c_2^4$	3	28	$2^6 \cdot 8$	(2^2)	$(21)^4$	7
36	b_1	α_1	1	1	$a_1 a_1 b$	(21^2)	$a_1 a_1^2 c_1^2 c_2$	15	16	$2^6 \cdot 2$	(21^2)	$(1^3)^2(21)^2$	5
37	b_2	α_1	1	α_1	$a_1 a_2 b$	(21^2)	$a_2 c_1^2 c_2 c_2^2$	7	24	$2^6 \cdot 2$	(21^2)	$(21)^2(21)^2$	5
38	c_1	1	α_1	1	$a_1 b^2$	(21^2)	$a_1 c_1 c_1^2 c_2^2$	11	20	$2^6 \cdot 2$	(21^2)	$(1^3)(21)(21)^2$	3
39	c_2	α_1	$\alpha_1\alpha_2$	1	$a_1 b^2$	(2^2)	$a_1 a_2 c_1^4$	11	20	$2^6 \cdot 4$	(2^2)	$(1^3)^2(21)^2$	3
40	c_3	α_1	$\alpha_1\alpha_2$	α_2	$a_2 b^2$	(2^2)	$c_2^2 c_2^4$	3	28	$2^6 \cdot 4$	(2^2)	$(21)^4$	3
41	d	α_2	$\alpha_1\alpha_2$	1	b^3	(2^2)	$c_1^3 c_2^3$	7	24	$2^6 \cdot 3$	(2^2)	$(1^3)(21)^3$	1

FAMILY Γ_4 (Continued).

FIRST BRANCH. Order 64.

Relation: $\alpha_2 = \beta_2$.

Group: Number	Group: Symbol	Defining Relations: α_1	Defining Relations: α_3^2	Defining Relations: α_4^2	Defining Relations: α_5^2	Generic Invariants: Y_2	Generic Invariants: Z_1	First Quotient Signal: Γ_2^3	First Quotient Signal: Γ_4^m	First Subgroup Signal: Γ_1	First Subgroup Signal: Γ_2^6	First Subgroup Signal: Γ_4^n	Order Structure: 2	Order Structure: 4	Automorphisms: $t_1 \cdot t_2$	Automorphisms: t_3	Self-centralizers: $32\Gamma_1$	Self-centralizers: $(16\Gamma_1)^4$
68	a_1	β_1	1	1	1	(1^4)	(1^3)	a_1^3	a_1^4	(1^5)	$a_1^3 c_1^3$	a_1^8	39	24	$2^8 \cdot 48$	6	(1^5)	$(1^4)(21^2)^3$
69	a_2	β_1	β_1	β_2	1			a_1^3	a_2^4	(2^21)	a_1^6	a_2^8	39	24	$2^8 \cdot 192$	24	(2^21)	$(1^4)^4$
70	a_3	β_1	β_1	β_2	β_1			$a_1 a_2^2$	a_3^4	(2^21)	$a_2^2 c_2^4$	a_3^8	7	56	$2^8 \cdot 64$	8	(2^21)	$(21^2)^4$
71	b_1	β_1	β_1	1	1	(1^4)	(1^3)	$a_1 a_1 b$	b_1^4	(21^3)	$a_1 a_1^2 c_1^2 c_2$	b_1^8	31	32	$2^8 \cdot 16$	2	(21^3)	$(1^4)^2(21^2)^2$
72	b_2	β_1	β_1	1	β_1			$a_1 a_2 b$	b_2^4	(21^3)	$a_2 c_1^2 c_2 c_2^2$	b_2^8	15	48	$2^8 \cdot 16$	2	(21^3)	$(21^2)^2(21^2)^2$
73	c_1	β_1	1	β_1	1	(1^4)	(1^3)	$a_1 b^2$	c_1^4	(21^3)	$a_1 c_1 c_1^2 c_2^2$	c_1^8	23	40	$2^8 \cdot 16$	2	(21^3)	$(1^4)(21^2)(21^2)^2$
74	c_2	β_1	β_1	$\beta_1\beta_2$	1			$a_1 b^2$	c_2^4	(2^21)	$a_1 a_2 c_1^4$	c_2^8	23	40	$2^8 \cdot 32$	4	(2^21)	$(1^4)^2(21^2)^2$
75	c_3	β_1	β_1	$\beta_1\beta_2$	β_2			$a_2 b^2$	c_3^4	(2^21)	$c_2^2 c_2^4$	c_3^8	7	56	$2^8 \cdot 32$	4	(2^21)	$(21^2)^4$
76	d	β_1	β_2	$\beta_1\beta_2$	1	(1^4)	(1^3)	b^3	d^4	(2^21)	$c_1^3 c_2^3$	d^8	15	48	$2^8 \cdot 24$	3	(2^21)	$(1^4)(21^2)^3$
77	e_1	β_1^2	β_1^2	1	1	(1^4)	(21)	$a_1 b^2$		(21^3)	$b e_1^4 f$	$a_1^2 b_1^2 b_2^2 c_1^2$	23	40	$2^8 \cdot 4$	4	(21^3)	$(21^2)^2(2^2)^2$
78	e_2	β_1^2	1	β_2	1			$a_1 b^2$		(2^21)	$b^2 e_1^4$	$a_2 a_3 b_1^4 c_2^2$	23	40	$2^8 \cdot 8$	8	(2^21)	$(21^2)^4$
79	e_3	β_1^2	1	β_2	β_2			$a_2 b^2$		(2^21)	$e_2^4 f^2$	$a_3^2 b_2^4 c_3^2$	7	56	$2^8 \cdot 8$	8	(2^21)	$(2^2)^4$
80	f	β_1^2	β_2	1	1	(1^4)	(21)	b b b		(2^21)	$e_1 e_1^2 e_2 f^2$	$b_1 b_2 c_1^2 c_2 c_3 d^2$	15	48	$2^8 \cdot 2$	2	(2^21)	$(21^2)^2(2^2)^2$
81	g_1	β_1	1	1	β_3	(21^2)	(1^3)	c_1^3	a_1^4	(1^5)	c_1^6		31	32	$2^9 \cdot 24$	24	(1^5)	$(21^2)^4$
82	g_2	β_1	β_1	β_2	β_3			c_2^3	$a_2 a_3^3$	(2^21)	c_2^6		7	56	$2^9 \cdot 24$	24	(2^21)	$(21^2)^4$
83	h	β_1	β_1	1	β_3	(21^2)	(1^3)	$c_1 c_2 f$	$b_1^2 b_2^2$	(21^3)	$c_1^2 c_2^2 h^2$		15	48	$2^9 \cdot 4$	4	(21^3)	$(21^2)^4$

FAMILY Γ_4 (Continued).
FIRST BRANCH (Continued).

Group Number	Group Symbol	Defining Relations				Generic Invariants		First Quotient Signal		First Subgroup Signal		Order Structure			Automorphisms		Self-centralizers	
		α_1	α_3^2	α_4^2	α_5^2	Y_2	Z_1	Γ_2^3	Γ_4^m	Γ_1	Γ_2^6	2	4	8	$t_1 \cdot t_2$	t_3	$32\Gamma_1$	$(16\Gamma_1)^4$
84	i_1	β_1	1	$\beta_2\beta_3$	1	(21^2)	(1^3)	$c_1e_1^2$	$a_1b_1^2c_1$	(21^3)	$a_1c_1c_1^2h^2$	23	40		$2^9 \cdot 2$	2	(21^3)	$(1^4)(21^2)(21^2)^2$
85	i_2	β_1	β_1	β_3	1			$c_1e_1^2$	$a_2b_1^2c_2$	(2^21)	$a_1c_1^4c_2$	23	40		$2^9 \cdot 4$	4	(2^21)	$(1^4)^2(21^2)^2$
86	i_3	β_1	1	$\beta_2\beta_3$	β_2			$c_2e_1^2$	$a_1b_2^2c_1$	(21^3)	$c_1c_1c_2^2h^2$	15	48		$2^9 \cdot 2$	2	(21^3)	$(21^2)(21^2)(21^2)^2$
87	i_4	β_1	β_1	β_3	β_1			$c_1e_2^2$	$a_3b_2^2c_2$	(2^21)	$a_2c_2h^4$	7	56		$2^9 \cdot 4$	4	(2^21)	$(21^2)^2(21^2)^2$
88	i_5	β_1	β_1	β_3	β_2			$c_2e_1e_2$	$a_3b_1b_2c_3$	(2^21)	$c_2c_2c_2^2h^2$	7	56		$2^9 \cdot 2$	2	(2^21)	$(21^2)^2(21^2)^2$
89	j_1	β_1	1	β_1	β_3	(21^2)	(1^3)	c_1f^2	c_1^4	(21^3)	$c_1^2h^4$	15	48		$2^9 \cdot 8$	8	(21^3)	$(21^2)^4$
90	j_2	β_1	β_1	$\beta_1\beta_2$	β_3			c_2f^2	$c_2^2c_3^2$	(2^21)	$c_2^2h^4$	7	56		$2^9 \cdot 8$	8	(2^21)	$(21^2)^4$
91	k_1	β_1	β_2	β_3	1	(21^2)	(1^3)	e_1e_1f	$b_1c_1c_2d$	(2^21)	$c_1c_1c_1c_2hh$	15	48		$2^9 \cdot 1$	1	(2^21)	$(1^4)(21^2)(21^2)(21^2)$
92	k_2	β_1	β_2	β_3	β_1			e_1e_2f	$b_2c_1c_3d$	(2^21)	c_2c_2hhhh	7	56		$2^9 \cdot 1$	1	(2^21)	$(21^2)(21^2)(21^2)(21^2)$
93	l	β_1	β_2	$\beta_1\beta_2$	β_3	(21^2)	(1^3)	f^3	d^4	(2^21)	h^6	7	56		$2^9 \cdot 12$	12	(2^21)	$(21^2)^4$
94	m_1	β_1^2	β_1^2	1	β_1	(21^2)	(21)	c_1d^2		(21^3)	$d^2j_1^4$	15	16	32	$2^7 \cdot 8$	8	(21^3)	$(31)^4$
95	m_2	β_1^2	1	β_2	β_1			c_2d^2		(2^21)	$d^2j_2^4$	7	24	32	$2^7 \cdot 8$	8	(2^21)	$(31)^4$
96	n_1	β_1^2	1	β_1	1	(21^2)	(21)	c_1g^2		(31^2)	bdj_1^4	15	16	32	$2^7 \cdot 4$	4	(31^2)	$(21^2)^2(31)^2$
97	n_2	β_1^2	1	β_1	β_2			c_2g^2		(31^2)	dfj_2^4	7	24	32	$2^7 \cdot 4$	4	(31^2)	$(2^2)^2(31)^2$
98	o	β_1^2	β_2	1	β_1	(21^2)	(21)	ddf		(2^21)	$i^2j_1^2j_2^2$	7	24	32	$2^7 \cdot 4$	4	(2^21)	$(31)^4$
99	p_1	β_1^2	β_1^{-1}	1	1	(21^2)	(21)	de_1g		(31^2)	$de_1ij_1j_1j_2$	11	20	32	$2^7 \cdot 1$	1	(31^2)	$(21^2)(2^2)(31)(31)$
100	p_2	β_1^2	β_1	β_2	1			de_1g		(32)	$d^2e_1j_1^2j_2$	11	20	32	$2^7 \cdot 2$	2	(32)	$(21^2)^2(31)^2$
101	p_3	β_1^2	β_1	β_2	β_2			de_2g		(32)	$e_2i^2j_2j_2^2$	3	28	32	$2^7 \cdot 2$	2	(32)	$(2^2)^2(31)^2$
102	q	β_1^2	β_2	β_1	1	(21^2)	(21)	fg^2		(32)	$fij_1^2j_2^2$	7	24	32	$2^7 \cdot 2$	2	(32)	$(21^2)(2^2)(31)^2$

FAMILY $\Gamma_5 = {}^{4}_{1}\mathbf{B}$. Rank 5. Class 2. u=720.

Commutators $\neq 1$: $[\alpha_3, \alpha_4] = [\alpha_2, \alpha_5] = \alpha_1$.

Square: $\alpha_1^2 = 1$.

Congruences (mod Z_1): $\alpha_2^2 \equiv \alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv 1$.

$H_2 \sim (1)$. $I_1 \sim (1^4)$.

Each group is the central product of two Γ_2 groups, in ten distinct ways.

Y_2 is the unique Γ_1 group in the first quotient signal.

STEM. Order 32.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

Relations: $\alpha_2^2 = \alpha_3^2 = \alpha_4^2 = \alpha_1$.

Group Number	Group Symbol	Defining Relation	First Quot. Sig.	First Subgroup Signal	Order Structure		Automorphisms	Self-centralizers	Central Factors
		α_5^2	Γ_1	Γ_2^{15}	2	4	$t_1 \cdot t_2$	$(8\Gamma_1)^{15}$	$\{8\Gamma_2 * 8\Gamma_2\}^{10}$
42	a_1	α_1	(1^4)	$a_1^9 b^6$	19	12	$2^4 \cdot 72$	$(1^3)^6(21)^9$	$\underline{\{a_1 * a_1\}}^9 \underline{\{a_2 * a_2\}}$
43	a_2	1	(1^4)	$a_2^5 b^{10}$	11	20	$2^4 \cdot 120$	$(21)^{15}$	$\{a_1 * a_2\}^{10}$

FAMILY Γ_5 (Continued).
FIRST BRANCH. Order 64.

Group Number	Group Symbol	Defining Relations α_1	α_2^2	α_3^2	α_4^2	α_5^2	Generic Invariants Y_2	Z_1	First Quot. Signal Γ_1	Γ_5^m	First Subgroup Signal Γ_2^{15}	Γ_5^n	Order Structure 2	4	8	Automorphisms $t_1 \cdot t_2$	t_3	Self-centralizers $(16\Gamma_1)^{15}$	Central Factors $\{16\Gamma_2 * 16\Gamma_2\}^{10}$
103	a_1	β_1	β_1	β_1	β_1	β_1	(1^5)	(1^2)	(1^5)	a_1^2	$a_1^9 b^6$	a_1^{16}	39	24		$2^5 \cdot 1152$	72	$(1^4)^6 (21^2)^9$	$\{\underline{a_1 * a_1}\}^9 \{\underline{a_2 * a_2}\}$
104	a_2	β_1	β_1	β_1	β_1	1			(1^5)	a_2^2	$a_2^5 b^{10}$	a_2^{16}	23	40		$2^5 \cdot 1920$	120	$(21^2)^{15}$	$\{a_1 * a_2\}^{10}$
105	b	β^2	1	1	1	1	(1^5)	(2)	(1^5)		b^{15}	$a_1^{10} a_2^6$	31	32		$2^5 \cdot 720$	720	$(21^2)^{15}$	$\{\underline{b * b}\}^{10}$
106	c_1	β_1	1	β_1	1	β_2	(21^3)	(1^2)	(21^3)	a_1^2	$a_1 c_1^4 e_1^8 f^2$		23	40		$2^8 \cdot 16$	16	$(1^4)^2 (21^2)(21^2)^8 (2^2)^4$	$\{a_1 * c_1\}^4 \{\underline{c_1 * c_1}\}^4 \{\underline{c_2 * c_2}\}^2$
107	c_2	β_1	β_1	β_1	β_1	β_2			(21^3)	$a_1 a_2$	$b c_2^3 e_1^6 e_2^2 f^3$		15	48		$2^8 \cdot 12$	12	$(21^2)^3 (21^2)^6 (2^2)^6$	$\{a_1 * c_2\}^3 \{a_2 * c_2\} \{c_1 * c_2\}^6$
108	c_3	β_1	1	β_1	β_1	β_2			(21^3)	a_2^2	$a_2 e_2^8 f^6$		7	56		$2^8 \cdot 48$	48	$(21^2)^3 (2^2)^{12}$	$\{a_2 * c_1\}^4 \{\underline{c_2 * c_2}\}^6$
109	d	β^2	1	1	1	β	(21^3)	(2)	(21^3)		$b d^6 g^8$		15	16	32	$2^5 \cdot 48$	48	$(21^2)^3 (31)^{12}$	$\{b * d\}^4 \{\underline{d * d}\}^6$

FAMILY $\Gamma_6={}^{3}_{1}\mathbf{C}_1$. Rank 5. Class 3. u=32.

Commutators $\neq 1$: $[\alpha_3,\alpha_4]=[\alpha_2,\alpha_5]=\alpha_1$, $[\alpha_4,\alpha_5]=\alpha_2$.

Squares: $\alpha_1^2=1$, $\alpha_2^2=\alpha_1$.

Congruences (mod Z_1): $\alpha_3^2\equiv\alpha_5^2\equiv 1$, $\alpha_4^2\equiv\alpha_2$.

$H_2\sim(2)$, $H_3\sim(1)$. $I_1\sim 16\Gamma_2 a_1$, $I_2\sim(1^2)$.

Z_2 is the verbally characteristic self-centralizer.

$Y_3=\widetilde{I}_1$ is the unique Γ_2 group in the first quotient signal.

H_2^* is the verbally characteristic Γ_2 group in the first subgroup signal.

STEM. Order 32.

$Z_1=H_3$, $Z_2\sim(21)$. $Y_2\sim(1^3)$, $Y_3=I_1$.

Relations: $\alpha_3^2=1$, $\alpha_4^2=\alpha_2^{-1}$.

Group Number	Group Symbol	Defining Relation	First Quot. Sig.	First Subgroup Signal			Order Structure			Automorphisms		Self-centralizers	
		α_5^2	Γ_2	Γ_2^2	Γ_2	Γ_3^4	2	4	8	$t_1\cdot t_2$	t_3	$8\Gamma_1$	$(8\Gamma_1)^2$
44	a_1	1	a_1	$a_1 b$	d	$a_1^2a_2^2$	15	8	8	$2^5\cdot 2$	8	(21)	$(3)^2$
45	a_2	α_1	a_1	$a_2 b$	d	$a_2^2a_3^2$	7	16	8	$2^5\cdot 2$	8	(21)	$(3)^2$

FAMILY Γ_6 (Continued).
FIRST BRANCH. Order 64.

Group		Defining Relations				Generic Invariants.		First Quotient Signal		First Subgroup Signal				Order Structure			Automorphisms		Self-centralizers	
Number	Symbol	α_1	α_3^2	α_4^2	α_5^2	Y_2	Z_1	Γ_2	Γ_6^m	Γ_2^2	Γ_2	Γ_3^4	Γ_6^n	2	4	8	$t_1 \cdot t_2$	t_3	$16\Gamma_1$	$(16\Gamma_1)^2$
110	a_1	β_1	1	α_2^{-1}	1	(1^4)	(1^2)	a_1	a_1^2	$a_1 b$	d	$a_1^2 a_2^2$	a_1^8	31	16	16	$2^6 \cdot 16$	8	(21^2)	$(31)^2$
111	a_2	β_1	1	α_2^{-1}	β_1			a_1	a_2^2	$a_2 b$	d	$a_2^2 a_3^2$	a_2^8	15	32	16	$2^6 \cdot 16$	8	(21^2)	$(31)^2$
112	b	β^2	1	α_2^{-1}	1	(1^4)	(2)	a_1		b^2	d	b^4	$a_1^4 a_2^4$	23	24	16	$2^6 \cdot 8$	32	(21^2)	$(31)^2$
113	c_1	β_1	1	$\alpha_2^{-1}\beta_2$	1	(21^2)	(1^2)	c_1	a_1^2	$a_1 f$	d	c_1^4		23	24	16	$2^8 \cdot 4$	16	(21^2)	$(31)^2$
114	c_2	β_1	β_1	$\alpha_2^{-1}\beta_2$	β_1			c_1	$a_1 a_2$	$b c_2$	d	$c_1^2 c_2^2$		15	32	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
115	c_3	β_1	1	$\alpha_2^{-1}\beta_2$	β_1			c_1	a_2^2	$a_2 f$	d	c_2^4		7	40	16	$2^8 \cdot 4$	16	(21^2)	$(31)^2$
116	d	β_1	1	α_2^{-1}	β_2	(21^2)	(1^2)	c_2	$a_1 a_2$	$c_2 f$	d	$d_1^2 d_2^2$		7	40	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
117	e_1	β_1	β_2	α_2^{-1}	1	(21^2)	(1^2)	e_1	a_1^2	e_1^2	i	$a_1 c_1^2 d_2$		19	28	16	$2^8 \cdot 2$	8	(2^2)	$(31)(31)$
118	e_2	β_1	β_2	α_2	1			e_1	$a_1 a_2$	$e_1 e_2$	i	$a_2 c_1 c_2 d_1$		11	36	16	$2^8 \cdot 1$	4	(2^2)	$(31)(31)$
119	e_3	β_1	β_2	α_2^{-1}	β_1			e_1	a_2^2	e_2^2	i	$a_3 c_2^2 d_2$		3	44	16	2^8 2	8	(2^2)	$(31)(31)$
120	f	β^2	1	$\alpha_2^{-1}\beta$	1	(21^2)	(2)	c_1		b d	f	e^4		15	32	16	$2^5 \cdot 4$	16	(21^2)	$(2^2)^2$
121	g	β^2	1	α_2^{-1}	β	(21^2)	(2)	c_2		d^2	d	f^4		7	8	48	$2^6 \cdot 8$	32	(21^2)	$(31)^2$
122	h	β^2	β	α_2^{-1}	1	(21^2)	(2)	e_1		g^2	i	$b e^2 f$		11	20	32	$2^6 \cdot 2$	8	(31)	$(2^2)(31)$

FAMILY $\Gamma_7 = {}^3_1\mathbf{C}_2$. Rank 5. Class 3. u=32.

Commutators ≠ 1: $[\alpha_3,\alpha_4]=[\alpha_2,\alpha_5]=\alpha_1$, $[\alpha_4,\alpha_5]=\alpha_2$.

Squares: $\alpha_1^2=\alpha_2^2=1$, $\alpha_5^2=\alpha_3$.

Congruences (mod Z_1): $\alpha_3^2\equiv\alpha_4^2\equiv1$.

$H_2\sim(1^2)$, $H_3\sim(1)$. $I_1\sim16\Gamma_2 c_1$, $I_2\sim(1^2)$.

Z_2 is the verbally characteristic self-centralizer.

$Y_3=\widetilde{I}_1$ is the unique Γ_2 group in the first quotient signal.

H_2^* is the verbally characteristic Γ_2 group in the first subgroup signal.

STEM. Order 32.

$Z_1=H_3$ $Y_2\sim(21)$, $Y_3=I_1$.

Group Number	Group Symbol	Defining Relations α_3^2	Defining Relations α_4^2	First Quot. Sig. Γ_2	First Subgroup Signal Γ_2	First Subgroup Signal Γ_2^2	Order Structure 2	4	8	Automorphisms $t_1\cdot t_2$	t_3	Self-centralizers $8\Gamma_1$	$(8\Gamma_1)^2$
46	a_1	1	1	c_1	a_1	c_1^2	11	20		$2^5\cdot2$	16	(1^3)	$(1^3)(21)$
47	a_2	α_1	1	c_1	a_1	d^2	11	4	16	$2^6\cdot2$	32	(21)	$(1^3)^2$
48	a_3	α_1	α_1	c_1	a_2	d^2	3	12	16	$2^6\cdot2$	32	(21)	$(21)^2$

FAMILY Γ_7 (Continued).

FIRST BRANCH. Order 64.

Group Number	Group Symbol	Defining Relations α_1	α_3^2	α_4^2	Generic Invariants Y_2	Z_1	First Quot. Signal Γ_2	Γ_7^m	First Subgroup Signal Γ_2	Γ_2^2	Γ_7^n	Order Structure 2	4	8	16	Automorphisms $t_1\cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(16\Gamma_1)^2$
123	a_1	β_1	1	1	(21^2)	(1^2)	c_1	a_1^2	a_1	c_1^2	a_1^4	23	40			$2^6\cdot8$	16	(1^4)	$(1^4)(21^2)$
124	a_2	β_1	β_1	1			c_1	a_2^2	a_1	d^2	a_2^4	23	8	32		$2^7\cdot8$	32	(21^2)	$(1^4)^2$
125	a_3	β_1	β_1	β_1			c_1	a_3^2	a_2	d^2	a_3^4	7	24	32		$2^7\cdot8$	32	(21^2)	$(21^2)^2$
126	b_1	β^2	1	1	(21^2)	(2)	c_1		b	f^2	a_1^4	15	48			$2^6\cdot8$	32	(21^2)	$(21^2)^2$
127	b_2	β^2	β^2	1			c_1		b	d^2	$a_2^2a_3^2$	15	16	32		$2^7\cdot4$	32	(21^2)	$(21^2)^2$
128	c_1	β_1	1	β_2	(2^2)	(1^2)	h	a_1^2	c_1	c_1^2		15	48			$2^8\cdot2$	32	(1^4)	$(21^2)^2$
129	c_2	β_1	β_1	β_2			h	a_2a_3	c_2	d^2		7	24	32		$2^8\cdot2$	32	(21^2)	$(21^2)^2$
130	d	β^2	1	β	(2^2)	(2)	h		d	df		7	24	32		$2^8\cdot1$	16	(21^2)	$(31)^2$
131	e_1	β_1	β_2	1	(31)	(1^2)	j_1	a_1a_2	c_1	j_1^2		15	16	32		$2^7\cdot2$	16	(21^2)	$(1^4)(21^2)$
132	e_2	β_1	β_2	β_1			j_1	a_1a_3	c_2	j_1^2		7	24	32		$2^7\cdot2$	16	(21^2)	$(21^2)(21^2)$
133	f	β^2	β	1	(31)	(2)	j_1		d	k^2		7	8	16	32	$2^6\cdot2$	16	(31)	$(21^2)(31)$

FAMILY $\Gamma_8 = {}^2_1\mathbf{D} = \Lambda_{(3)}$. Rank 5. Class 4. u=32.

Commutators ≠ 1: $[\alpha_2, \alpha_5] = \alpha_1$, $[\alpha_3, \alpha_5] = \alpha_2$, $[\alpha_4, \alpha_5] = \alpha_3$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_1$, $\alpha_3^2 = \alpha_2^{-1}$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_3^{-1}$, $\alpha_5^2 \equiv 1$.

$H_2 \sim (3)$, $H_3 \sim (2)$, $H_4 \sim (1)$. $I_1 \sim 16\Gamma_3 a_1$, $I_2 \sim 8\Gamma_2 a_1$, $I_3 \sim (1^2)$.

$Y_4 = \tilde{I}_1$ is the unique Γ_3 group in the first quotient signal.

STEM. Order 32.

$Z_1 = H_4$, $Z_2 = H_3$, $Z_3 = H_2$. $Y_2 = I_3$, $Y_3 = I_2$, $Y_4 = I_1$.

Group		Defining Relations		First Quot. Sig.	First Subgroup Signal		Order Structure				Automorphisms	
Number	Symbol	α_4^2	α_5^2	Γ_3	Γ_1	Γ_3^2	2	4	8	16	$t_1 \cdot t_2$	t_3
49	a_1	α_3^{-1}	1	a_1	(4)	a_1^2	17	2	4	8	$2^6 \cdot 2$	32
50	a_2	α_3^3	1	a_1	(4)	$a_1 a_3$	9	10	4	8	$2^6 \cdot 1$	16
51	a_3	α_3^{-1}	α_1	a_1	(4)	a_3^2	1	18	4	8	$2^6 \cdot 2$	32

FAMILY Γ_8 (Continued).

FIRST BRANCH. Order 64.

$Z_2 \sim (21)$, $Z_3 \sim (31)$.

Group		Defining Relations			Generic Invariants		First Quot. Signal		First Subgroup Signal			Order Structure				Automorphisms		Y_3
Number	Symbol	α_1	α_4^2	α_5^2	Y_2	Z_1	Γ_3	Γ_8^m	Γ_1	Γ_3^2	Γ_8^n	2	4	8	16	$t_1 \cdot t_2$	t_3	$16\Gamma_2$
134	a_1	β_1	α_3^{-1}	1	(1^3)	(1^2)	a_1	a_1^2	(41)	a_1^2	a_1^4	35	4	8	16	$2^7 \cdot 8$	32	a_1
135	a_2	β_1	α_3^3	1			a_1	a_2^2	(41)	$a_1 a_3$	a_2^4	19	20	8	16	$2^7 \cdot 4$	16	a_1
136	a_3	β_1	α_3^{-1}	β_1			a_1	a_3^2	(41)	a_3^2	a_3^4	3	36	8	16	$2^7 \cdot 8$	32	a_1
137	b	β^2	α_3^{-1}	1	(1^3)	(2)	a_1		(41)	b^2	$a_1 a_2^2 a_3$	19	20	8	16	$2^7 \cdot 2$	32	a_1
138	c_1	β_1	$\alpha_3^{-1}\beta_2$	1	(21)	(1^2)	c_1	$a_1 a_2$	(41)	$a_1 d_1$		19	20	8	16	$2^8 \cdot 1$	16	c_1
139	c_2	β_1	$\alpha_3^{-1}\beta_2$	β_1			c_1	$a_2 a_3$	(41)	$a_3 d_1$		3	36	8	16	$2^8 \cdot 1$	16	c_1
140	d_1	β_1	α_3^{-1}	β_2	(21)	(1^2)	d_1	$a_1 a_3$	(41)	d_1^2		3	36	8	16	$2^8 \cdot 2$	32	c_2
141	d_2	β_1	α_3^3	β_2			d_1	a_2^2	(41)	d_1^2		3	36	8	16	$2^8 \cdot 2$	32	c_2
142	e	β^2	$\alpha_3^{-1}\beta$	1	(21)	(2)	c_1		(41)	bf		11	12	24	16	$2^7 \cdot 1$	16	c_1
143	f	β^2	α_3^{-1}	β	(21)	(2)	d_1		(41)	f^2		3	4	40	16	$2^7 \cdot 2$	32	c_2

FAMILY $\Gamma_9 = {}^3_3\mathbf{B}$. Rank 6. Class 2. u = 168.

Commutators ≠ 1: $[\alpha_4, \alpha_5] = \alpha_1$, $[\alpha_4, \alpha_6] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$.

$H_2 \sim (1^3)$. $I_1 \sim (1^3)$.

STEM. Order 64.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

n is the number of self-conjugate subgroups of order 8.

Group		Defining Relations			First Quotient Signal	First Subgroup Signal	Order Structure		Automorphisms	Self-centralizers	n
Number	Symbol	α_4^2	α_5^2	α_6^2	Γ_4^7	Γ_2^7	2	4	$t_1 \cdot t_2$	$(16\Gamma_1)^7$	
144	a_1	1	1	1	$a_1^3 a_2 b_1^3$	$a_1^3 c_1^3 h$	31	32	$2^9 \cdot 6$	$(1^4)^3(21^2)(21^2)^3$	13
145	a_2	α_1	α_1	$\alpha_2\alpha_3$	$a_1 a_3^3 b_2^3$	$a_2 c_2^3 h^3$	7	56	$2^9 \cdot 6$	$(21^2)(21^2)^3(21^2)^3$	13
146	b_1	1	1	$\alpha_2\alpha_3$	$a_1 b_1 b_1^2 c_1^2 c_2$	$a_1 c_1^2 c_1^2 c_2 h$	23	40	$2^9 \cdot 2$	$(1^4)^2(21^2)(21^2)(21^2)(21^2)^2$	9
147	b_2	α_1	α_1	1	$a_1 b_2^3 c_2^3$	$a_2 c_1^3 h^3$	15	48	$2^9 \cdot 6$	$(1^4)(21^2)^3(21^2)^3$	9
148	b_3	$\alpha_1\alpha_2$	$\alpha_1\alpha_3$	1	$a_2 b_2^3 c_1^3$	$c_1^3 c_2^3 h$	15	48	$2^9 \cdot 6$	$(1^4)(21^2)^3(21^2)^3$	9
149	b_4	α_2	$\alpha_1\alpha_3$	α_3	$a_3 b_1 b_2^2 c_1 c_3^2$	$c_2 c_2 c_2^2 h h^2$	7	56	$2^9 \cdot 2$	$(21^2)(21^2)(21^2)(21^2)^2(21^2)^2$	9
150	c	α_1	α_3	α_2	$b_1^3 b_2^3 d$	$c_1^3 c_2^3 h$	15	48	$2^9 \cdot 3$	$(1^4)(21^2)^3(21^2)^3$	9
151	d_1	1	α_2	α_1	$b_1 c_1 c_1^2 c_2 d^2$	$c_1 c_1^2 c_2 h h^2$	15	48	$2^9 \cdot 2$	$(1^4)(21^2)(21^2)(21^2)^2(21^2)^2$	5
152	d_2	α_2	$\alpha_2\alpha_3$	α_1	$b_2 c_1 c_2 c_3^2 d^2$	$c_2 c_2 h h^2 h^2$	7	56	$2^9 \cdot 2$	$(21^2)(21^2)(21^2)(21^2)^2(21^2)^2$	5
153	e	$\alpha_1\alpha_3$	$\alpha_2\alpha_3$	$\alpha_1\alpha_2$	d^7	h^7	7	56	$2^9 \cdot 21$	$(21^2)^7$	1

FAMILY $\Gamma_{10} = {}^{4}_{2}\mathbf{B}_1$. Rank 6. Class 2. u=72.

Commutators $\neq 1$: $[\alpha_3, \alpha_4] = \alpha_1$, $[\alpha_5, \alpha_6] = \alpha_2$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$.

Congruences (mod Z_1): $\alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$.

$H_2 \sim (1^2)$. $I_1 \sim (1^4)$.

Each group is the central product of two Γ_2 groups, in one and only one way.

STEM. Order 64.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

n is the number of self-conjugate subgroups of order 4.

Group Number	Group Symbol	Defining Relations α_3^2	α_4^2	α_5^2	α_6^2	First Quot. Signal Γ_2^2	Γ_5	First Subgroup Signal Γ_2^6	Γ_4^9	Order Structure 2	4	Automorphisms $t_1 \cdot t_2$	Self-centralizers $(16\Gamma_1)^9$	Central Factors $\{16\Gamma_2 * 16\Gamma_2\}$	n
154	a_1	α_1	1	α_2	1	a_1^2	a_1	$a_1^4 e_1^2$	$a_1^4 a_2 b_1^4$	35	28	$2^8 \cdot 8$	$(1^4)^4(21^2)^4(2^2)$	$\{a_1 * a_1\}$	13
155	a_2	α_1	α_1	α_2	1	$a_1 a_2$	a_2	$a_2^2 e_1^3 e_2$	$a_3^3 b_2^6$	11	52	$2^8 \cdot 12$	$(21^2)^6(2^2)^3$	$\{a_1 * a_2\}$	13
156	a_3	α_1	α_1	α_2	α_2	a_2^2	a_1	e_2^6	a_3^9	3	60	$2^8 \cdot 72$	$(2^2)^9$	$\{a_2 * a_2\}$	13
157	b_1	α_1	1	1	α_1	$a_1 b$	a_1	$a_1 b\ c_1^2 e_1 e_1$	$a_1^2 b_1 b_1^2 b_2 c_1^2 c_2$	27	36	$2^8 \cdot 2$	$(1^4)^2(21^2)(21^2)(21^2)^2(21^2)^2(2^2)$	$\{a_1 * c_1\}$	9
158	b_2	α_1	α_1	α_2	α_1	$a_1 b$	a_1	$b^2 e_1^3 e_2$	$a_2^3 b_1^6$	27	36	$2^8 \cdot 12$	$(21^2)^6(2^2)^3$	$\{a_2 * c_2\}$	9
159	b_3	α_1	α_1	1	α_1	$a_1 b$	a_2	$a_2 b\ e_1^3 e_2$	$b_1^3 b_2^3 c_2^3$	19	44	$2^8 \cdot 6$	$(21^2)^3(21^2)^3(2^2)^3$	$\{a_2 * c_1\}$	9
160	b_4	α_1	1	α_2	α_1	$a_1 b$	a_2	$b^2 c_2^2 e_1 e_1$	$a_3 b_1^2 b_2^2 c_1^4$	19	44	$2^8 \cdot 4$	$(21^2)^2(21^2)^2(21^2)^4(2^2)$	$\{a_1 * c_2\}$	9
161	b_5	α_1	1	α_2	$\alpha_1\alpha_2$	$a_2 b$	a_1	$c_2^2 e_1 e_1^2 e_2$	$a_3 b_2^2 b_2^4 c_3^2$	11	52	$2^8 \cdot 4$	$(21^2)^2(21^2)^4(2^2)(2^2)^2$	$\{a_1 * c_2\}$	9
162	b_6	α_1	α_1	α_2	$\alpha_1\alpha_2$	$a_2 b$	a_2	$e_2 e_2^2 e_2^3$	$a_3^3 c_3^6$	3	60	$2^8 \cdot 12$	$(2^2)^3(2^2)^6$	$\{a_2 * c_2\}$	9
163	c_1	1	α_2	1	α_1	b^2	a_1	$c_1^2 e_1^2 f^2$	$a_1 b_2^2 c_1 c_1^2 c_2 d^2$	19	44	$2^8 \cdot 2$	$(1^4)(21^2)(21^2)^2(21^2)^2(2^2)(2^2)^2$	$\{c_1 * c_1\}$	5
164	c_2	α_1	$\alpha_1\alpha_2$	1	α_1	bb	a_1	$c_2 e_1 e_1 e_1^2 f$	$b_1 b_1 b_1^2 c_1^2 c_3 d^2$	19	44	$2^8 \cdot 2$	$(21^2)(21^2)(21^2)^2(21^2)^2(2^2)(2^2)^2$	$\{c_1 * c_2\}$	5
165	c_3	α_1	$\alpha_1\alpha_2$	α_2	$\alpha_1\alpha_2$	b^2	a_1	$e_1^4 e_2^2$	$a_2 c_1^4 c_2^4$	19	44	$2^8 \cdot 8$	$(21^2)^4(2^2)(2^2)^4$	$\{c_2 * c_2\}$	5
166	c_4	α_1	α_2	α_2	α_1	b^2	a_1	$e_1^2 f^4$	$a_3 b_1^4 c_2^4$	19	44	$2^8 \cdot 8$	$(21^2)^4(2^2)(2^2)^4$	$\{c_2 * c_2\}$	5
167	c_5	α_1	α_2	1	α_1	bb	a_2	$c_2 e_1 e_2 f f^2$	$b_2 b_2 c_1^2 c_3 c_3^2 d^2$	11	52	$2^8 \cdot 2$	$(21^2)(21^2)(21^2)^2(2^2)(2^2)^2(2^2)^2$	$\{c_1 * c_2\}$	5
168	c_6	α_1	$\alpha_1\alpha_2$	α_2	α_1	bb	a_2	$e_1 e_2 e_2^2 f^2$	$a_3 b_2^2 c_2^2 d^4$	11	52	$2^8 \cdot 4$	$(21^2)^2(2^2)(2^2)^2(2^2)^4$	$\{c_2 * c_2\}$	5

FAMILY $\Gamma_{11} = {}^{4}_{2}\mathbf{B}_2$. Rank 6. Class 2. $u = 96$.

Commutators $\neq 1$: $[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1$, $[\alpha_5, \alpha_6] = \alpha_2$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$.

Congruences (mod Z_1): $\alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$.

$H_2 \sim (1^2)$. $I_1 \sim (1^4)$.

The meet of the three Γ_2 groups of the first subgroup signal is the verbally characteristic self-centralizer.

STEM. Order 64.

$Z_1 = H_2$. $Y_2 = I_1$. $t_2 = t_3$.

n is the number of self-conjugate subgroups of order 4.

Group Number	Group Symbol	Defining Relations α_3^2	α_4^2	α_5^2	α_6^2	First Quot. Signal Γ_2	Γ_5^2	First Subgroup Signal Γ_2^3	Γ_4^{12}	Order Structure 2	4	Automorphisms $t_1 \cdot t_2$	Self-centralizers $16\Gamma_1$	$(16\Gamma_1)^6$	n
169	a_1	1	1	α_2	1	a_1	a_1^2	$a_1^2 c_1$	$a_1^4 b_1^4 c_1^4$	31	32	$2^8 \cdot 8$	(1^4)	$(1^4)^2(21^2)^2(21^2)^2$	7
170	a_2	1	α_1	1	1	a_1	a_1^2	$a_1 b f$	$a_1^4 a_2^2 b_1^4 c_2^2$	31	32	$2^8 \cdot 8$	(21^2)	$(1^4)^2(21^2)^2(2^2)^2$	7
171	a_3	1	α_1	α_2	1	a_1	$a_1 a_2$	$b^2 c_2$	$b_1^4 b_1^4 b_2^4$	23	40	$2^8 \cdot 8$	(21^2)	$(21^2)^2(21^2)^4$	7
172	a_4	1	α_1	α_1	1	a_1	a_2^2	$a_2 b f$	$a_3^2 b_2^4 c_1^4 c_2^2$	15	48	$2^8 \cdot 8$	(21^2)	$(21^2)^2(21^2)^2(2^2)^2$	7
173	a_5	1	1	α_2	α_2	a_2	a_1^2	c_1^3	b_2^{12}	15	48	$2^8 \cdot 24$	(1^4)	$(21^2)^6$	7
174	a_6	1	α_1	α_2	α_2	a_2	$a_1 a_2$	$c_2 f^2$	$a_3^4 b_2^4 c_3^4$	7	56	$2^8 \cdot 8$	(21^2)	$(21^2)^2(2^2)^4$	7
175	b_1	1	α_2	1	1	b	$a_1 a_1$	$c_1 e_1^2$	$a_1^2 b_1 b_1^2 b_2 c_1^2 c_2^2 d^2$	23	40	$2^8 \cdot 2$	(21^2)	$(1^4)(21^2)(21^2)^2(2^2)^2$	3
176	b_2	α_1	α_2	1	α_2	b	a_1^2	$e_1^2 f$	$a_2 b_1^2 b_1^4 c_1^4 c_3$	23	40	$2^8 \cdot 4$	(2^2)	$(21^2)(21^2)^4(2^2)$	3
177	b_3	1	α_2	α_1	1	b	$a_1 a_2$	$c_2 e_1^2$	$b_1 b_2 b_2^2 c_1^2 c_1^2 c_3^2 d^2$	15	48	$2^8 \cdot 2$	(21^2)	$(21^2)(21^2)(21^2)^2(2^2)^2$	3
178	b_4	α_1	α_2	1	1	b	$a_1 a_2$	$e_1 e_2 f$	$a_3 b_1 b_2 b_2^2 c_1^2 c_2 c_2^2 d^2$	15	48	$2^8 \cdot 2$	(2^2)	$(21^2)(21^2)^2(2^2)(2^2)^2$	3
179	b_5	α_1	α_2	1	$\alpha_1 \alpha_2$	b	a_2^2	$e_2^2 f$	$a_3 b_2^2 c_3 c_3^4 d^4$	7	56	$2^8 \cdot 4$	(2^2)	$(21^2)(2^2)(2^2)^4$	3

FAMILY $\Gamma_{12} = {}^4_2\mathbf{B}_3$. Rank 6. Class 2. u=96.

Commutators ≠ 1: $[\alpha_4,\alpha_5]=[\alpha_3,\alpha_6]=\alpha_1$, $[\alpha_5,\alpha_6]=\alpha_2$.

Squares: $\alpha_1^2=1$, $\alpha_2^2=\alpha_1$, $\alpha_5^2=\alpha_3$, $\alpha_6^2=\alpha_4$.

Congruences (mod Z_1): $\alpha_3^2\equiv\alpha_4^2\equiv 1$.

$H_2\sim(2)$. $I_1\sim(2^2)$.

The meet of the three Γ_2 groups of the first subgroup signal is the verbally characteristic self-centralizer.

STEM. Order 64.

$Z_1=H_2$. $Y_2=I_1$.

Group Number	Group Symbol	Defining Relations α_3^2	Defining Relations α_4^2	First Quot. Sig. Γ_2	First Subgroup Signal Γ_2^3	Order Structure 2	4	8	16	Automorphisms $t_1 \cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(16\Gamma_1)^6$
180	a_1	α_1	1	h	df^2	7	40	16		$2^8 \cdot 2$	32	(21^2)	$(2^2)^4(31)^2$
181	a_2	α_1	α_1	h	d^3	7	8	48		$2^8 \cdot 6$	96	(21^2)	$(31)^6$
182	b	1	α_2	i	ik^2	3	12	16	32	$2^6 \cdot 2$	8	(31)	$(2^2)(31)(4)^4$

FAMILY $\Gamma_{13} = {}^4_2\mathbf{B}_4$. Rank 6. Class 2. u=360.

Commutators ≠ 1: $[\alpha_3,\alpha_5]=\alpha_1$, $[\alpha_4,\alpha_5]=[\alpha_3,\alpha_6]=\alpha_1\alpha_2$, $[\alpha_4,\alpha_6]=\alpha_2$.

Squares: $\alpha_1^2=\alpha_2^2=1$.

Congruences (mod Z_1): $\alpha_3^2\equiv\alpha_4^2\equiv\alpha_5^2\equiv\alpha_6^2\equiv 1$.

$H_2\sim(1^2)$. $I_1\sim(1^4)$.

STEM. Order 64.

$Z_1=H_2$. $Y_2=I_1$. $t_2=t_3$.

Relation: $\alpha_3^2=\alpha_1$.

There is only one self-conjugate subgroup of order 4.

Group Number	Group Symbol	Defining Relations α_4^2	α_5^2	α_6^2	First Quot. Sig. Γ_5^3	First Subgroup Signal Γ_4^{15}	Order Structure 2	4	Automorphisms $t_1 \cdot t_2$	Self-centralizers $(16\Gamma_1)^5$
183	a_1	α_2	1	1	a_1^3	$a_1^6c_2^9$	27	36	$2^8 \cdot 36$	$(1^4)^2(2^2)^3$
184	a_2	$\alpha_1\alpha_2$	1	1	a_1^3	$a_1^3a_2b_1^6c_1^3d^2$	27	36	$2^8 \cdot 6$	$(1^4)(21^2)^3(2^2)$
185	a_3	α_2	$\alpha_1\alpha_2$	1	$a_1^2a_2$	$b_1^4b_2^4c_1^4c_2^2c_3$	19	44	$2^8 \cdot 4$	$(21^2)^4(2^2)$
186	a_4	α_2	1	α_1	$a_1a_2^2$	$a_3^2b_2^4c_1^2c_2c_3^2d^4$	11	52	$2^8 \cdot 4$	$(21^2)^2(2^2)(2^2)^2$
187	a_5	α_2	α_2	$\alpha_1\alpha_2$	a_2^3	c_3^{15}	3	60	$2^8 \cdot 60$	$(2^2)^5$

FAMILY $\Gamma_{14} = {}^{3}_{2}\mathbf{C}_{1} = \Lambda_{(21)}$. Rank 6. Class 3. u=64.

Commutators≠1: $[\alpha_3, \alpha_6] = \alpha_1$, $[\alpha_4, \alpha_6] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_3^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_6^2 \equiv 1$, $\alpha_5^2 \equiv \alpha_3$.

$H_2 \sim (21)$, $H_3 \sim (1)$. $I_1 \sim 16\Gamma_2 a_1$, $I_2 \sim (1^2)$.

Y_3 is the unique Γ_4 group in the first quotient signal.

STEM. Order 64.

$Z_1 \sim (1^2)$. $Y_2 \sim (1^3)$.

Group Number	Group Symbol	Defining Relations α_4^2	α_5^2	α_6^2	First Quot. Signal Γ_3^2	Γ_4	First Subgroup Signal Γ_1	Γ_3^4	Γ_4^2	Order Structure 2	4	8	Automorphisms $t_1 \cdot t_2$	t_3	Z_2 $16\Gamma_1$
188	a_1	α_2	α_3^{-1}	1	a_1^2	a_2	(32)	a_1^4	a_2^2	35	12	16	$2^9 \cdot 8$	64	(2^2)
189	a_2	α_2	α_3	1	a_2^2	a_2	(32)	a_2^4	$a_2 a_3$	19	28	16	$2^9 \cdot 4$	32	(2^2)
190	a_3	α_2	α_3^{-1}	α_2	$a_1 a_3$	a_3	(32)	d_1^4	a_3^2	3	44	16	$2^8 \cdot 8$	32	(2^2)
191	a_4	α_2	α_3	α_2	a_2^2	a_3	(32)	d_2^4	a_3^2	3	44	16	$2^8 \cdot 8$	32	(2^2)
192	a_5	α_2	α_3^{-1}	α_1	a_3^2	a_2	(32)	a_3^4	a_3^2	3	44	16	$2^9 \cdot 8$	64	(2^2)
193	b_1	α_2	$\alpha_2\alpha_3^{-1}$	1	$a_1 a_2$	c_2	(32)	c_1^4	$a_2 a_3$	19	28	16	$2^8 \cdot 4$	16	(2^2)
194	b_2	α_2	$\alpha_2\alpha_3^{-1}$	α_1	$a_2 a_3$	c_2	(32)	c_2^4	$a_3 a_3$	3	44	16	$2^8 \cdot 4$	16	(2^2)
195	c_1	1	α_3^{-1}	1	$a_1 b$	b_1	(31^2)	$a_1 c_1^2 d_1$	b_1^2	23	24	16	$2^8 \cdot 2$	8	(21^2)
196	c_2	1	α_3	1	$a_2 b$	b_1	(31^2)	$a_2 c_1 c_2 d_2$	$b_1 b_2$	15	32	16	$2^8 \cdot 1$	4	(21^2)
197	c_3	1	α_3^{-1}	α_1	$a_3 b$	b_1	(31^2)	$a_3 c_2^2 d_1$	b_2^2	7	40	16	$2^8 \cdot 2$	8	(21^2)
198	d_1	$\alpha_1\alpha_2$	α_3^{-1}	1	b^2	a_2	(32)	$a_1 a_2^2 a_3$	c_2^2	19	28	16	$2^9 \cdot 2$	16	(2^2)
199	d_2	$\alpha_1\alpha_2$	α_3^{-1}	α_2	b^2	a_3	(32)	$d_1^2 d_2^2$	c_3^2	3	44	16	$2^8 \cdot 4$	16	(2^2)
200	e	$\alpha_1\alpha_2$	$\alpha_2\alpha_3^{-1}$	1	b^2	c_2	(32)	$c_1^2 c_2^2$	$c_2 c_3$	11	36	16	$2^8 \cdot 2$	8	(2^2)

FAMILY $\Gamma_{15} = {}^3_2\mathbf{C}_2$. Rank 6. Class 3. $u = 16$.

Commutators $\neq 1$: $[\alpha_3, \alpha_6] = \alpha_1$, $[\alpha_4, \alpha_5] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_3^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_6^2 \equiv 1$, $\alpha_5^2 \equiv \alpha_3$.

$H_2 \sim (21)$, $H_3 \sim (1)$. $I_1 \sim 16\Gamma_2 a_1$, $I_2 \sim (1^2)$.

Z_2 is the verbally characteristic self-centralizer.

Y_3 is the unique Γ_4 group in the first quotient signal.

H_2^* is the Γ_2' group in the first subgroup signal.

STEM. Order 64.

$Z_1 \sim (1^2)$. $Y_2 \sim (1^3)$.

Group		Defining Relations			First Quotient Signal			First Subgroup Signal				Order Structure			Automorphisms		Self-centralizers	
Number	Symbol	α_4^2	α_5^2	α_6^2	Γ_3	Γ_4	Γ_6	Γ_2	Γ_2'	Γ_3^4	Γ_4	2	4	8	$t_1 \cdot t_2$	t_3	$16\Gamma_1$	$(16\Gamma_1)^2$
201	a_1	1	α_3^{-1}	1	a_1	a_1	a_1	a_1	j_1	$a_1^2 c_1^2$	b_1	31	16	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
202	a_2	1	α_3	1	a_2	a_1	a_1	a_1	j_1	$a_2^2 c_1^2$	b_2	23	24	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
203	a_3	1	α_3	α_1	a_2	a_1	a_2	a_2	j_1	$a_2^2 c_2^2$	b_1	15	32	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
204	a_4	1	α_3^{-1}	α_1	a_3	a_1	a_2	a_2	j_1	$a_3^2 c_2^2$	b_2	7	40	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
205	b_1	α_2	α_3^{-1}	1	a_1	b_1	a_1	e_1	j_2	$a_1^2 c_1^2$	a_2	27	20	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
206	b_2	α_2	α_3	α_1	a_2	b_1	a_1	e_2	j_2	$a_2^2 c_1^2$	a_2	19	28	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
207	b_3	α_2	$\alpha_2\alpha_3^{-1}$	1	a_1	b_2	a_2	e_1	j_2	$c_1^2 d_1^2$	a_3	11	36	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
208	b_4	α_2	α_3	1	a_2	b_1	a_2	e_1	j_2	$a_2^2 c_2^2$	a_3	11	36	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
209	b_5	α_2	$\alpha_2\alpha_3$	1	a_2	b_2	a_1	e_1	j_2	$c_1^2 d_2^2$	a_3	11	36	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
210	b_6	α_2	$\alpha_2\alpha_3$	α_1	a_2	b_2	a_2	e_2	j_2	$c_2^2 d_2^2$	a_3	3	44	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
211	b_7	α_2	α_3^{-1}	α_1	a_3	b_1	a_2	e_2	j_2	$a_3^2 c_2^2$	a_3	3	44	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
212	b_8	α_2	$\alpha_2\alpha_3^{-1}$	α_1	a_3	b_2	a_1	e_2	j_2	$c_2^2 d_1^2$	a_3	3	44	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
213	c_1	1	α_3^{-1}	α_2	a_1	c_1	a_2	c_2	j_1	$c_1^2 d_1^2$	b_1	15	32	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
214	c_2	1	α_3	$\alpha_1\alpha_2$	a_2	c_1	a_1	c_2	j_1	$c_1^2 d_2^2$	b_1	15	32	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
215	c_3	1	α_3	α_2	a_2	c_1	a_2	c_2	j_1	$c_2^2 d_2^2$	b_2	7	40	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
216	c_4	1	α_3^{-1}	$\alpha_1\alpha_2$	a_3	c_1	a_1	c_2	j_1	$c_2^2 d_1^2$	b_2	7	40	16	$2^8 \cdot 2$	8	(21^2)	$(31)^2$
217	d_1	α_1	α_3^{-1}	1	b	a_1	a_1	b	j_1	$a_1 a_2 c_1 c_2$	b_1	23	24	16	$2^8 \cdot 1$	4	(21^2)	$(31)(31)$
218	d_2	α_1	α_3^{-1}	α_1	b	a_1	a_2	b	j_1	$a_2 a_3 c_1 c_2$	b_2	15	32	16	$2^8 \cdot 1$	4	(21^2)	$(31)(31)$
219	e_1	$\alpha_1\alpha_2$	α_3^{-1}	1	b	b_1	a_1	e_1	j_2	$a_1 a_2 c_1 c_2$	c_2	19	28	16	$2^8 \cdot 1$	4	(2^2)	$(31)(31)$
220	e_2	$\alpha_1\alpha_2$	α_3^{-1}	α_1	b	b_1	a_2	e_2	j_2	$a_2 a_3 c_1 c_2$	c_2	11	36	16	$2^8 \cdot 1$	4	(2^2)	$(31)(31)$
221	e_3	$\alpha_1\alpha_2$	$\alpha_2\alpha_3^{-1}$	1	b	b_2	a_1	e_1	j_2	$c_1 c_1 d_1 d_2$	c_3	11	36	16	$2^8 \cdot 1$	4	(2^2)	$(31)(31)$
222	e_4	$\alpha_1\alpha_2$	$\alpha_2\alpha_3^{-1}$	α_1	b	b_2	a_2	e_2	j_2	$c_2 c_2 d_1 d_2$	c_3	3	44	16	$2^8 \cdot 1$	4	(2^2)	$(31)(31)$
223	f_1	α_1	α_3^{-1}	$\alpha_1\alpha_2$	b	c_1	a_1	f	j_1	$c_1 c_1 d_1 d_2$	b_1	15	32	16	$2^8 \cdot 1$	4	(21^2)	$(31)(31)$
224	f_2	α_1	α_3^{-1}	α_2	b	c_1	a_2	f	j_1	$c_2 c_2 d_1 d_2$	b_2	7	40	16	$2^8 \cdot 1$	4	(21^2)	$(31)(31)$

FAMILY $\Gamma_{16} = {}^{3}_{2}\mathbf{C}_3$. Rank 6. Class 3. $u = 64$.

Commutators $\neq 1$: $[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1$, $[\alpha_4, \alpha_6] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_3^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_6^2 \equiv 1$, $\alpha_5^2 \equiv \alpha_3$.

$H_2 \sim (21)$, $H_3 \sim (1)$. $I_1 \sim 16\Gamma_2 a_1$, $I_2 \sim (1^2)$.

Z_2 is the verbally characteristic self-centralizer.

Y_3 is the unique Γ_4 group in the first quotient signal.

H_2^* is the unique Γ_2 group in the first subgroup signal.

STEM. Order 64.

$Z_1 \sim (1^2)$. $Y_2 \sim (1^3)$.

Group Number	Group Symbol	Defining Relations α_4^2	α_5^2	α_6^2	First Quotient Signal Γ_4	Γ_6^2	First Subgroup Signal Γ_2	Γ_3^4	Γ_4^2	Order Structure 2	4	8	Automorphisms $t_1 \cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(16\Gamma_1)^2$
225	a_1	α_2	α_3^{-1}	1	a_2	a_1^2	i	$a_1^2 a_2^2$	$a_2 c_2$	27	20	16	$2^9 \cdot 2$	16	(2^2)	$(31)^2$
226	a_2	α_2	α_3^{-1}	α_1	a_2	a_2^2	i	$a_2^2 a_3^2$	$a_3 c_2$	11	36	16	$2^9 \cdot 2$	16	(2^2)	$(31)^2$
227	a_3	α_2	α_3^{-1}	α_2	a_3	$a_1 a_2$	i	$d_1^2 d_2^2$	$a_3 c_3$	3	44	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
228	b_1	1	α_3^{-1}	1	b_1	a_1^2	d	$a_1 c_1^2 d_2$	b_1^2	23	24	16	$2^8 \cdot 2$	8	(21^2)	$(31)(31)$
229	b_2	1	α_3^{-1}	α_2	b_1	$a_1 a_2$	d	$a_2 c_1 c_2 d_1$	$b_1 b_2$	15	32	16	$2^8 \cdot 1$	4	(21^2)	$(31)(31)$
230	b_3	1	α_3^{-1}	α_1	b_1	a_2^2	d	$a_3 c_2^2 d_2$	b_2^2	7	40	16	$2^8 \cdot 2$	8	(21^2)	$(31)(31)$
231	c_1	α_2	$\alpha_2 \alpha_3^{-1}$	1	c_2	a_1^2	i	c_1^4	$a_2 c_3$	19	28	16	$2^8 \cdot 4$	16	(2^2)	$(31)^2$
232	c_2	α_2	$\alpha_2 \alpha_3^{-1}$	α_2	c_2	$a_1 a_2$	i	$c_1^2 c_2^2$	$a_3 c_2$	11	36	16	$2^8 \cdot 2$	8	(2^2)	$(31)^2$
233	c_3	α_2	$\alpha_2 \alpha_3^{-1}$	α_1	c_2	a_2^2	i	c_2^4	$a_3 c_3$	3	44	16	$2^8 \cdot 4$	16	(2^2)	$(31)^2$

FAMILY $\Gamma_{17} = {}^{3}_{2}\mathbf{C}_4$. Rank 6. Class 3. u=32.

Commutators ≠1: $[\alpha_3, \alpha_5] = \alpha_1$, $[\alpha_3, \alpha_6] = \alpha_1\alpha_2$, $[\alpha_4, \alpha_5] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_3^2 = \alpha_1$, $\alpha_6^2 = \alpha_4$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_5^2 \equiv 1$.

$H_2 \sim (21)$, $H_3 \sim (1^2)$. $I_1 \sim 16\Gamma_2 c_1$, $I_2 \sim (1^2)$.

$Z_2 = H_2^*$ is the unique self-centralizer.

STEM. Order 64.

$Z_1 = H_3$. $Y_2 \sim (21)$, $Y_3 = I_1$.

Group Number	Group Symbol	Defining Relations α_4^2	Defining Relations α_5^2	First Quot. Signal Γ_3^2	First Quot. Signal Γ_7	First Subgroup Signal Γ_2^2	First Subgroup Signal Γ_4	Order Structure 2	Order Structure 4	Order Structure 8	Auto-morphisms $t_1 \cdot t_2$	Auto-morphisms t_3	Z_2 $16\Gamma_1$
234	a_1	α_2	1	c_1^2	a_2	j_2^2	a_2	19	12	32	$2^8 \cdot 2$	32	(2^2)
235	a_2	α_2	α_2	c_1c_2	a_3	j_2j_2	a_3	3	28	32	$2^8 \cdot 1$	16	(2^2)
236	a_3	α_2	α_1	c_2^2	a_2	j_2^2	a_3	3	28	32	$2^8 \cdot 2$	32	(2^2)
237	b_1	1	1	c_1e	a_1	hj_1	b_1	15	32	16	$2^7 \cdot 1$	8	(21^2)
238	b_2	1	α_1	c_2e	a_1	hj_1	b_2	7	40	16	$2^7 \cdot 1$	8	(21^2)
239	c_1	$\alpha_1\alpha_2$	1	e^2	a_2	i^2	c_2	11	20	32	$2^7 \cdot 2$	16	(2^2)
240	c_2	$\alpha_1\alpha_2$	α_2	e^2	a_3	i^2	c_3	3	28	32	$2^7 \cdot 2$	16	(2^2)

FAMILY $\Gamma_{18} = {}^{4}_{1}\mathbf{C}$. Rank 6. Class 3. $u=192$.

Commutators $\neq 1$: $[\alpha_4, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1$, $[\alpha_3, \alpha_6] = \alpha_2$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_3^2 \equiv \alpha_2$, $\alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$.

$H_2 \sim (2)$, $H_3 \sim (1)$. $I_1 \sim 32\Gamma_2 a_1$, $I_2 \sim (1^2)$.

$Y_3 = \widetilde{I}_1$ is the unique Γ_2 group in the first quotient signal.

H_2^* is the unique Γ_2 group in the first subgroup signal.

Each group is the central product of a Γ_2 group and a Γ_3 group in four distinct ways.

STEM. Order 64.

$Z_1 = H_3$, $Z_2 \sim 16\Gamma_2 b$. $Y_2 \sim (1^4)$, $Y_3 = I_1$. $H_2^* \sim 32\Gamma_2 g$.

Relations: $\alpha_4^2 = \alpha_5^2 = \alpha_1$.

Group		Defining Relations		First Quot. Sig.	First Subgroup Signal				Order Structure			Auto-morphisms		Self-central-izers	Central Factors
Number	Symbol	α_3^2	α_6^2	Γ_2	Γ_2	Γ_3^6	Γ_5^2	Γ_8^6	2	4	8	$t_1 \cdot t_2$	t_3	$(16\Gamma_1)^3$	$\{8\Gamma_2 * 16\Gamma_3\}^4$
241	a_1	α_2^{-1}	α_1	a_1	g	$a_1^3 b^3$	a_1^2	a_1^6	31	16	16	$2^6 \cdot 12$	48	$(31)^3$	$\{8\Gamma_2 a_1 * 16\Gamma_3 a_1\}^3 \{8\Gamma_2 a_2 * 16\Gamma_3 a_3\}$
242	a_2	α_2	1	a_1	g	$a_2^3 b^3$	$a_1 a_2$	$a_1^3 a_2^3$	23	24	16	$2^6 \cdot 6$	24	$(31)^3$	$\{8\Gamma_2 a_1 * 16\Gamma_3 a_2\}^3 \{8\Gamma_2 a_2 * 16\Gamma_3 a_2\}$
243	a_3	α_2^{-1}	1	a_1	g	$a_3^3 b^3$	a_2^2	a_2^6	15	32	16	$2^6 \cdot 12$	48	$(31)^3$	$\{8\Gamma_2 a_1 * 16\Gamma_3 a_3\}^3 \{8\Gamma_2 a_2 * 16\Gamma_3 a_1\}$

FAMILY $\Gamma_{19} = {}^{3}_{1}\mathbf{D}_1$. Rank 6. Class 4. u = 128.

Commutators ≠ 1: $[\alpha_4,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1$, $[\alpha_3,\alpha_6]=\alpha_2$, $[\alpha_5,\alpha_6]=\alpha_3$.

Squares: $\alpha_1^2=1$, $\alpha_2^2=\alpha_1$, $\alpha_3^2=\alpha_2^{-1}$.

Congruences (mod Z_1): $\alpha_4^2\equiv\alpha_6^2\equiv 1$, $\alpha_5^2\equiv\alpha_3^{-1}$.

$H_2\sim(3)$, $H_3\sim(2)$, $H_4\sim(1)$. $I_1\sim 32\Gamma_3 a_1$, $I_2\sim 8\Gamma_2 a_1$, $I_3\sim(1^2)$.

Z_3 is the verbally characteristic self-centralizer.

$Y_4=\widetilde{I}_1$ is the unique Γ_3 group in the first quotient signal.

H_2^* is the unique Γ_2 group in the first subgroup signal.

STEM. Order 64.

$Z_1=H_4$, $Z_2\sim(21)$, $Z_3\sim(31)$. $Y_2\sim(1^3)$, $Y_3\sim 16\Gamma_2 a_1$, $Y_4=I_1$. $H_2^*\sim 32\Gamma_2 k$.

Relations: $\alpha_4^2=1$, $\alpha_5^2=\alpha_3^{-1}$.

Group Number	Group Symbol	Defining Relation	First Quot. Sig.	First Subgroup Signal			Order Structure				Automorphisms		Self-centralizers	
		α_6^2	Γ_3	Γ_2	Γ_3^2	Γ_8^4	2	4	8	16	$t_1\cdot t_2$	t_3	$16\Gamma_1$	$(16\Gamma_1)^2$
244	a_1	1	a_1	k	$a_1 b$	$a_1^2 a_2^2$	27	12	8	16	$2^7\cdot 2$	32	(31)	$(4)^2$
245	a_2	α_1	a_1	k	$a_3 b$	$a_2^2 a_3^2$	11	28	8	16	$2^7\cdot 2$	32	(31)	$(4)^2$

FAMILY $\Gamma_{20} = {}^{3}_{1}\mathbf{D}_2$. Rank 6. Class 4. u = 128.

Commutators ≠ 1: $[\alpha_4,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1$, $[\alpha_3,\alpha_6]=\alpha_2$, $[\alpha_5,\alpha_6]=\alpha_3$.

Squares: $\alpha_1^2=1$, $\alpha_2^2=\alpha_1$, $\alpha_3^2=\alpha_2$, $\alpha_6^2=\alpha_4$.

Congruences (mod Z_1): $\alpha_4^2\equiv 1$, $\alpha_5^2\equiv\alpha_3$.

$H_2\sim(3)$, $H_3\sim(2)$, $H_4\sim(1)$. $I_1\sim 32\Gamma_3 d_2$, $I_2\sim 8\Gamma_2 a_1$, $I_3\sim(1^2)$.

Z_3 is the verbally characteristic self-centralizer.

$Y_4=\widetilde{I}_1$ is the unique Γ_3 group in the first quotient signal.

H_2^* is the unique Γ_2 group in the first subgroup signal.

STEM. Order 64.

$Z_1=H_4$, $Z_2\sim(21)$, $Z_3\sim(31)$. $Y_2\sim(21)$, $Y_3\sim 16\Gamma_2 c_2$, $Y_4=I_1$. $H_2^*\sim 32\Gamma_2 k$.

Relations: $\alpha_4^2=1$, $\alpha_5^2=\alpha_3$.

Group Number	Group Symbol	First Quot. Sig.	First Subgroup Signal		Order Structure				Automorphisms		Self-centralizers	
		Γ_3	Γ_2	Γ_3^2	2	4	8	16	$t_1\cdot t_2$	t_3	$16\Gamma_1$	$(16\Gamma_1)^2$
246	a	d_2	k	$d_2 f$	3	20	24	16	$2^7\cdot 1$	32	(31)	$(4)^2$

FAMILY $\Gamma_{21} = {}^3_1\mathbf{D}_3$. Rank 6. Class 4. u = 64.

Commutators ≠ 1: $[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1$, $[\alpha_3, \alpha_6] = [\alpha_4, \alpha_6] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_1$, $\alpha_3^2 = \alpha_2^{-1}$, $\alpha_5^2 = \alpha_4$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_2$, $\alpha_6^2 \equiv 1$.

$H_2 \sim (3)$, $H_3 \sim (2)$, $H_4 \sim (1)$. $I_1 \sim 32\Gamma_3 c_1$, $I_2 \sim 8\Gamma_2 a_1$, $I_3 \sim (1^2)$.

$Z_3 = H_2^*$ *is the unique self-centralizer.*

$Y_4 = \tilde{I}_1$ is the unique Γ_3 group in the first quotient signal.

H_3^* is the unique Γ_2 group in the first subgroup signal.

In the first subgroup signal Γ_3 and Γ_3' are such that $\Gamma_3/Z_1 \sim 16\Gamma_2 a_1$, $\Gamma_3'/Z_1 \sim 16\Gamma_2 c_2$.

STEM. Order 64.

$Z_1 = H_4$, $Z_2 \sim (21)$, $Z_3 \sim (31)$. $Y_2 \sim (21)$, $Y_3 \sim 16\Gamma_2 c_1$, $Y_4 = I_1$. $H_3^* \sim 32\Gamma_2 k$.

Group Number	Group Symbol	Defining Relations α_4^2	Defining Relations α_6^2	First Quot. Sig. Γ_3	First Subgroup Signal Γ_2	Γ_3	Γ_3'	Order Structure 2	4	8	16	Auto-morphisms $t_1 \cdot t_2$	t_3
247	a_1	α_2^{-1}	1	c_1	k	a_1	f	19	4	24	16	$2^8 \cdot 1$	64
248	a_2	α_2	1	c_1	k	b	d_2	11	28	8	16	$2^7 \cdot 1$	32
249	a_3	α_2^{-1}	α_1	c_1	k	a_3	f	3	20	24	16	$2^8 \cdot 1$	64

FAMILY $\Gamma_{22} = {}^3_1\mathbf{D}_4$. Rank 6. Class 4. u = 64.

Commutators ≠ 1: $[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1$, $[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_2$, $[\alpha_4, \alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$, $\alpha_6^2 = \alpha_5$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_5^2 \equiv 1$.

$H_2 \sim (1^3)$, $H_3 \sim (1^2)$, $H_4 \sim (1)$. $I_1 \sim 32\Gamma_7 a_1$, $I_2 \sim 16\Gamma_2 c_1$, $I_3 \sim (1^2)$.

$Y_4 = \tilde{I}_1$ is the unique Γ_7 group in the first quotient signal.

H_2^* is the unique self-centralizer of index 4. $G/H_2^* \sim (2)$.

H_3^* is the unique Γ_4 group in the first subgroup signal.

STEM. Order 64.

$Z_1 = H_4$, $Z_2 = H_3$, $Z_3 \sim 16\Gamma_2 a_1$. $Y_2 \sim (21)$, $Y_3 = I_2$, $Y_4 = I_1$.

Relation: $\alpha_5^2 = 1$.

Group Number	Group Symbol	Defining Relation α_4^2	First Quot. Sig. Γ_7	First Subgroup Signal Γ_4	Γ_7^2	Order Structure 2	4	8	Auto-morphisms $t_1 \cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(8\Gamma_1)^2$
250	a_1	1	a_1	a_1	$a_1 a_2$	19	28	16	$2^7 \cdot 1$	32	(1^4)	$(1^3)(21)$
251	a_2	α_1	a_1	c_1	$a_1 a_2$	11	36	16	$2^7 \cdot 1$	32	(21^2)	$(1^3)(21)$

FAMILY $\Gamma_{23} = {}^3_1\mathbf{D}_5$. Rank 6. Class 4. u = 64.

Commutators ≠ 1: $[\alpha_3,\alpha_5] = [\alpha_2,\alpha_6] = \alpha_1$, $[\alpha_4,\alpha_5] = [\alpha_3,\alpha_6] = \alpha_2$, $[\alpha_4,\alpha_6] = \alpha_2\alpha_3$.

Squares: $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_3^2 = \alpha_1$, $\alpha_6^2 = \alpha_5$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_2$, $\alpha_5^2 \equiv 1$.

$H_2 \sim (21)$, $H_3 \sim (1^2)$, $H_4 \sim (1)$. $I_1 \sim 32\Gamma_7 a_1$, $I_2 \sim 16\Gamma_2 c_1$, $I_3 \sim (1^2)$.

$Y_4 = \tilde{I}_1$ is the unique Γ_7 group in the first quotient signal.

H_2^* is the unique self-centralizer of index 4. $G/H_2^* \sim (2)$.

H_3^* is the unique Γ_4 group in the first subgroup signal.

STEM. Order 64.

$Z_1 = H_4$, $Z_2 = H_3$. $Y_2 \sim (21)$, $Y_3 = I_2$, $Y_4 = I_1$. $H_2^* \sim (2^2)$.

Group Number	Group Symbol	Defining Relations α_4^2	Defining Relations α_5^2	First Quot. Sig. Γ_7	First Subgroup Signal Γ_4	First Subgroup Signal Γ_7^2	Order Structure 2	Order Structure 4	Order Structure 8	Automorphisms $t_1 \cdot t_2$	Automorphisms t_3	Self-centralizers $16\Gamma_1$	Self-centralizers $(8\Gamma_1)^2$	Z_3 $16\Gamma_2$
252	a_1	α_2	1	a_1	a_2	a_1^2	19	44		$2^7 \cdot 2$	64	(2^2)	$(1^3)^2$	a_1
253	a_2	$\alpha_1\alpha_2$	1	a_1	c_2	a_1^2	11	52		$2^7 \cdot 2$	64	(2^2)	$(1^3)^2$	a_1
254	a_3	$\alpha_1\alpha_2$	α_1	a_1	c_2	a_3^2	11	20	32	$2^7 \cdot 2$	64	(2^2)	$(21)^2$	a_2
255	a_4	α_2	α_1	a_1	a_3	a_3^2	3	28	32	$2^7 \cdot 2$	64	(2^2)	$(21)^2$	a_2

FAMILY $\Gamma_{24} = {}^3_1\mathbf{E}_1$. Rank 6. Class 3. u = 64.

Commutators ≠ 1: $[\alpha_3,\alpha_5] = [\alpha_2,\alpha_6] = \alpha_1$, $[\alpha_4,\alpha_6] = \alpha_2$, $[\alpha_5,\alpha_6] = \alpha_3$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_3^2 = \alpha_1$.

Congruences (mod Z_1): $\alpha_4^2 \equiv \alpha_2$, $\alpha_5^2 \equiv 1$, $\alpha_6^2 \equiv \alpha_3$.

$H_2 \sim (21)$, $H_3 \sim (1)$. $I_1 \sim 32\Gamma_4 b_1$, $I_2 \sim (1^3)$.

$Y_3 = \tilde{I}_1$ is the unique Γ_4 group in the first quotient signal.

H_2^* is the unique self-centralizer. $G/H_2^* \sim (1^2)$.

In the first subgroup signal Γ_3 and Γ_3' are such that $\Gamma_3/Z_1 \sim 16\Gamma_2 a_1$, $\Gamma_3'/Z_1 \sim 16\Gamma_2 c_2$.

STEM. Order 64.

$Z_1 = H_3$, $Z_2 = H_2$. $Y_2 = I_2$, $Y_3 = I_1$. $H_2^* \sim (31)$.

Relation: $\alpha_5^2 = \alpha_1$.

Group Number	Group Symbol	Defining Relations α_4^2	Defining Relations α_6^2	First Quot. Sig. Γ_4	First Subgroup Signal Γ_2	First Subgroup Signal Γ_3	First Subgroup Signal Γ_3	First Subgroup Signal Γ_6^2	First Subgroup Signal Γ_7^2	Order Structure 2	Order Structure 4	Order Structure 8	Automorphisms $t_1 \cdot t_2$	Automorphisms t_3
256	a_1	$\bar{\alpha}_2^1$	$\bar{\alpha}_3^1$	b_1	g	a_1	f	a_1^2	a_2^2	23	8	32	$2^7 \cdot 2$	32
257	a_2	α_2	α_3	b_1	g	a_2	f	a_1a_2	a_2a_3	15	16	32	$2^7 \cdot 1$	16
258	[illegible]	$\bar{\alpha}_2^1$	[illegible]	b_1	g	[illegible]	f	[illegible]	[illegible]	7	24	32	$2^7 \cdot 2$	32

FAMILY $\Gamma_{25} = {}^{3}_{1}\mathbf{E}_2$ Rank 6. Class 3. u=192.

Commutators ≠1: $[\alpha_3,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1$, $[\alpha_4,\alpha_5]=\alpha_2$, $[\alpha_4,\alpha_6]=\alpha_3$

Squares: $\alpha_1^2=\alpha_2^2=\alpha_3^2=1$.

Congruences (mod Z_1): $\alpha_4^2\equiv\alpha_5^2\equiv\alpha_6^2\equiv1$.

$H_2\sim(1^3)$, $H_3\sim(1)$. $I_1\sim32\Gamma_4a_1$, $I_2\sim(1^3)$.

$Y_3=\tilde{I}_1$ is the unique Γ_4 group in the first quotient signal.

H_2^* is the unique self-centralizer of index 4. $G/H_2^*\sim(1^2)$.

STEM. Order 64.

$Z_1=H_3$, $Z_2=H_2$. $Y_2=I_2$, $Y_3=I_1$.

Relations: $\alpha_5^2=\alpha_6^2=1$.

Group Number	Group Symbol	Defining Relation α_4^2	First Quot. Sig. Γ_4	First Subgroup Signal Γ_4^3	Γ_5	Γ_7^3	Order Structure 2	4	Automorphisms $t_1\cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(8\Gamma_1)^6$
259	a_1	1	a_1	a_1^3	a_1	a_1^3	27	36	$2^6\cdot6$	48	(1^4)	$(1^3)^3(21)^3$
260	a_2	α_1	a_1	c_1^3	a_1	a_1^3	19	44	$2^6\cdot6$	48	(21^2)	$(1^3)^3(21)^3$

FAMILY $\Gamma_{26} = {}^{3}_{1}\mathbf{E}_3$. Rank 6. Class 3. u=64.

Commutators ≠1: $[\alpha_3,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1$, $[\alpha_4,\alpha_5]=\alpha_2$, $[\alpha_4,\alpha_6]=\alpha_2\alpha_3$.

Squares: $\alpha_1^2=\alpha_2^2=1$, $\alpha_3^2=\alpha_1$.

Congruences (mod Z_1): $\alpha_4^2\equiv\alpha_2$, $\alpha_5^2\equiv\alpha_6^2\equiv1$.

$H_2\sim(21)$, $H_3\sim(1)$. $I_1\sim32\Gamma_4a_1$, $I_2\sim(1^3)$.

$Y_3=\tilde{I}_1$ is the unique Γ_4 group in the first quotient signal.

H_2^* is the unique self-centralizer of index 4. $G/H_2^*\sim(1^2)$.

STEM. Order 64.

$Z_1=H_3$, $Z_2=H_2$. $Y_2=I_2$, $Y_3=I_1$. $H_2^*\sim(2^2)$.

Relation: $\alpha_6^2=1$.

Group Number	Group Symbol	Defining Relations α_4^2	α_5^2	First Quot. Sig. Γ_4	First Subgroup Signal Γ_3^2	Γ_4	Γ_5	Γ_6^2	Γ_7	Order Structure 2	4	8	Automorphisms $t_1\cdot t_2$	t_3	Self-centralizers $16\Gamma_1$	$(8\Gamma_1)^2$
261	a_1	α_2	1	a_1	e^2	a_2	a_1	a_1^2	a_2	27	20	16	$2^7\cdot2$	32	(2^2)	$(1^3)^2$
262	a_2	$\alpha_1\alpha_2$	1	a_1	e^2	c_2	a_1	a_2^2	a_2	19	28	16	$2^7\cdot2$	32	(2^2)	$(1^3)^2$
263	a_3	$\alpha_1\alpha_2$	α_1	a_1	e^2	c_2	a_2	a_1^2	a_3	19	28	16	$2^7\cdot2$	32	(2^2)	$(21)^2$
264	a_4	α_2	α_1	a_1	e^2	a_3	a_2	a_2^2	a_3	11	36	16	$2^7\cdot2$	32	(2^2)	$(21)^2$

FAMILY $\Gamma_{27} = {}^{2}_{1}\mathbf{F} = \Lambda_{(4)}$. Rank 6. Class 5. u=128.

Commutators ≠ 1: $[\alpha_2, \alpha_6] = \alpha_1$, $[\alpha_3, \alpha_6] = \alpha_2$, $[\alpha_4, \alpha_6] = \alpha_3$, $[\alpha_5, \alpha_6] = \alpha_4$.

Squares: $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_1$, $\alpha_3^2 = \alpha_2^{-1}$, $\alpha_4^2 = \alpha_3^{-1}$.

Congruences (mod Z_1): $\alpha_5^2 \equiv \alpha_4^{-1}$, $\alpha_6^2 \equiv 1$.

$H_2 \sim (4)$, $H_3 \sim (3)$, $H_4 \sim (2)$, $H_5 \sim (1)$. $I_1 \sim 32\Gamma_8 a_1$, $I_2 \sim 16\Gamma_3 a_1$, $I_3 \sim 8\Gamma_2 a_1$, $I_4 \sim (1^2)$.

$Y_5 = \tilde{I}_1$ is the unique Γ_8 group in the first quotient signal.

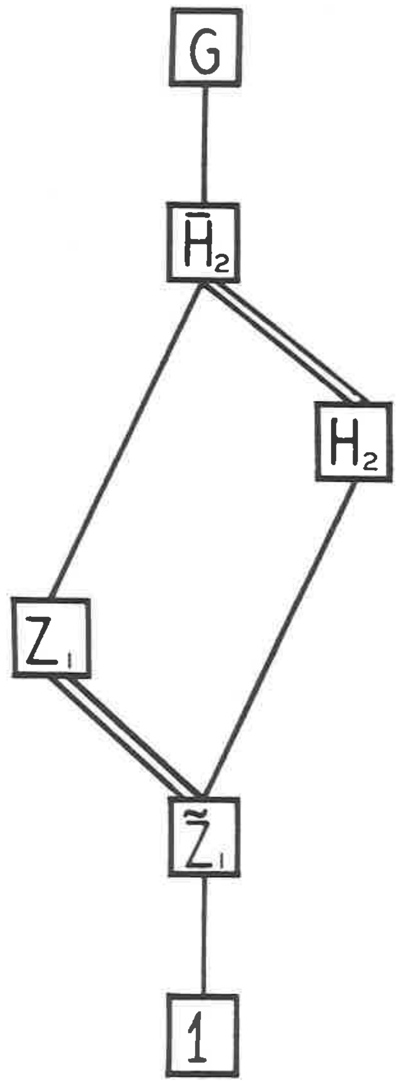
STEM. Order 64.

$Z_1 = H_5$, $Z_2 = H_4$, $Z_3 = H_3$, $Z_4 = H_2$ $Y_2 = I_4$, $Y_3 = I_3$, $Y_4 = I_2$, $Y_5 = I_1$.

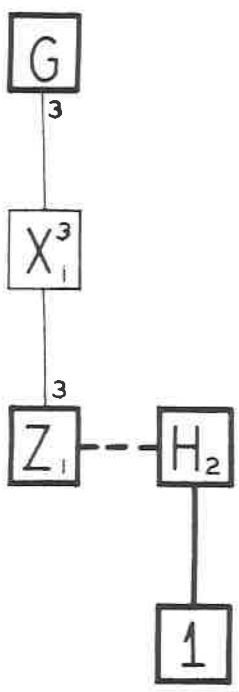
Group Number	Group Symbol	Defining Relations α_5^2	Defining Relations α_6^2	First Quot. Sig. Γ_8	First Subgroup Signal Γ_1	First Subgroup Signal Γ_8^2	Order Structure 2	4	8	16	32	Automorphisms $t_1 \cdot t_2$	t_3
265	a_1	α_4^{-1}	1	a_1	(5)	a_1^2	33	2	4	8	16	$2^8 \cdot 2$	128
266	a_2	α_4^7	1	a_1	(5)	$a_1 a_3$	17	18	4	8	16	$2^8 \cdot 1$	64
267	a_3	α_4^{-1}	α_1	a_1	(5)	a_3^2	1	34	4	8	16	$2^8 \cdot 2$	128

FAMILY DIAGRAMS

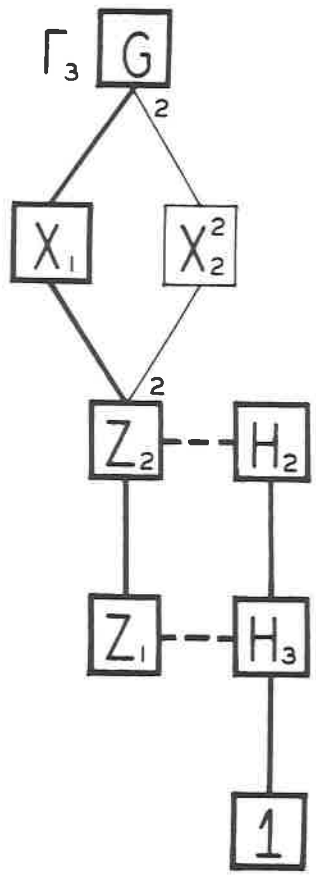
A



Γ_2



Γ_3



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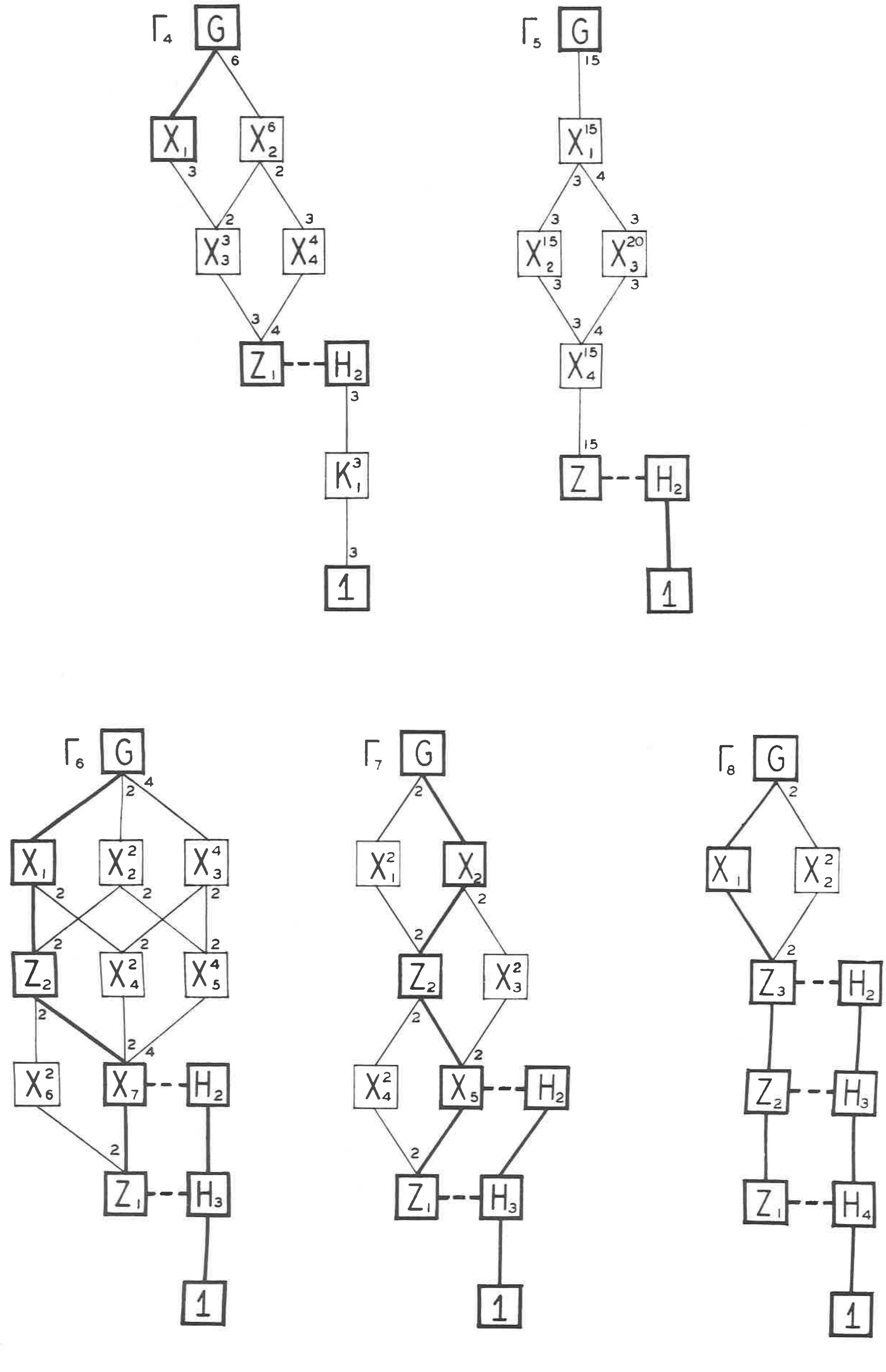

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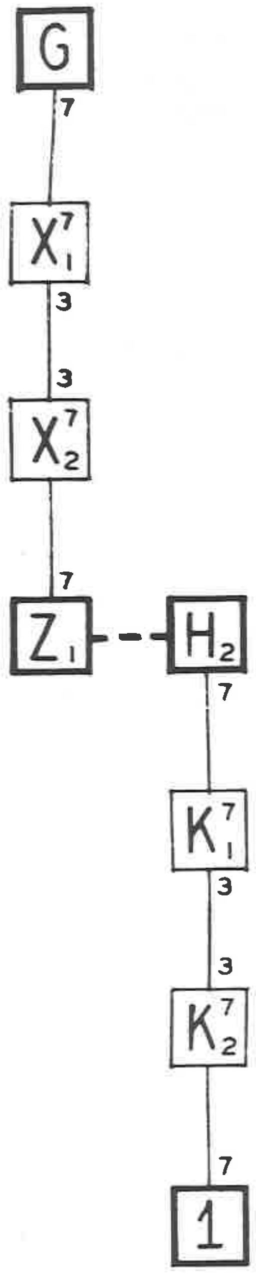
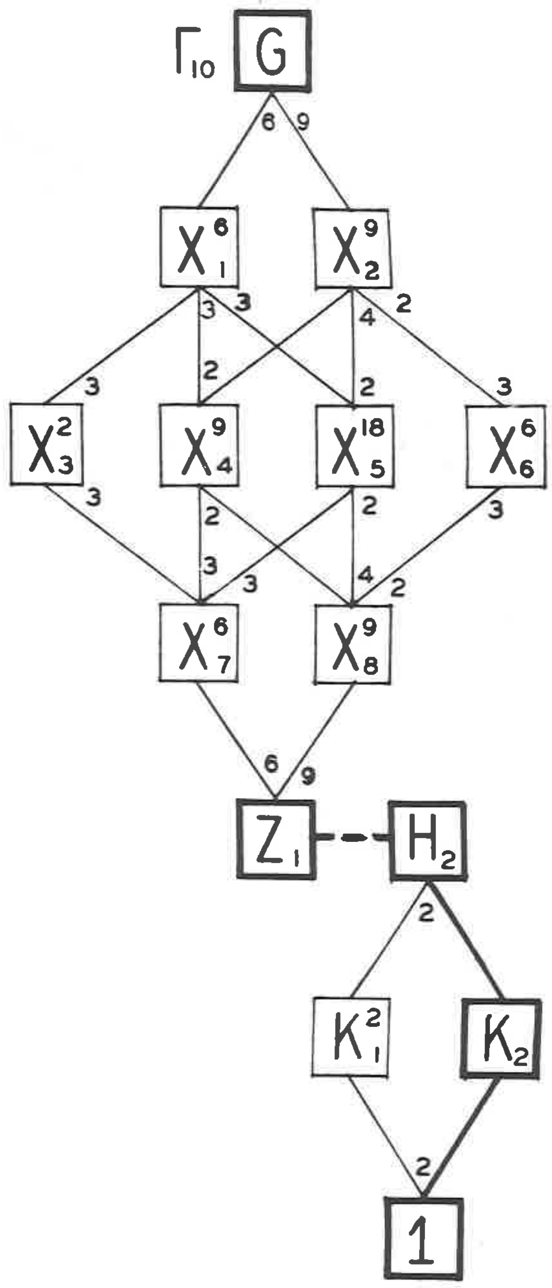
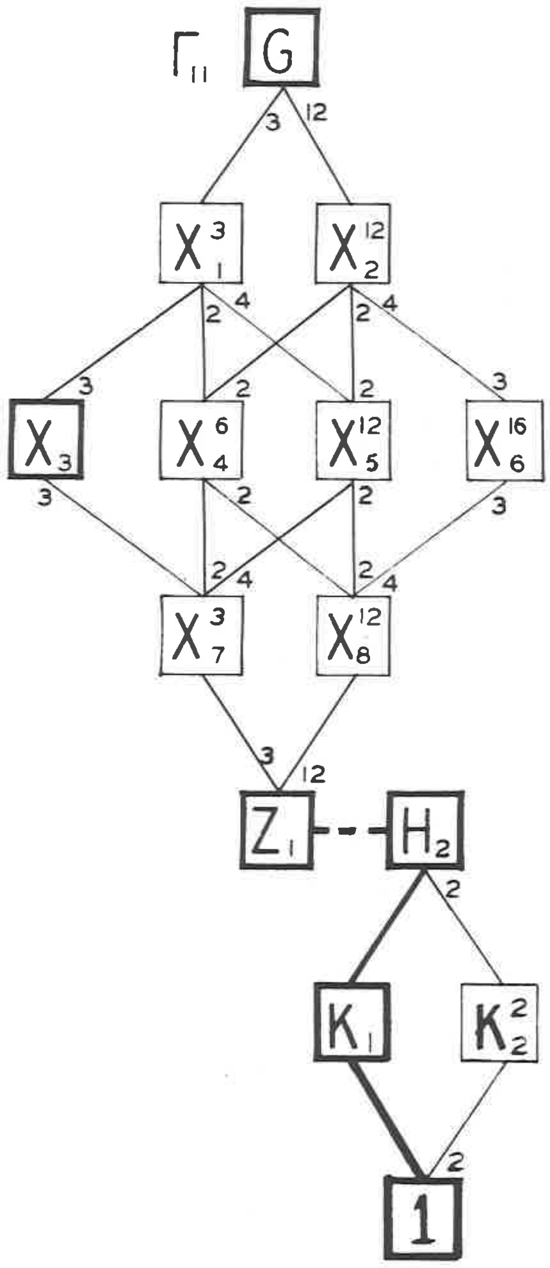
Γ_9



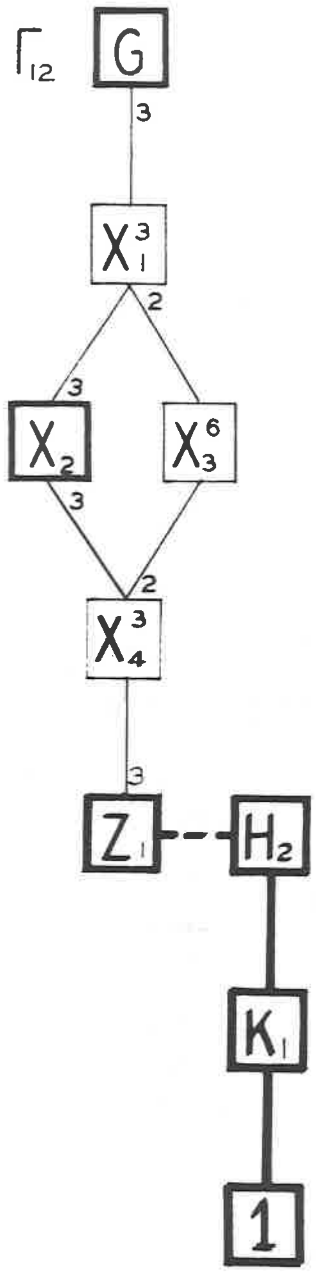
Γ_{10}



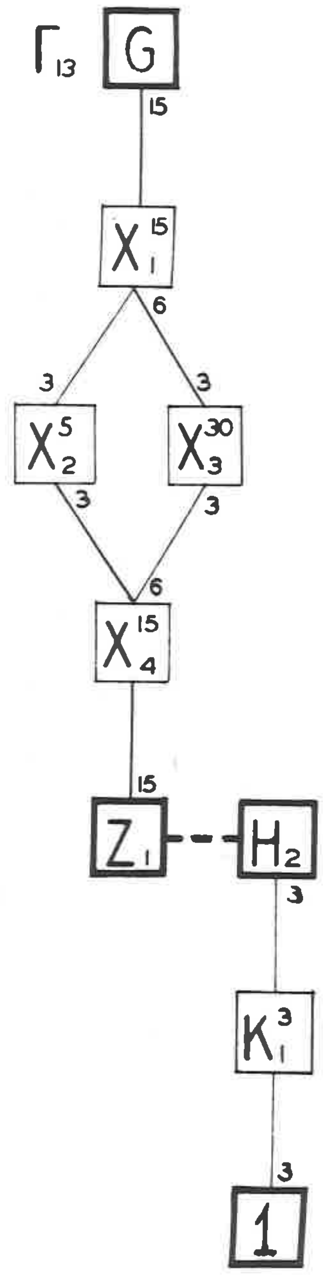
Γ_{11}



Γ_{12}

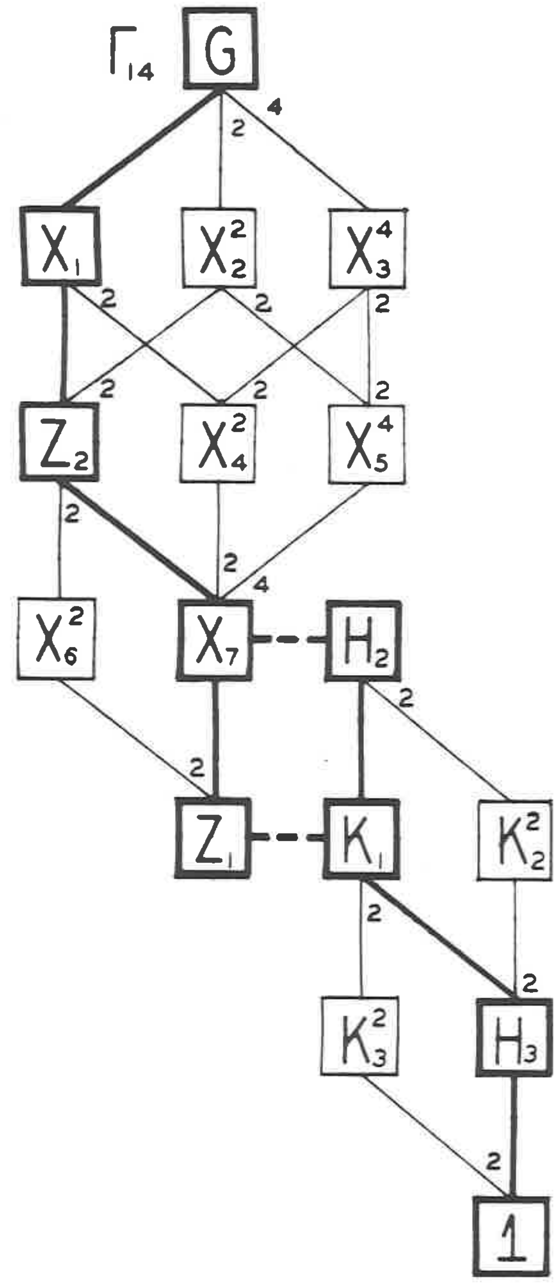


Γ_{13}

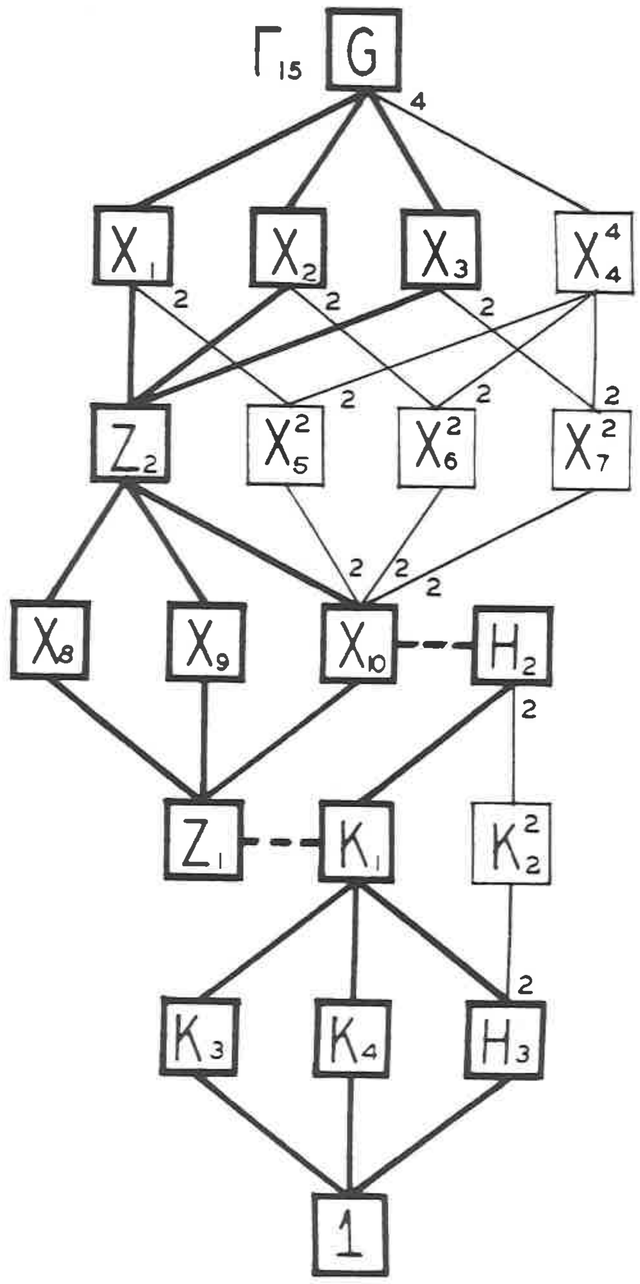


D

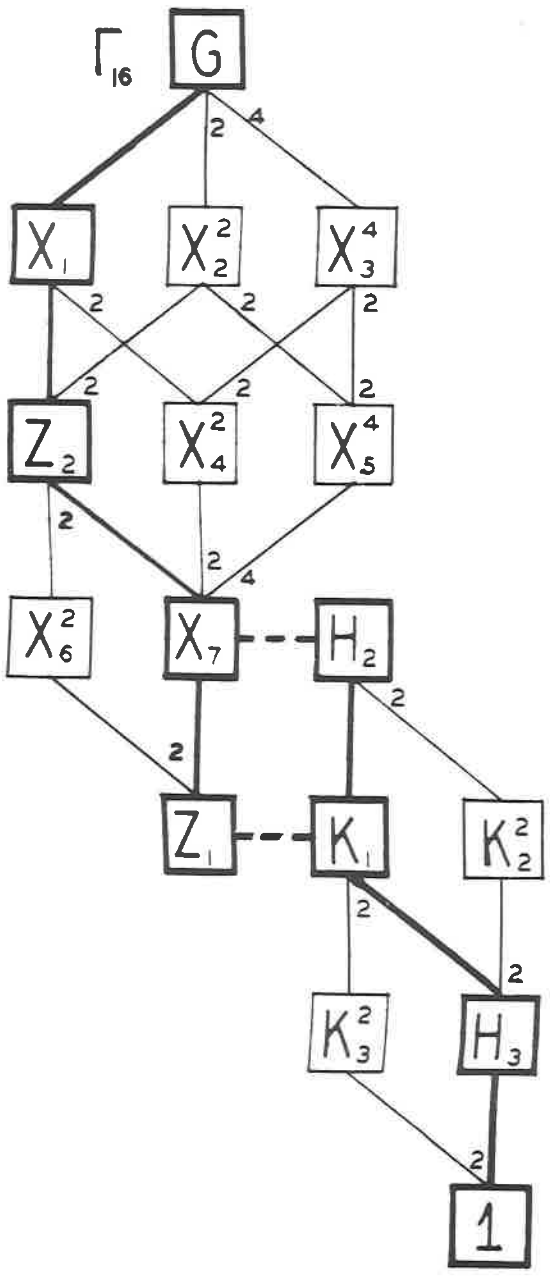
Γ_{14}



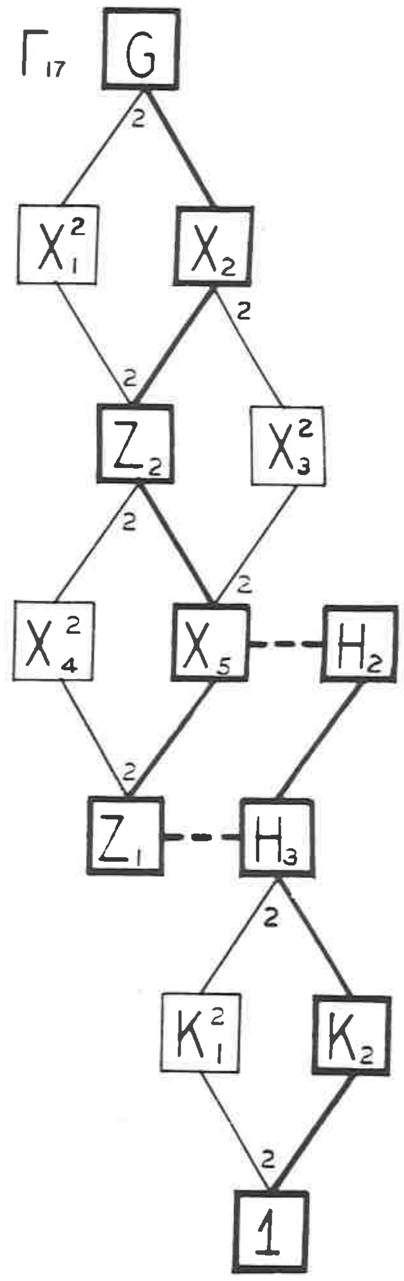
Γ_{15}



Γ_{16}

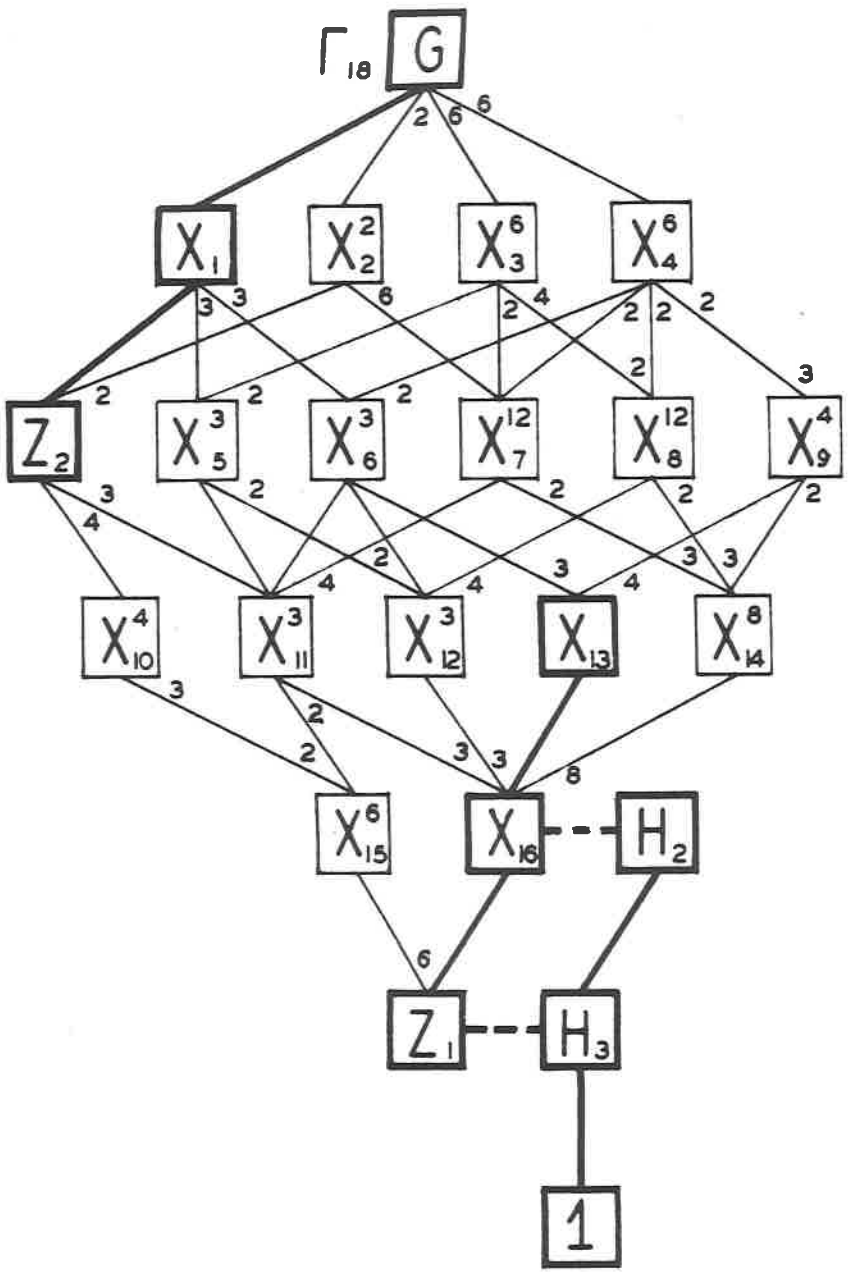


Γ_{17}

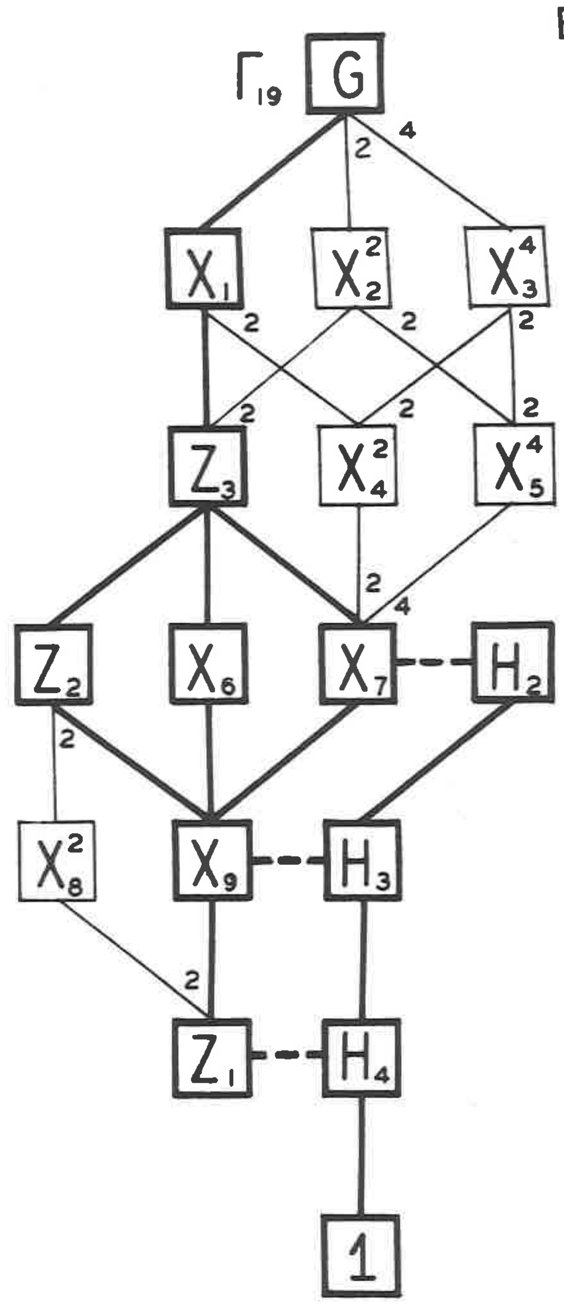


E

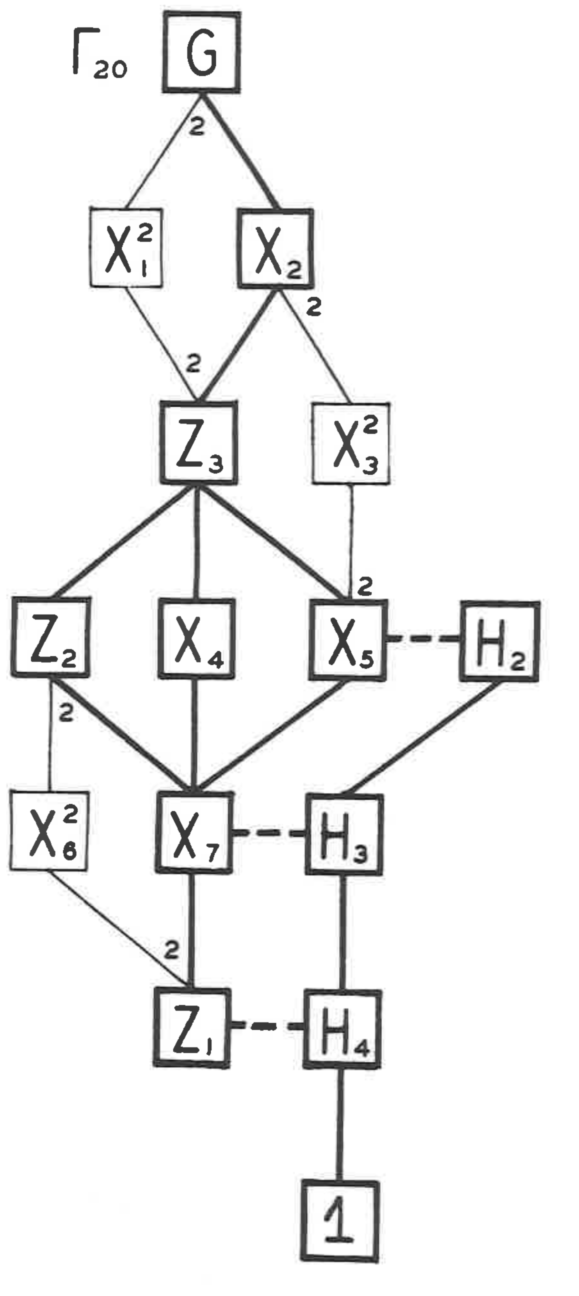
Γ_{18}



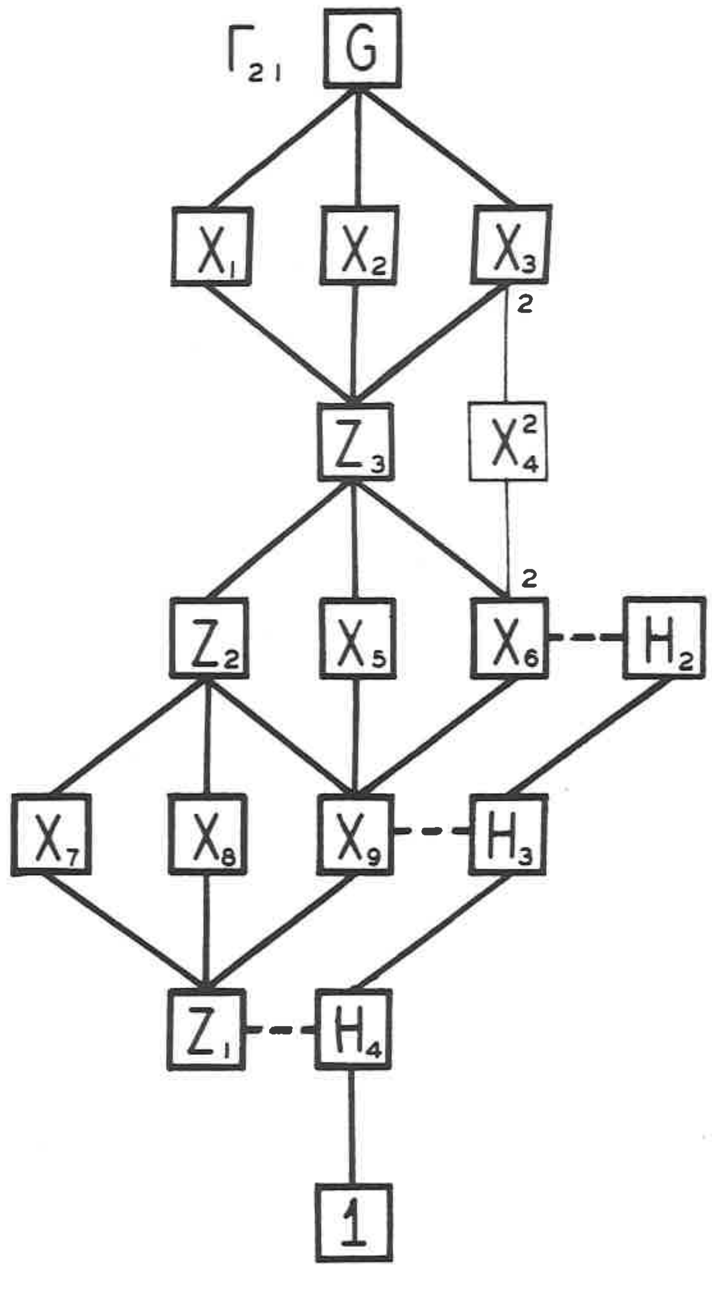
Γ_{19}



Γ_{20}

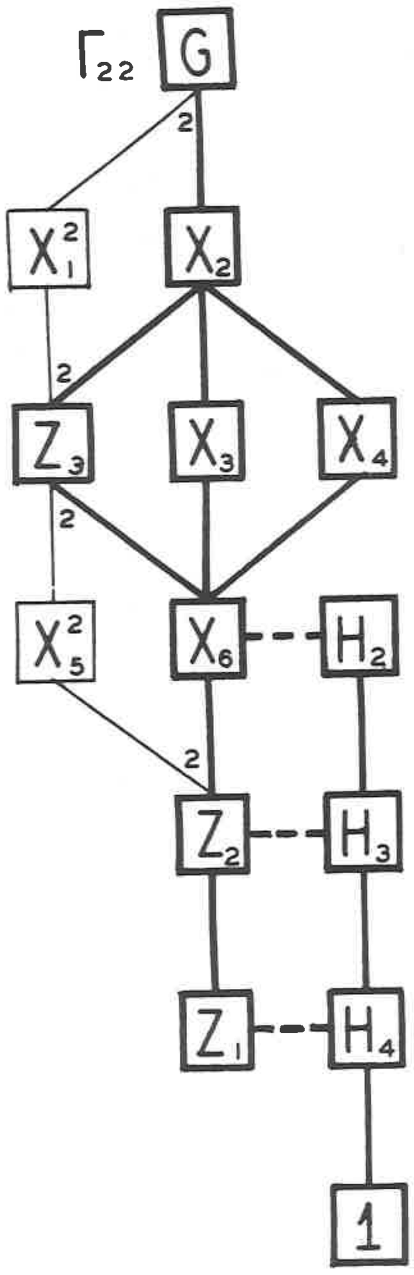


Γ_{21}

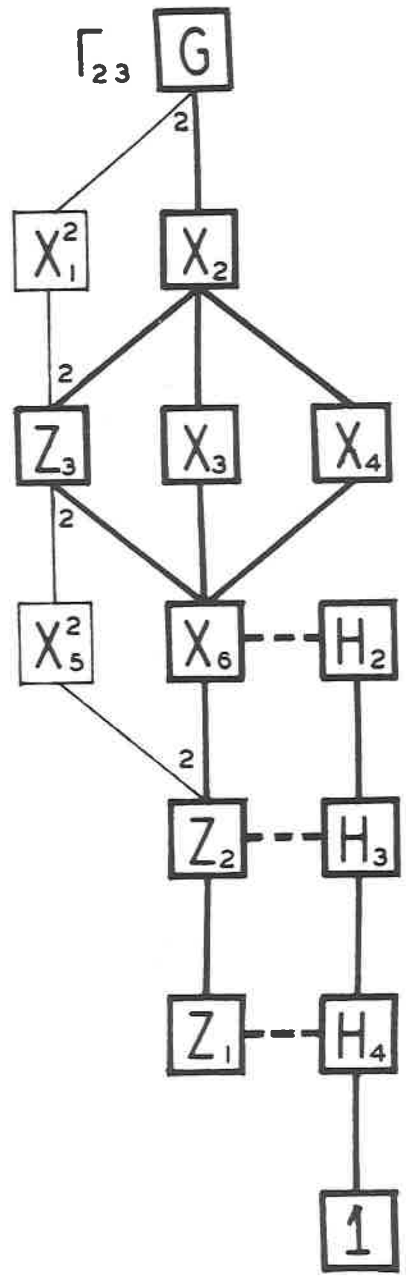


F

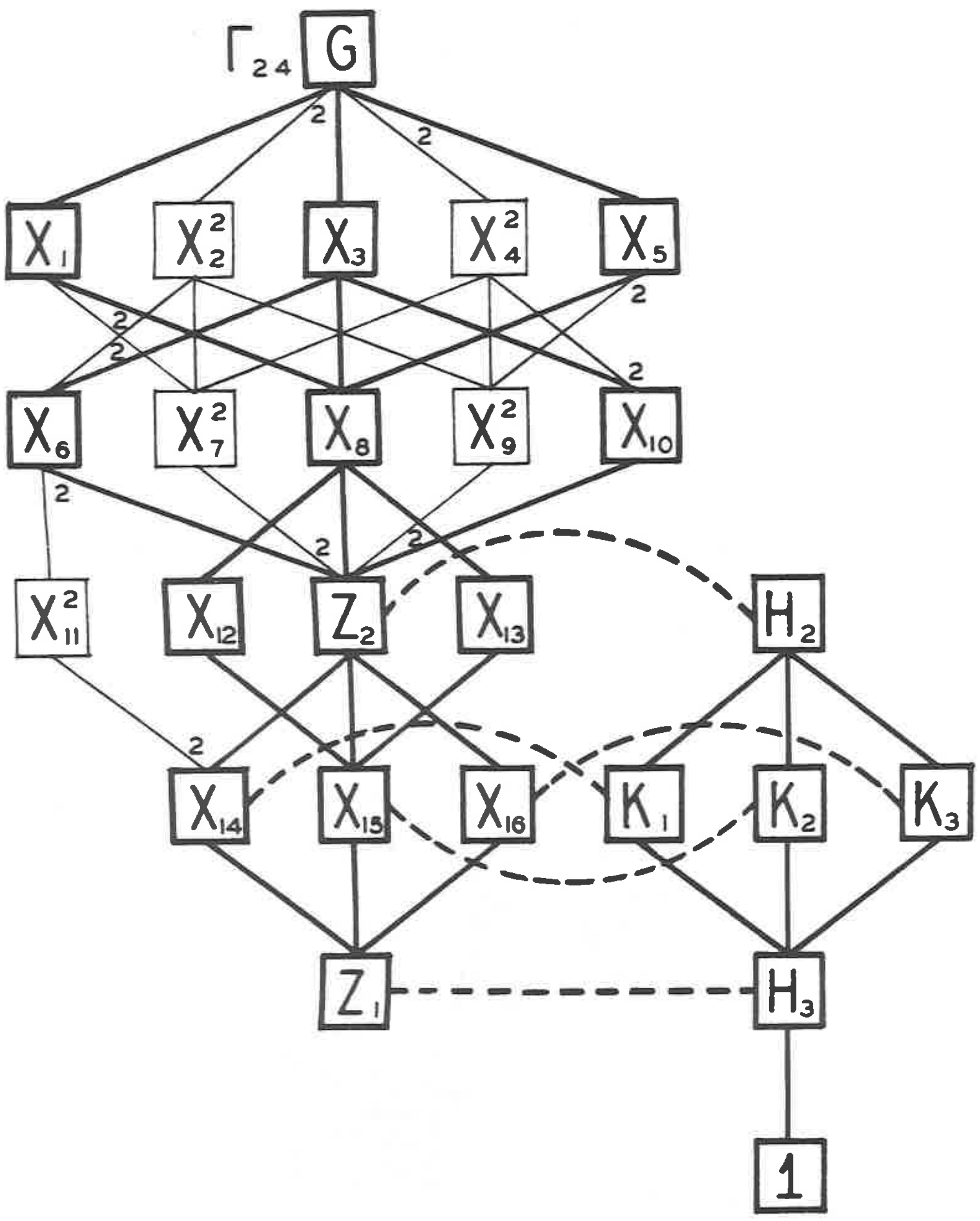
Γ_{22}



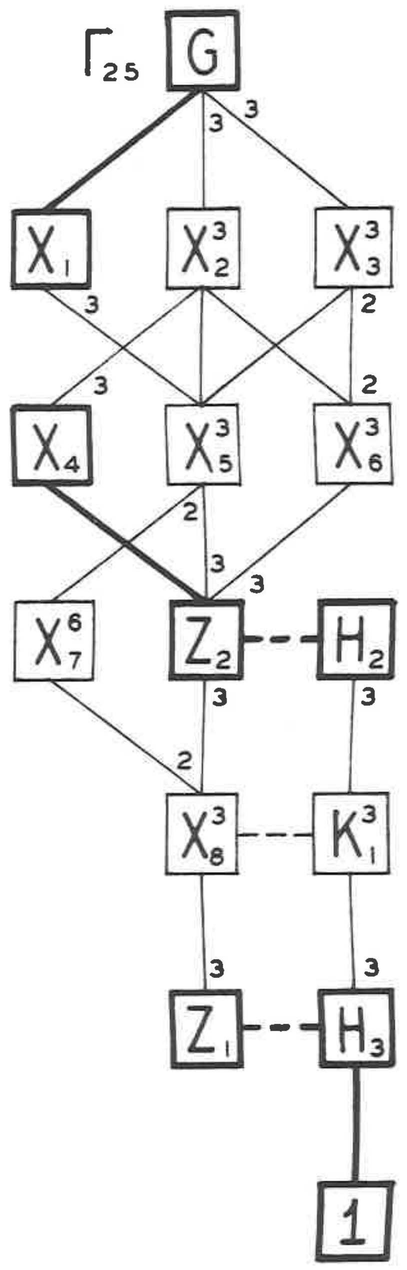
Γ_{23}



Γ_{24}

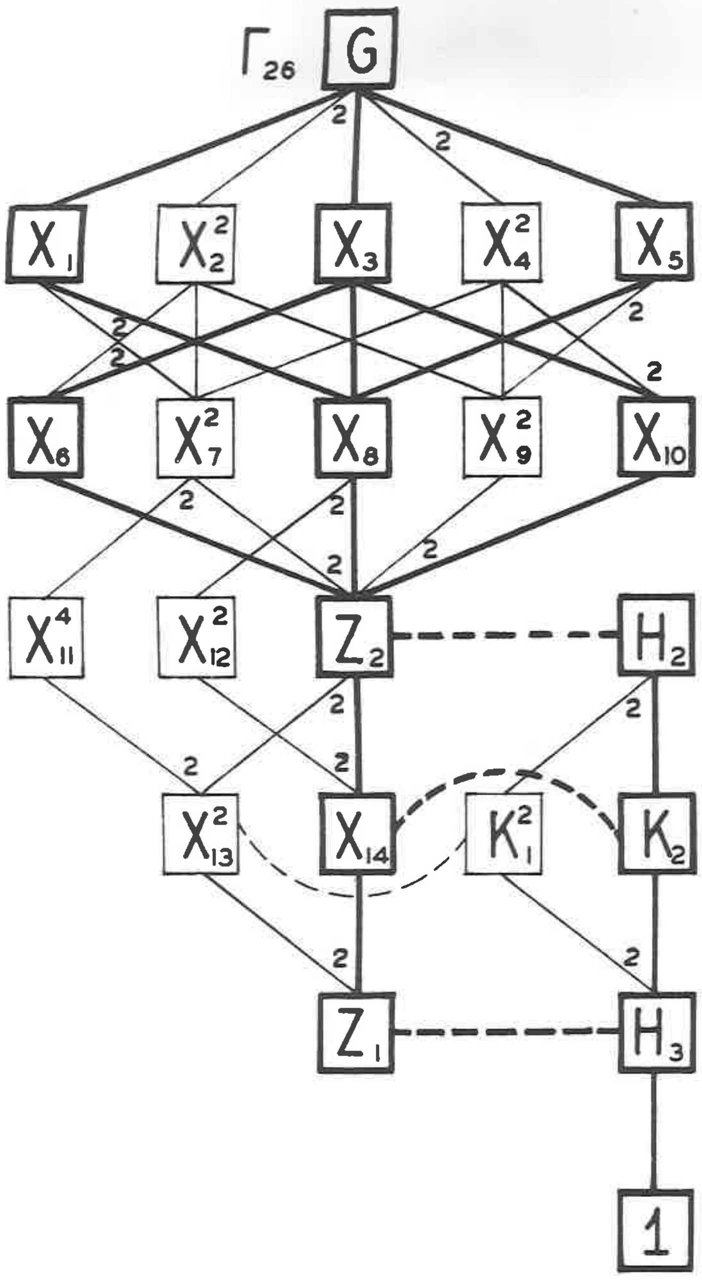


Γ_{25}

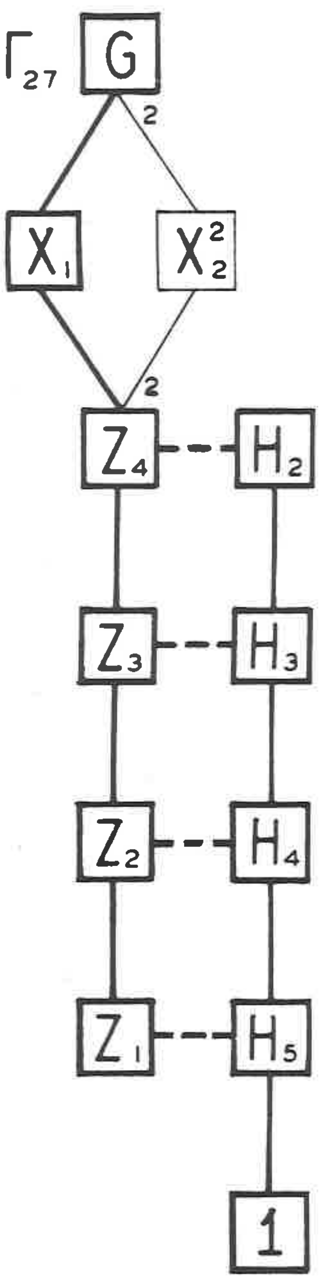


G

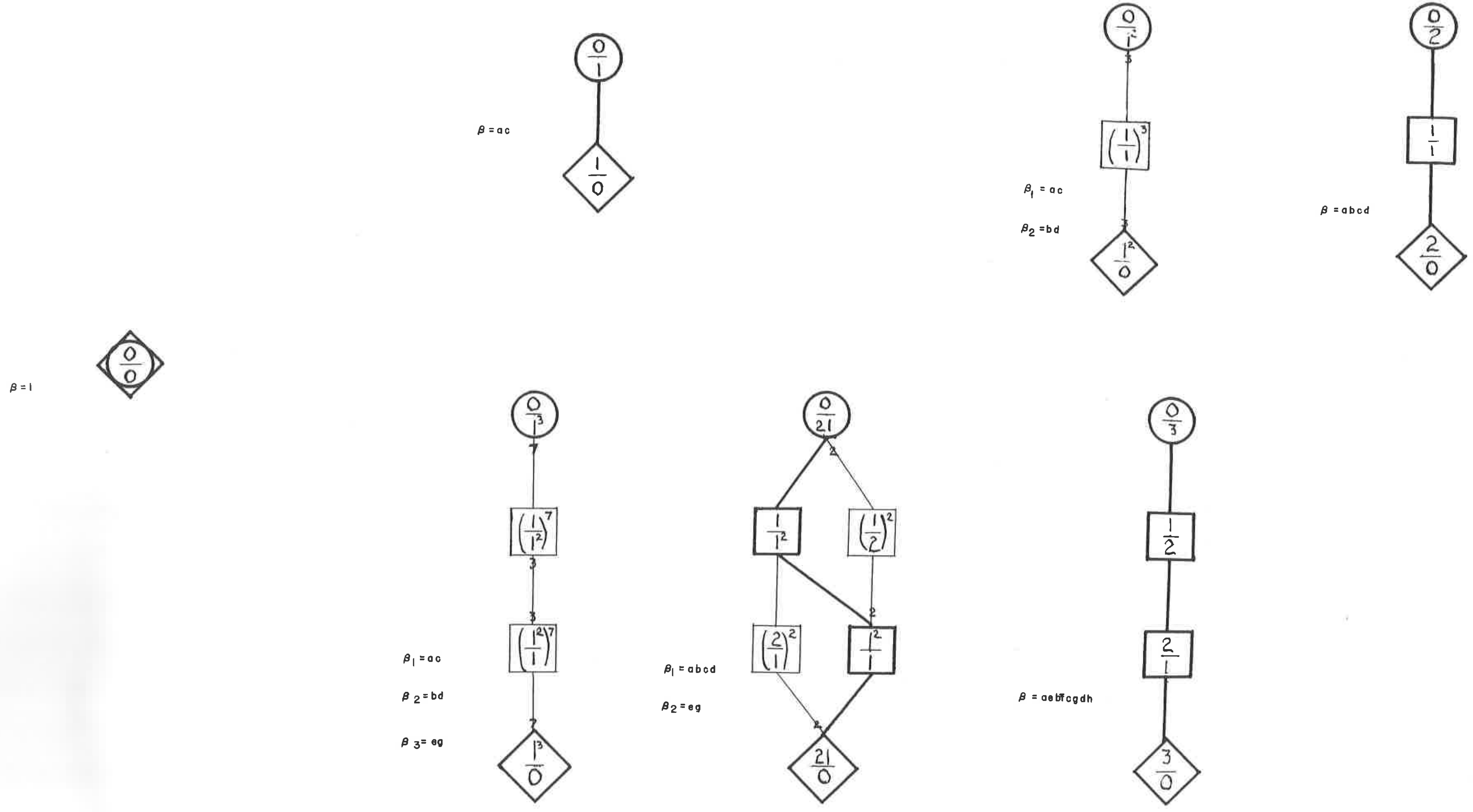
Γ_{26}

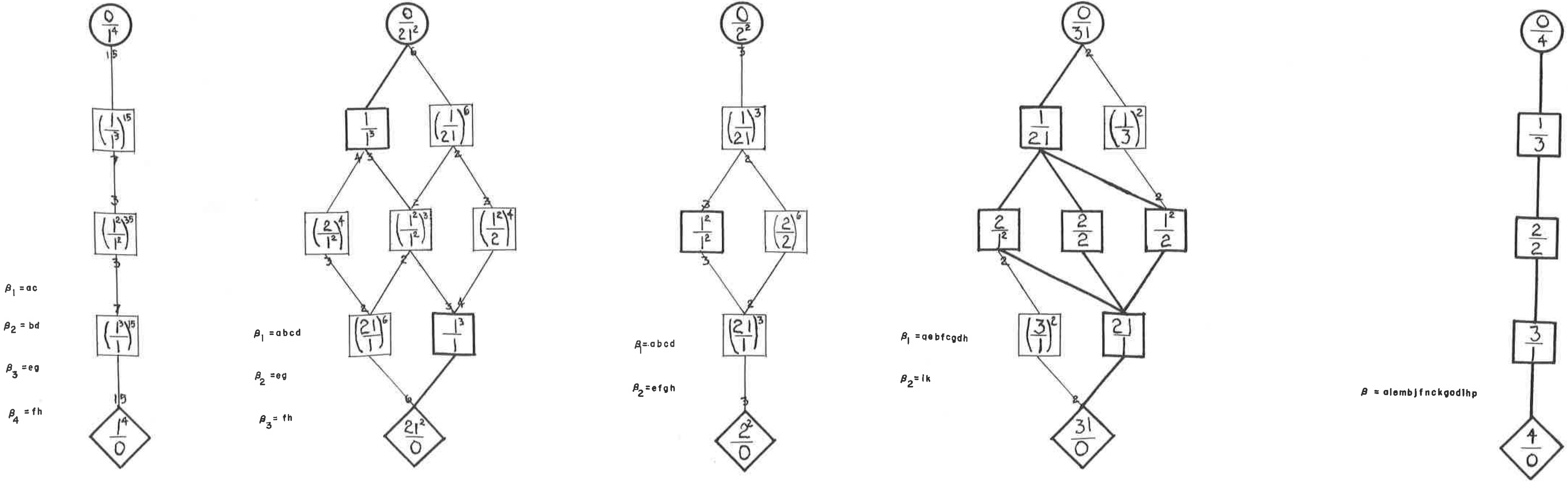


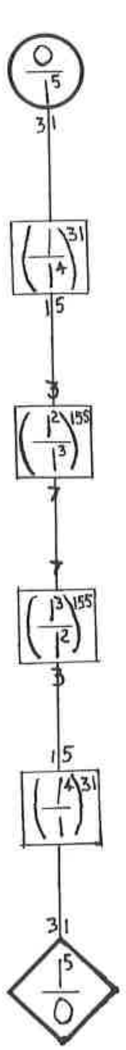
Γ_{27}



INDIVIDUAL GROUP DIAGRAMS




β1 = ac
β2 = bd
β3 = eg
β4 = fh
β1 = abcd
β2 = eg
β3 = fh
β1 = abcd
β2 = efgh
β1 = aebfcgdh
β2 = ik
β = alembjfnckgodlhp



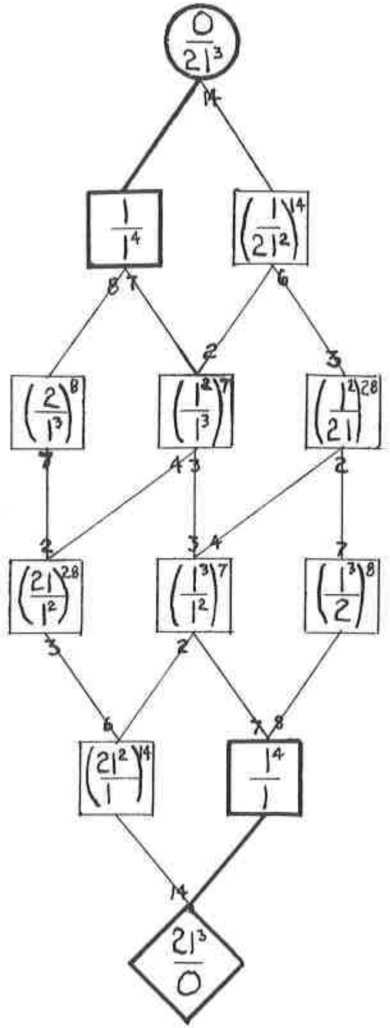
β_1 = ac

β_2 = bd

β_3 = eg

β_4 = fh

β_5 = ik

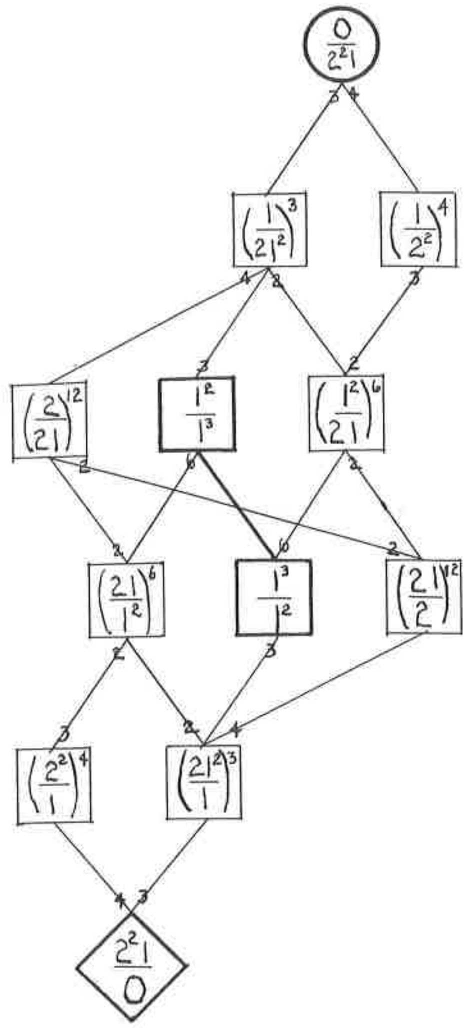


β_1 = abcd

β_2 = eg

β_3 = fh

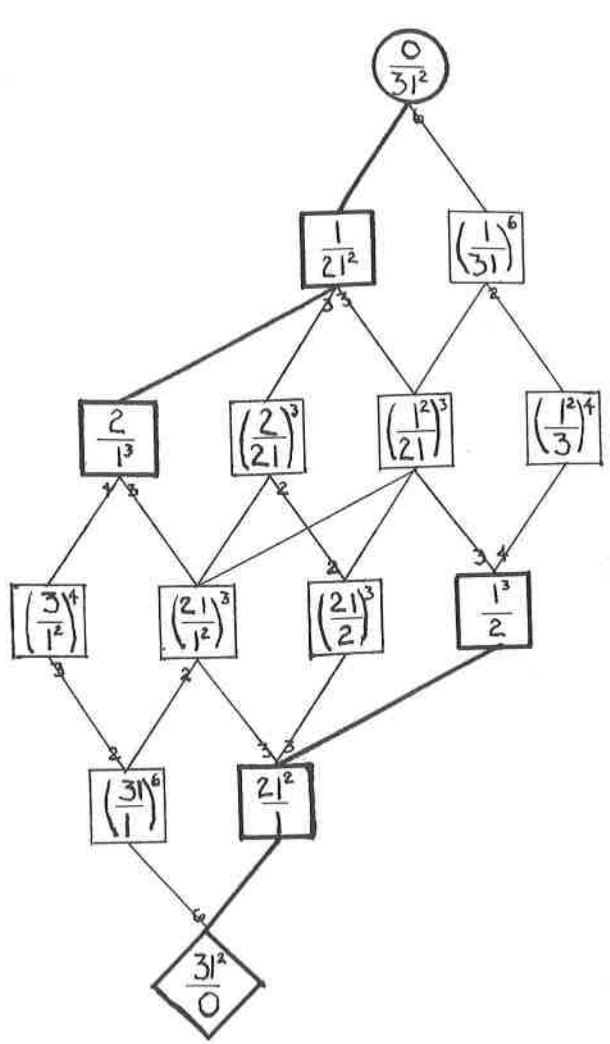
β_4 = ik



β_1 = abcd

β_2 = efgh

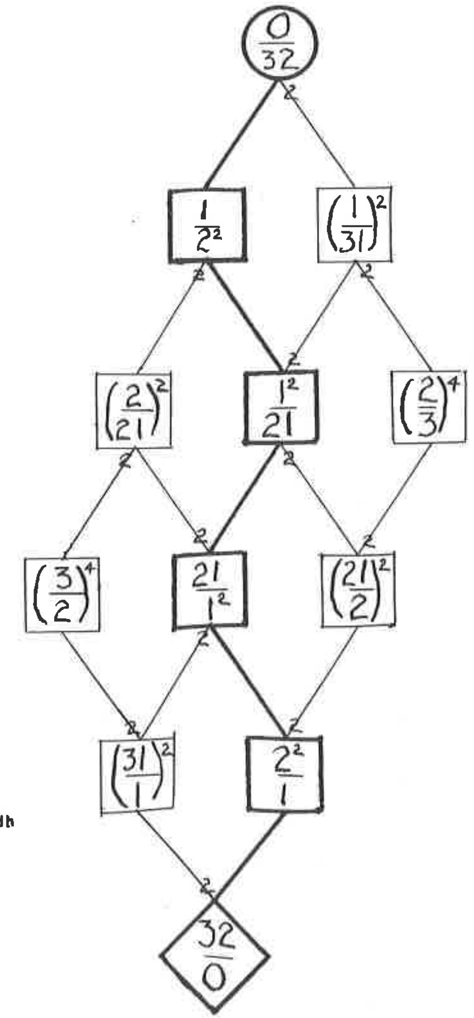
β_3 = ik



β_1 = aebfcgdh

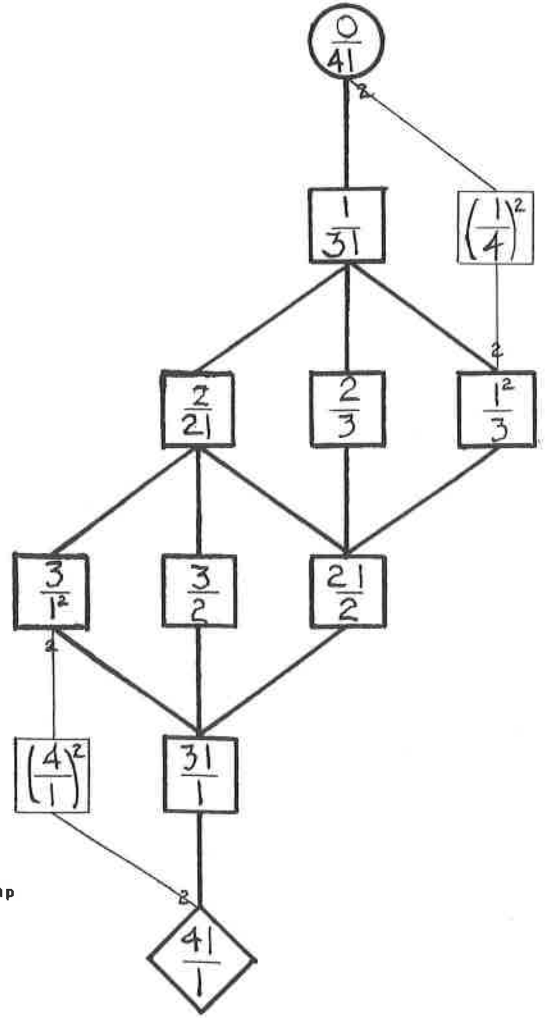
β_2 = ik

β_3 = jl



β_1 = aebfcgdh

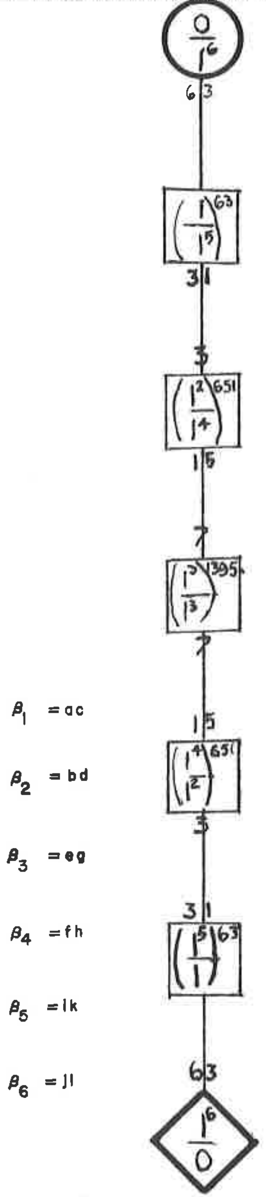
β_2 = ijkl

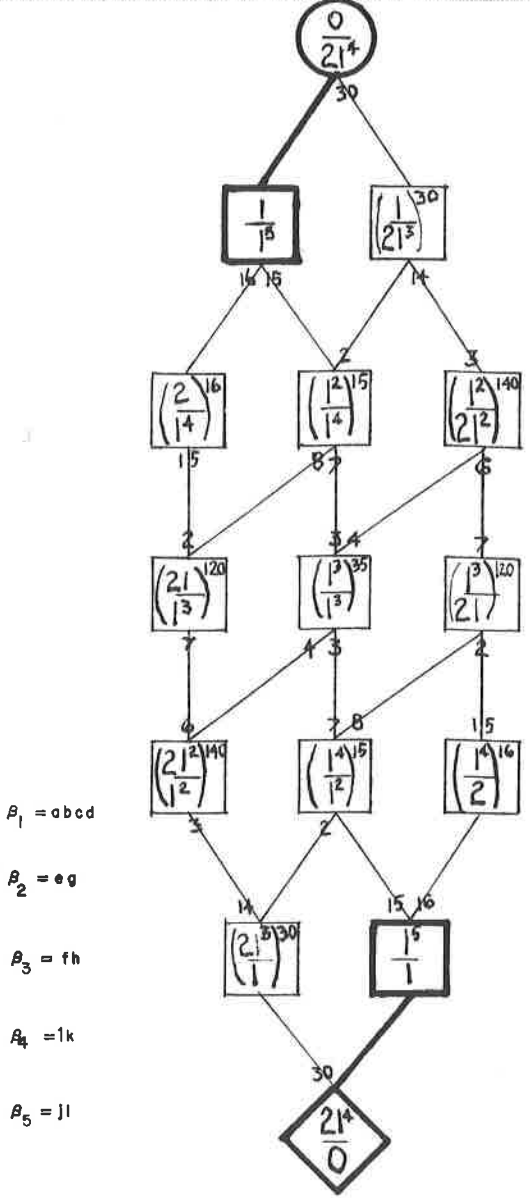


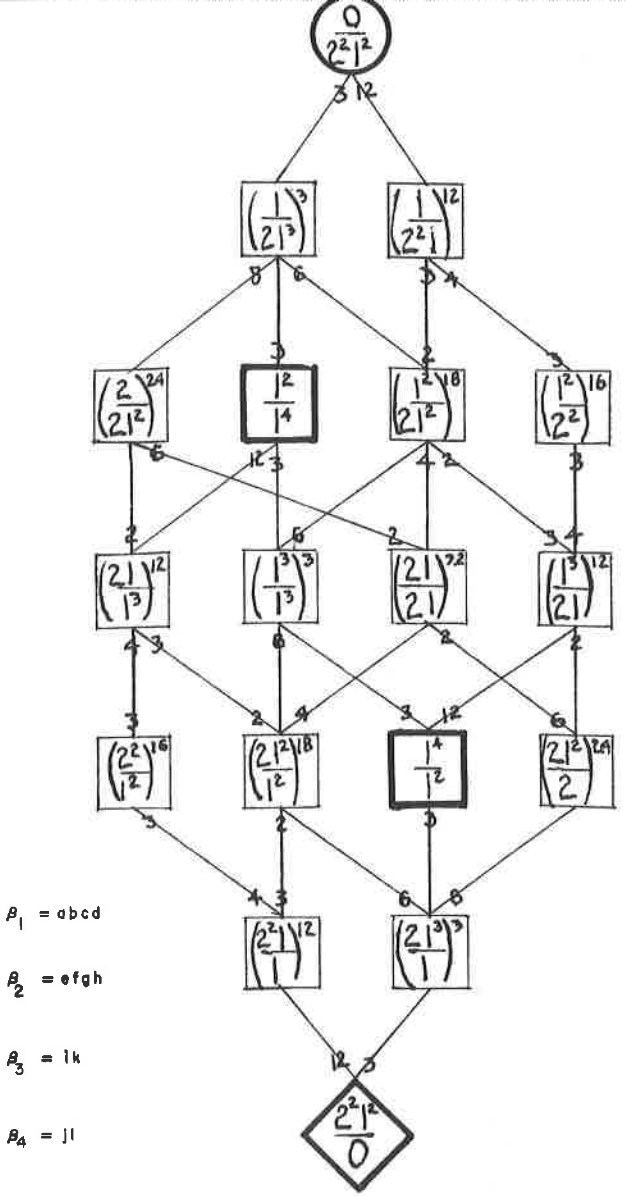
β_1 = aiembj fnckgodlhp

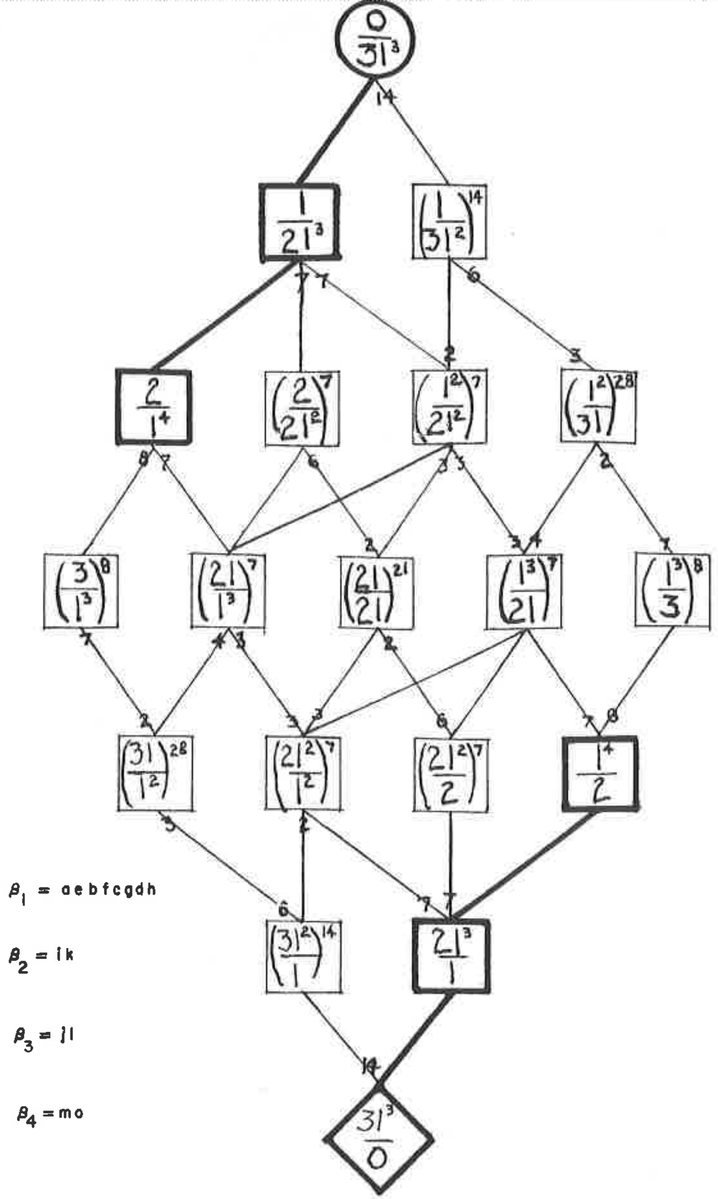
β_2 = qs

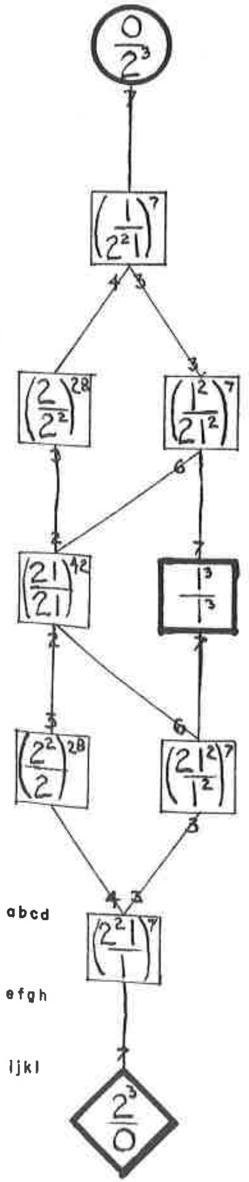
β = aAiIeEmMbBjJ fF nNcCkKgGoOdDlL hHpP


β1 = ac
β2 = bd
β3 = eg
β4 = fh
β5 = lk
β6 = jl


β1 = abcd
β2 = eg
β3 = fh
β4 = lk
β5 = jl


β1 = abcd
β2 = efgh
β3 = lk
β4 = jl

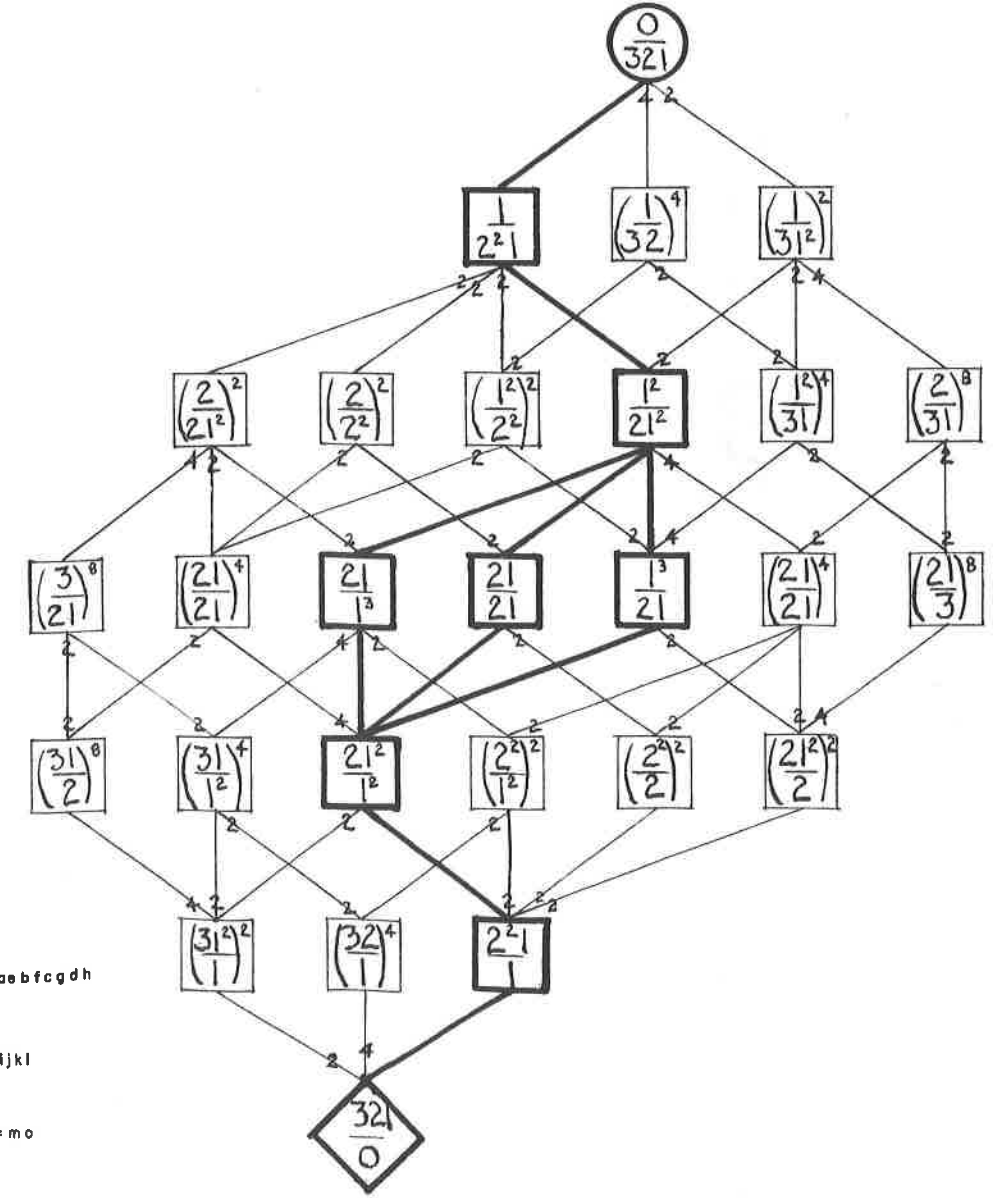

β1 = aebfcgdh
β2 = lk
β3 = jl
β4 = mo



β_1 = abcd

β_2 = efgh

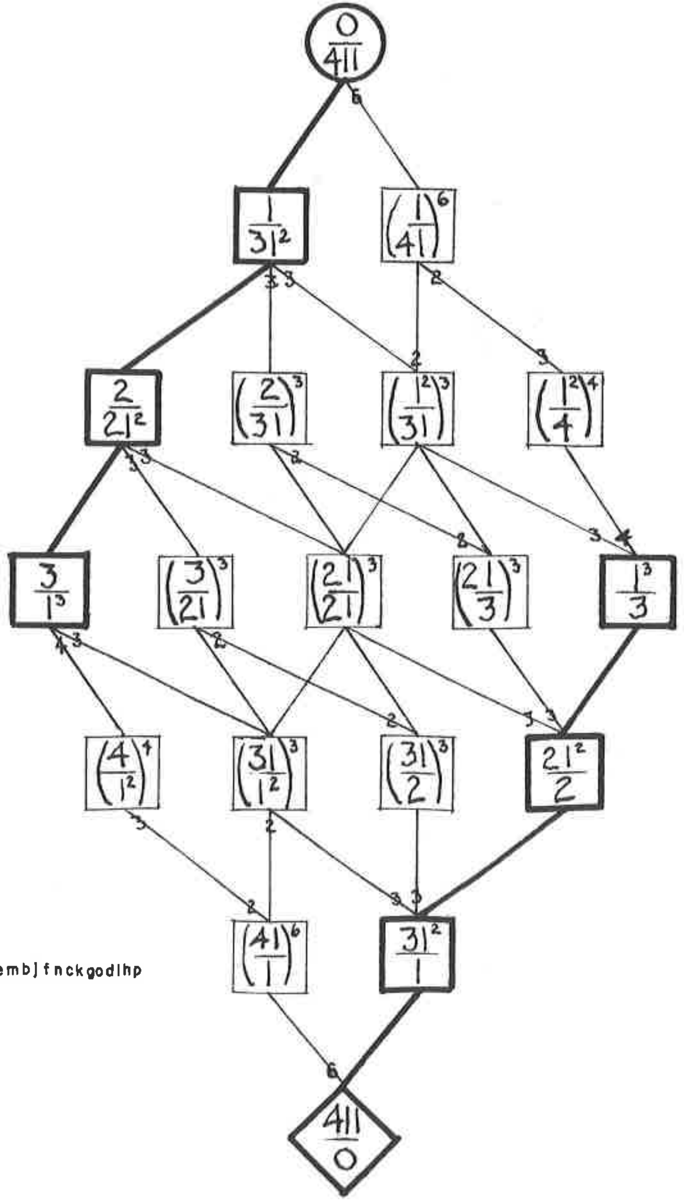
β_3 = ijkl



β_1 = aebfcgdh

β_2 = ijkl

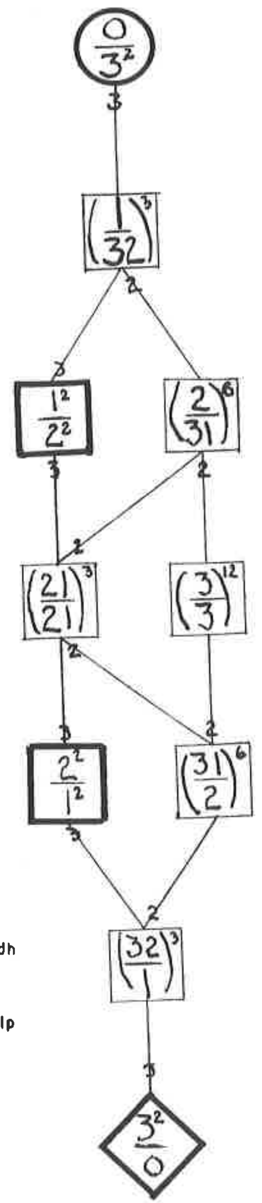
β_3 = mo



β_1 = aiembjfnckgodlhp

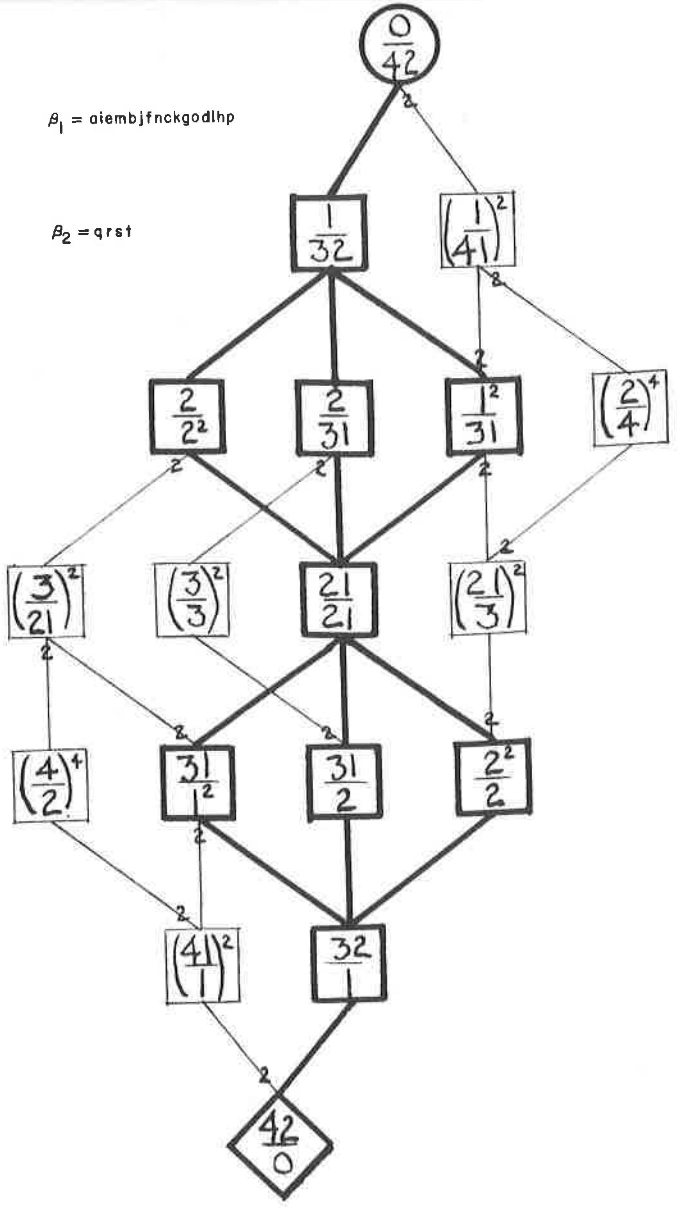
β_2 = qs

β_3 = rt



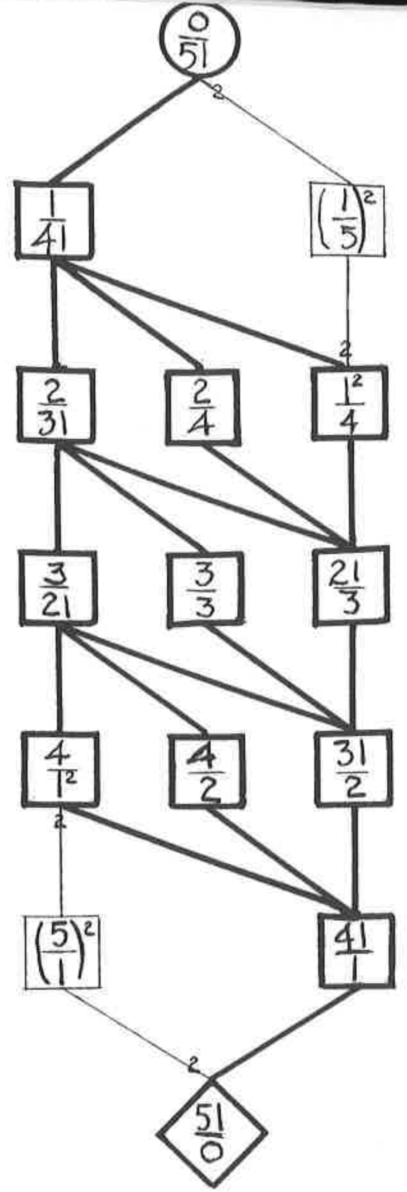
β_1 = aebfcgdh

β_2 = imjnkolp



β_1 = aiembjfnckgodlhp

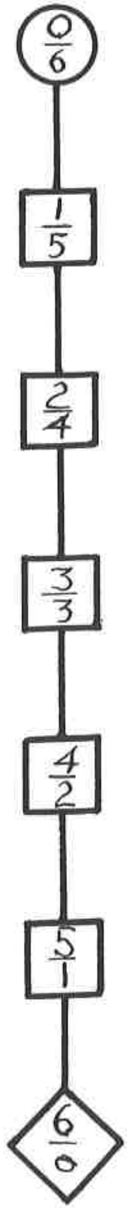
β_2 = qrst

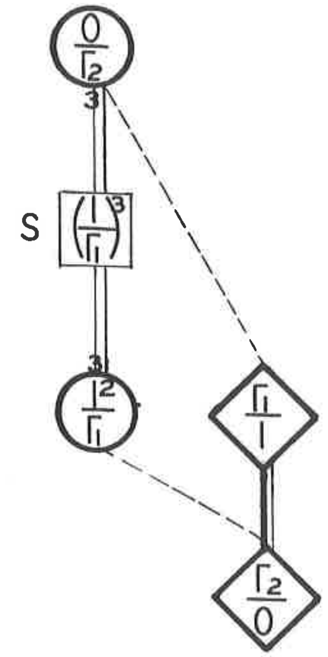


β_1 = aAiIeEmMbBjJfFnN–
cCkKgGoOdDlLhHpP

β_2 = qs

$\beta = aa'AA'ii'II'ee'EE'mm'MM'-$
$bb'BB'jj'JJ'ff'FF'nn'NN'-$
$cc'CC'kk'KK'gg'GG'oo'OO'-$
$dd'DD'll'LL'hh'HH'pp'PP'$

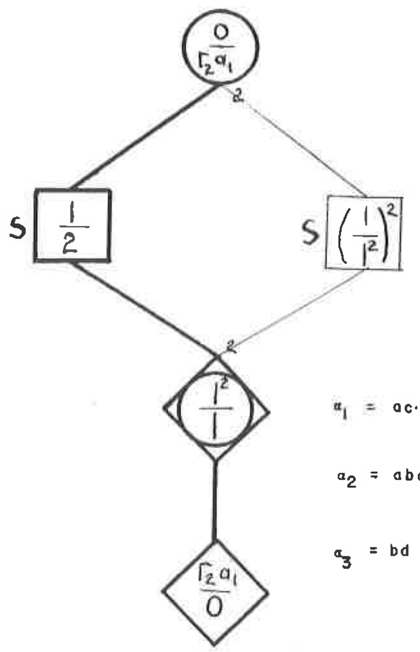




$a_1^2 = 1$

$a_2^2 \equiv a_3^2 \equiv 1$

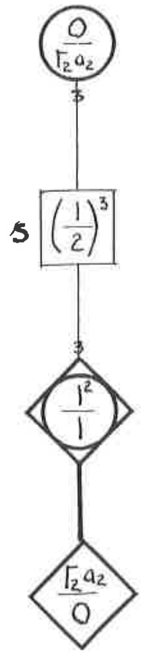
$[a_2, a_3] = a_1$



$a_1 = ac \cdot bd$

$a_2 = abcd$

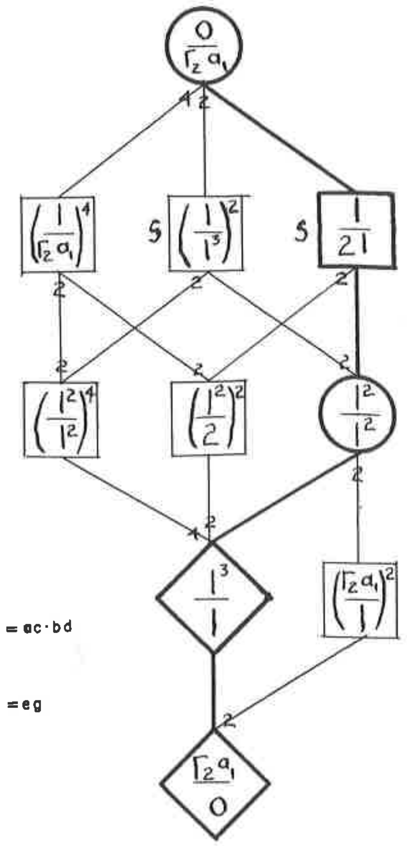
$a_3 = bd$



$a_1 = ac \cdot bd \cdot eg \cdot fh$

$a_2 = abcd \cdot efgh$

$a_3 = aecg \cdot bhdf$



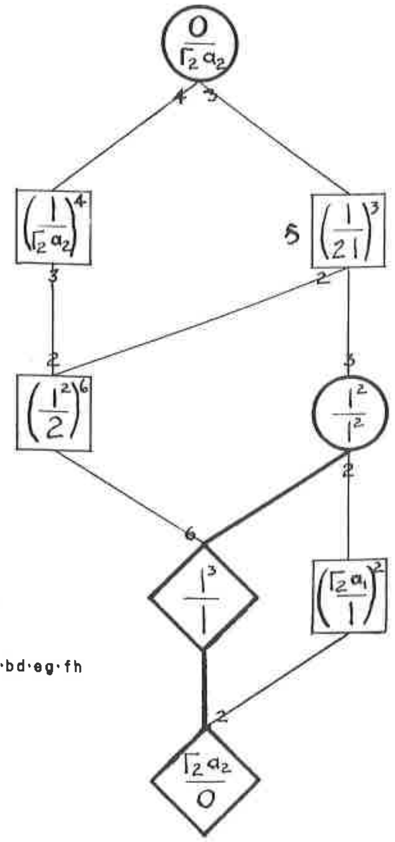
$\beta_1 = ac \cdot bd$

$\beta_2 = eg$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = abcd$

$\alpha_3 = bd$



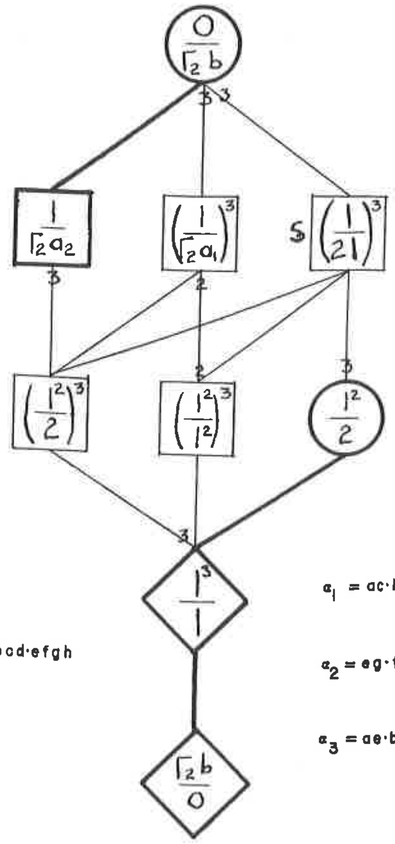
$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aecg \cdot bhdf$

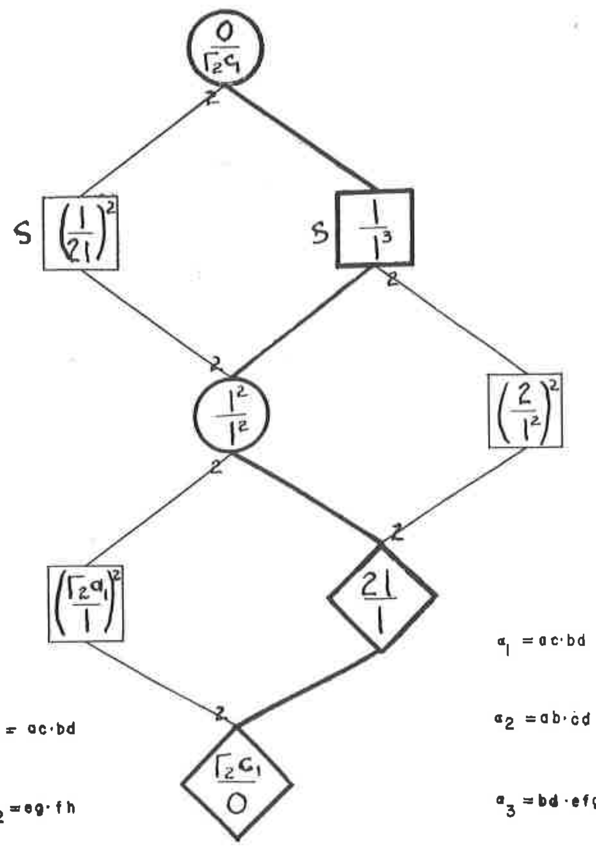


$\beta = abcd \cdot efgh$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ae \cdot bf \cdot cg \cdot dh$



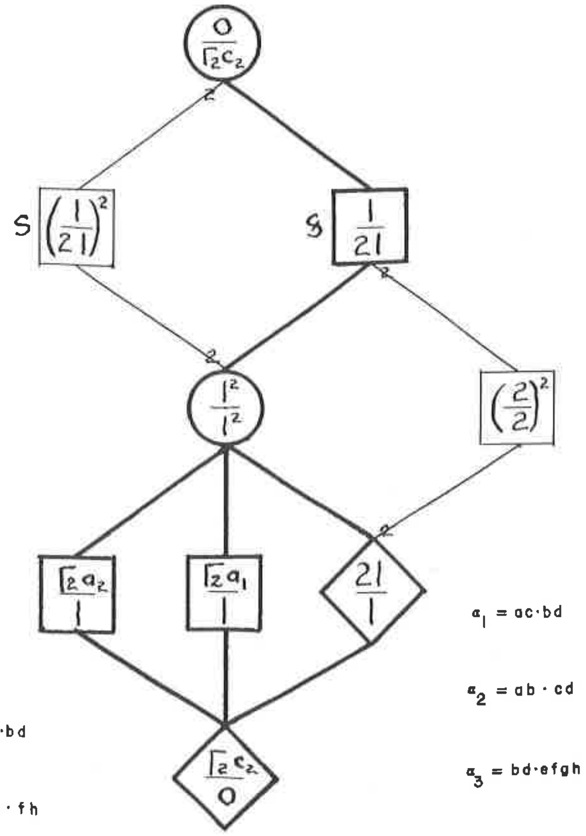
$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd$

$\alpha_3 = bd \cdot efgh$



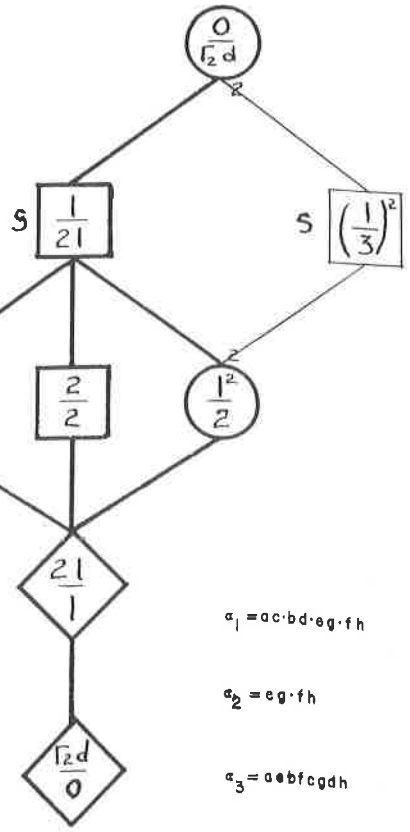
$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd$

$\alpha_3 = bd \cdot efgh$

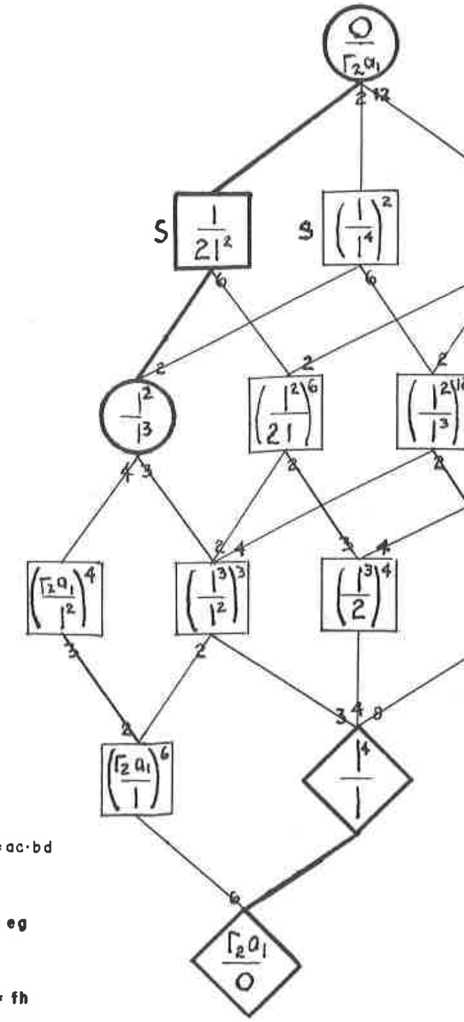


$\beta = abcd \cdot efgh$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = aebfcgdh$



$\beta_1 = ac \cdot bd$

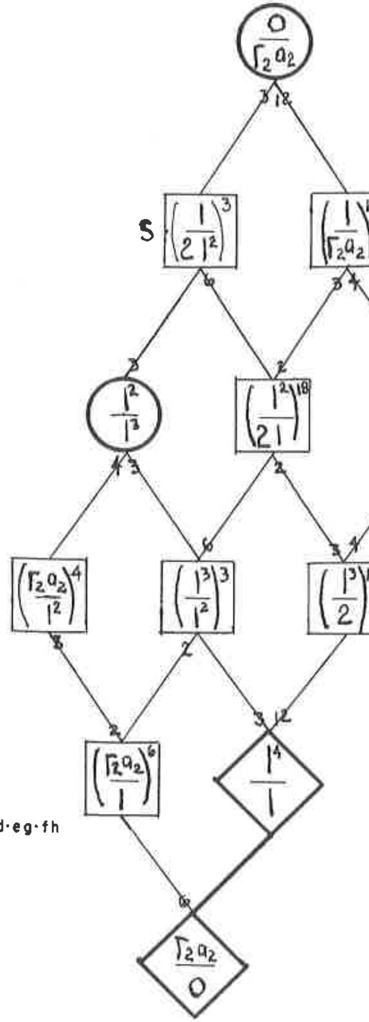
$\beta_2 = eg$

$\beta_3 = fh$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = abcd$

$\alpha_3 = bd$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

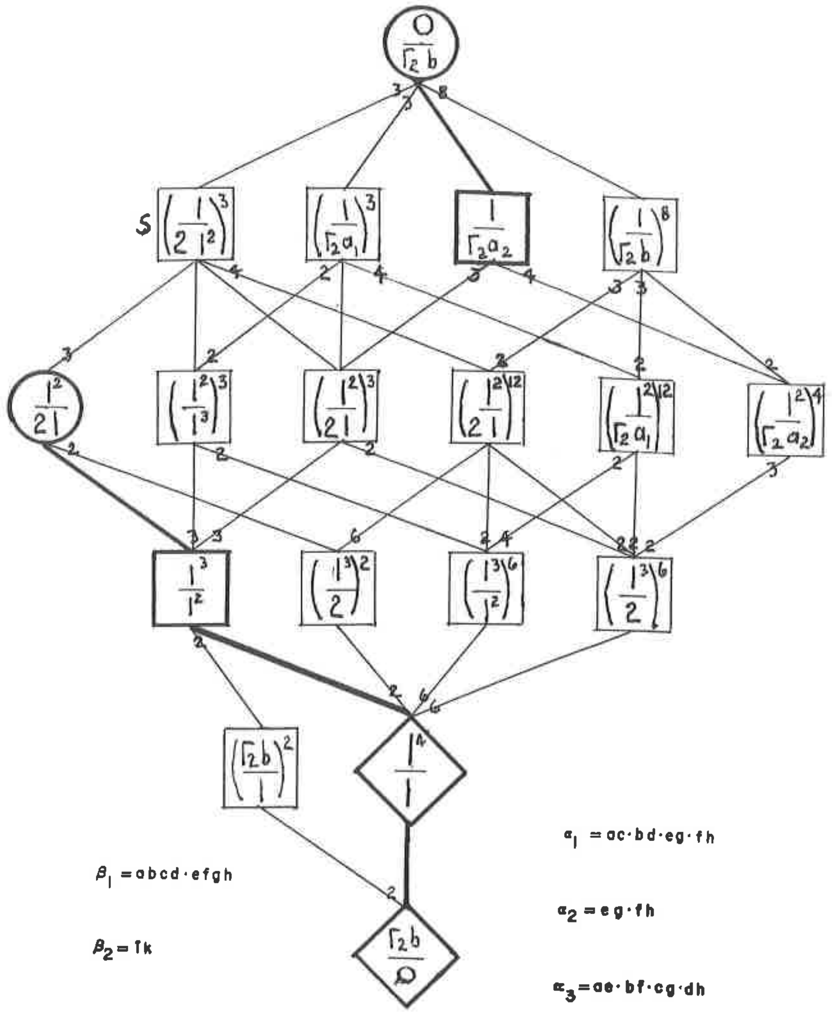
$\beta_2 = ik$

$\beta_3 = jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aecg \cdot bhdf$



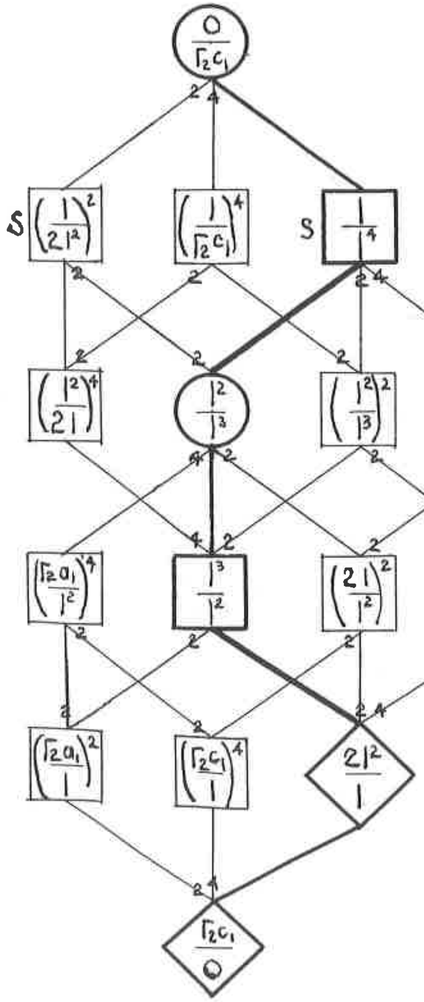
$\beta_1 = abcd \cdot efgh$

$\beta_2 = ik$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ae \cdot bf \cdot cg \cdot dh$



$\beta_1 = ac \cdot bd$

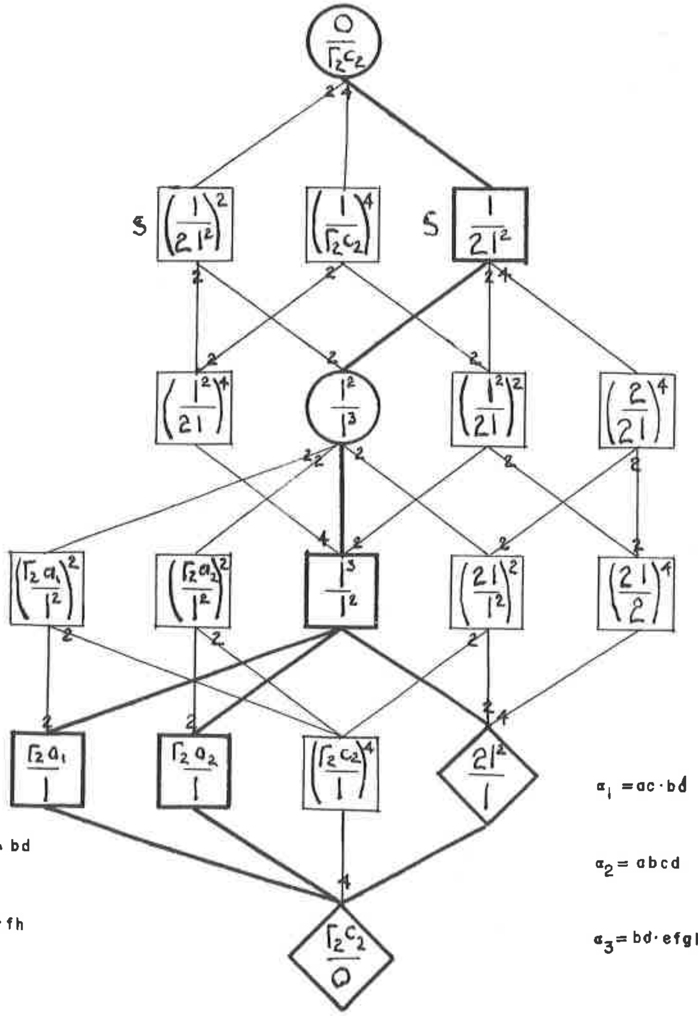
$\beta_2 = eg \cdot fh$

$\beta_3 = ik$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd$

$\alpha_3 = bd \cdot efgh$



$\beta_1 = ac \cdot bd$

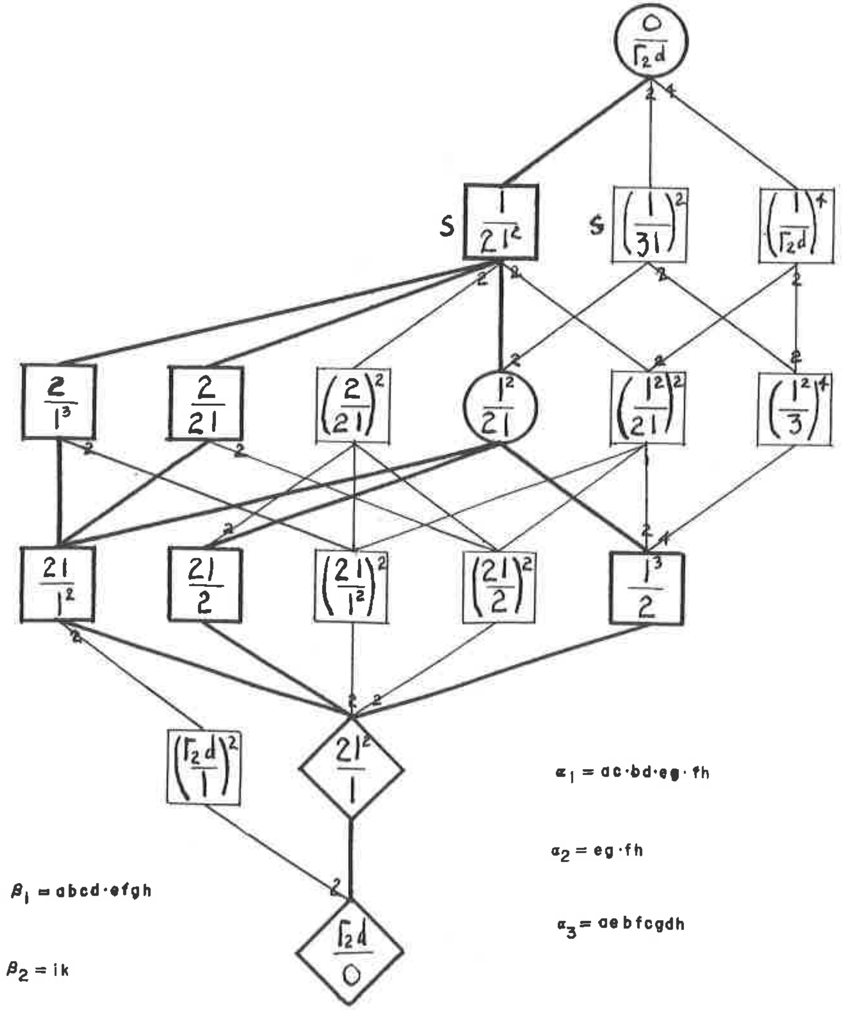
$\beta_2 = eg \cdot fh$

$\beta_3 = ik$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = abcd$

$\alpha_3 = bd \cdot efgh$



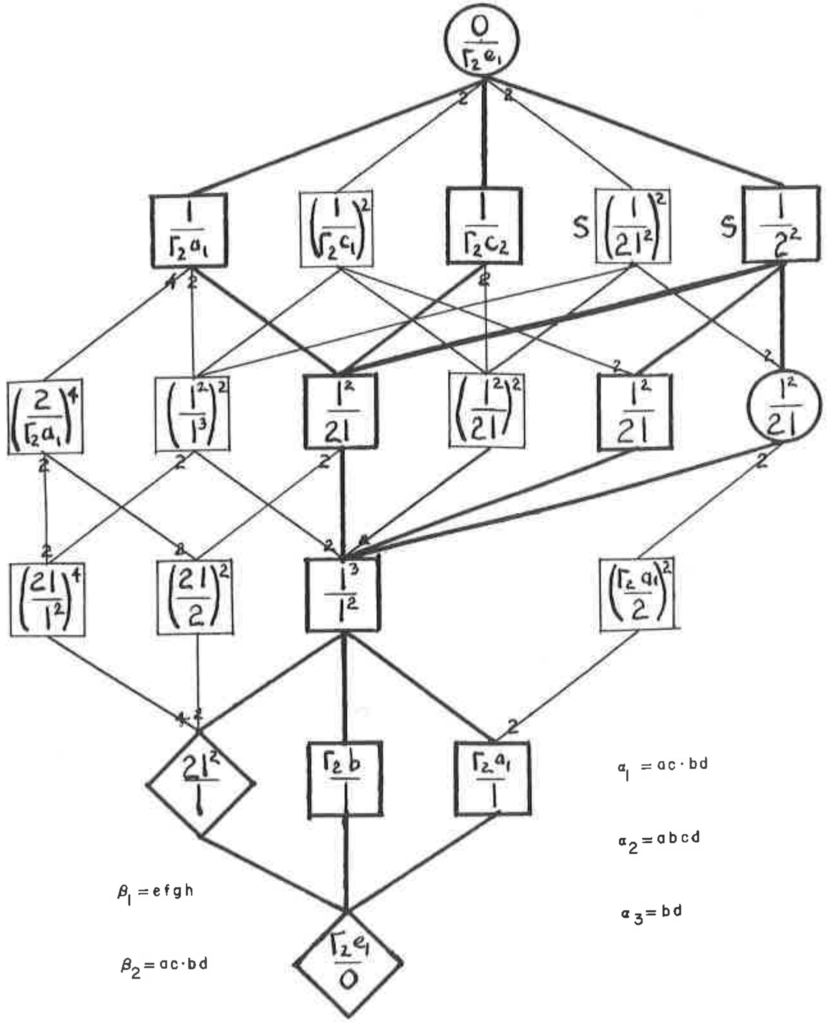
$\beta_1 = abcd \cdot efgh$

$\beta_2 = ik$

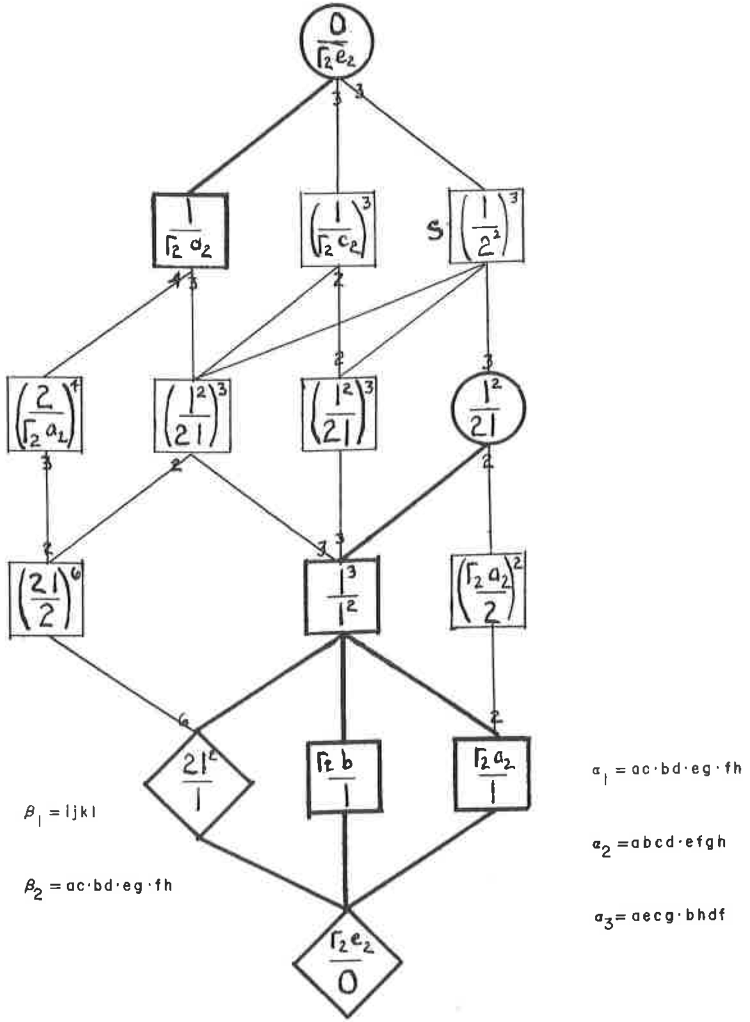
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

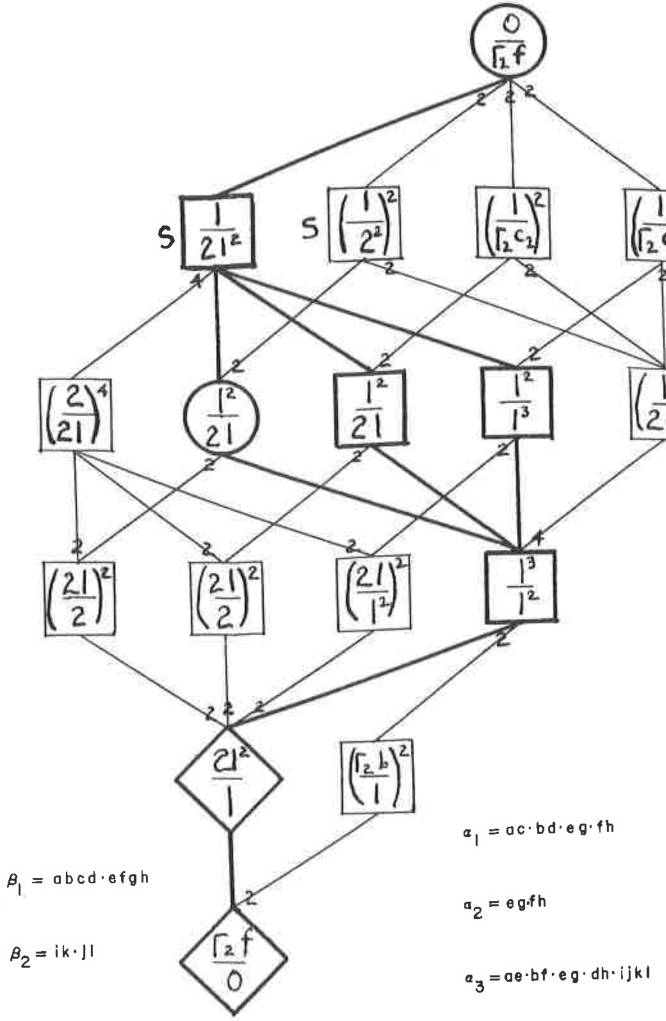
$\alpha_3 = aebfcgdh$



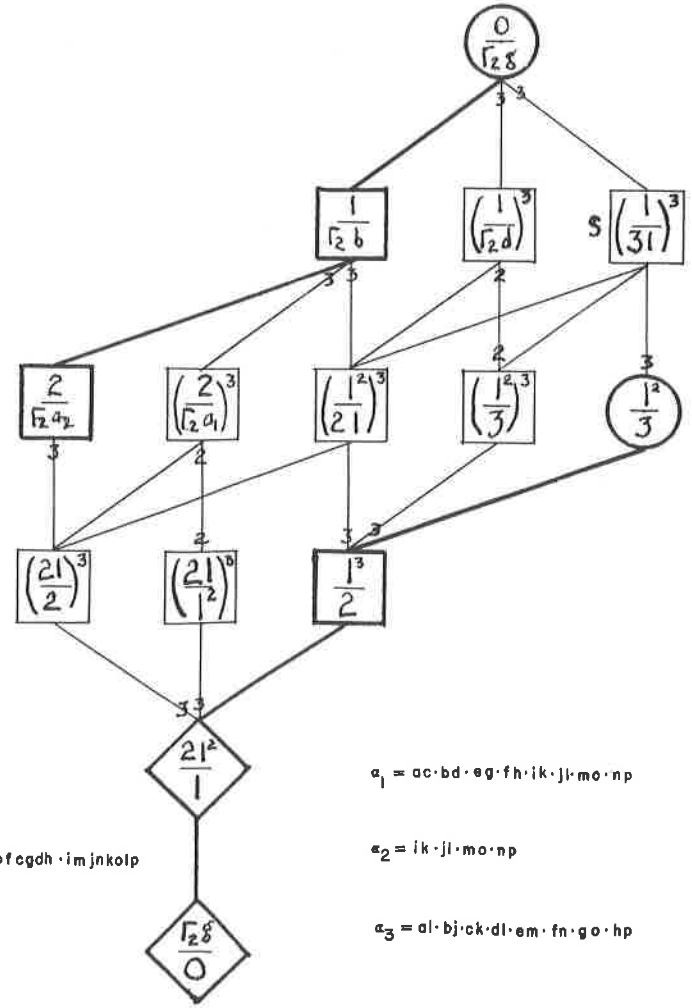
α1 = ac·bd
α2 = abcd
α3 = bd
β1 = efgh
β2 = ac·bd



α1 = ac·bd·eg·fh
α2 = abcd·efgh
α3 = aecg·bhdf
β1 = ijkl
β2 = ac·bd·eg·fh



α1 = ac·bd·eg·fh
α2 = eg·fh
α3 = ae·bf·eg·dh·ijkl
β1 = abcd·efgh
β2 = ik·jl

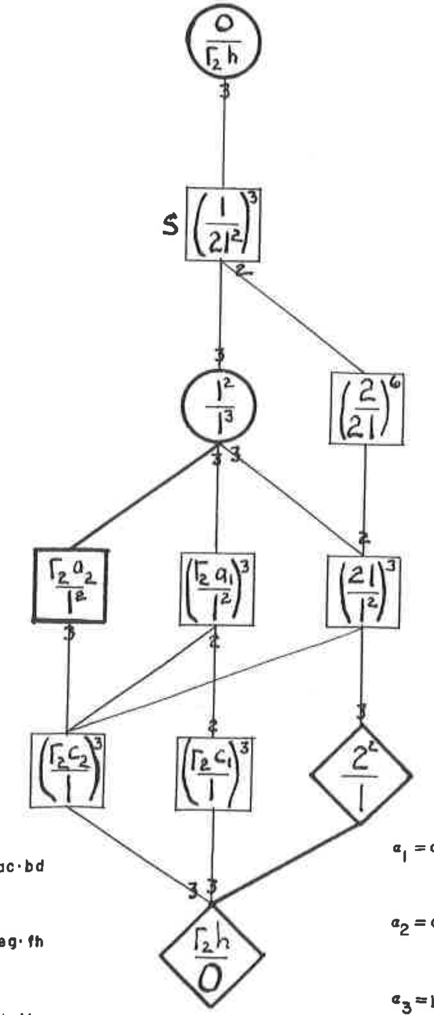


$\beta = aebfcgdh \cdot imjnkolp$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_2 = ik \cdot jl \cdot mo \cdot np$

$\alpha_3 = al \cdot bj \cdot ck \cdot dl \cdot em \cdot fn \cdot go \cdot hp$



$\beta_1 = ac \cdot bd$

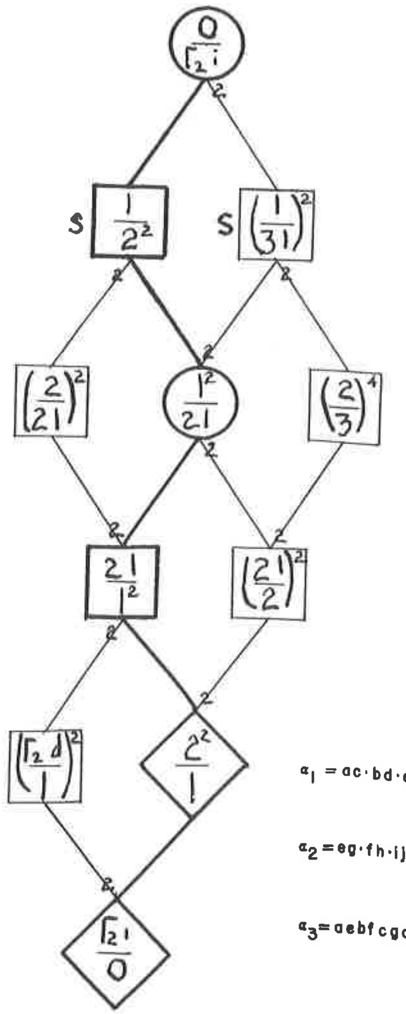
$\beta_2 = eg \cdot fh$

$\beta_3 = ik \cdot jl$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd \cdot efgh$

$\alpha_3 = bd \cdot ijkl$



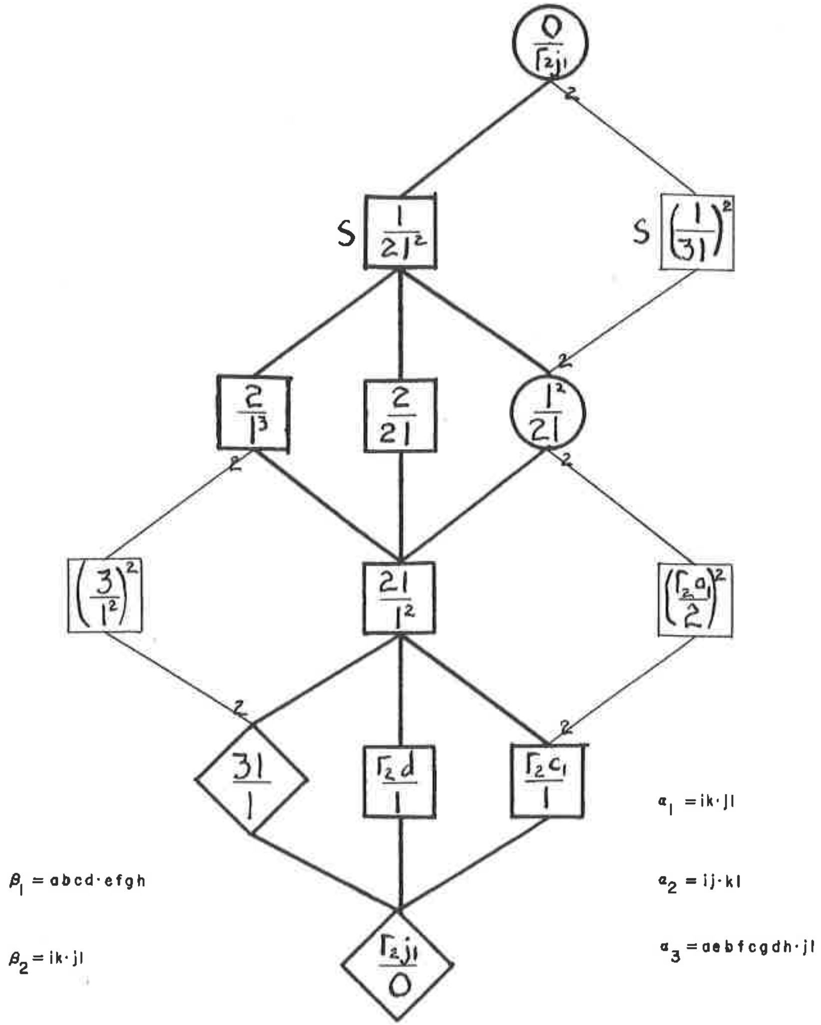
$\beta_1 = abcd \cdot efgh$

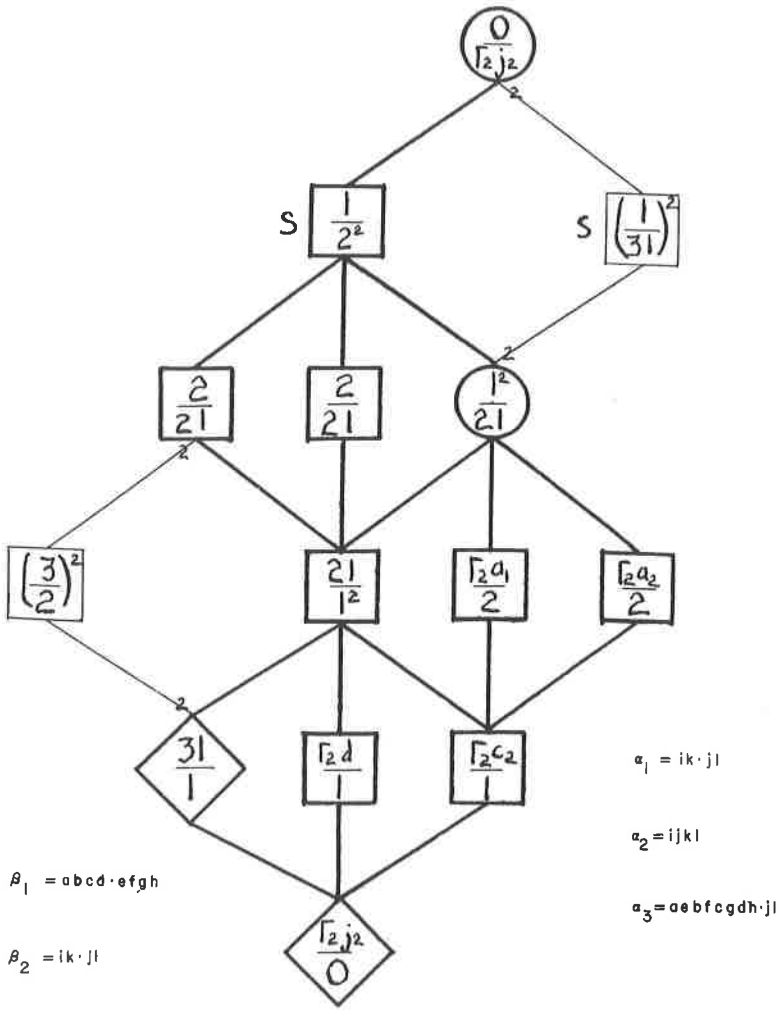
$\beta_2 = ik \cdot jl$

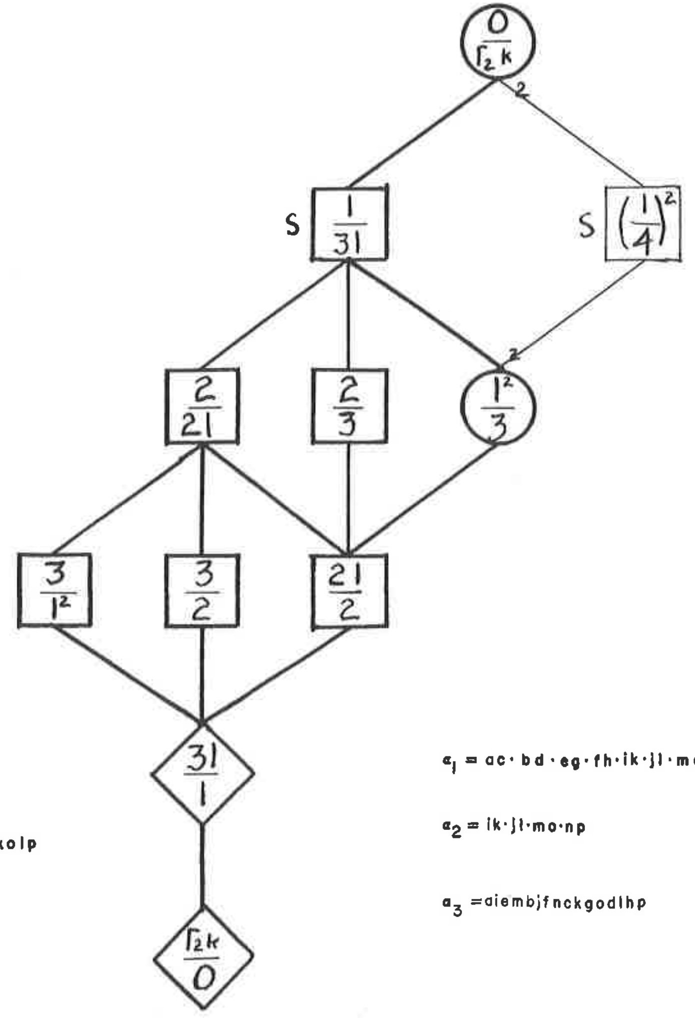
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh \cdot ijkl$

$\alpha_3 = aebfcgdh$


β_1 = abcd·efgh
β_2 = ik·jl
α_1 = ik·jl
α_2 = ij·kl
α_3 = aebfcgdh·jl


β_1 = abcd·efgh
β_2 = ik·jl
α_1 = ik·jl
α_2 = ijkl
α_3 = aebfcgdh·jl

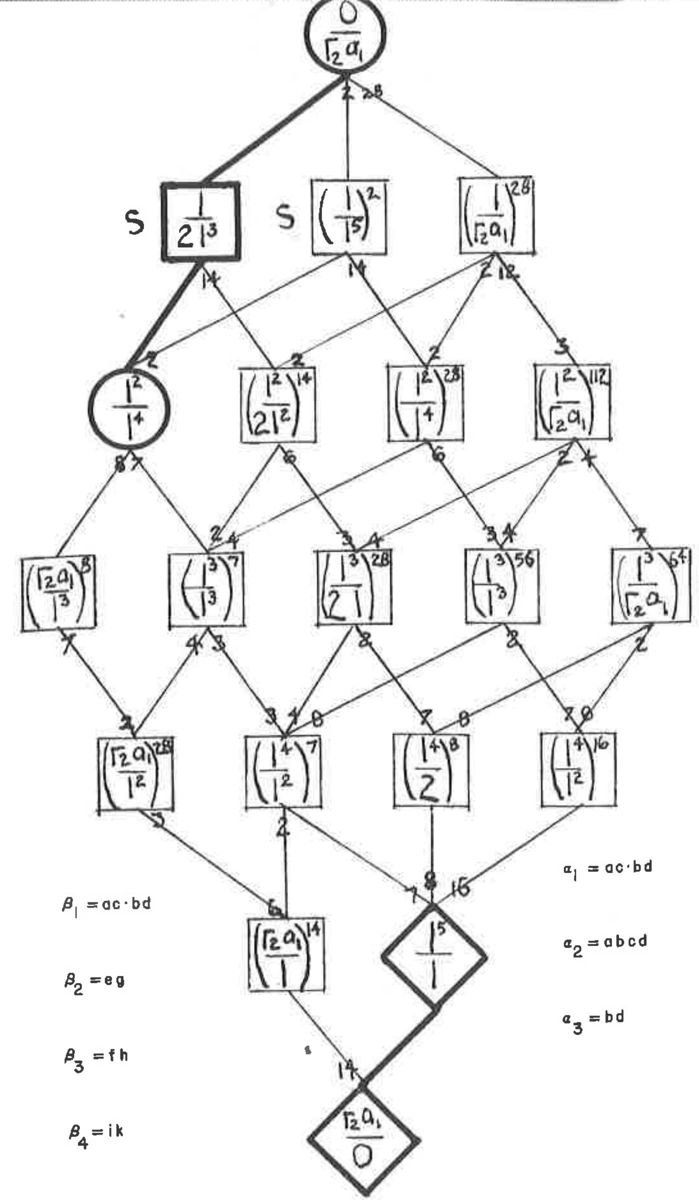


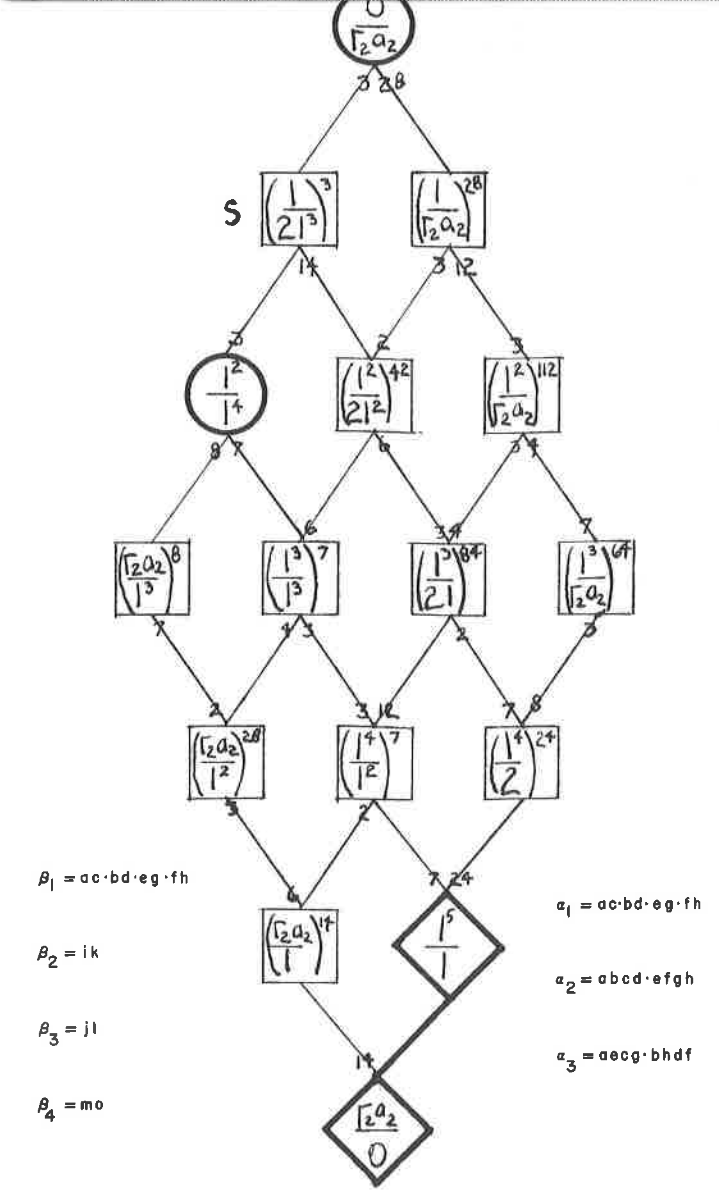
$\beta = aebfcgdh \cdot lmjnkolp$

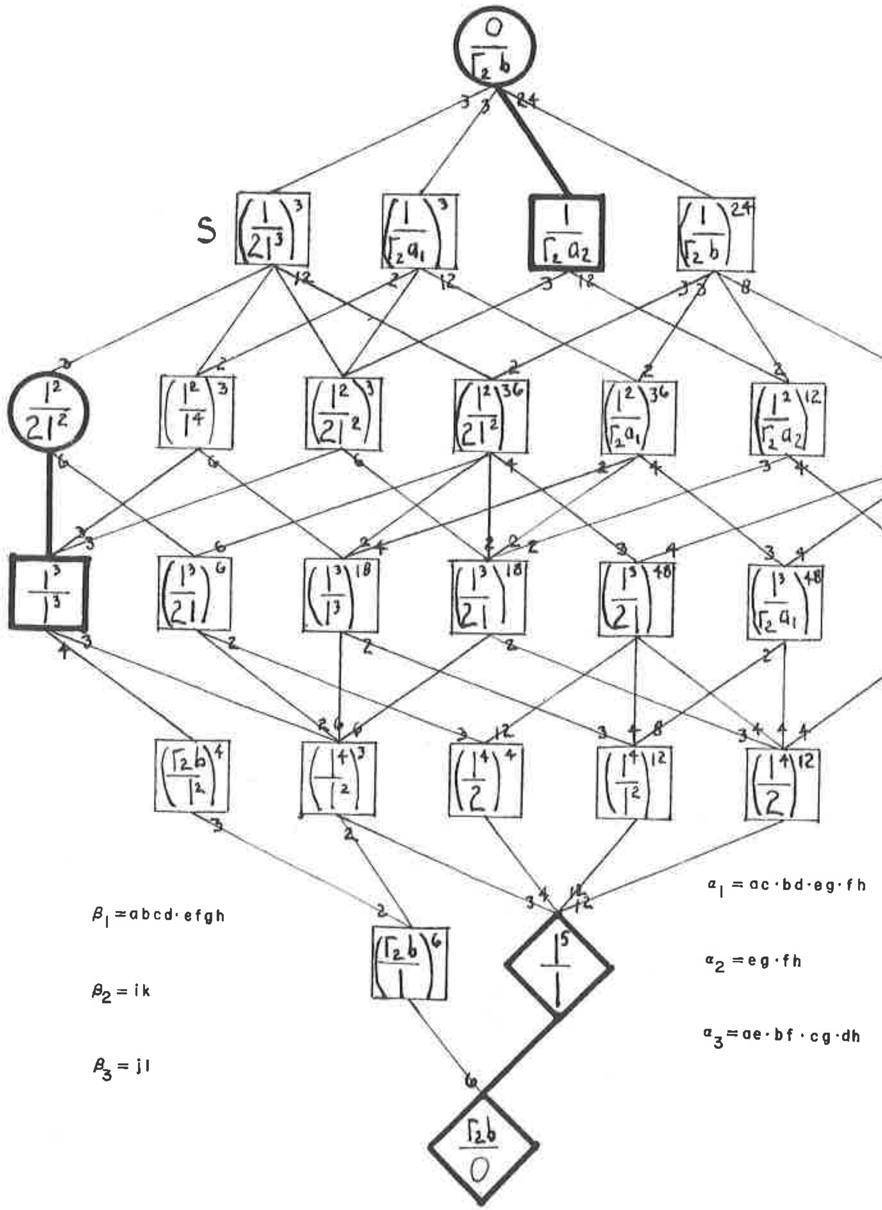
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

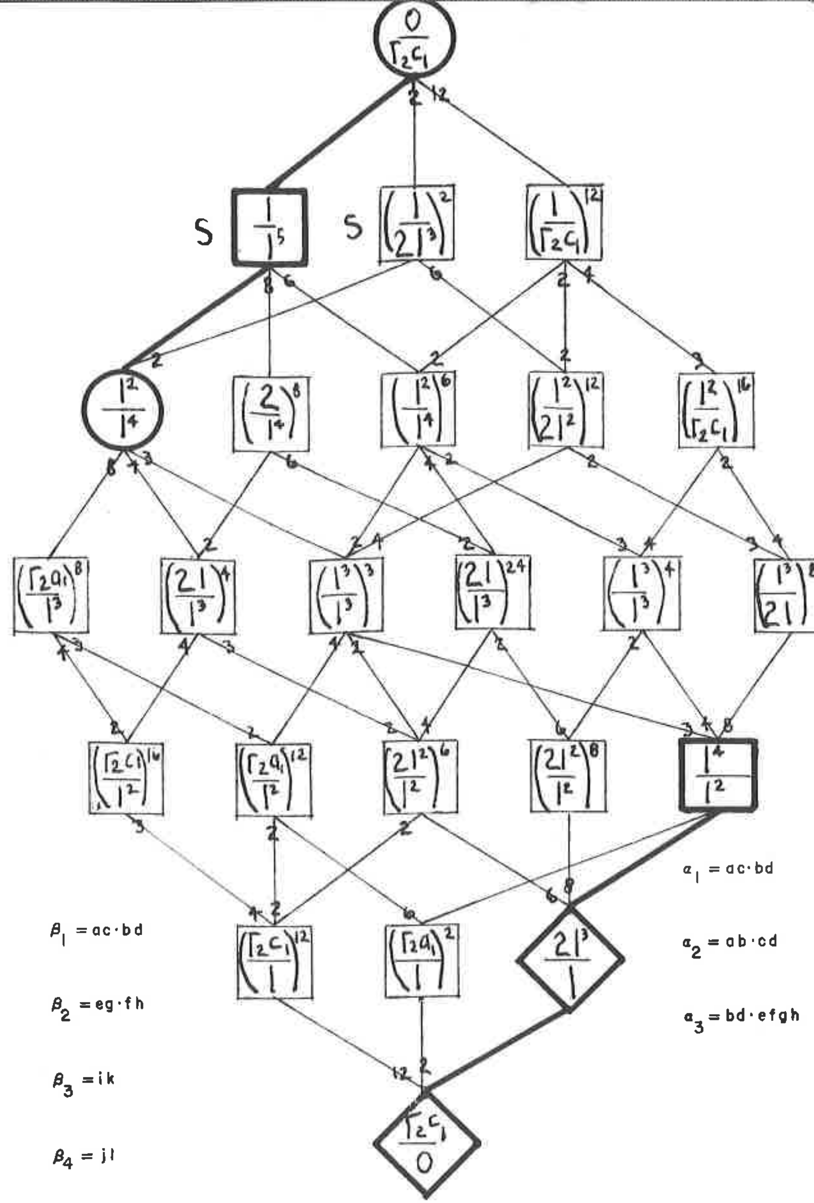
$\alpha_2 = ik \cdot jl \cdot mo \cdot np$

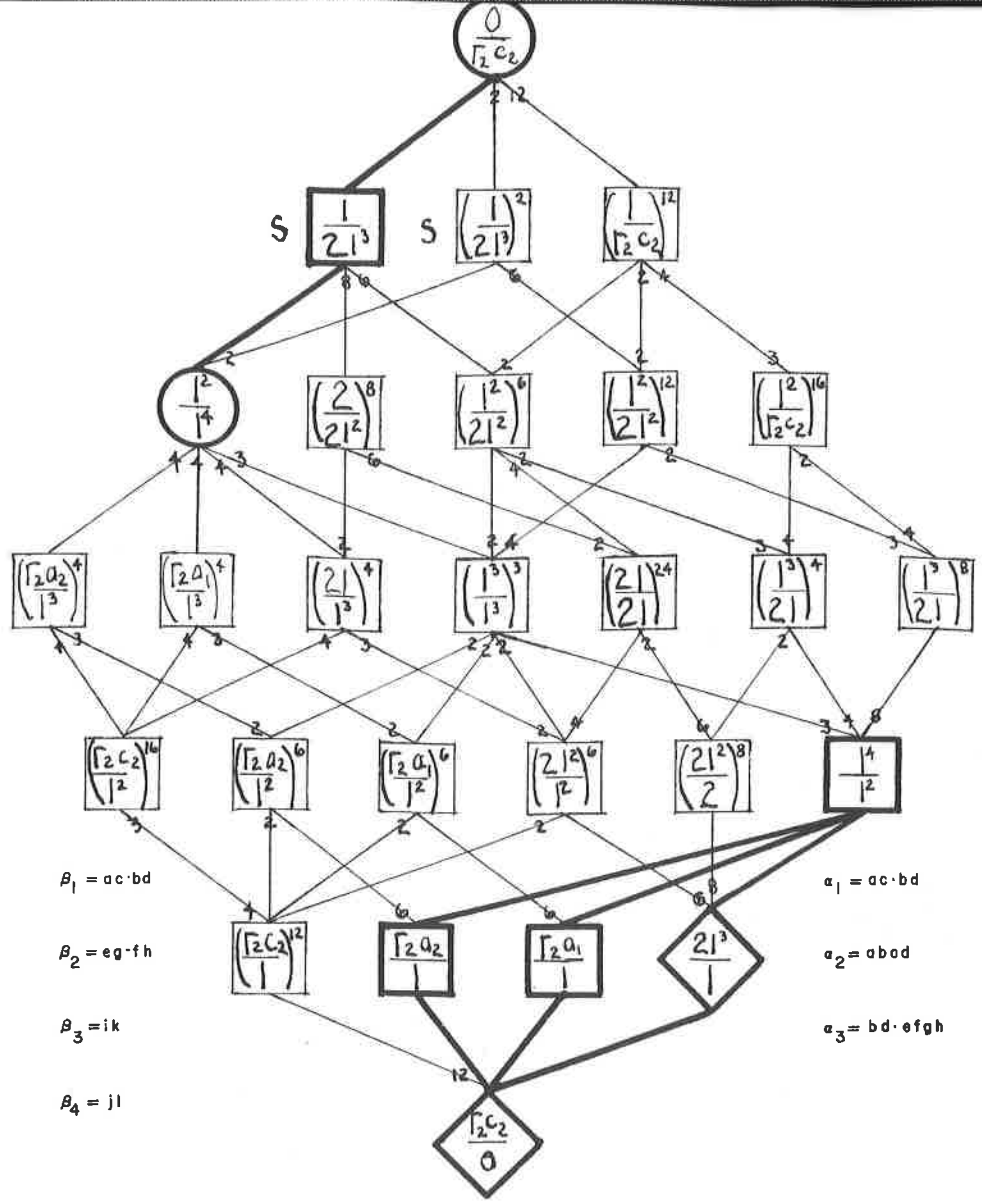
$\alpha_3 = aiembjfnckgodlhp$

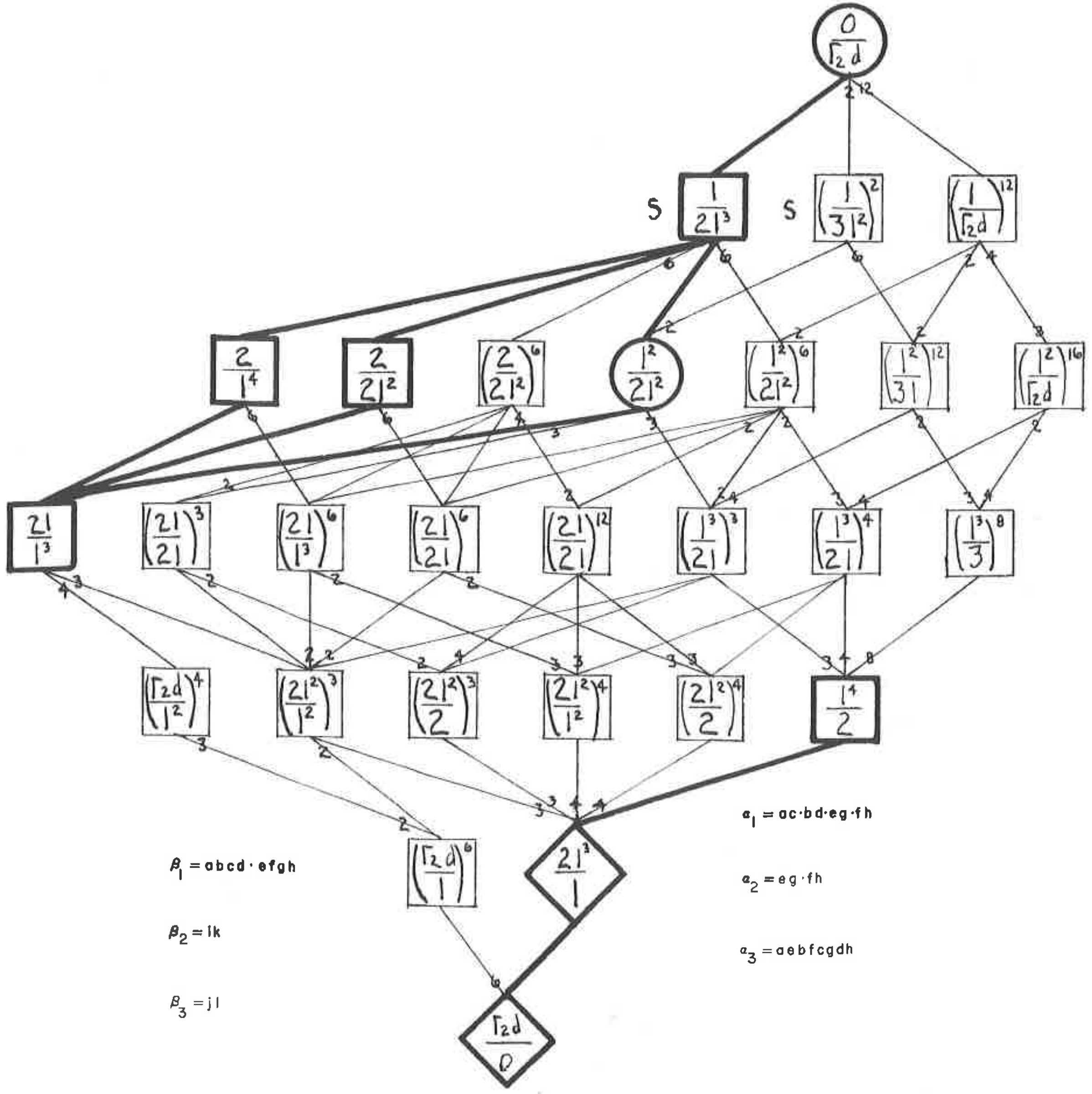

β1 = ac·bd
β2 = eg
β3 = fh
β4 = ik
α1 = ac·bd
α2 = abcd
α3 = bd


β1 = ac·bd·eg·fh
β2 = ik
β3 = jl
β4 = mo
α1 = ac·bd·eg·fh
α2 = abcd·efgh
α3 = aecg·bhdf


S
β1 = abcd·efgh
β2 = ik
β3 = jl
α1 = ac·bd·eg·fh
α2 = eg·fh
α3 = ae·bf·cg·dh


β1 = ac·bd
β2 = eg·fh
β3 = ik
β4 = jl
α1 = ac·bd
α2 = ab·cd
α3 = bd·efgh


β1 = ac·bd
β2 = eg·fh
β3 = ik
β4 = jl
α1 = ac·bd
α2 = abcd
α3 = bd·efgh



$\beta_1 = abcd \cdot efgh$

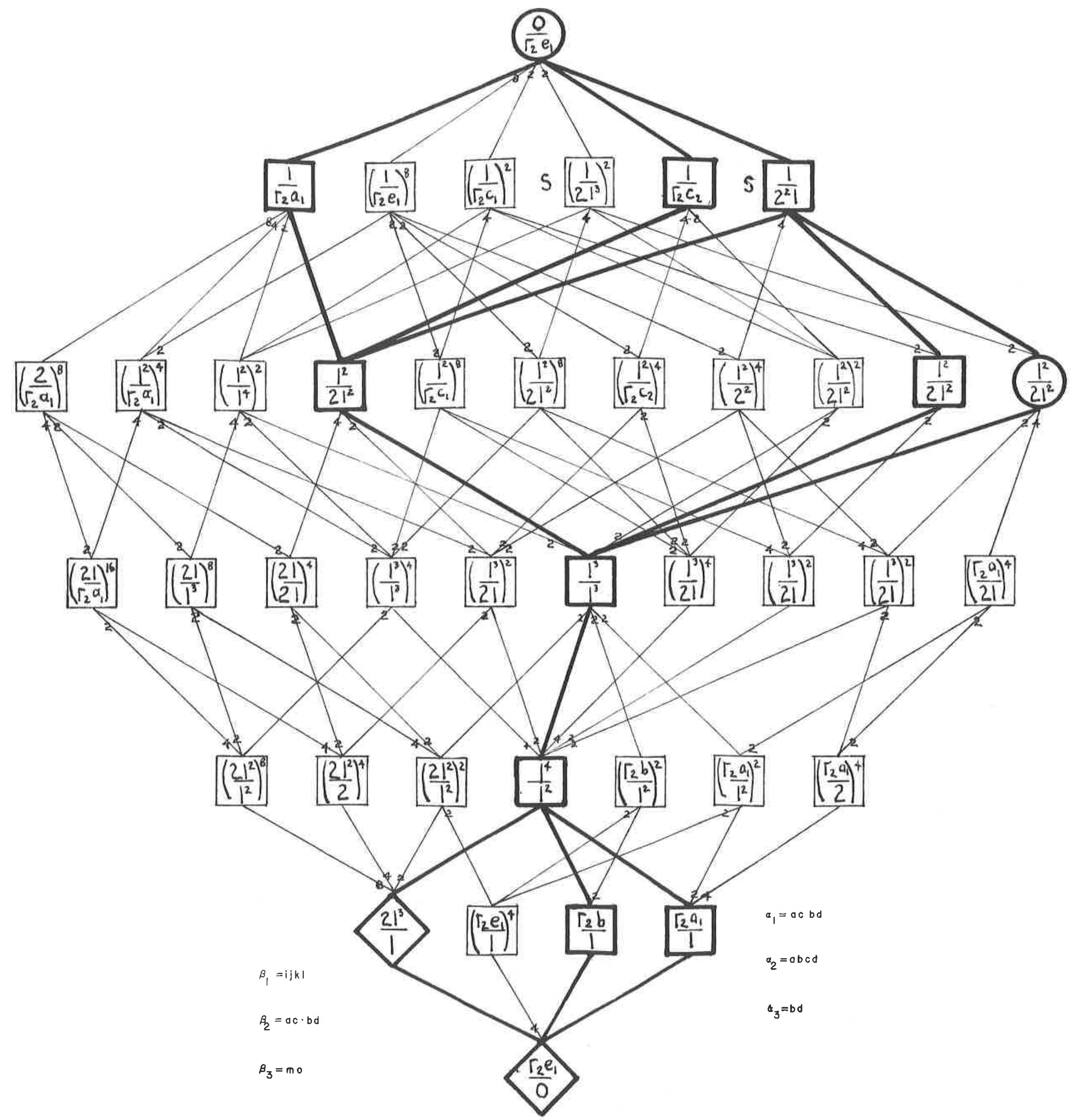
$\beta_2 = lk$

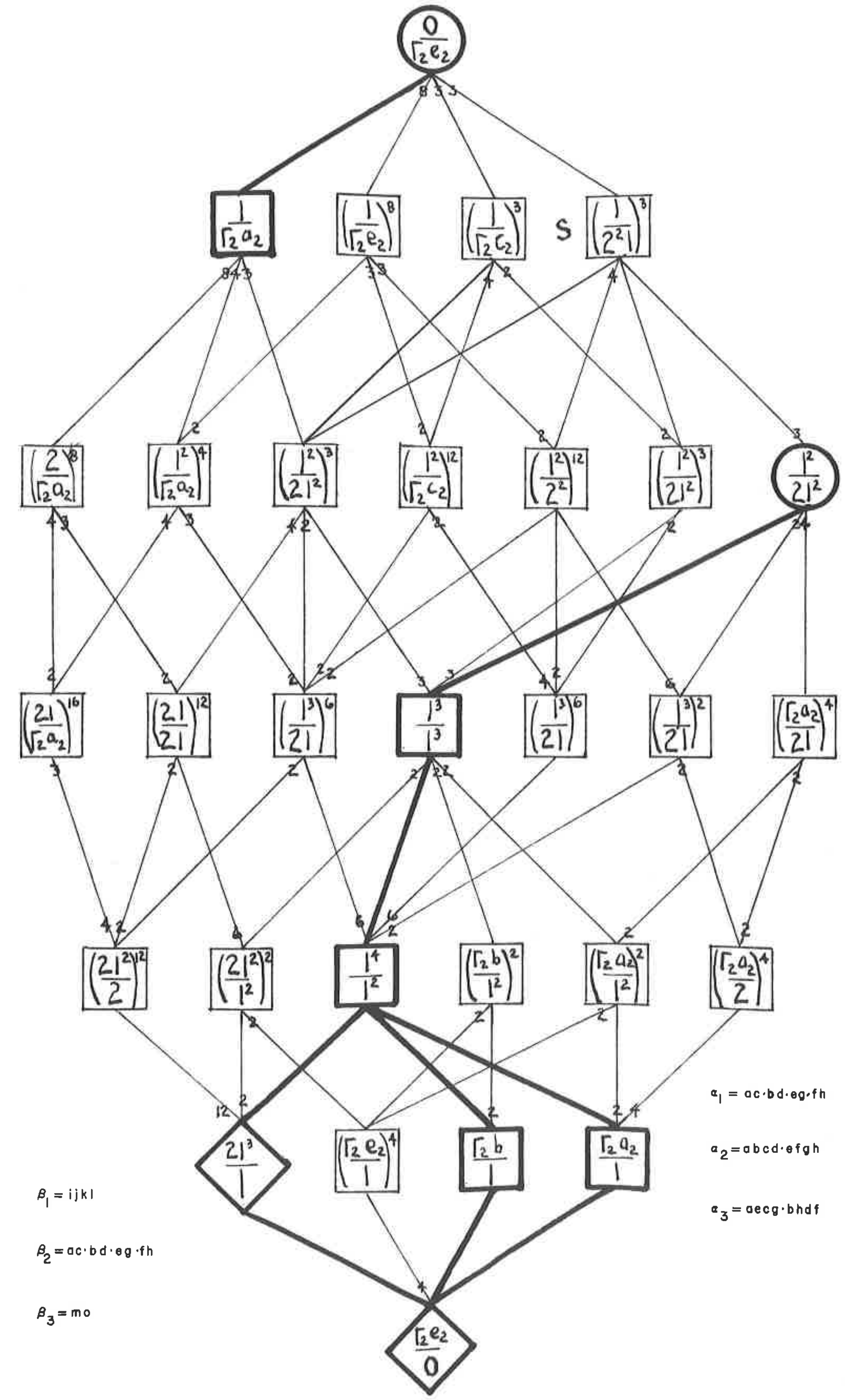
$\beta_3 = jl$

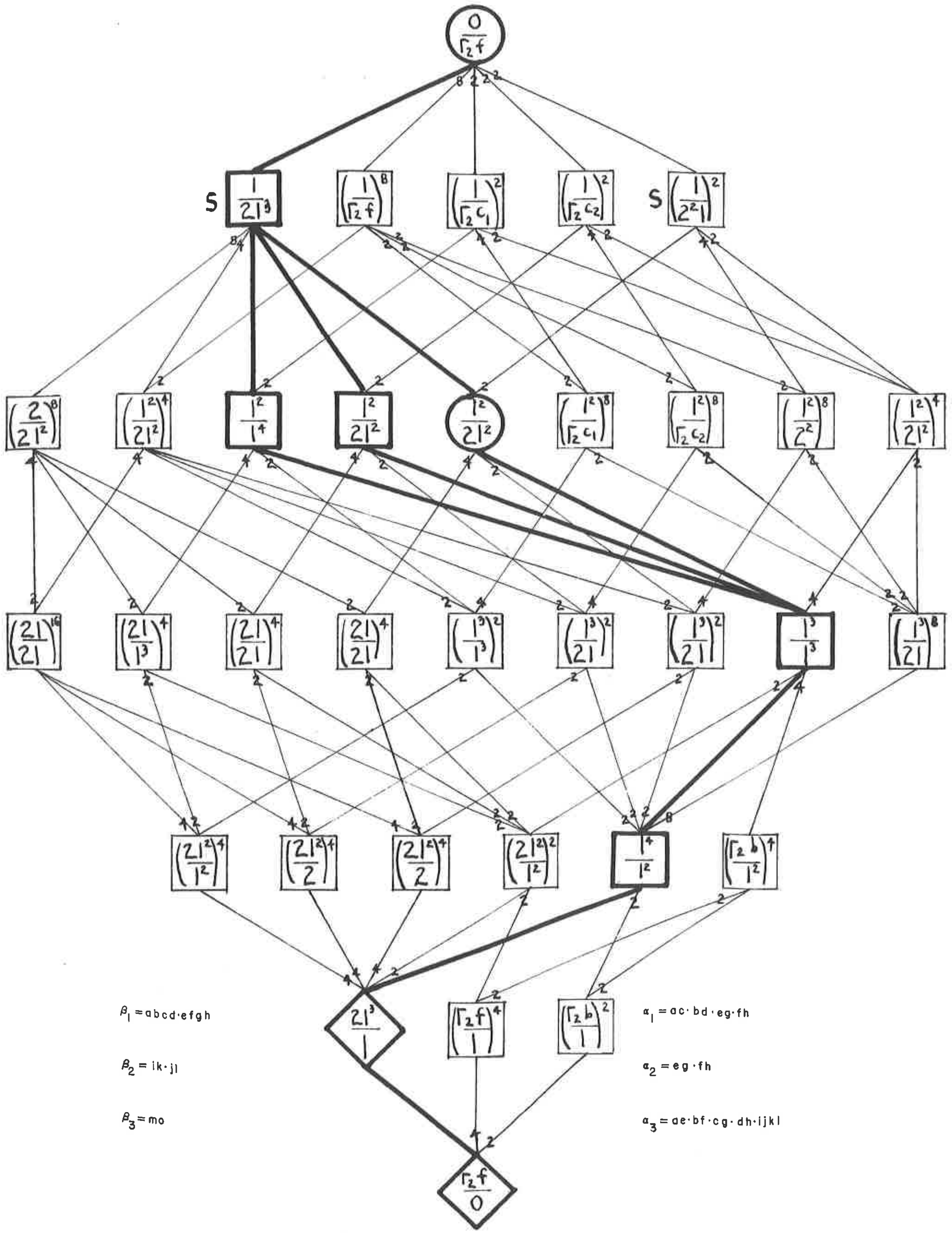
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = aebfcgdh$


β1 = ijkl
β2 = ac · bd
β3 = mo
α1 = ac bd
α2 = abcd
α3 = bd


β1 = ijkl
β2 = ac·bd·eg·fh
β3 = mo
α1 = ac·bd·eg·fh
α2 = abcd·efgh
α3 = aecg·bhdf



$\beta_1 = abcd \cdot efgh$

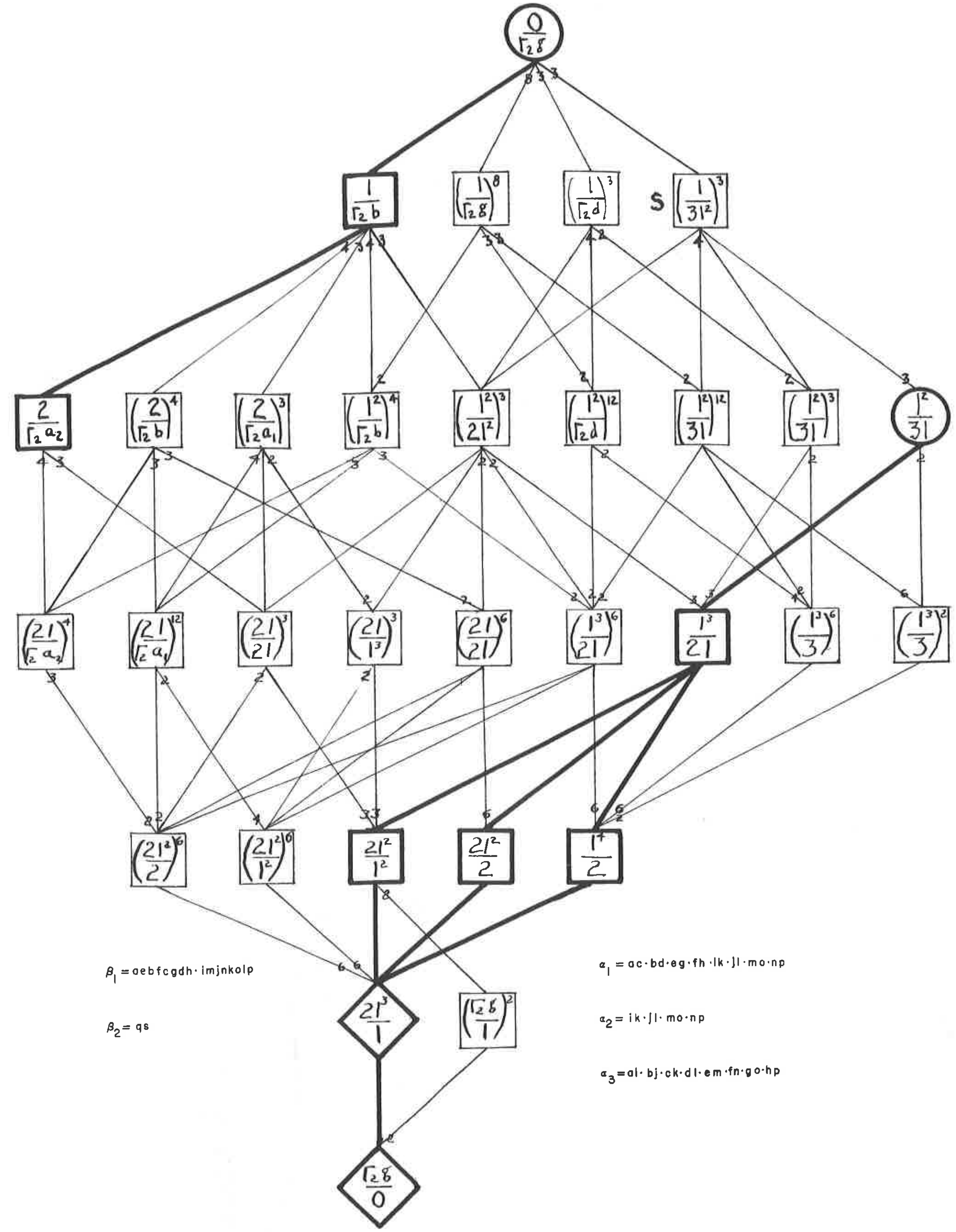
$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ae \cdot bf \cdot cg \cdot dh \cdot ijkl$



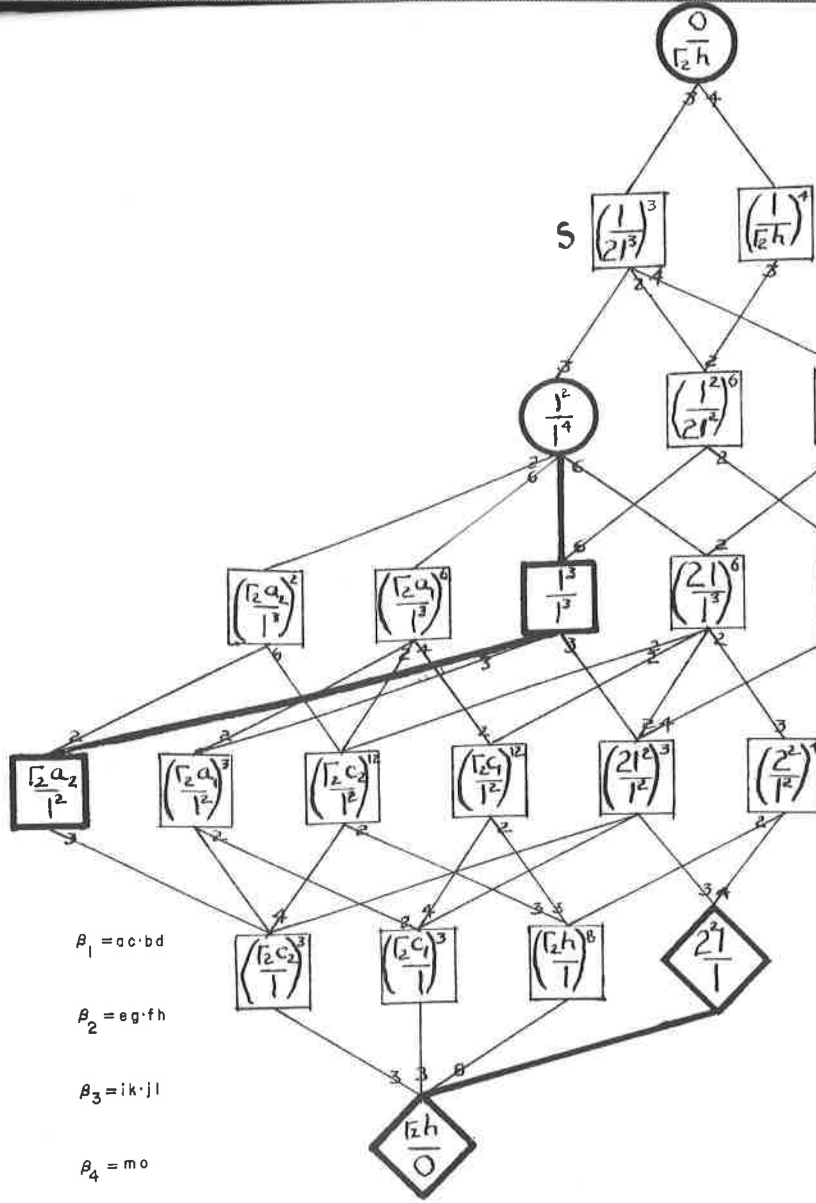
$\beta_1 = aebfcgdh \cdot imjnkolp$

$\beta_2 = qs$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_2 = ik \cdot jl \cdot mo \cdot np$

$\alpha_3 = al \cdot bj \cdot ck \cdot dl \cdot em \cdot fn \cdot go \cdot hp$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = ik \cdot jl$

$\beta_4 = mo$

$\alpha_1 = ac \cdot bd$

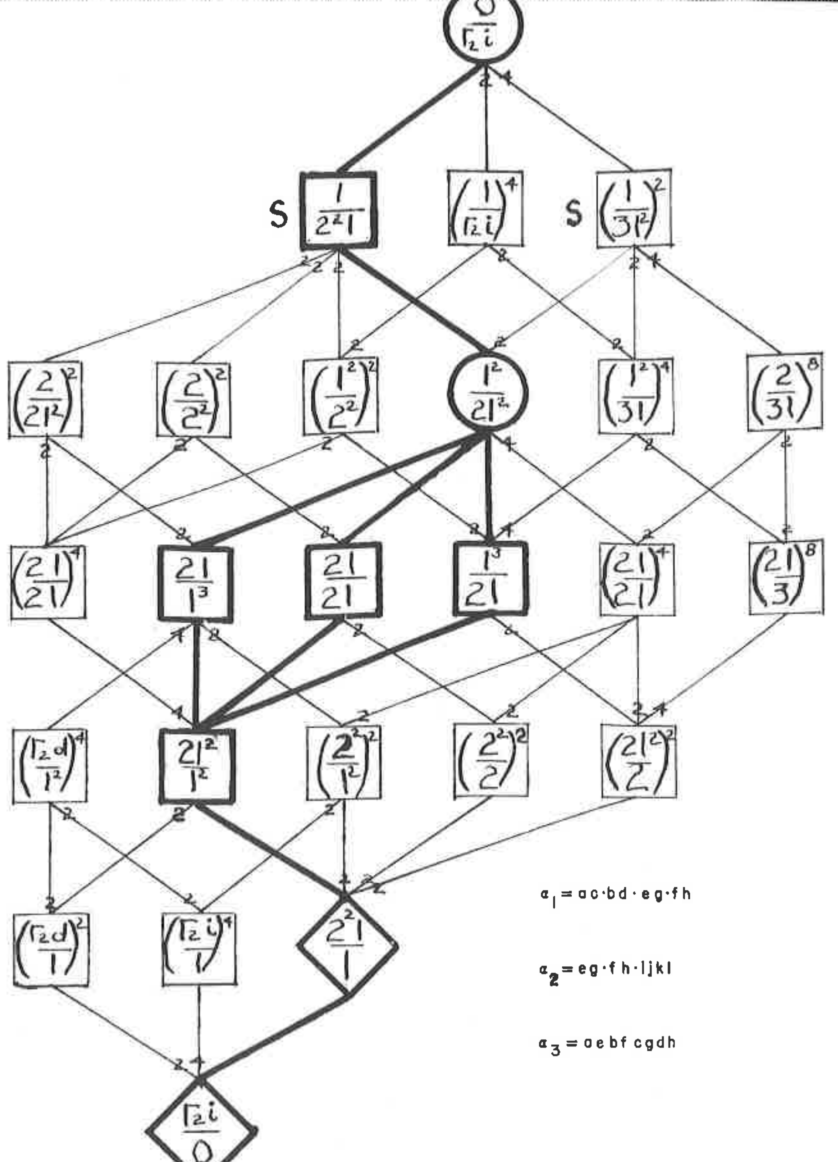
$\alpha_2 = ab \cdot cd \cdot efgh$

$\alpha_3 = bd \cdot ijkl$

$\beta_1 = abcd \cdot efgh$

$\beta_2 = ik \cdot jl$

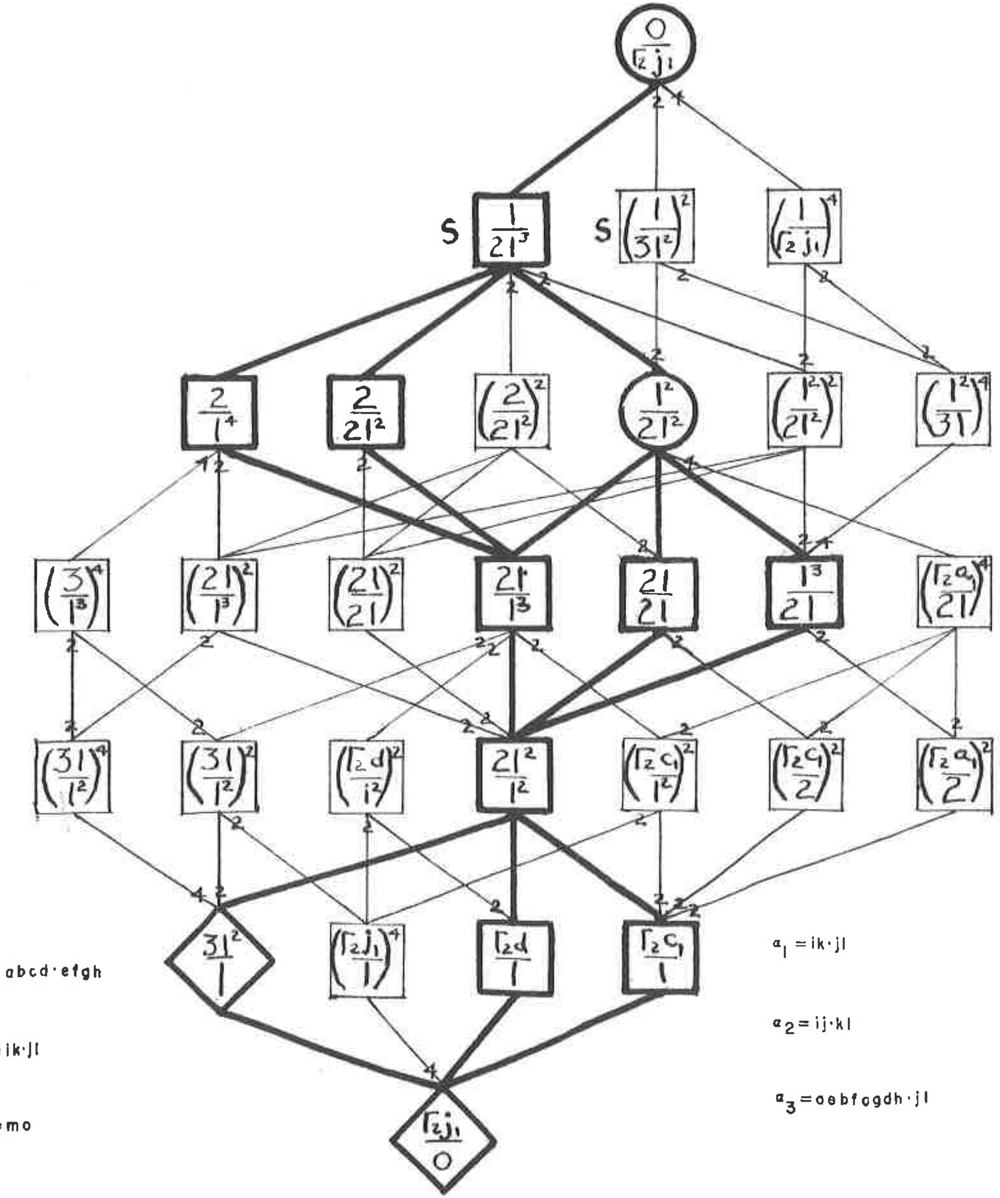
$\beta_3 = mo$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = eg \cdot fh \cdot ijkl$

$\alpha_3 = ae\ bf\ cgdh$



$\beta_1 = abcd \cdot efgh$

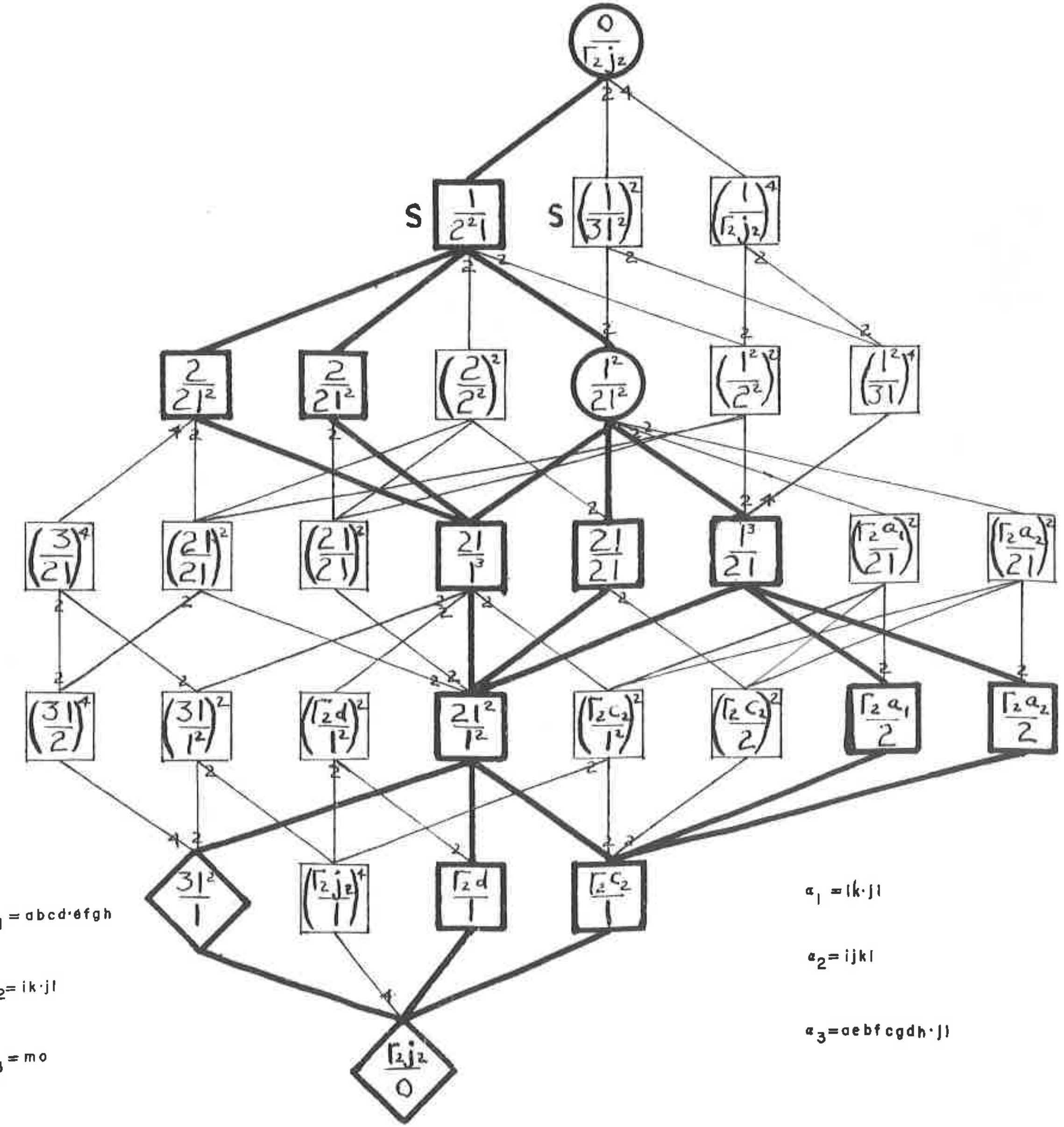
$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

$\alpha_1 = ik \cdot jl$

$\alpha_2 = ij \cdot kl$

$\alpha_3 = aebfcgdh \cdot jl$



$\beta_1 = abcd \cdot efgh$

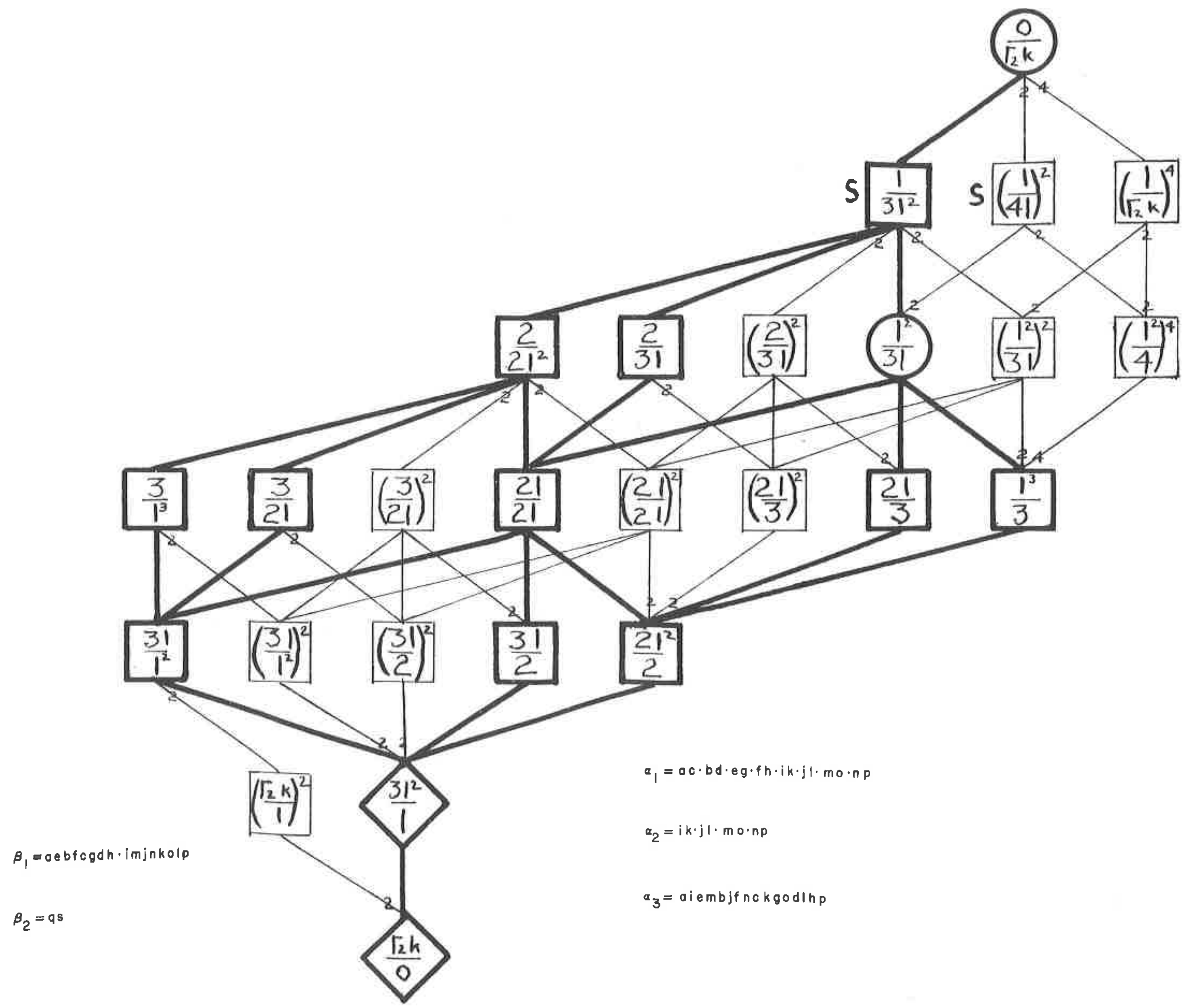
$\beta_2 = ik \cdot jl$

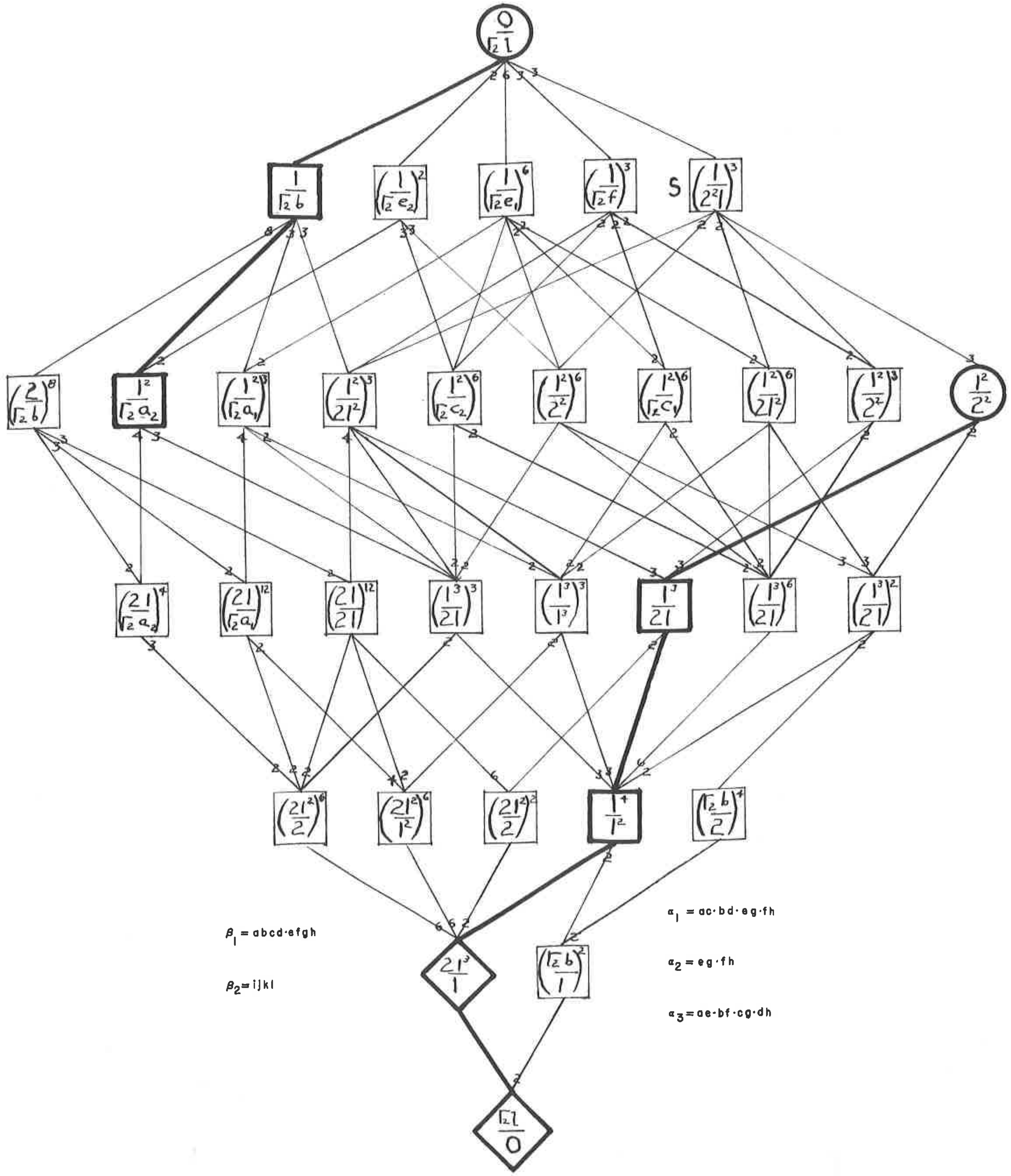
$\beta_3 = mo$

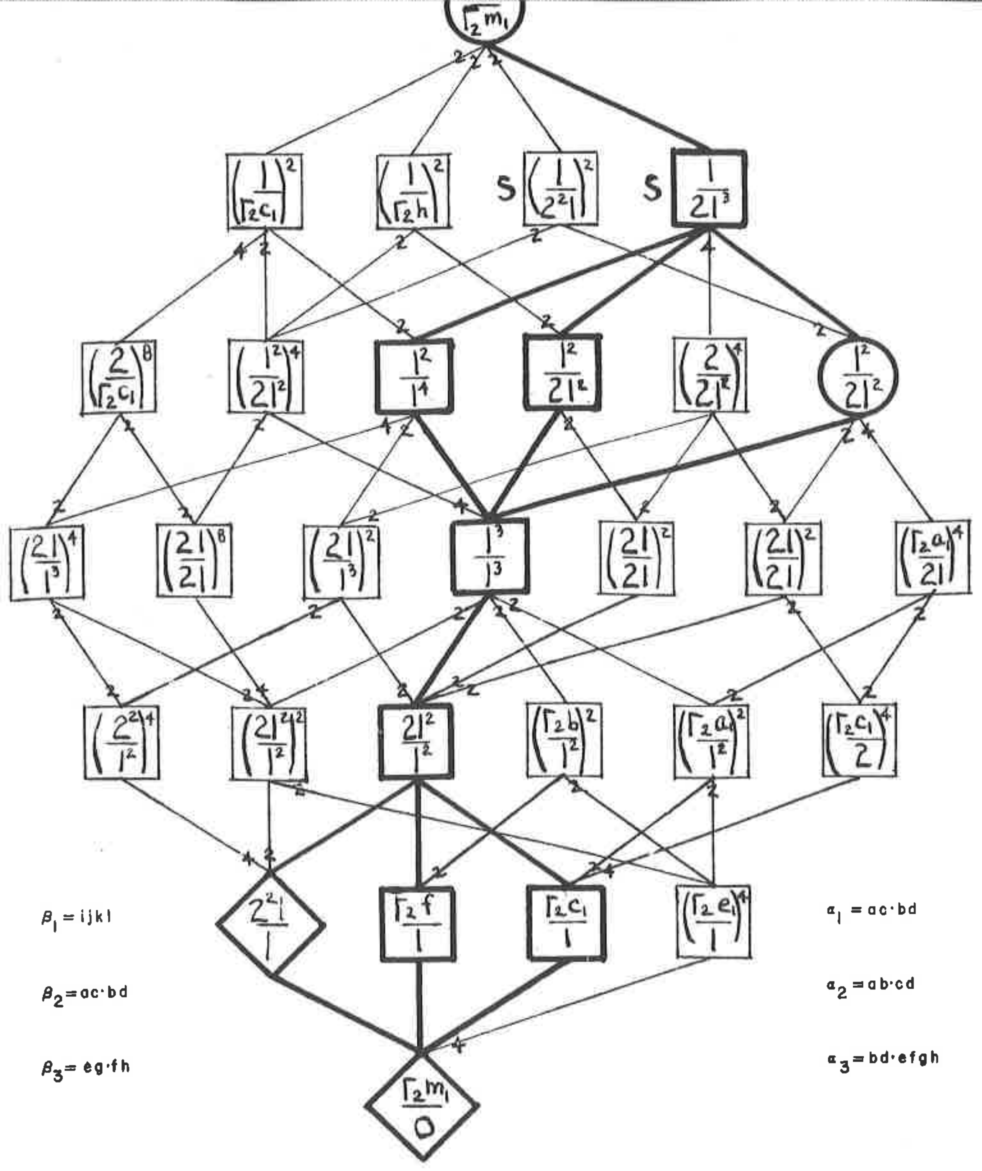
$\alpha_1 = ik \cdot jl$

$\alpha_2 = ijkl$

$\alpha_3 = aebfcgdh \cdot jl$


S
S
α1 = ac·bd·eg·fh·ik·jl·mo·np
α2 = ik·jl·mo·np
α3 = aiembjfnckgodlhp
β1 = aebfcgdh·imjnkolp
β2 = qs


β1 = abcd·efgh
β2 = ijkl
α1 = ac·bd·eg·fh
α2 = eg·fh
α3 = ae·bf·cg·dh



$\beta_1 = ijkl$

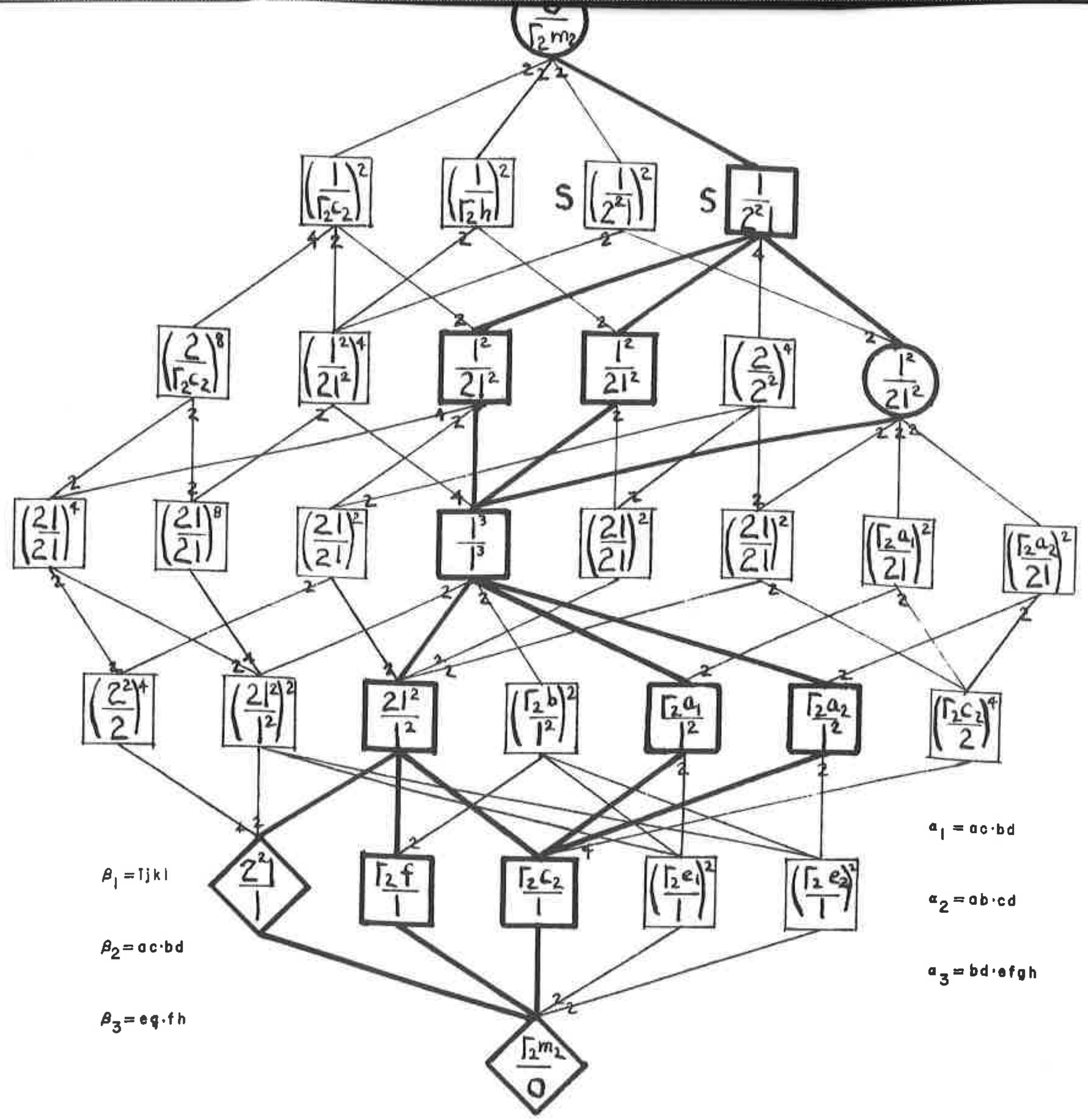
$\beta_2 = ac \cdot bd$

$\beta_3 = eg \cdot fh$

$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd$

$\alpha_3 = bd \cdot efgh$



$\beta_1 = ijkl$

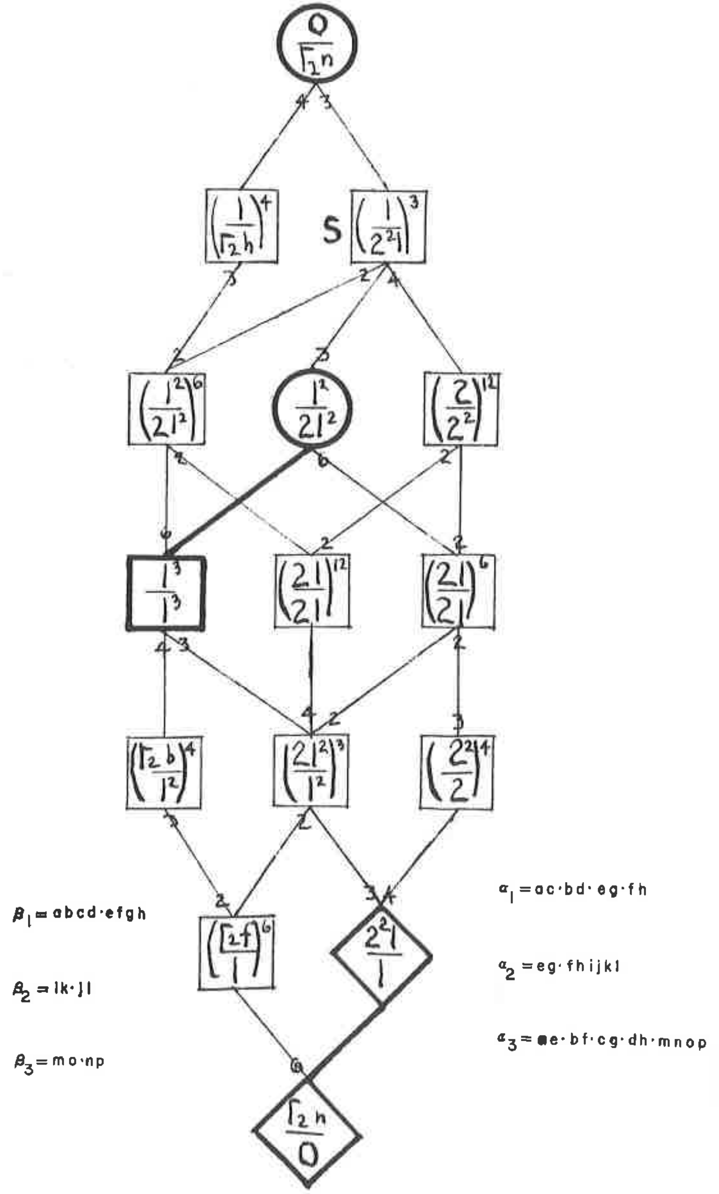
$\beta_2 = ac \cdot bd$

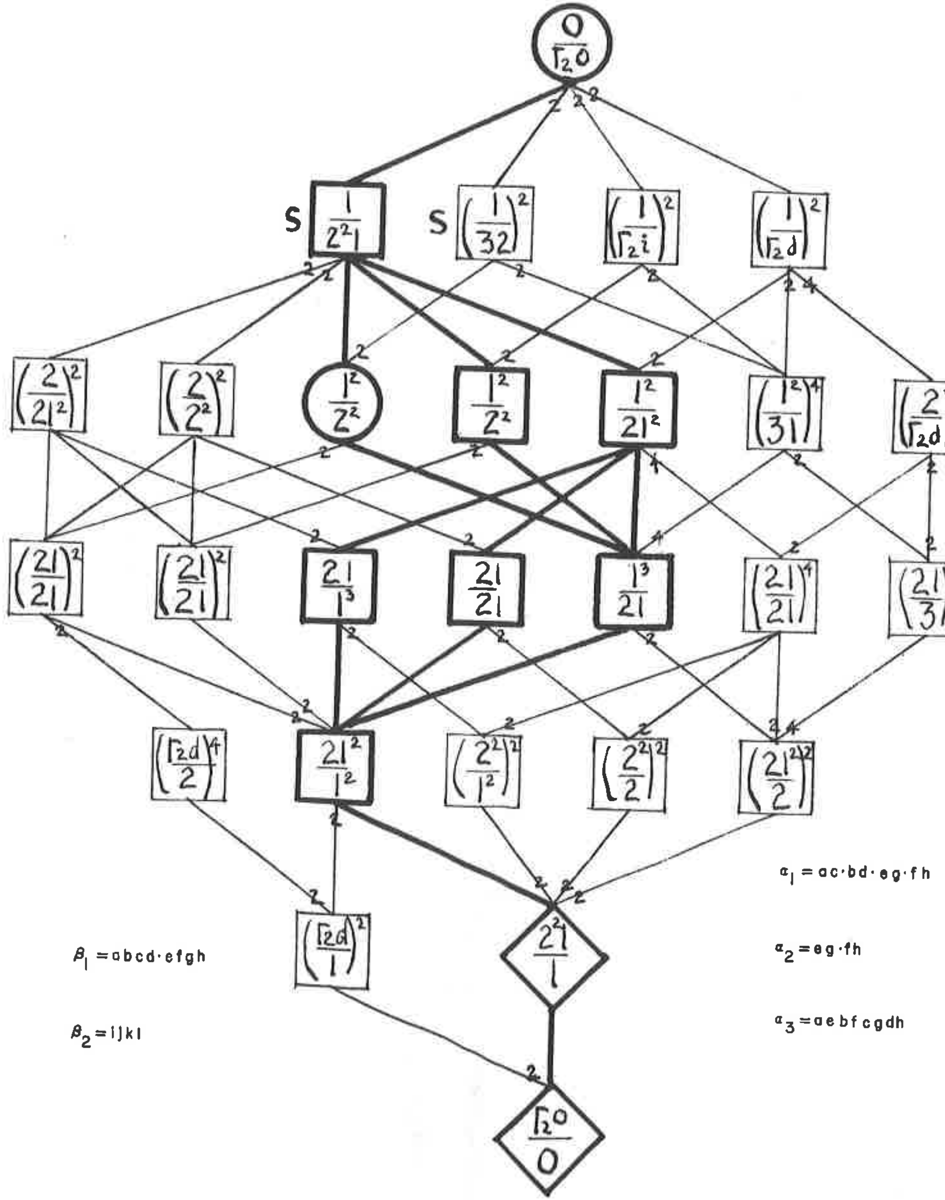
$\beta_3 = eg \cdot fh$

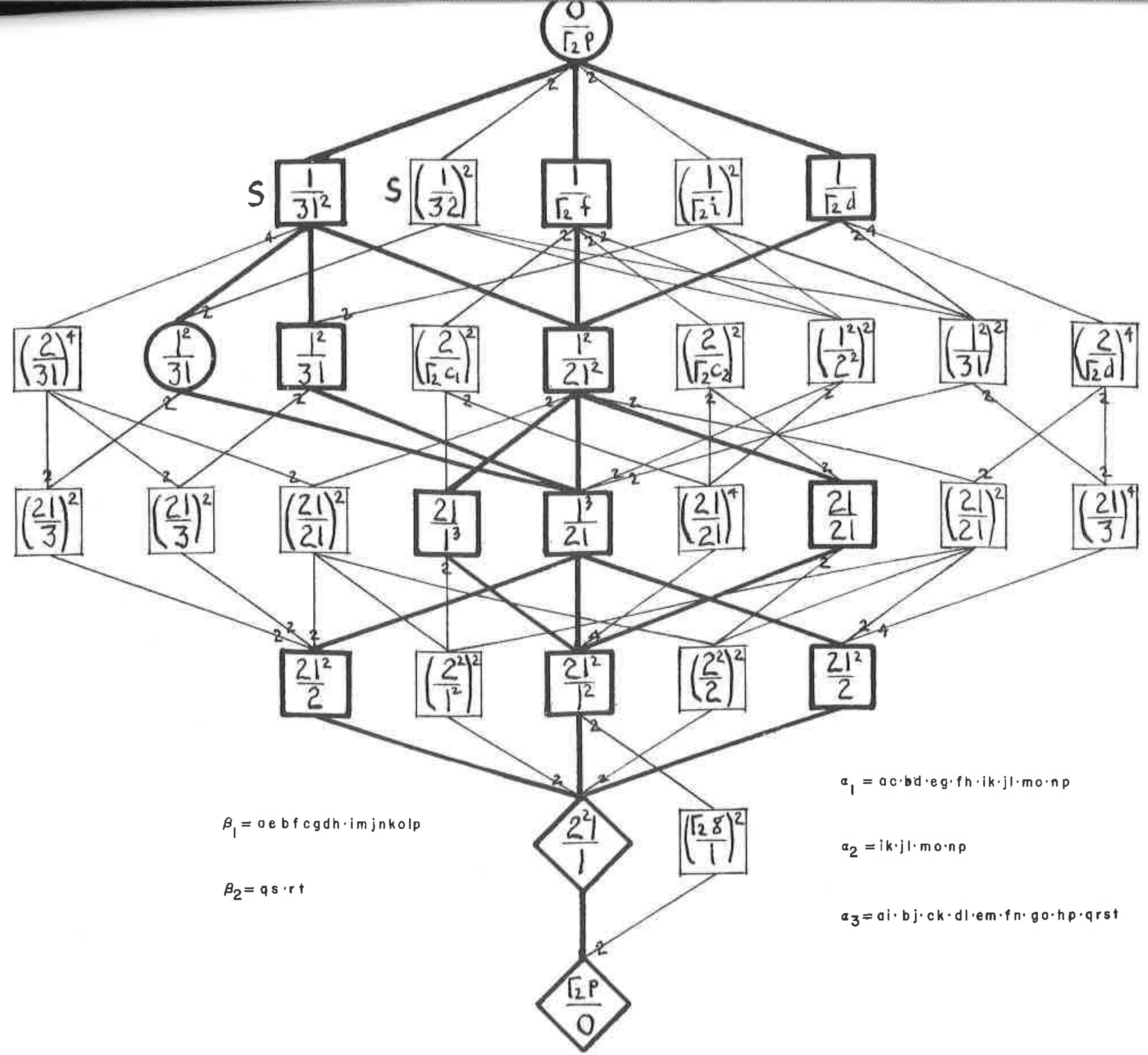
$\alpha_1 = ac \cdot bd$

$\alpha_2 = ab \cdot cd$

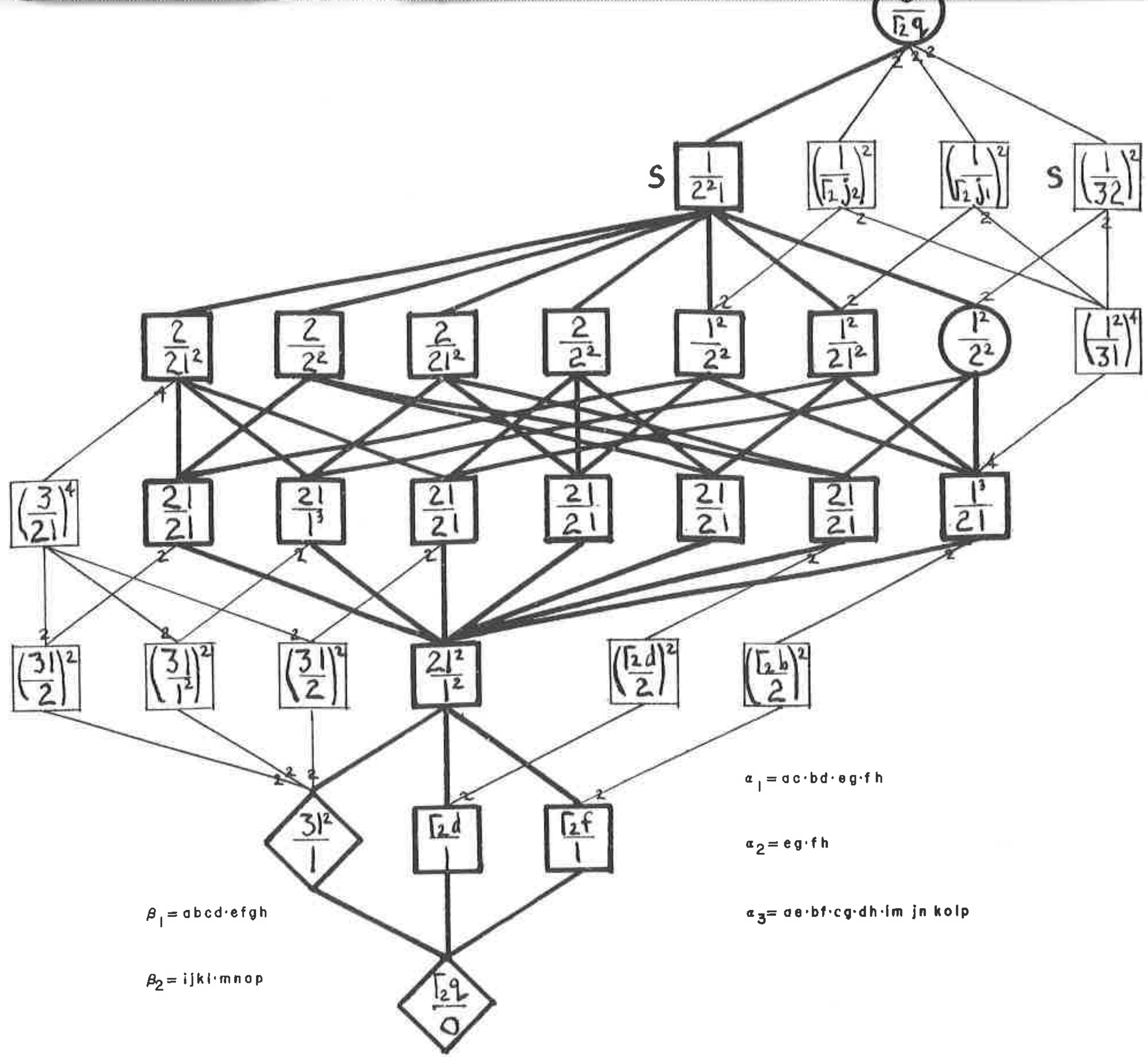
$\alpha_3 = bd \cdot efgh$


β1 = abcd·efgh
β2 = lk·jl
β3 = mo·np
α1 = ac·bd·eg·fh
α2 = eg·fhijkl
α3 = ae·bf·cg·dh·mnop

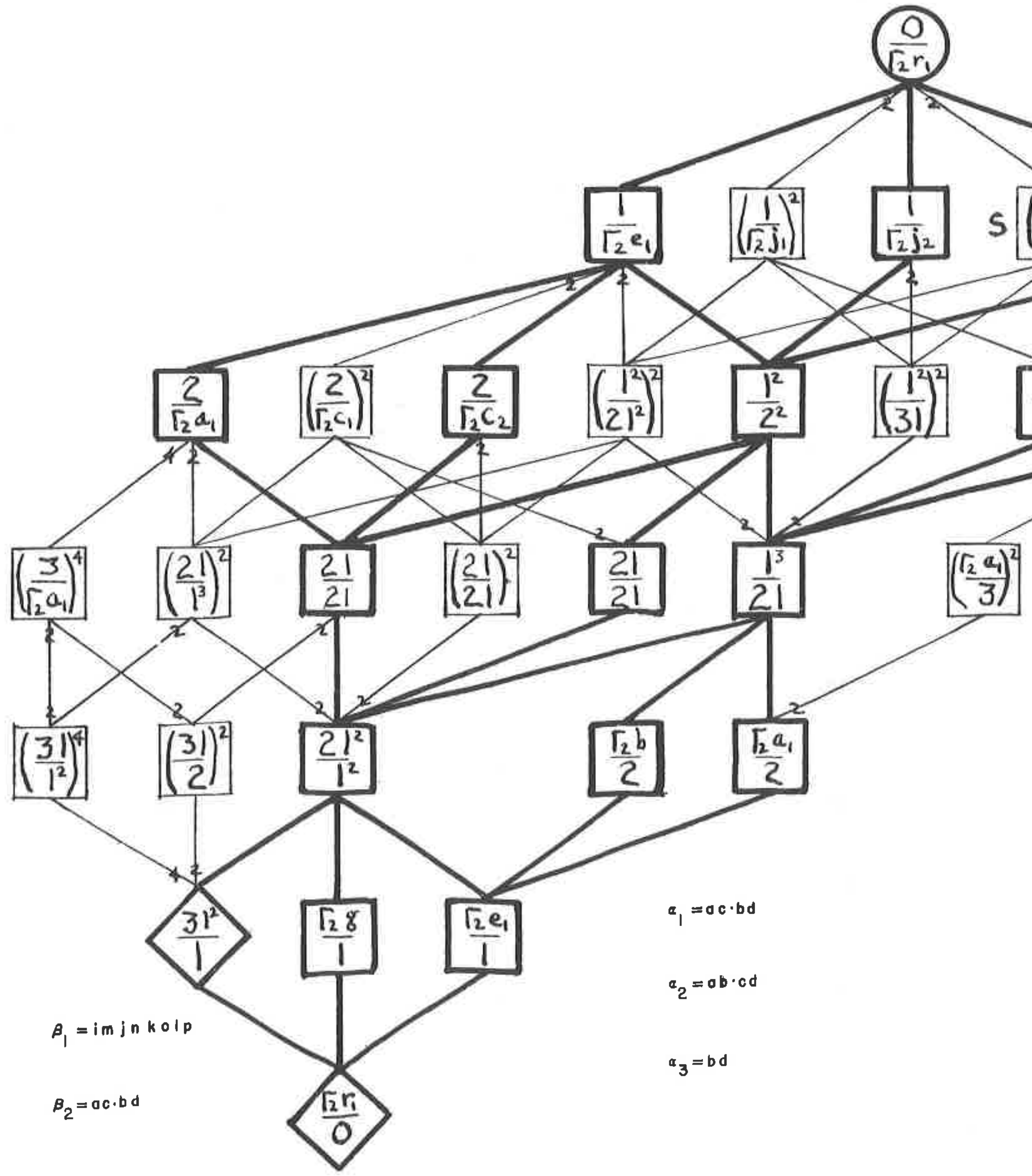

β1 = abcd·efgh
β2 = ijkl
α1 = ac·bd·eg·fh
α2 = eg·fh
α3 = ae bf cgdh

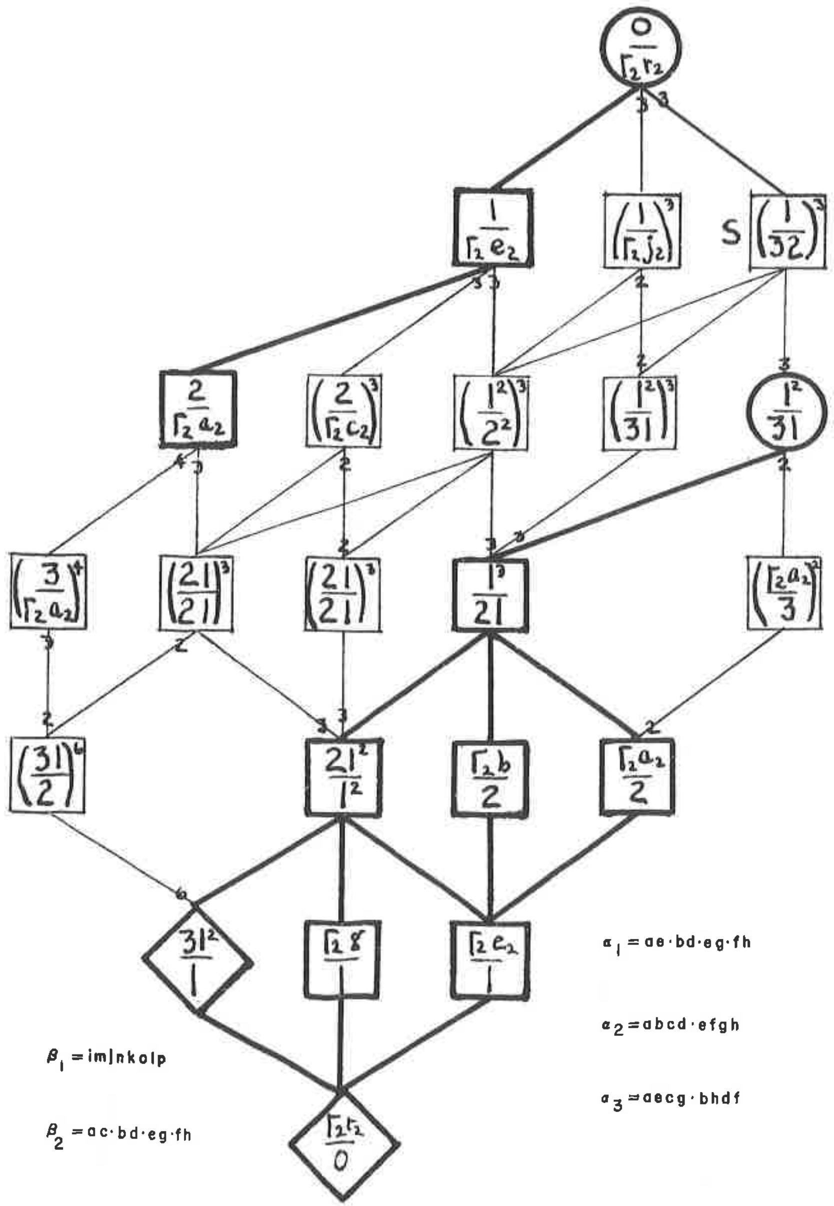


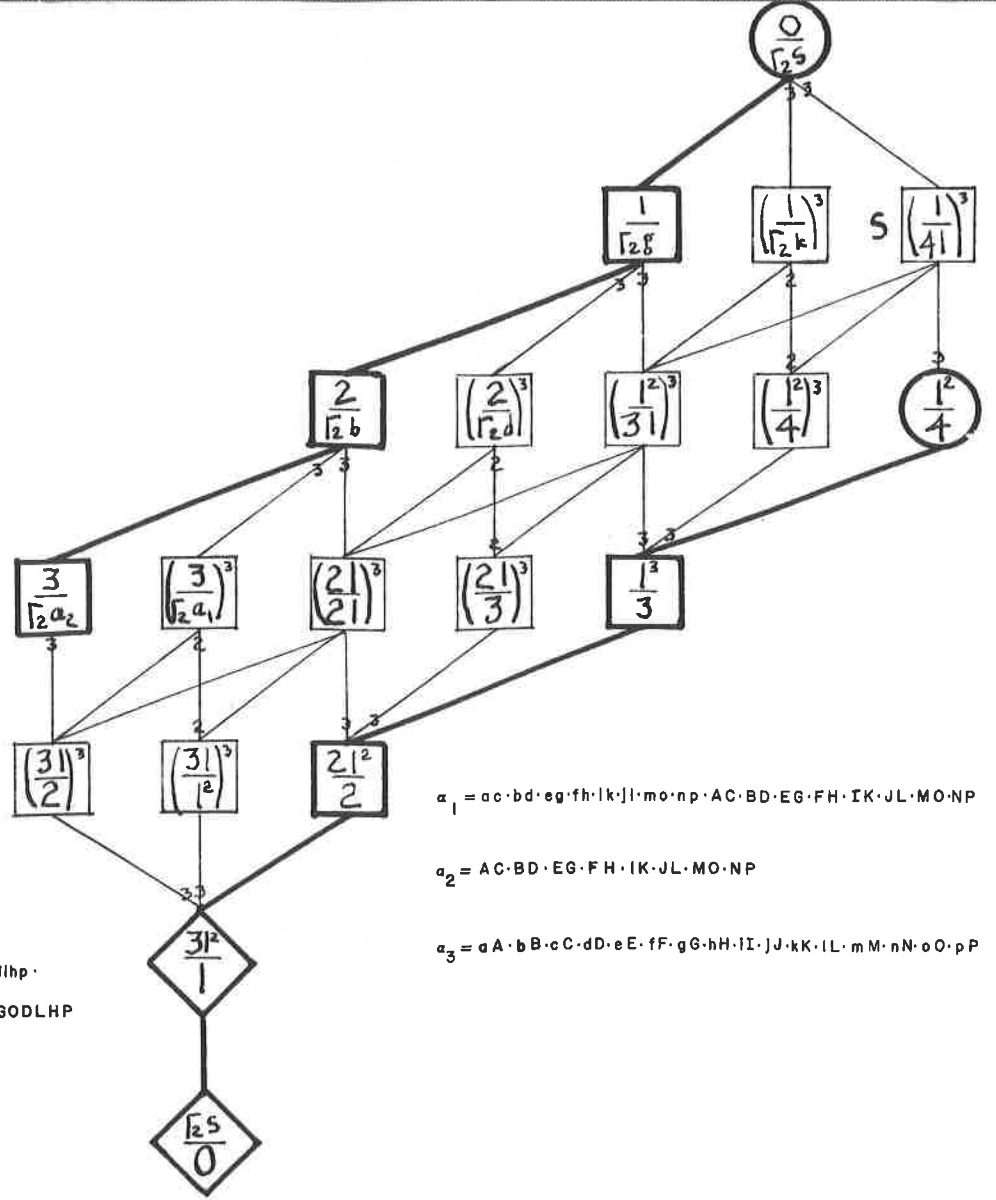
β_1 = ae bf cgdh · imjnkolp
β_2 = qs · rt
α_1 = ac · bd · eg · fh · ik · jl · mo · np
α_2 = ik · jl · mo · np
α_3 = ai · bj · ck · dl · em · fn · go · hp · qrst



β_1 = abcd · efgh
β_2 = ijkl · mnop
α_1 = ac · bd · eg · fh
α_2 = eg · fh
α_3 = ae · bf · cg · dh · lm jn kolp


β_1 = im jn kolp
β_2 = ac·bd
α_1 = ac·bd
α_2 = ab·cd
α_3 = bd


β_1 = imjnkolp
β_2 = ac·bd·eg·fh
α_1 = ae·bd·eg·fh
α_2 = abcd·efgh
α_3 = aecg·bhdf



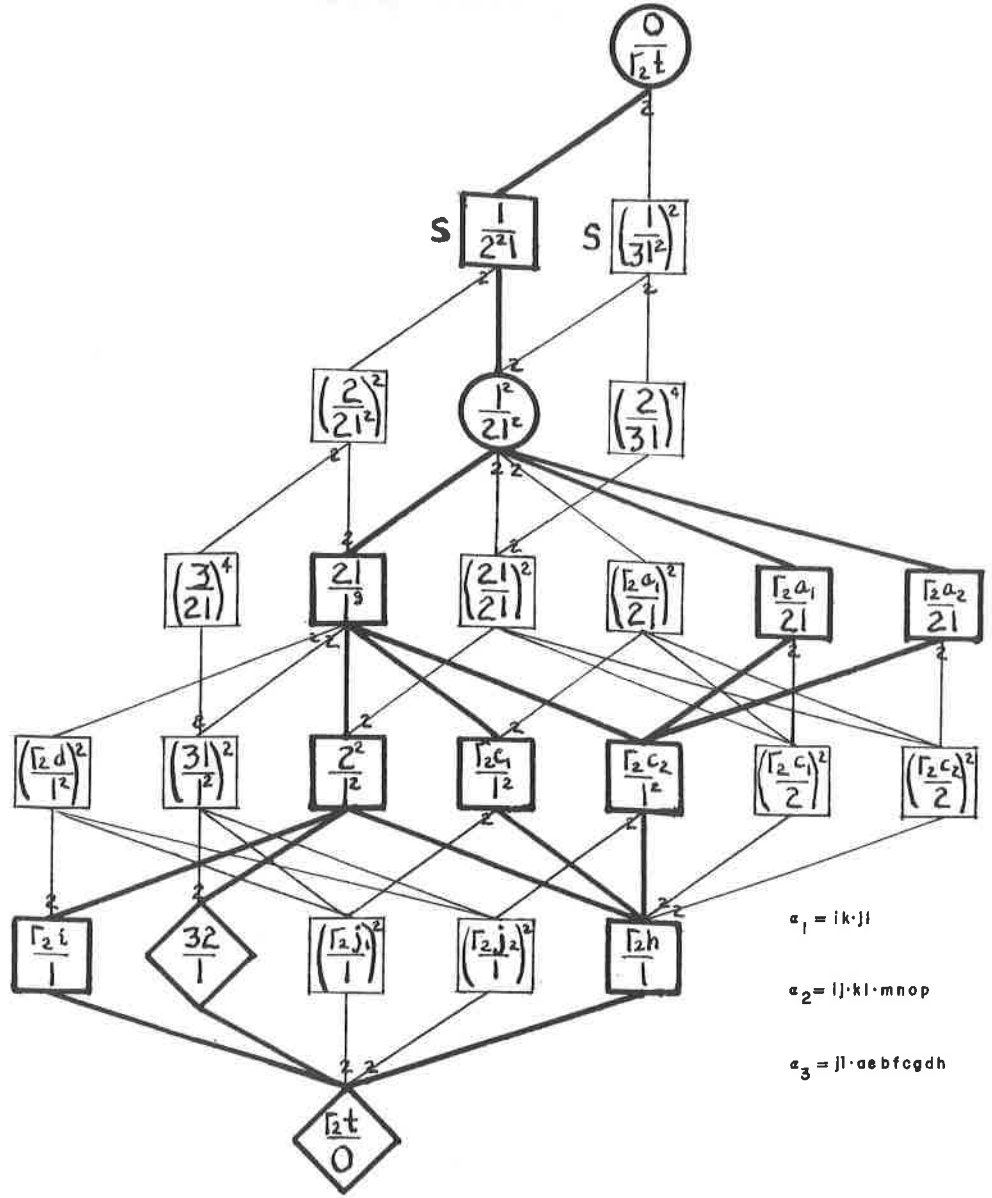
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np \cdot AC \cdot BD \cdot EG \cdot FH \cdot IK \cdot JL \cdot MO \cdot NP$

$\alpha_2 = AC \cdot BD \cdot EG \cdot FH \cdot IK \cdot JL \cdot MO \cdot NP$

$\alpha_3 = aA \cdot bB \cdot cC \cdot dD \cdot eE \cdot fF \cdot gG \cdot hH \cdot iI \cdot jJ \cdot kK \cdot lL \cdot mM \cdot nN \cdot oO \cdot pP$

$\beta = aiembjfnckgodlhp \cdot$

$AIEMBJFNCKGODLHP$



$\beta_1 = abcd \cdot efgh$

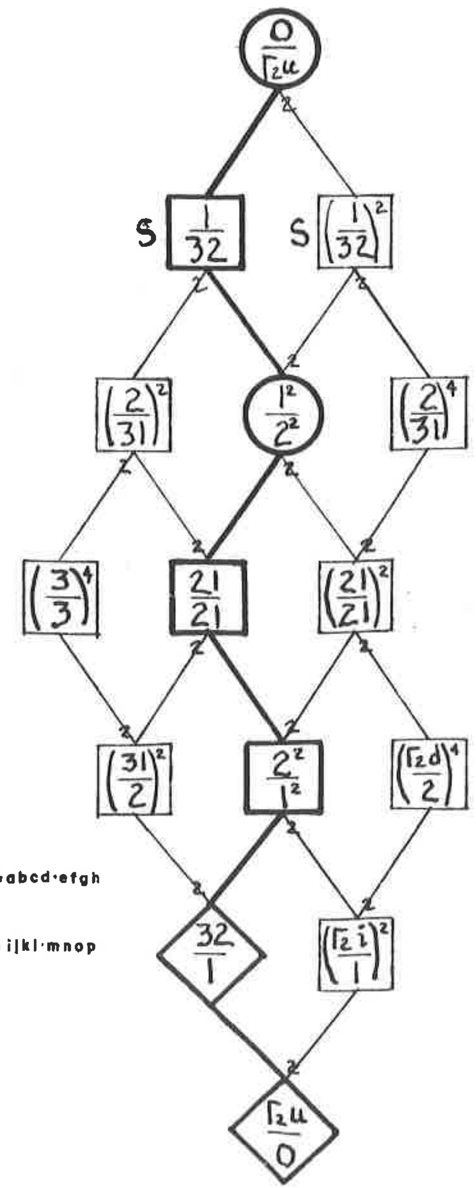
$\beta_2 = ik \cdot jl$

$\beta_3 = mo \cdot np$

$\alpha_1 = ik \cdot jl$

$\alpha_2 = ij \cdot kl \cdot mnop$

$\alpha_3 = jl \cdot aebfcgdh$



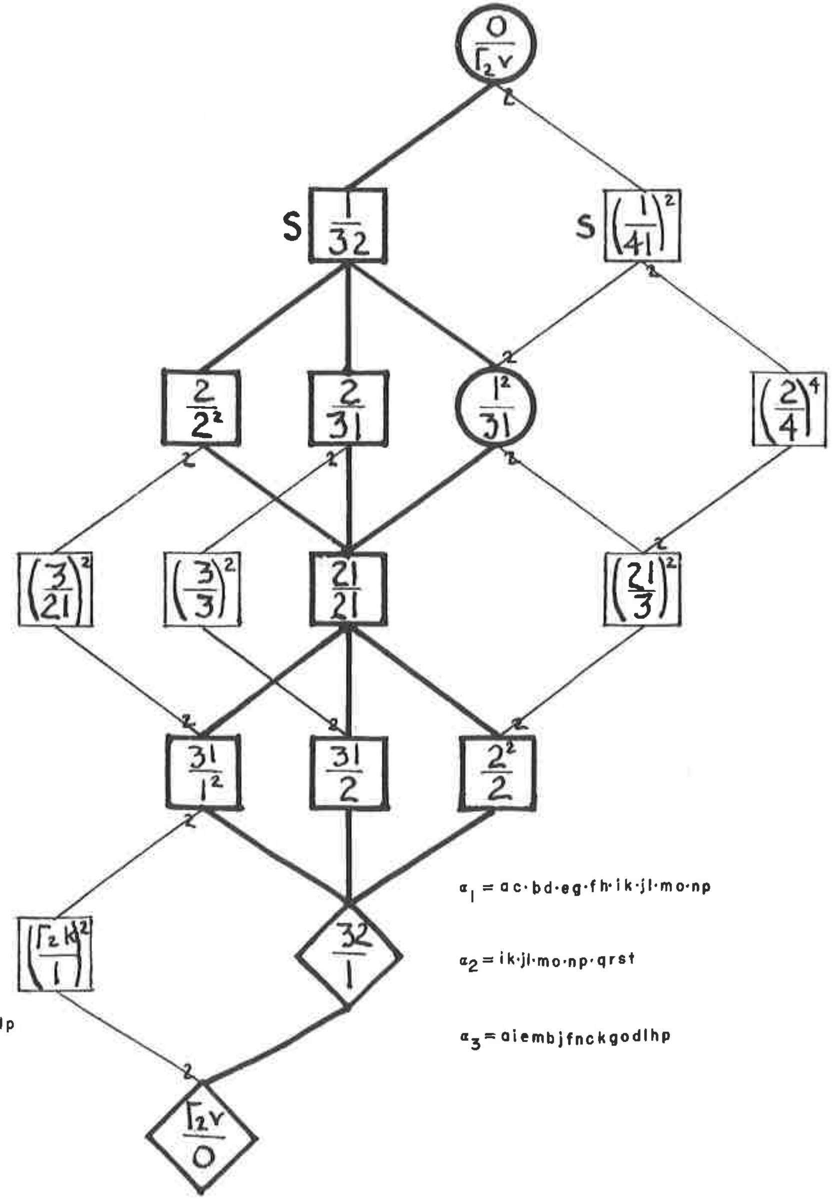
$\beta_1 = abcd \cdot efgh$

$\beta_2 = ijkl \cdot mnop$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = aebfcgdh$

$\alpha_3 = eg \cdot fh \cdot lmjnkolp$



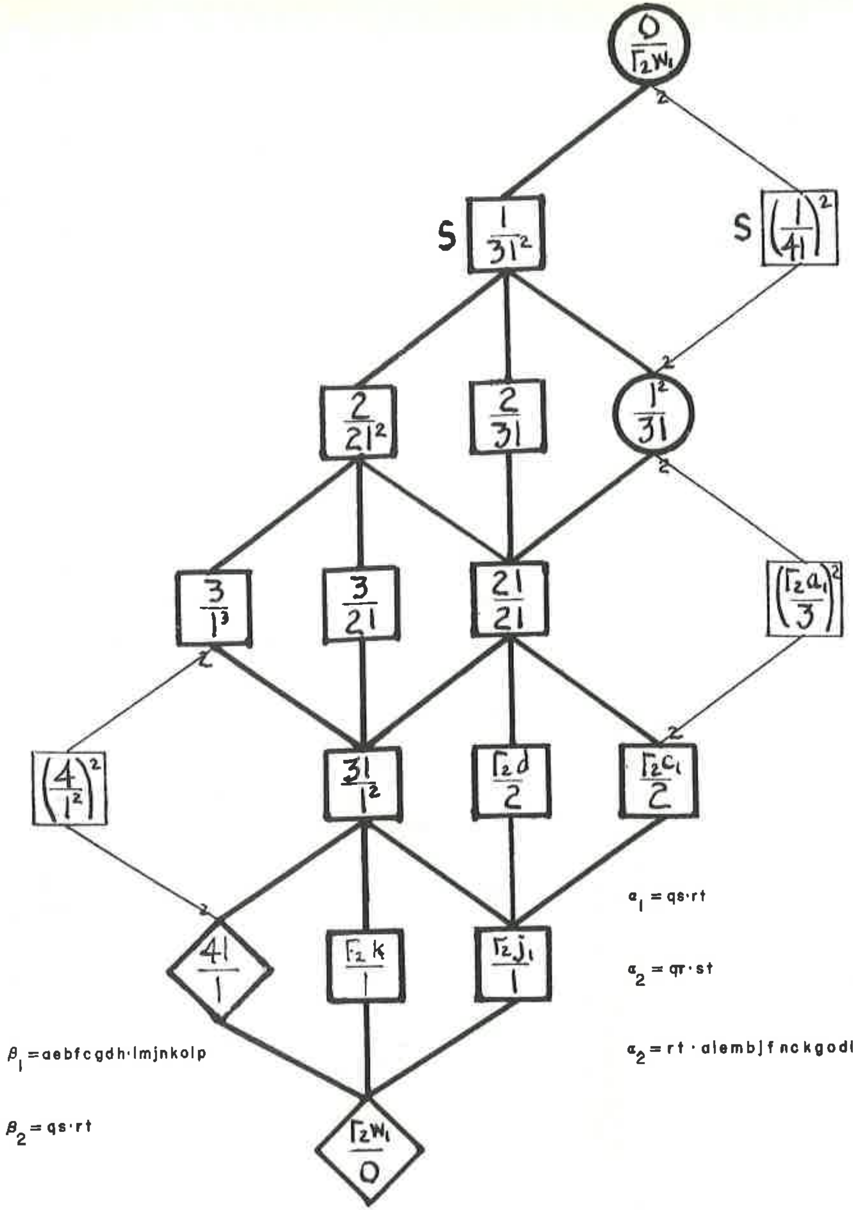
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_2 = ik \cdot jl \cdot mo \cdot np \cdot qrst$

$\alpha_3 = aiembjfnckgodlhp$

$\beta_1 = aebfcgdh \cdot imjnkolp$

$\beta_2 = qs \cdot rt$



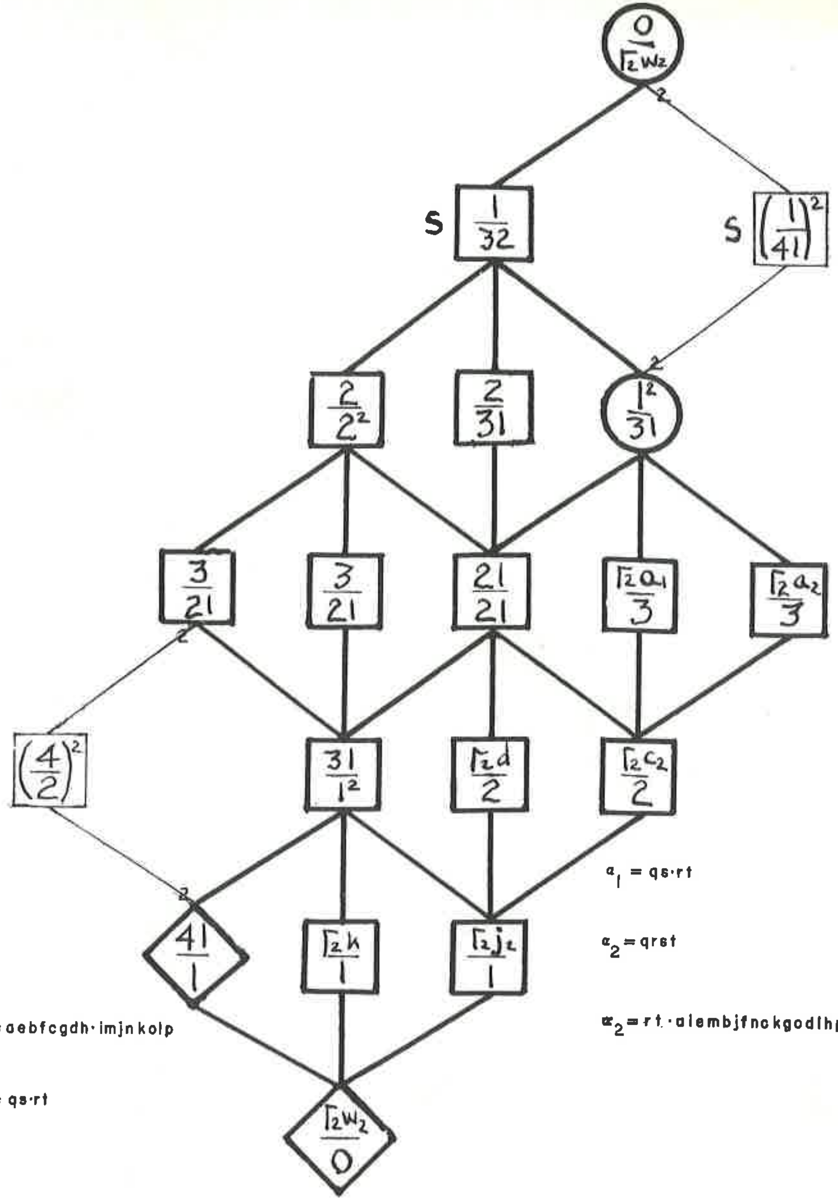
$\alpha_1 = qs \cdot rt$

$\alpha_2 = qr \cdot st$

$\alpha_2 = rt \cdot alembjfnckgodlhp$

$\beta_1 = aebfcgdh \cdot lmjnkolp$

$\beta_2 = qs \cdot rt$



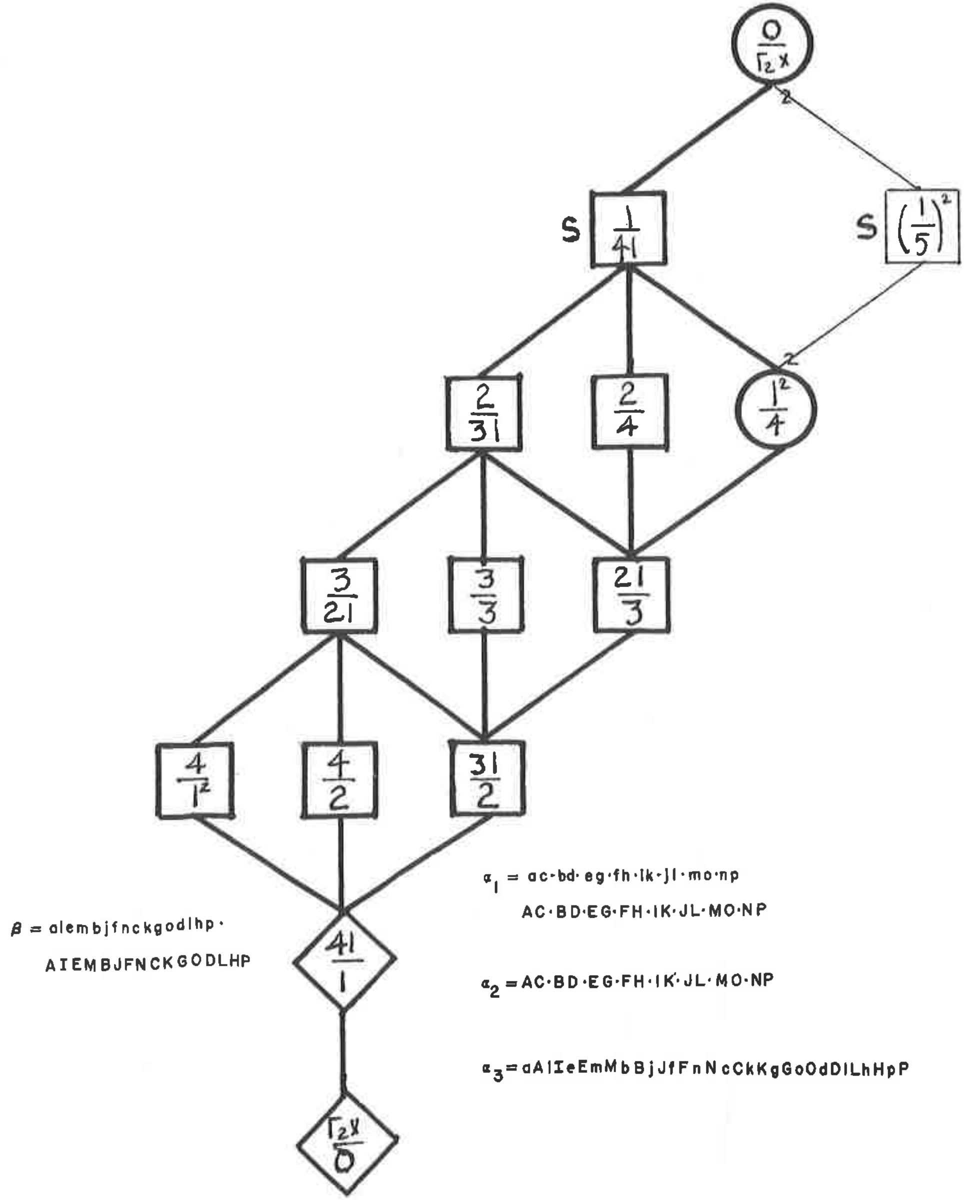
$\alpha_1 = qs \cdot rt$

$\alpha_2 = qrst$

$\alpha_2 = rt \cdot alembjfnckgodlhp$

$\beta_1 = aebfcgdh \cdot lmjnkolp$

$\beta_2 = qs \cdot rt$

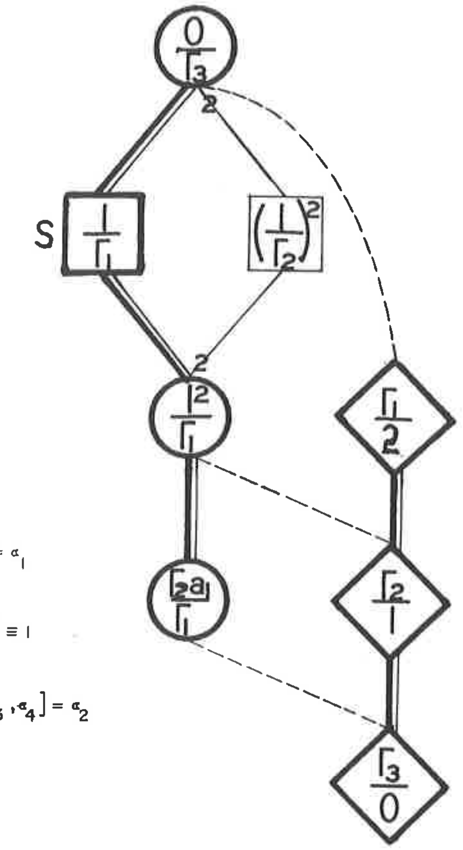


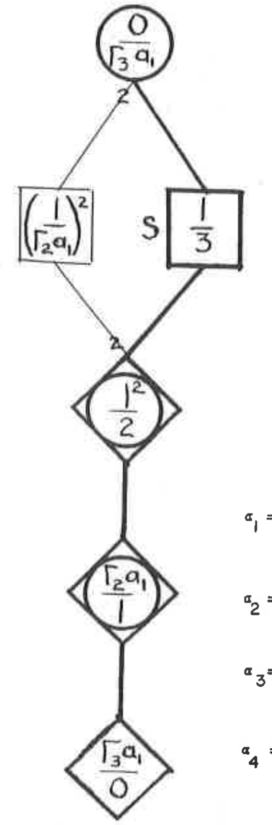
0 / Γ2x
S 1/41
S (1/5)²
2/31
2/4
1²/4
3/21
3/3
21/3
4/1²
4/2
31/2
41/1
Γ2x / 0
β = alembjfnckgodlhp·
AIEMBJFNCKGODLHP
α1 = ac·bd·eg·fh·ik·jl·mo·np
AC·BD·EG·FH·IK·JL·MO·NP
α2 = AC·BD·EG·FH·IK·JL·MO·NP
α3 = aAiIeEmMbBjJfFnNcCkKgGoOdDlLhHpP

$\alpha_1^2 = 1$ $\qquad$ $\alpha_2^2 = \alpha_1$

$\alpha_3^2 \equiv \alpha_2$ $\qquad$ $\alpha_4^2 \equiv 1$

$[\alpha_2, \alpha_4] = \alpha_1$ $\qquad$ $[\alpha_3, \alpha_4] = \alpha_2$



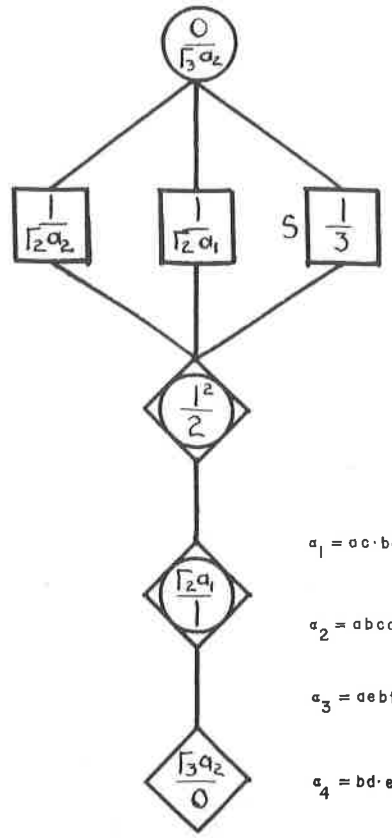


$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot eh \cdot fg$

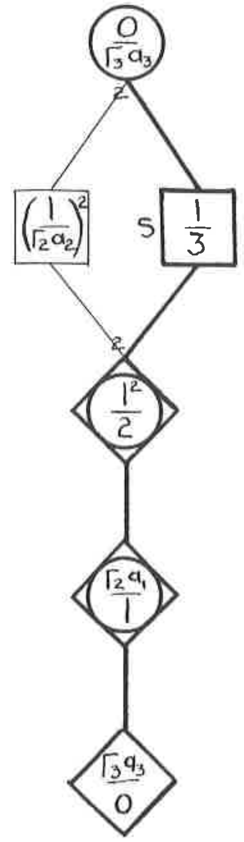


$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot ef \cdot gh$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

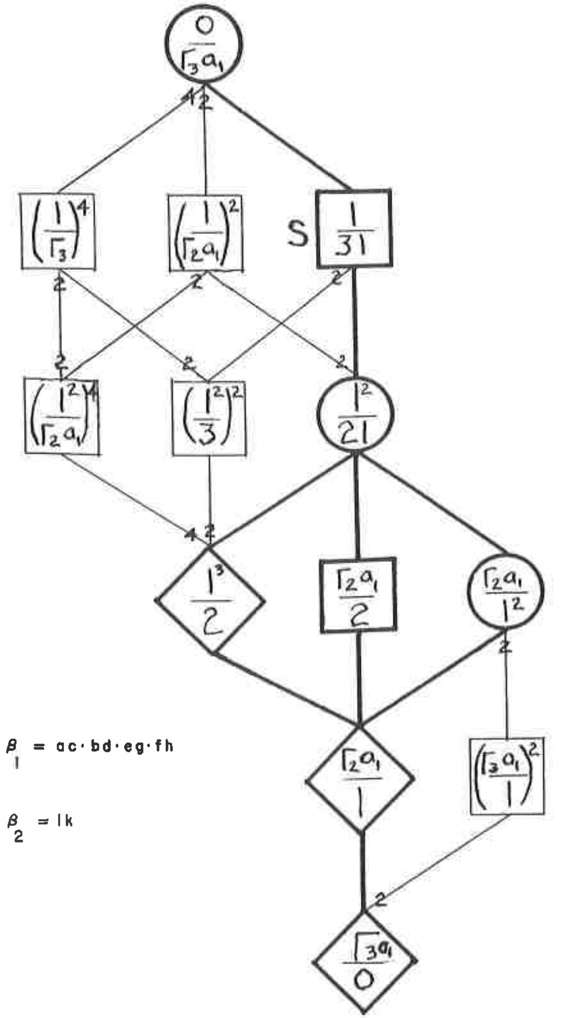
$\alpha_2 = adcb \cdot ehgf \cdot ilkj \cdot mpon$

$\alpha_3 = aebfcgdh \cdot imjnkolp$

$\alpha_4 = aick \cdot bldj \cdot epgn \cdot fohm$

β_1 = ac·bd·eg·fh

β_2 = lk



α_1 = ac·bd·eg·fh

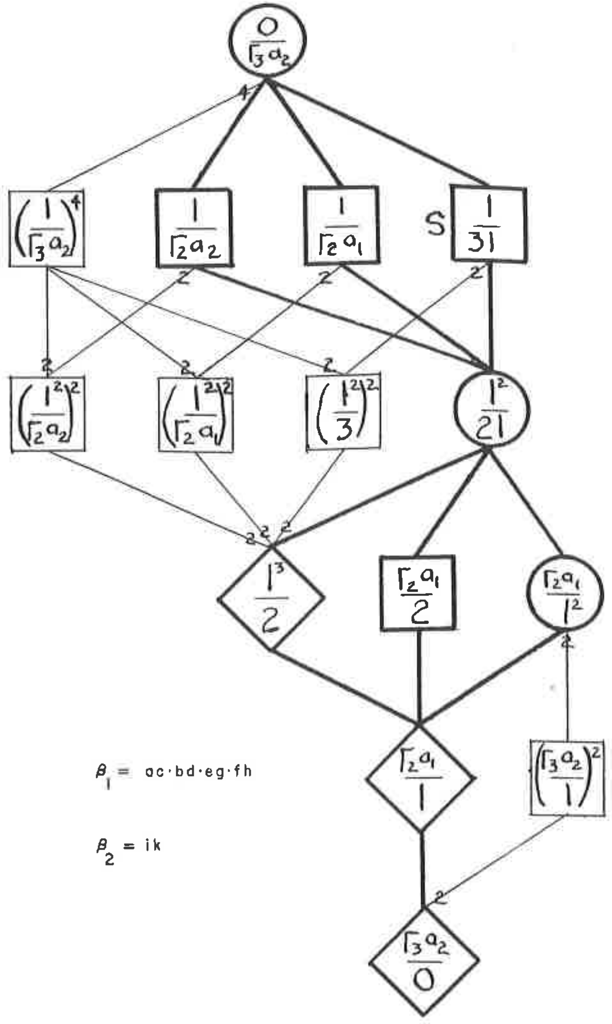
α_2 = adcb · ehgf

α_3 = aebfcgdh

α_4 = bd·eh·fg

β_1 = ac·bd·eg·fh

β_2 = ik

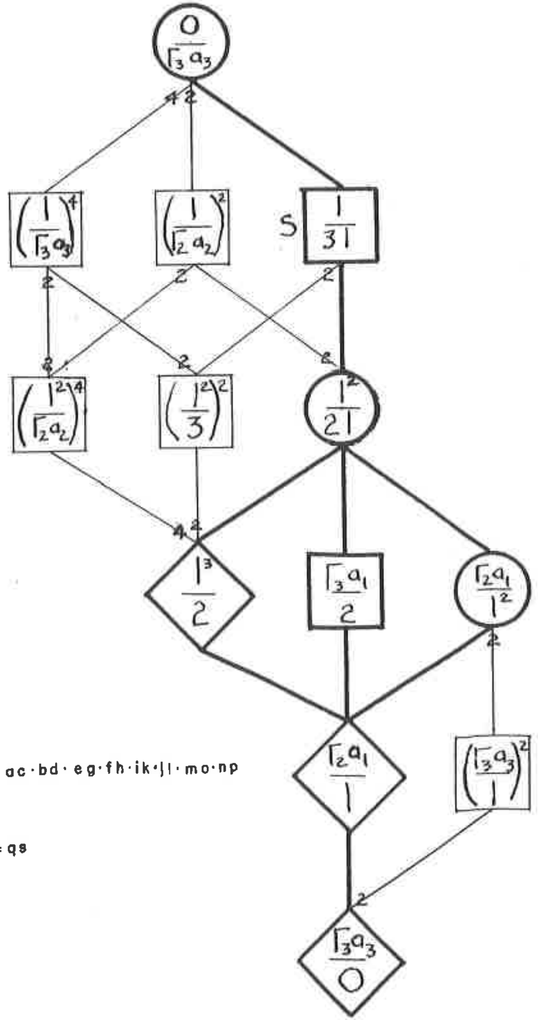


α_1 = ac·bd·eg·fh

α_2 = abcd·efgh

α_3 = aebfcgdh

α_4 = bd·ef·gh



β_1 = ac · bd · eg · fh · ik · jl · mo · np

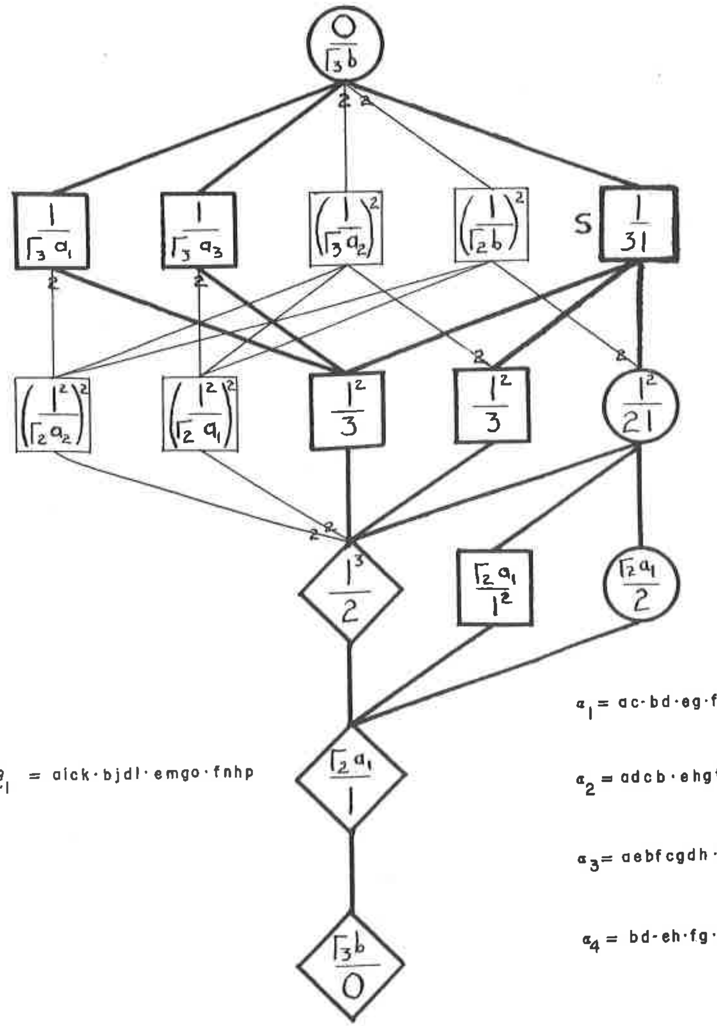
β_2 = qs

α_1 = ac · bd · eg · fh · ik · jl · mo · np

α_2 = adcb · ehgf · ilkj · mpon

α_3 = aebfcgdh · imjnkolp

α_4 = alck · bldj · epgn · fohm



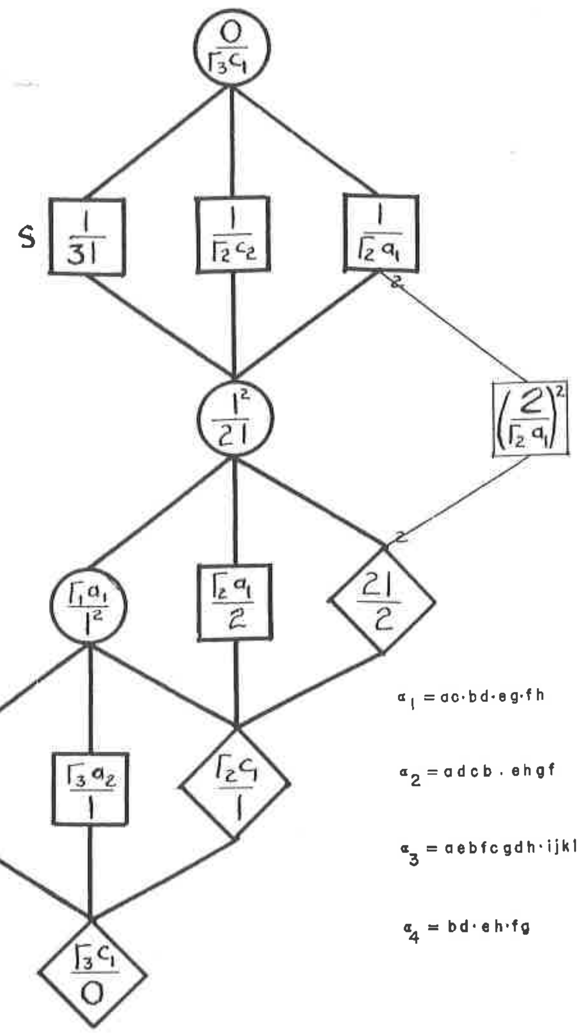
β_1 = alck · bjdl · emgo · fnhp

α_1 = ac · bd · eg · fh · ik · jl · mo · np

α_2 = adcb · ehgf · ilkj · mpon

α_3 = aebfcgdh · lmjnkolp

α_4 = bd · eh · fg · jl · mp · no



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

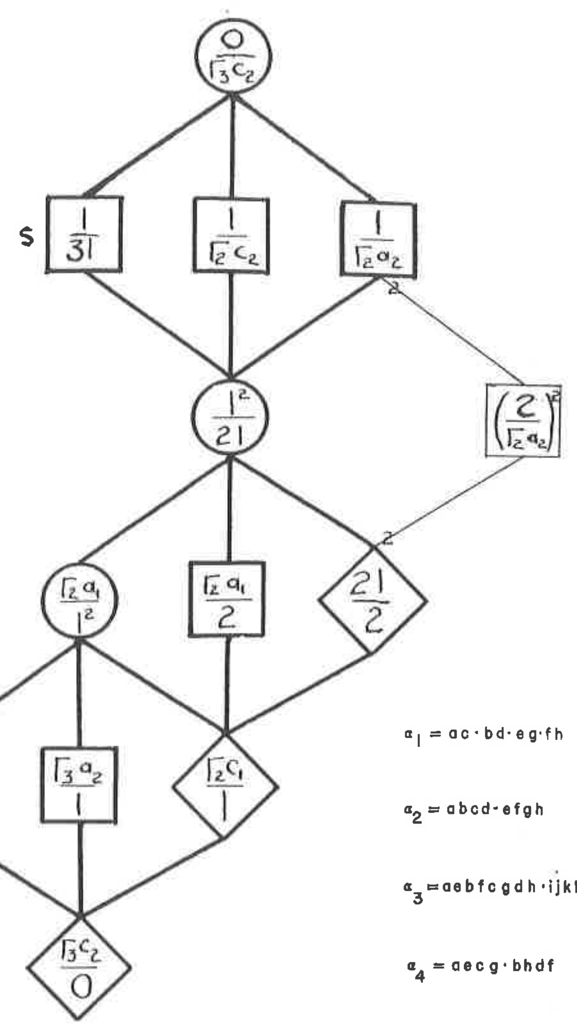
$\beta_2 = lk \cdot jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh \cdot ijkl$

$\alpha_4 = bd \cdot eh \cdot fg$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

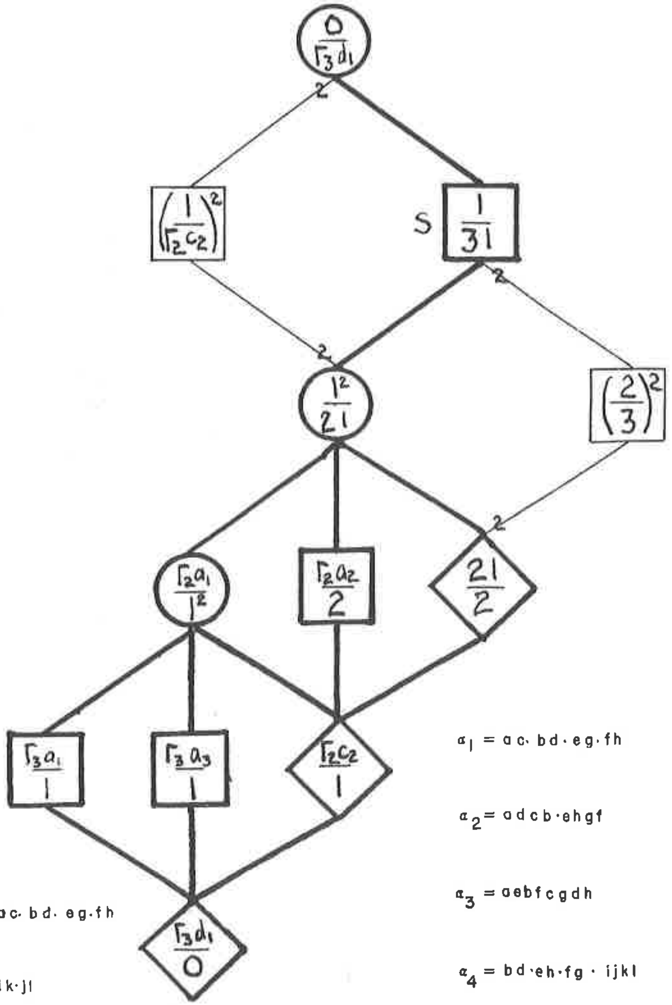
$\beta_2 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh \cdot ijkl$

$\alpha_4 = aecg \cdot bhdf$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

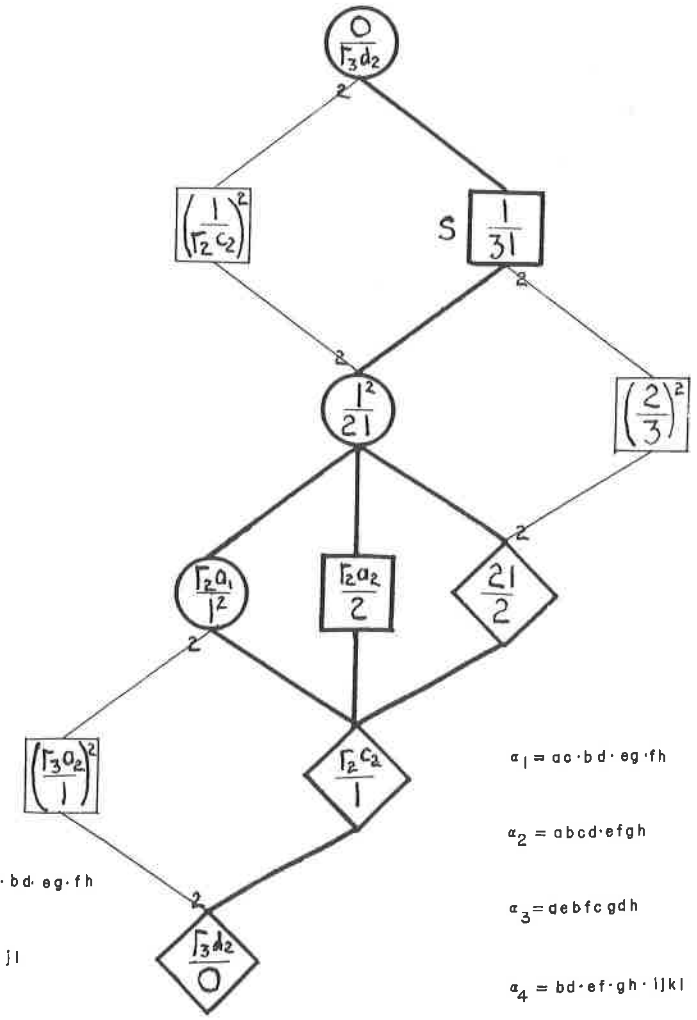
$\beta_2 = ik \cdot jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot eh \cdot fg \cdot ijkl$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

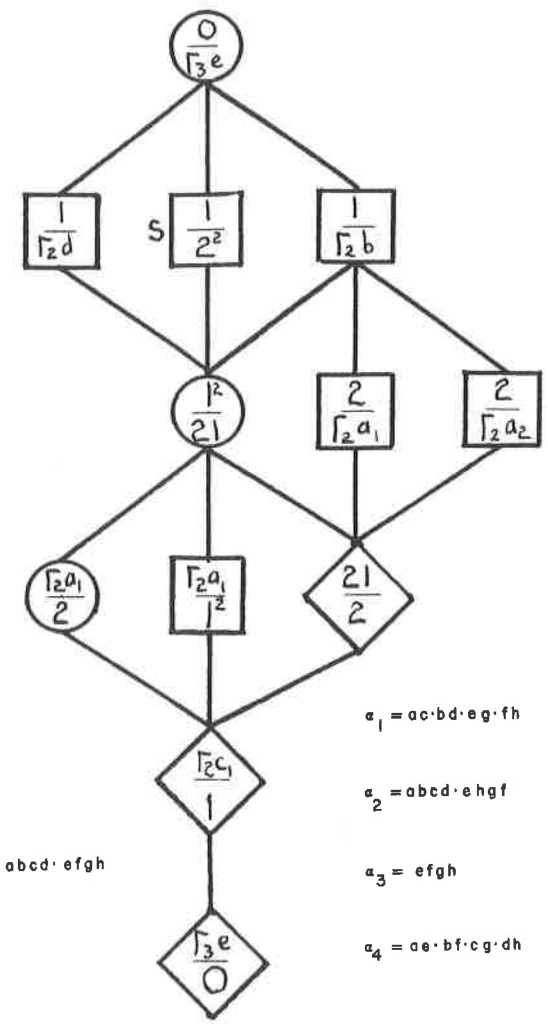
$\beta_2 = ik \cdot jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot ef \cdot gh \cdot ijkl$



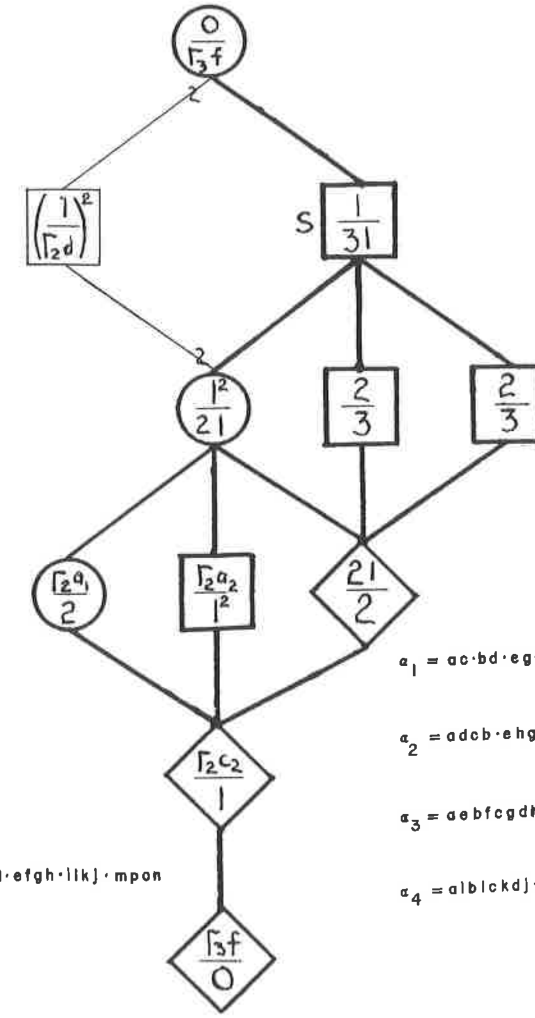
$\beta = abcd \cdot efgh$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot ehgf$

$\alpha_3 = efgh$

$\alpha_4 = ae \cdot bf \cdot cg \cdot dh$



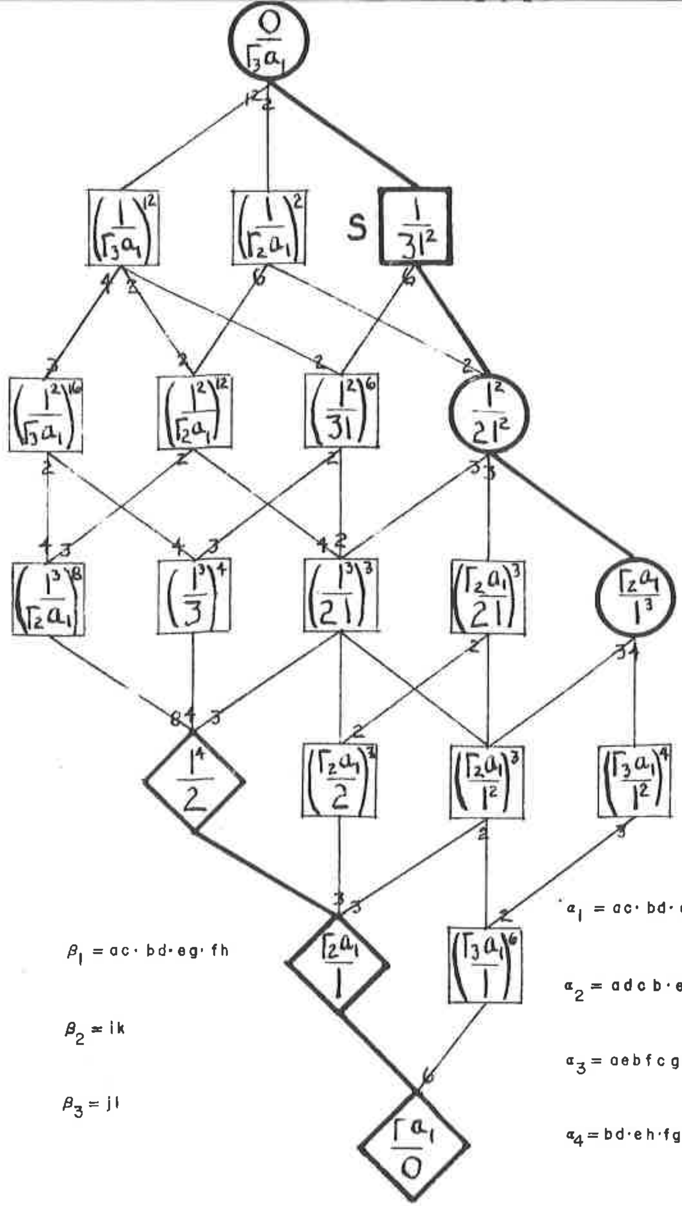
$\beta = abcd \cdot efgh \cdot ilkj \cdot mpon$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_2 = adcb \cdot ehgf \cdot ilkj \cdot mpon$

$\alpha_3 = aebfcgdh \cdot lmjnkolp$

$\alpha_4 = alblckdj \cdot epfognhm$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik$

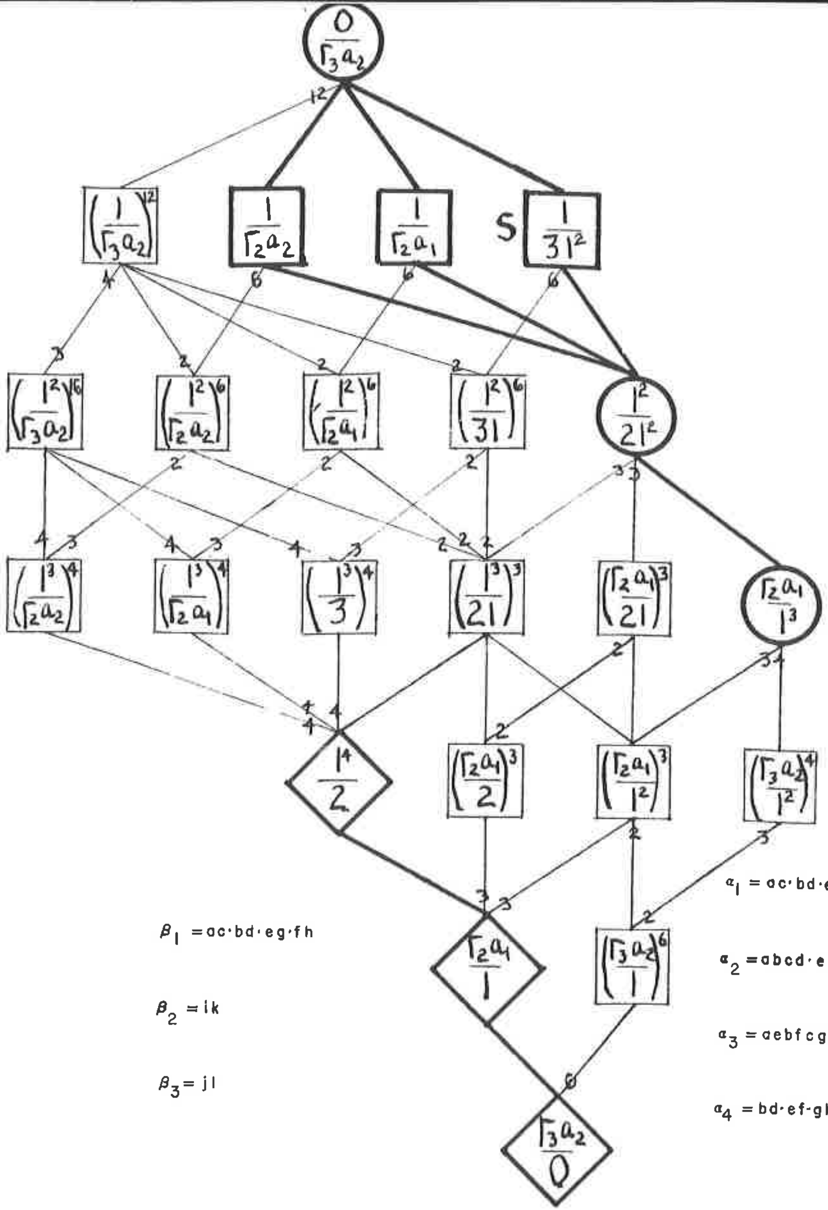
$\beta_3 = jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot eh \cdot fg$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik$

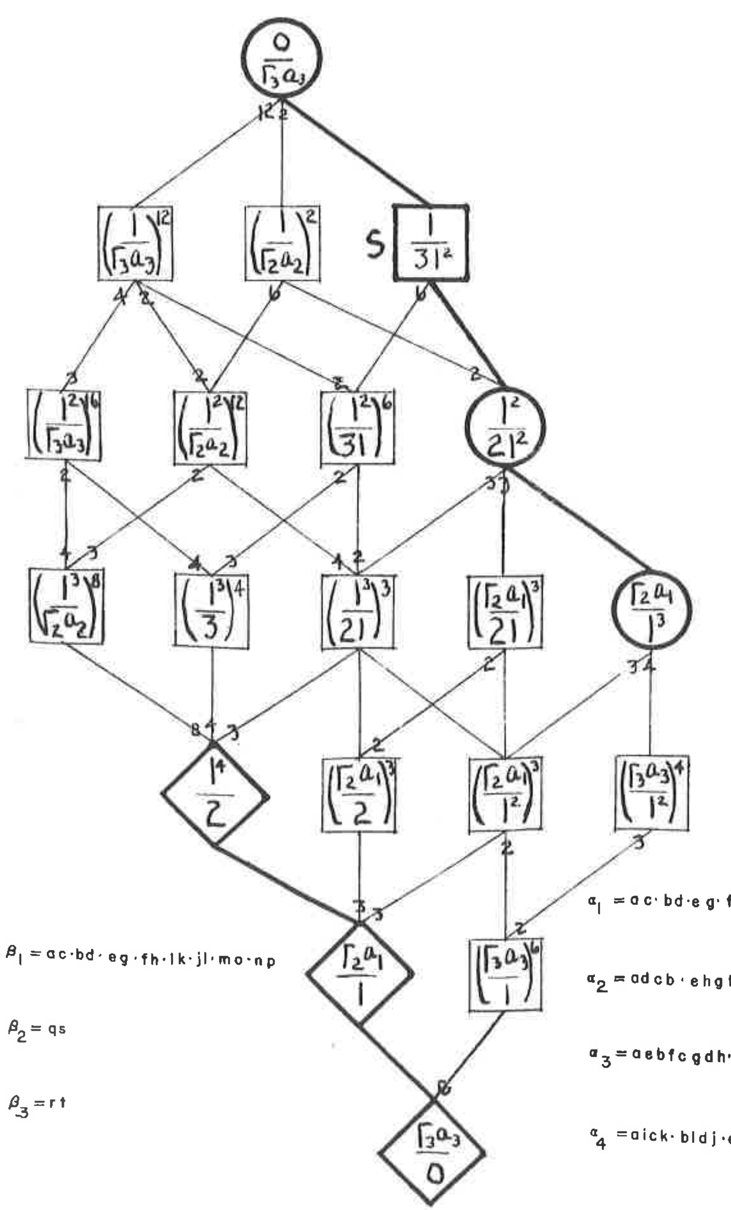
$\beta_3 = jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot ef \cdot gh$



β_1 = ac·bd·eg·fh·lk·jl·mo·np

β_2 = qs

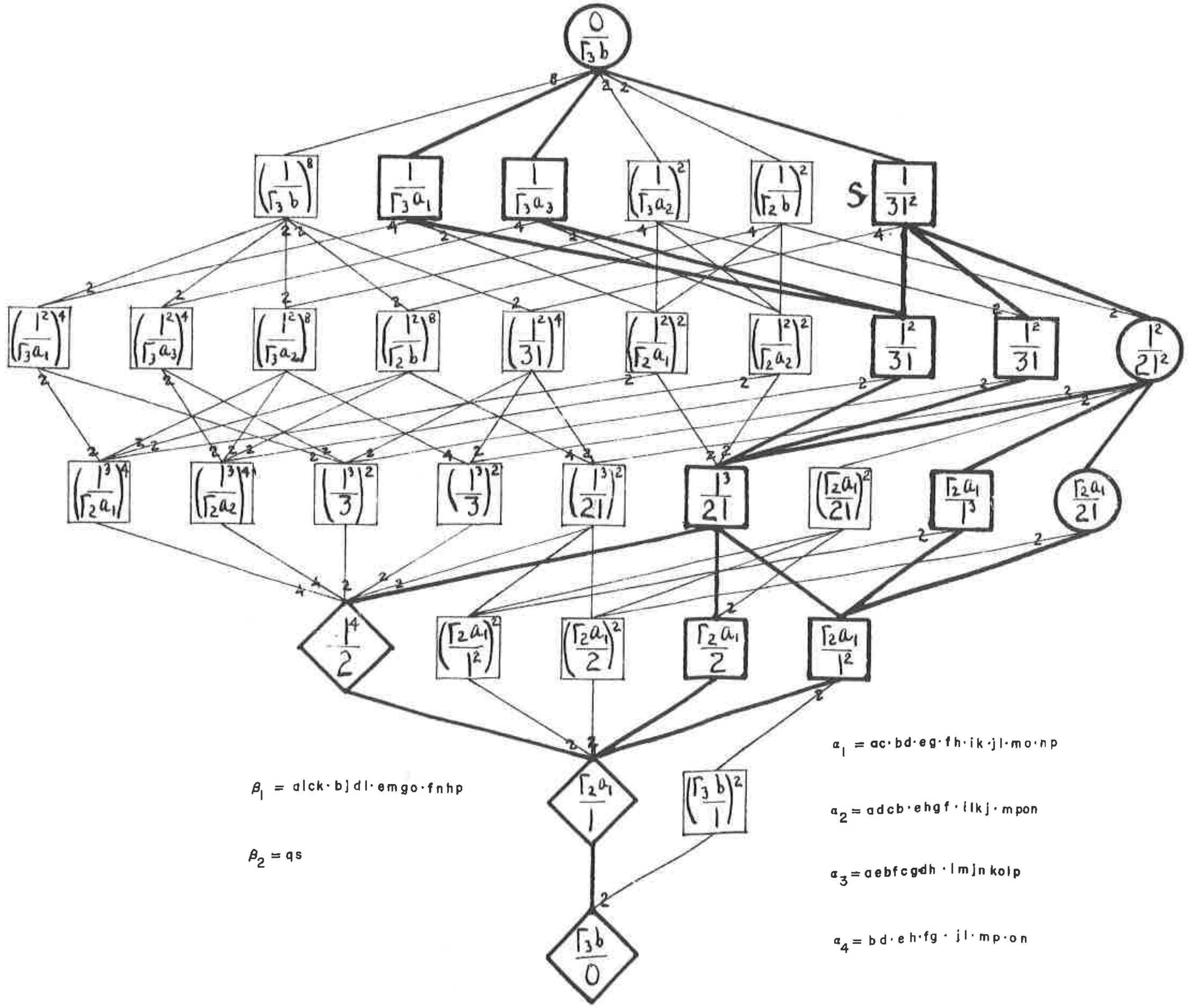
β_3 = rt

α_1 = ac·bd·eg·fh·ik·jl·mo·np

α_2 = adcb·ehgf·ilkj·mpon

α_3 = aebfcgdh·lmjnkolp

α_4 = aick·bldj·epgn·fohm



β_1 = alck·bjdl·emgo·fnhp

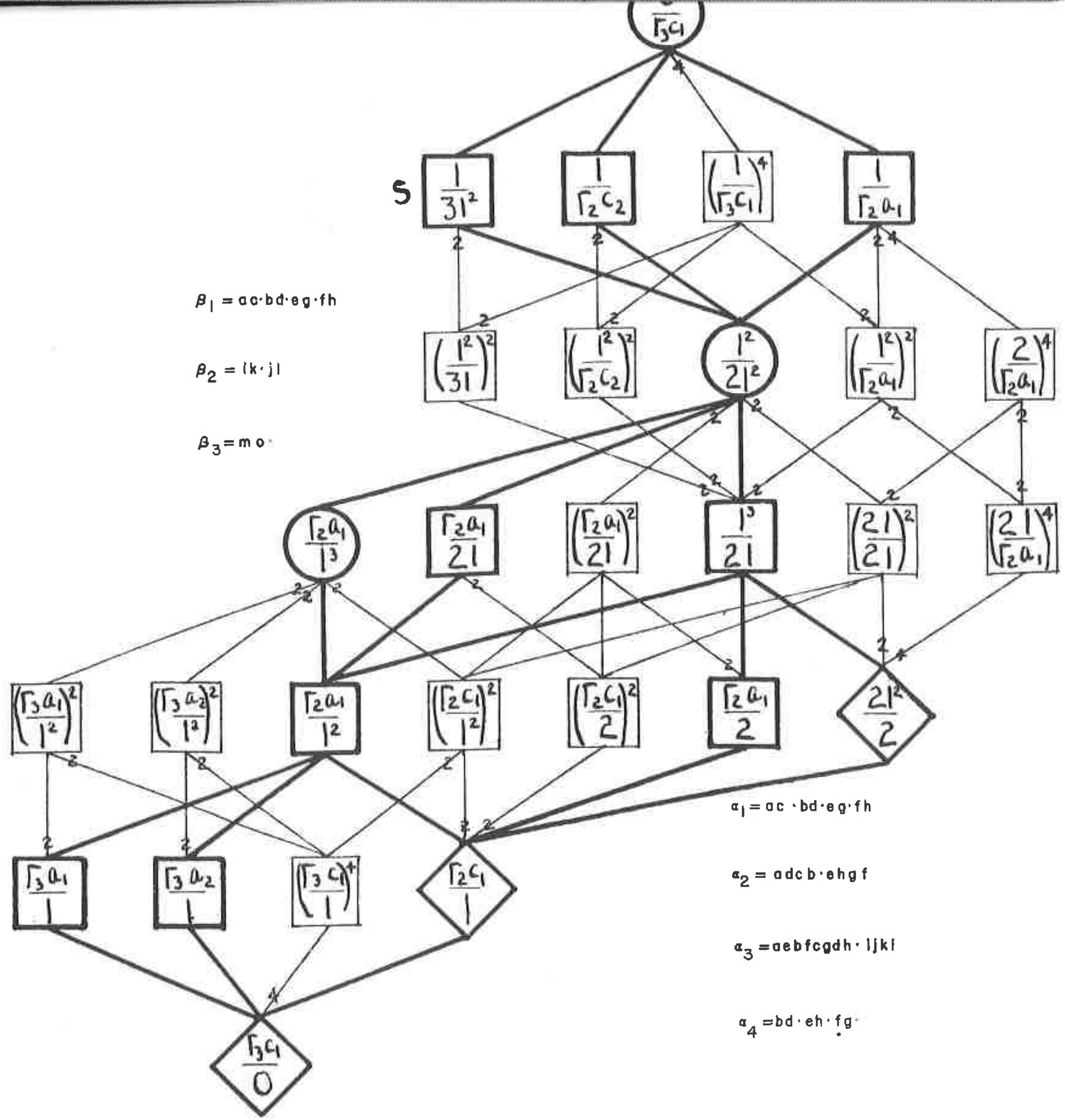
β_2 = qs

α_1 = ac·bd·eg·fh·ik·jl·mo·np

α_2 = adcb·ehgf·ilkj·mpon

α_3 = aebfcgdh·lmjnkolp

α_4 = bd·eh·fg·jl·mp·on



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = lk \cdot jl$

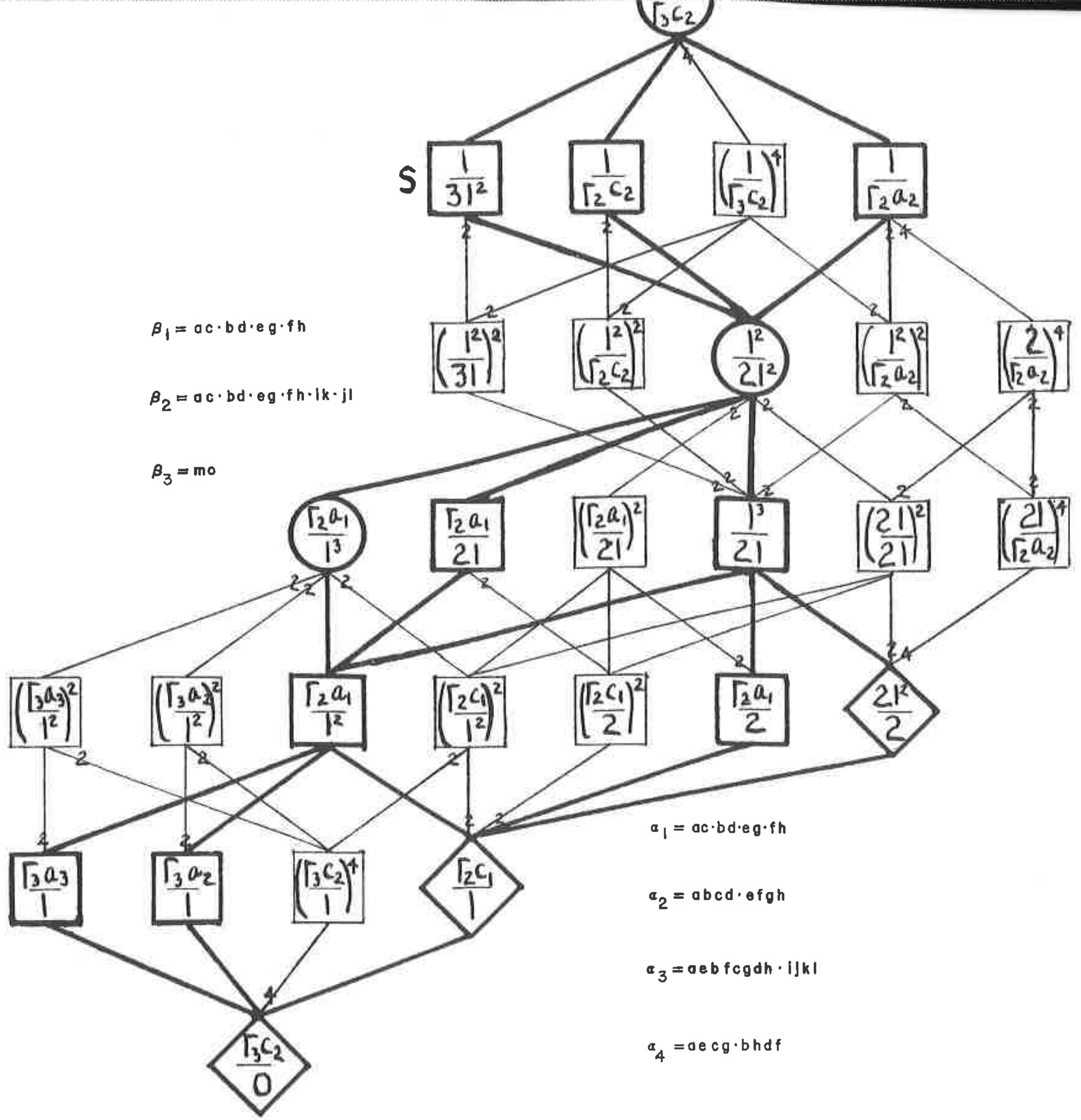
$\beta_3 = mo$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh \cdot ljkl$

$\alpha_4 = bd \cdot eh \cdot fg$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ac \cdot bd \cdot eg \cdot fh \cdot lk \cdot jl$

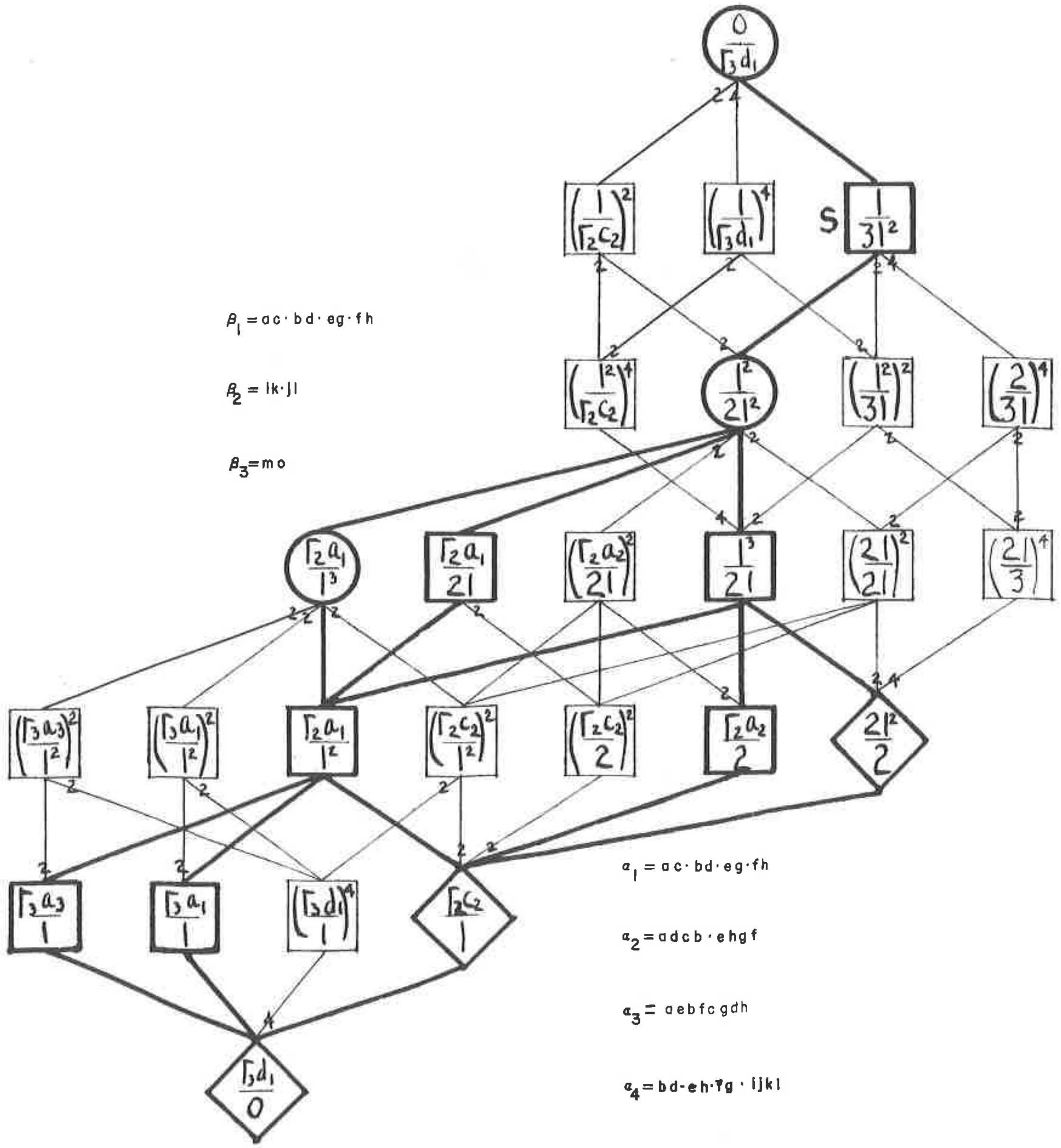
$\beta_3 = mo$

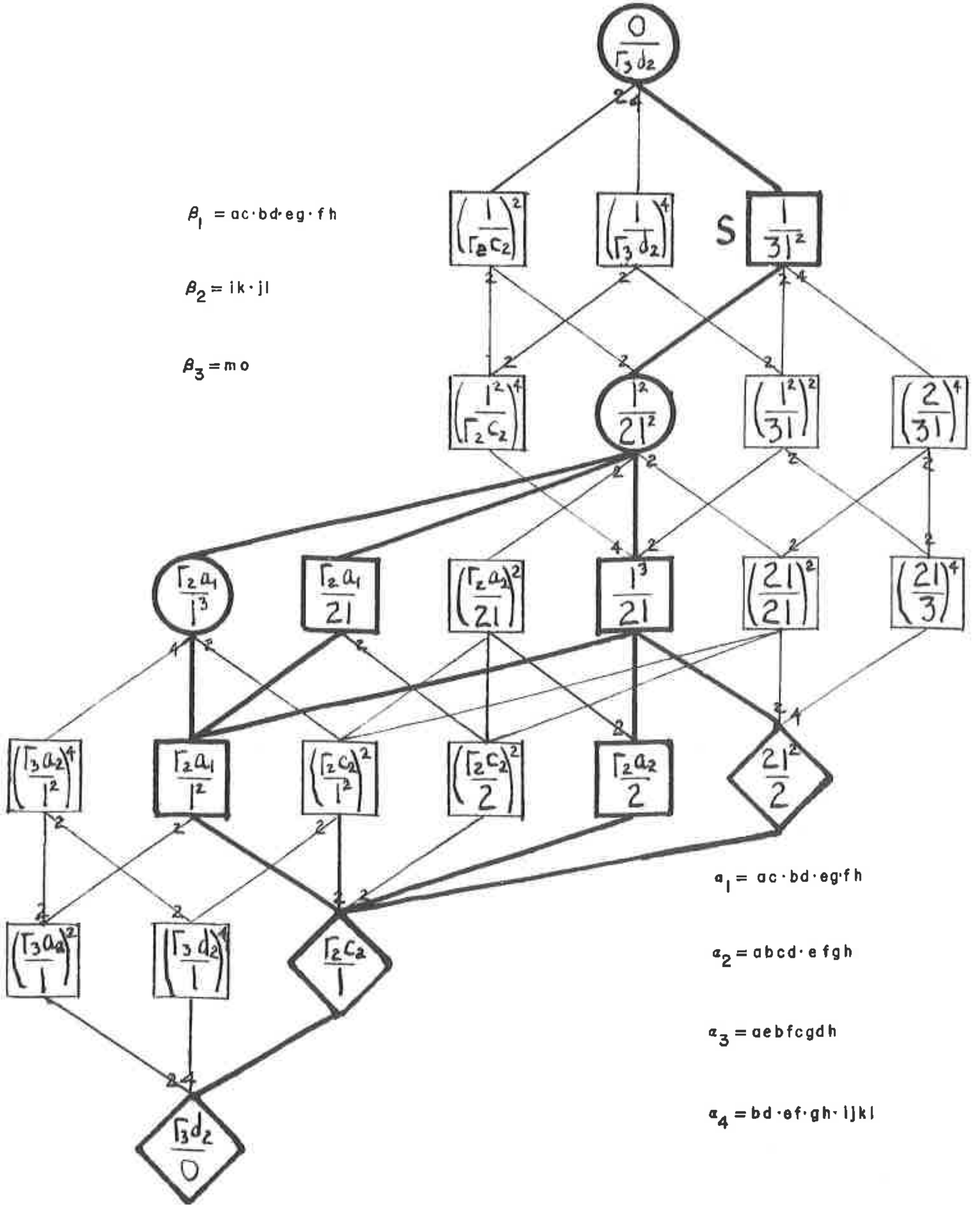
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

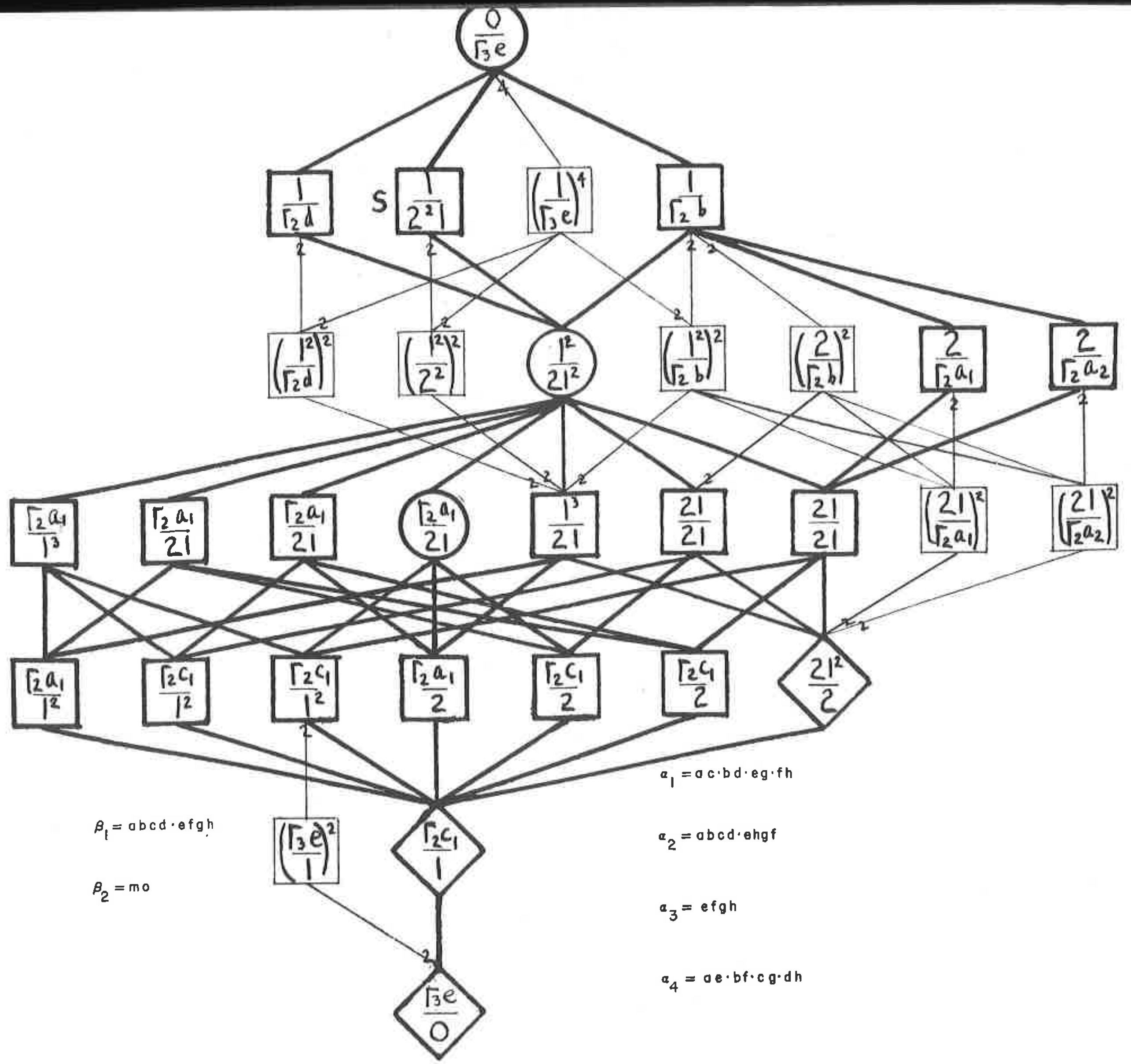
$\alpha_2 = abcd \cdot efgh$

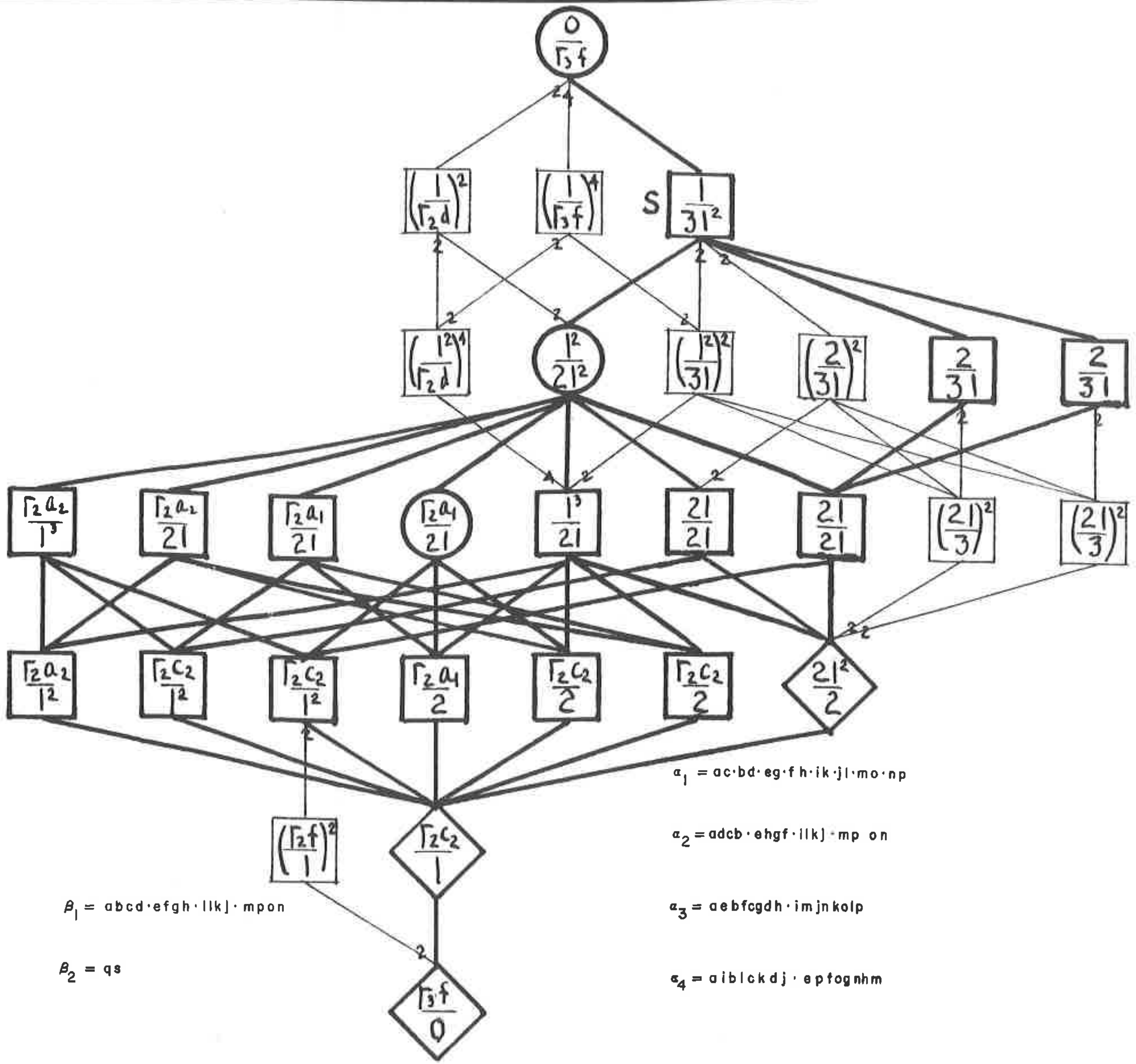
$\alpha_3 = aebfcgdh \cdot ljkl$

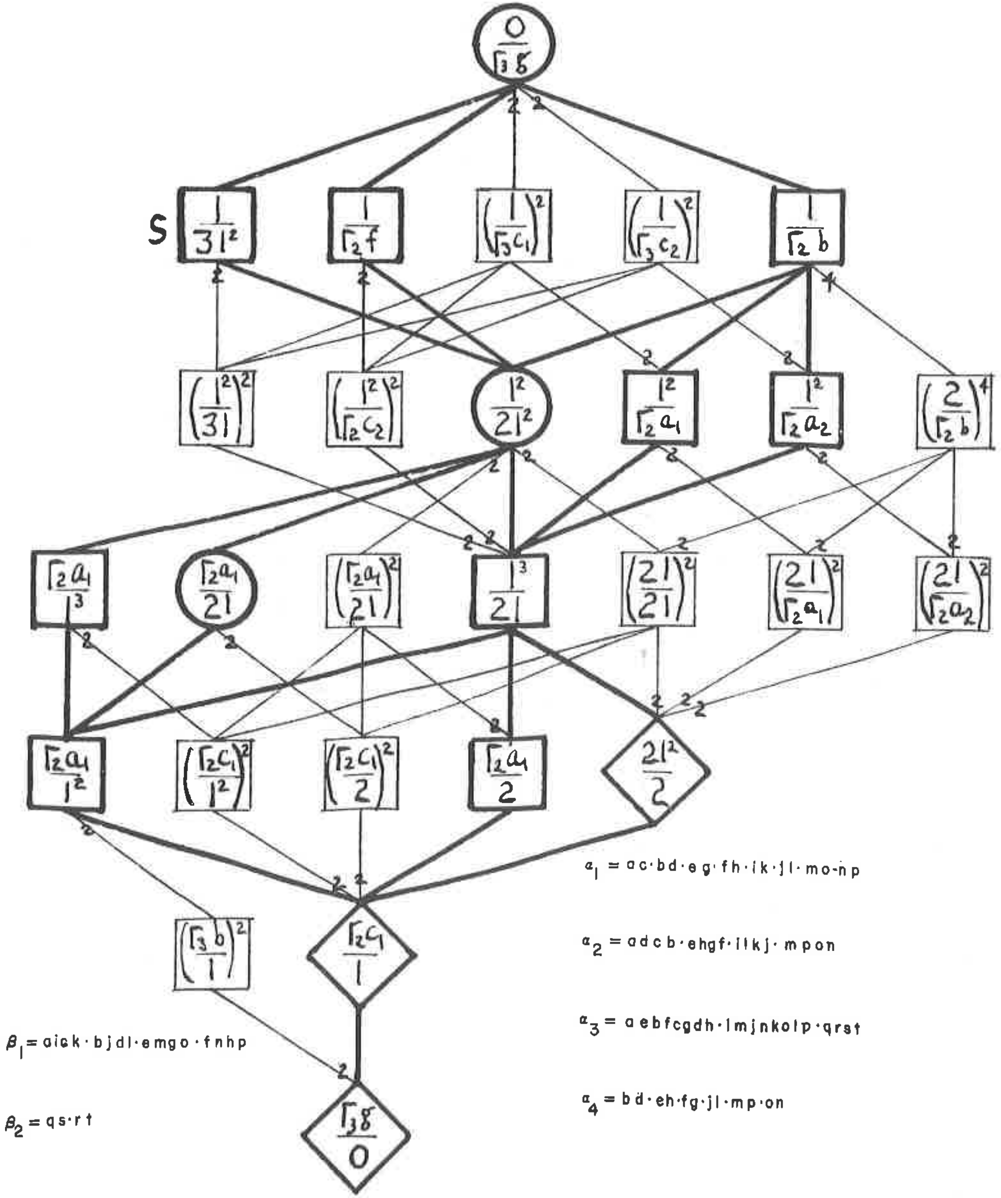
$\alpha_4 = aecg \cdot bhdf$


β_1 = ac·bd·eg·fh
β_2 = ik·jl
β_3 = mo
α_1 = ac·bd·eg·fh
α_2 = adcb·ehgf
α_3 = aebfcgdh
α_4 = bd·eh·fg·ijkl


β_1 = ac·bd·eg·fh
β_2 = ik·jl
β_3 = mo
α_1 = ac·bd·eg·fh
α_2 = abcd·efgh
α_3 = aebfcgdh
α_4 = bd·ef·gh·ijkl


S
β₁ = abcd·efgh
β₂ = mo
α₁ = ac·bd·eg·fh
α₂ = abcd·ehgf
α₃ = efgh
α₄ = ae·bf·cg·dh


S
β₁ = abcd·efgh·llkj·mpon
β₂ = qs
α₁ = ac·bd·eg·fh·ik·jl·mo·np
α₂ = adcb·ehgf·llkj·mp on
α₃ = aebfcgdh·imjnkolp
α₄ = aiblckdj·epfognhm



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

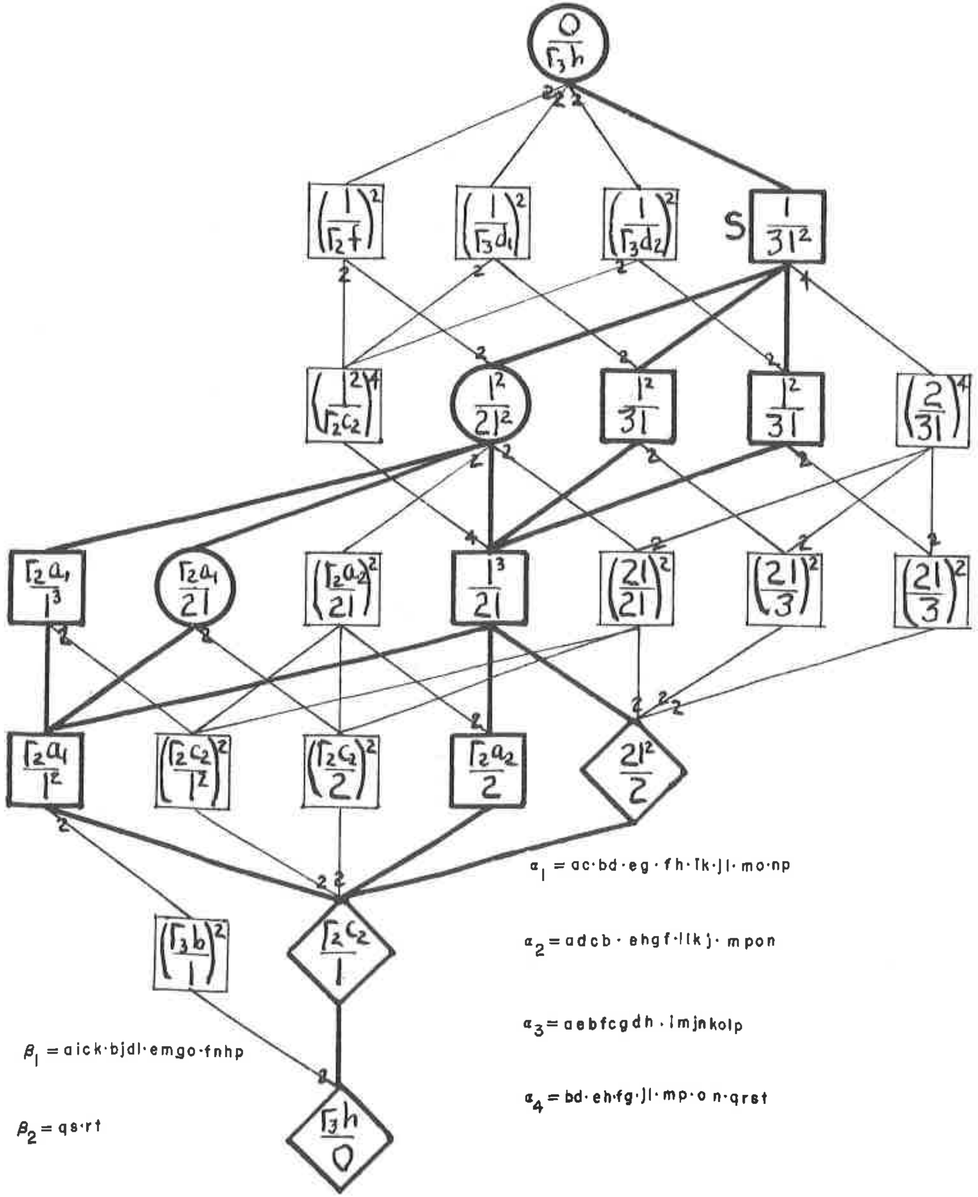
$\alpha_2 = adcb \cdot ehgf \cdot ilkj \cdot mpon$

$\alpha_3 = aebfcgdh \cdot imjnkolp \cdot qrst$

$\alpha_4 = bd \cdot eh \cdot fg \cdot jl \cdot mp \cdot on$

$\beta_1 = aick \cdot bjdl \cdot emgo \cdot fnhp$

$\beta_2 = qs \cdot rt$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

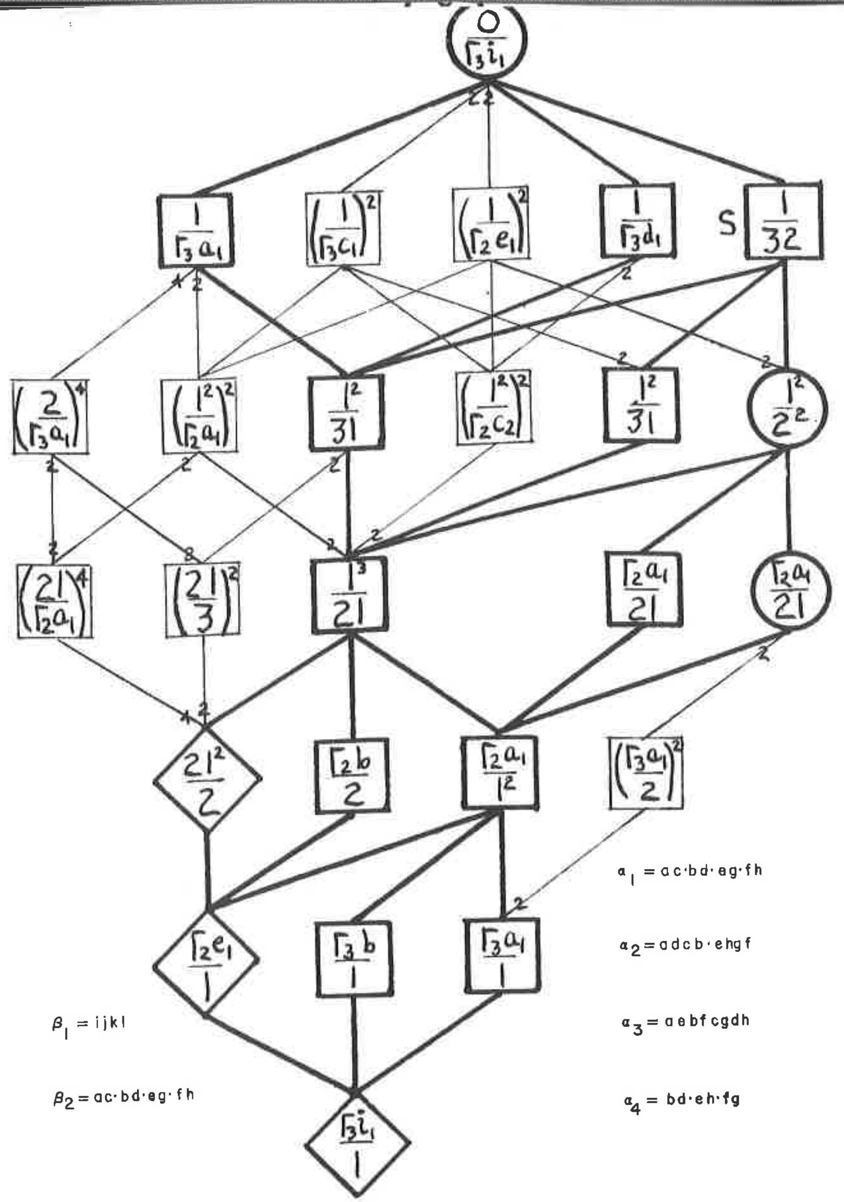
$\alpha_2 = adcb \cdot ehgf \cdot ilkj \cdot mpon$

$\alpha_3 = aebfcgdh \cdot imjnkolp$

$\alpha_4 = bd \cdot eh \cdot fg \cdot jl \cdot mp \cdot on \cdot qrst$

$\beta_1 = aick \cdot bjdl \cdot emgo \cdot fnhp$

$\beta_2 = qs \cdot rt$



$\beta_1 = ijkl$

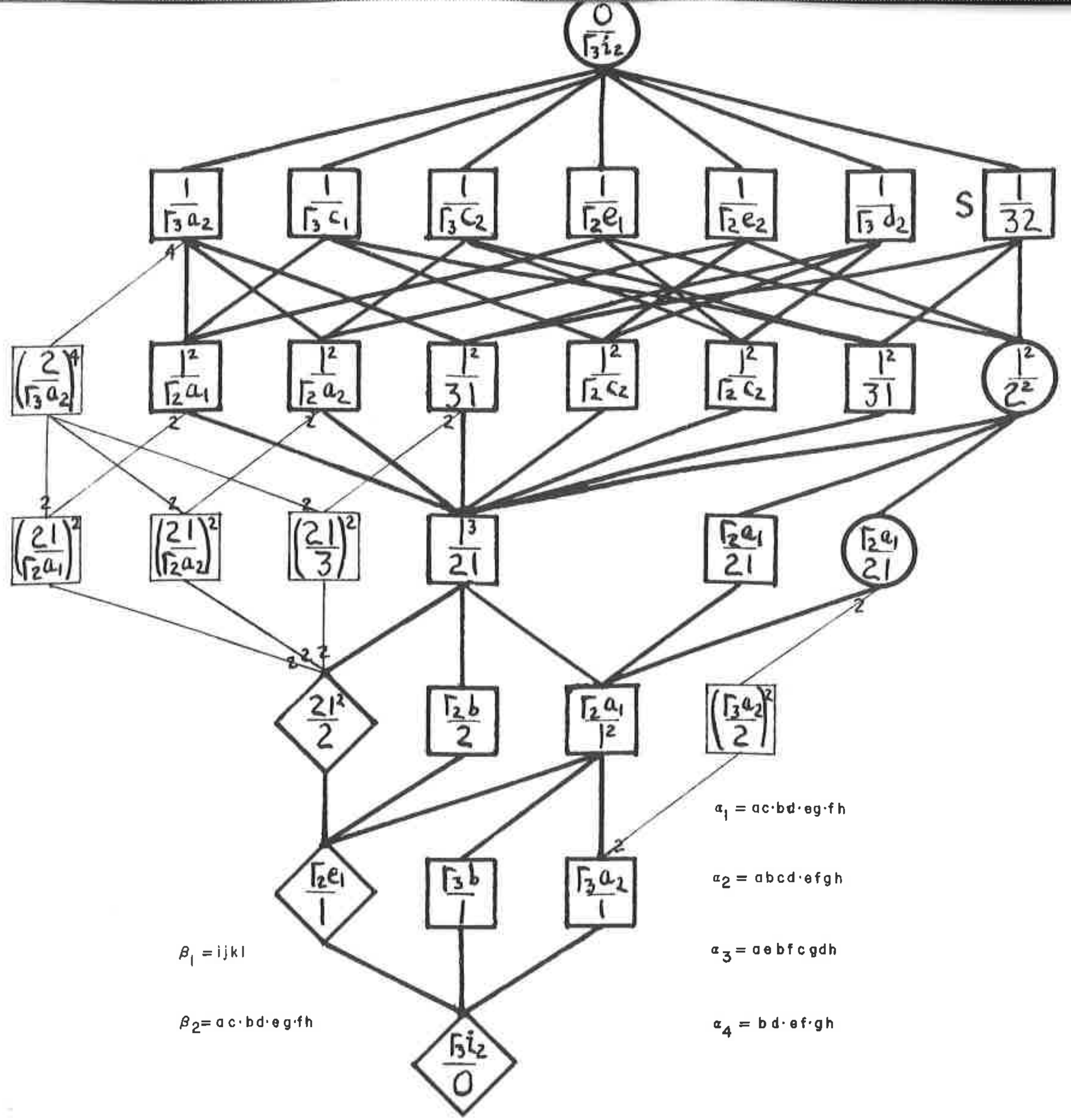
$\beta_2 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot eh \cdot fg$



$\beta_1 = ijkl$

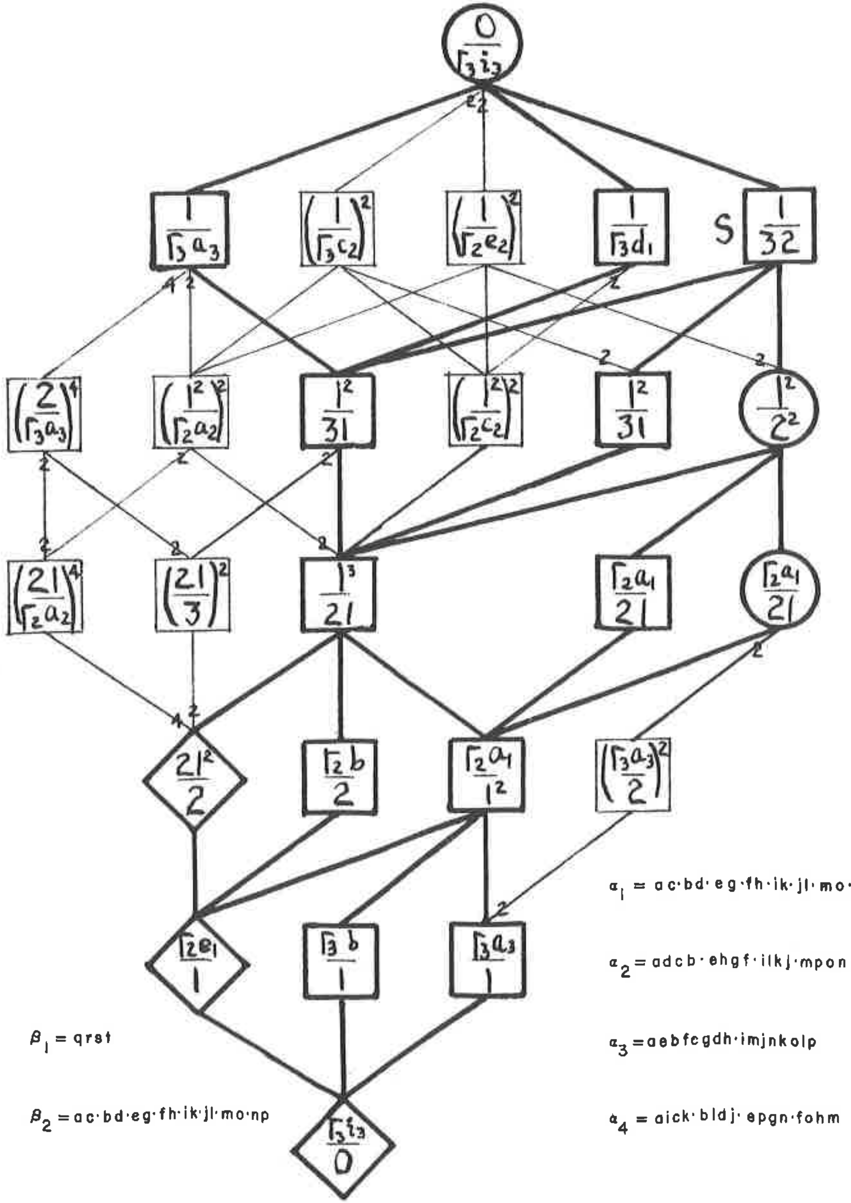
$\beta_2 = ac \cdot bd \cdot eg \cdot fh$

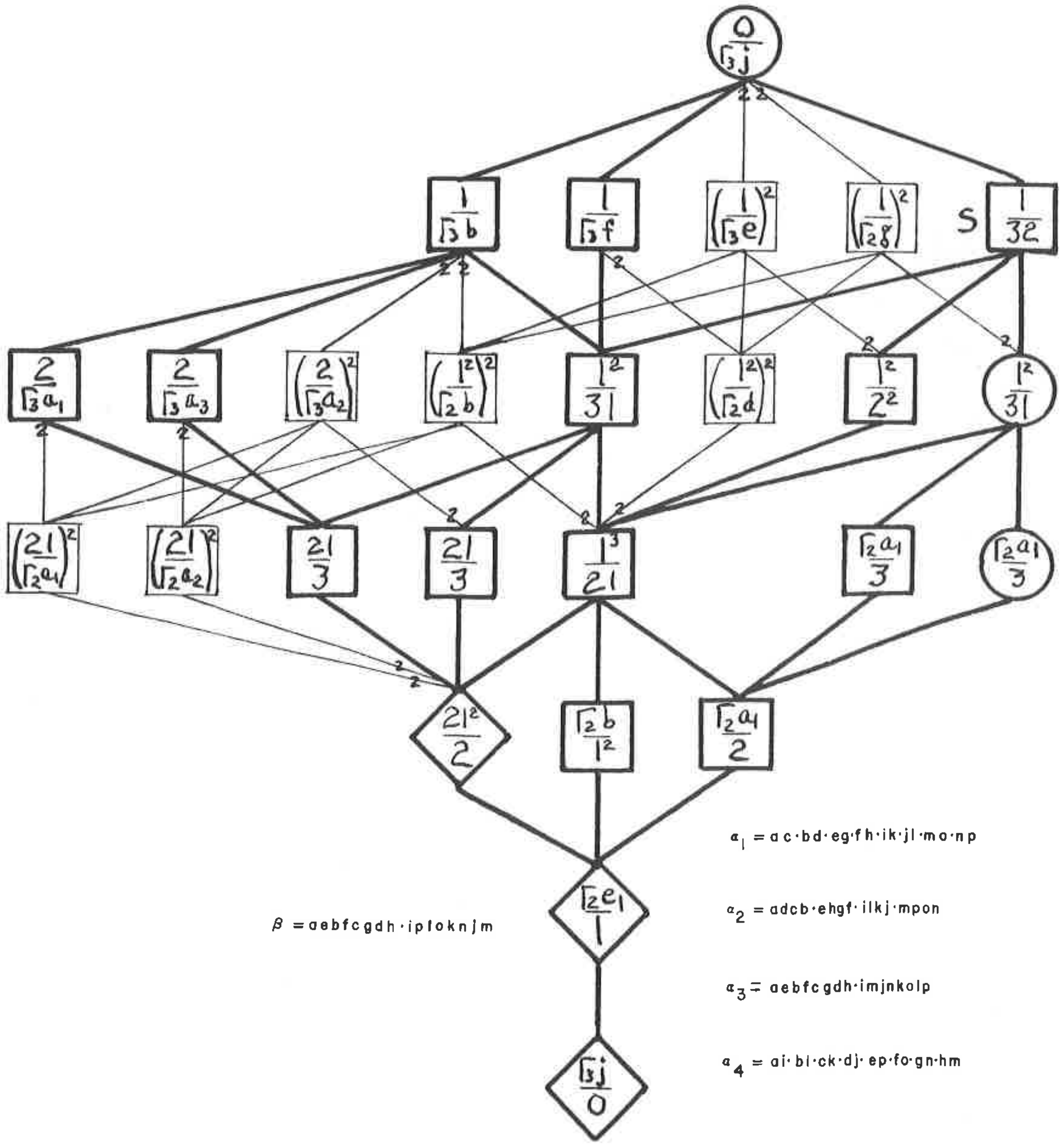
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot ef \cdot gh$

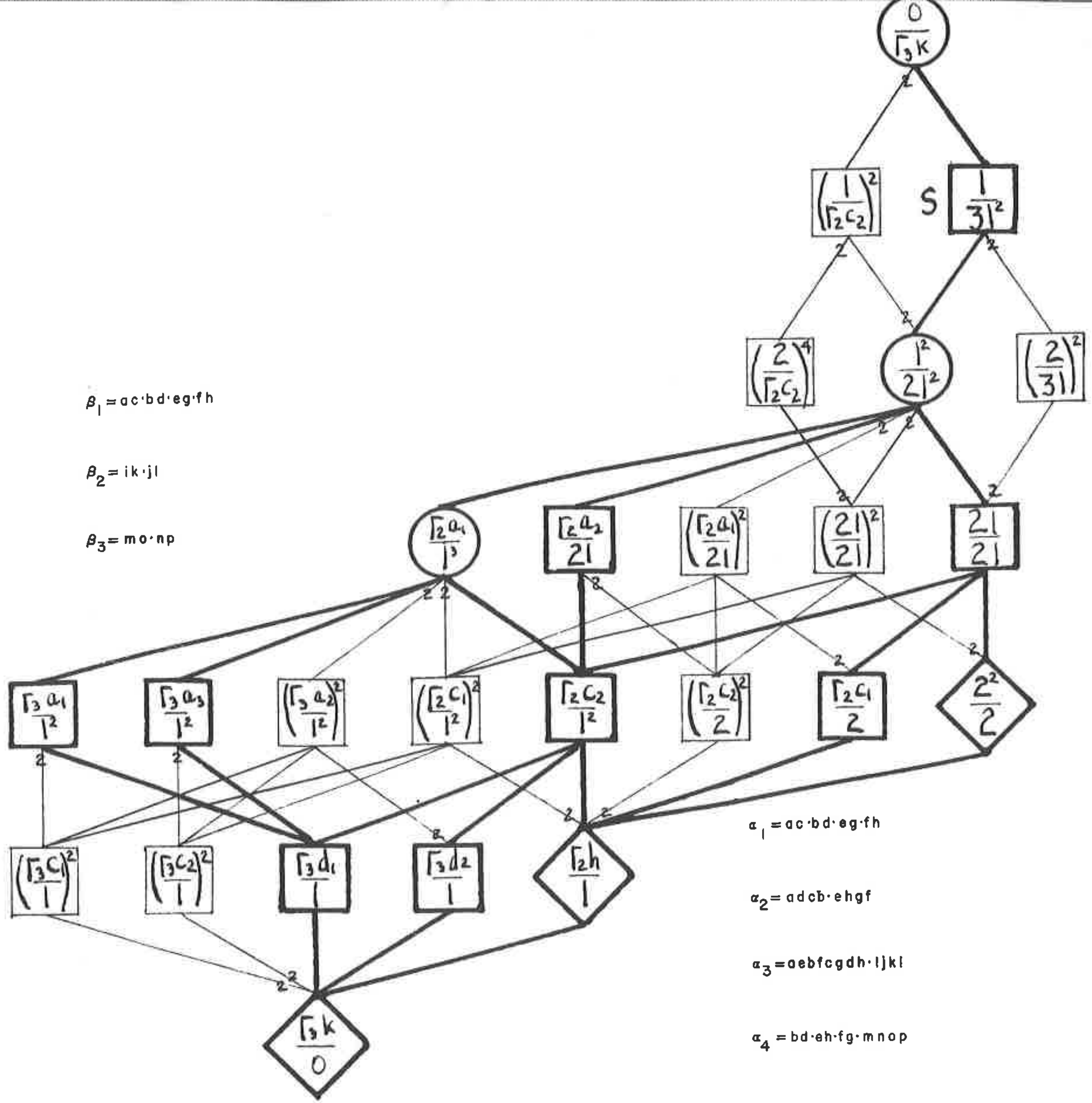

β_1 = qrst
β_2 = ac·bd·eg·fh·ik·jl·mo·np
α_1 = ac·bd·eg·fh·ik·jl·mo·np
α_2 = adcb·ehgf·ilkj·mpon
α_3 = aebfcgdh·imjnkolp
α_4 = aick·bldj·epgn·fohm


β = aebfcgdh·ipfoknjm
α_1 = ac·bd·eg·fh·ik·jl·mo·np
α_2 = adcb·ehgf·ilkj·mpon
α_3 = aebfcgdh·imjnkolp
α_4 = ai·bl·ck·dj·ep·fo·gn·hm

$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik \cdot jl$

$\beta_3 = mo \cdot np$

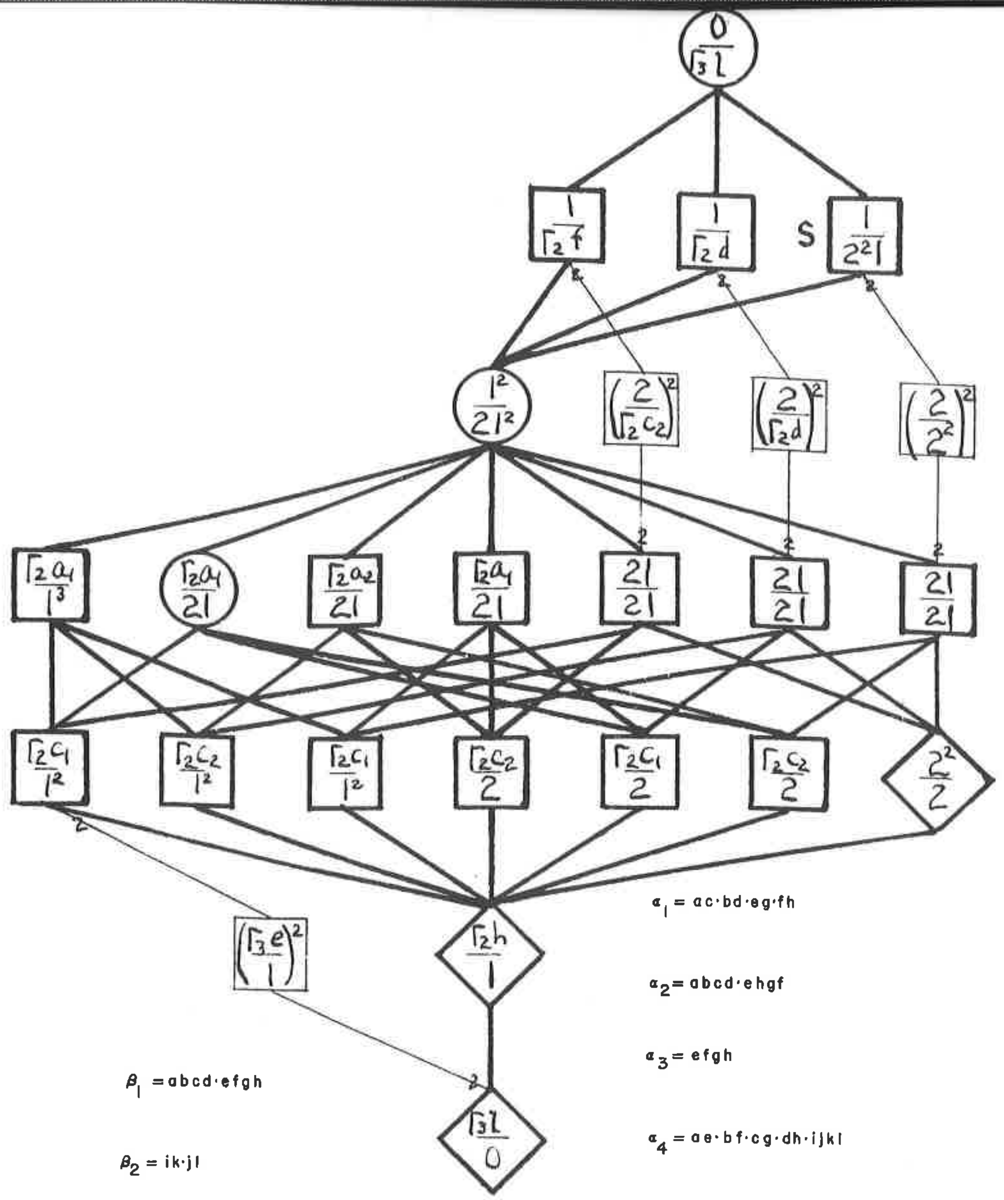


$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh \cdot ijkl$

$\alpha_4 = bd \cdot eh \cdot fg \cdot mnop$



$\beta_1 = abcd \cdot efgh$

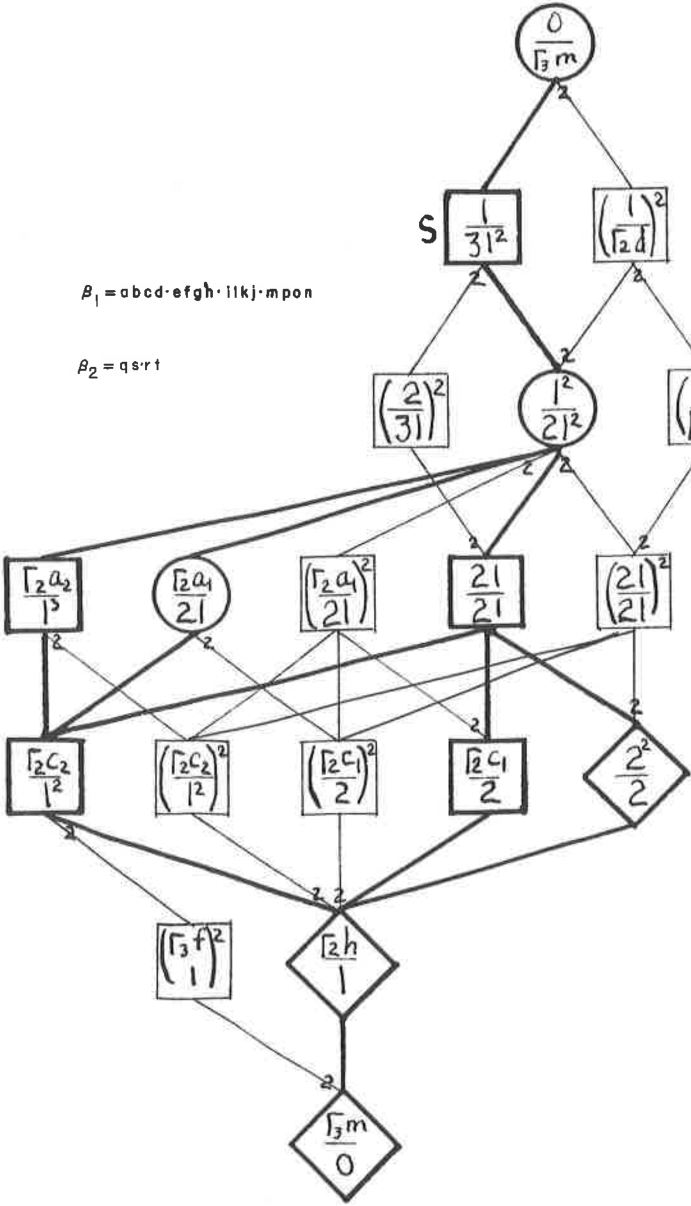
$\beta_2 = ik \cdot jl$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = abcd \cdot ehgf$

$\alpha_3 = efgh$

$\alpha_4 = ae \cdot bf \cdot cg \cdot dh \cdot ijkl$



$\beta_1 = abcd \cdot efgh \cdot ilkj \cdot mpon$

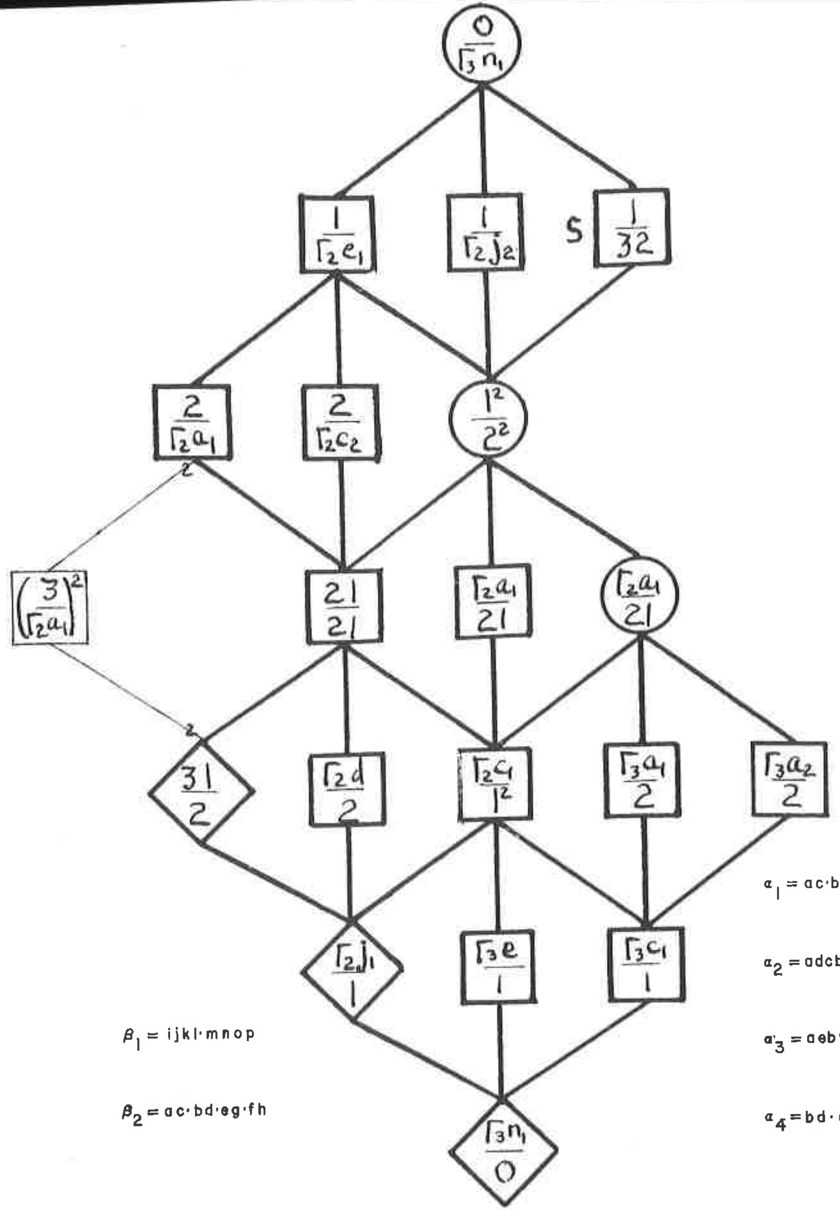
$\beta_2 = qs \cdot rt$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_2 = adcb \cdot ehgf \cdot ilkj \cdot mpon$

$\alpha_3 = aebfcgdh \cdot imjnkolp \cdot qrst$

$\alpha_4 = alblckdj \cdot epfognhm$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

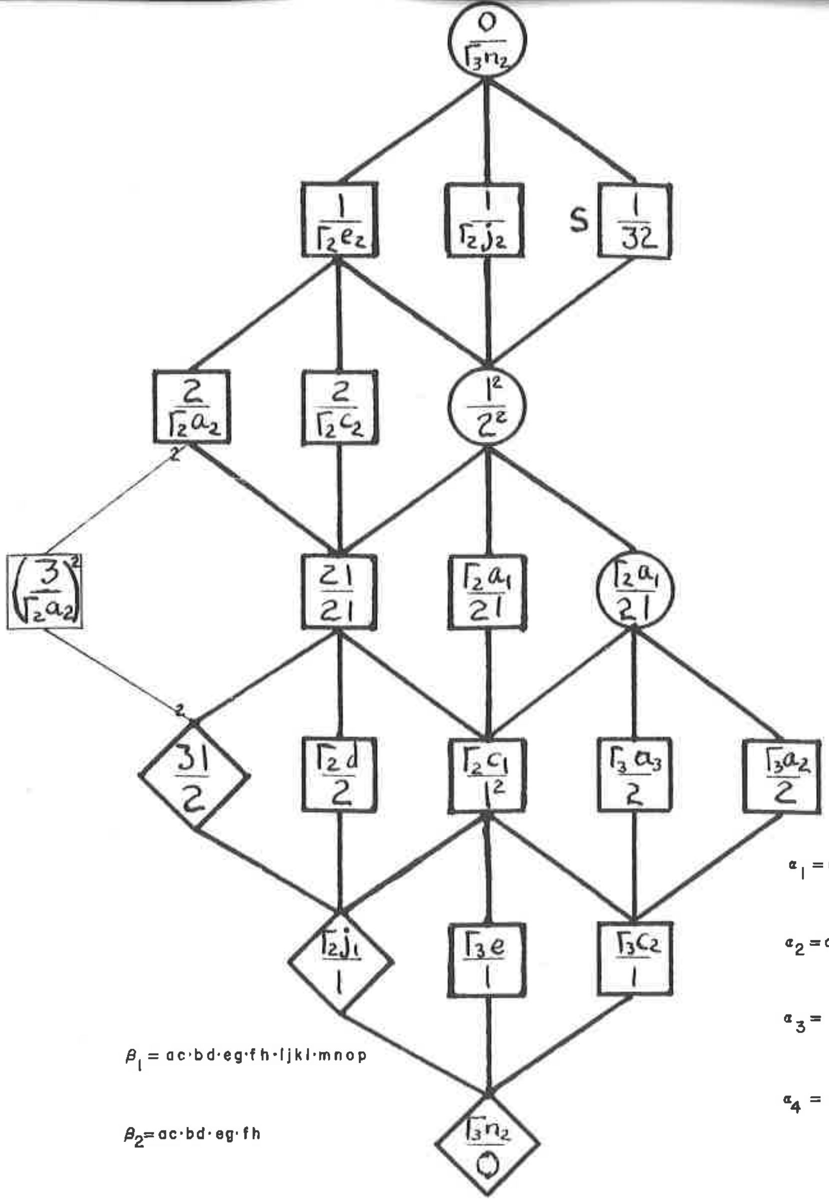
$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh \cdot imjnkolp$

$\alpha_4 = bd \cdot eh \cdot fg$

$\beta_1 = ijkl \cdot mnop$

$\beta_2 = ac \cdot bd \cdot eg \cdot fh$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

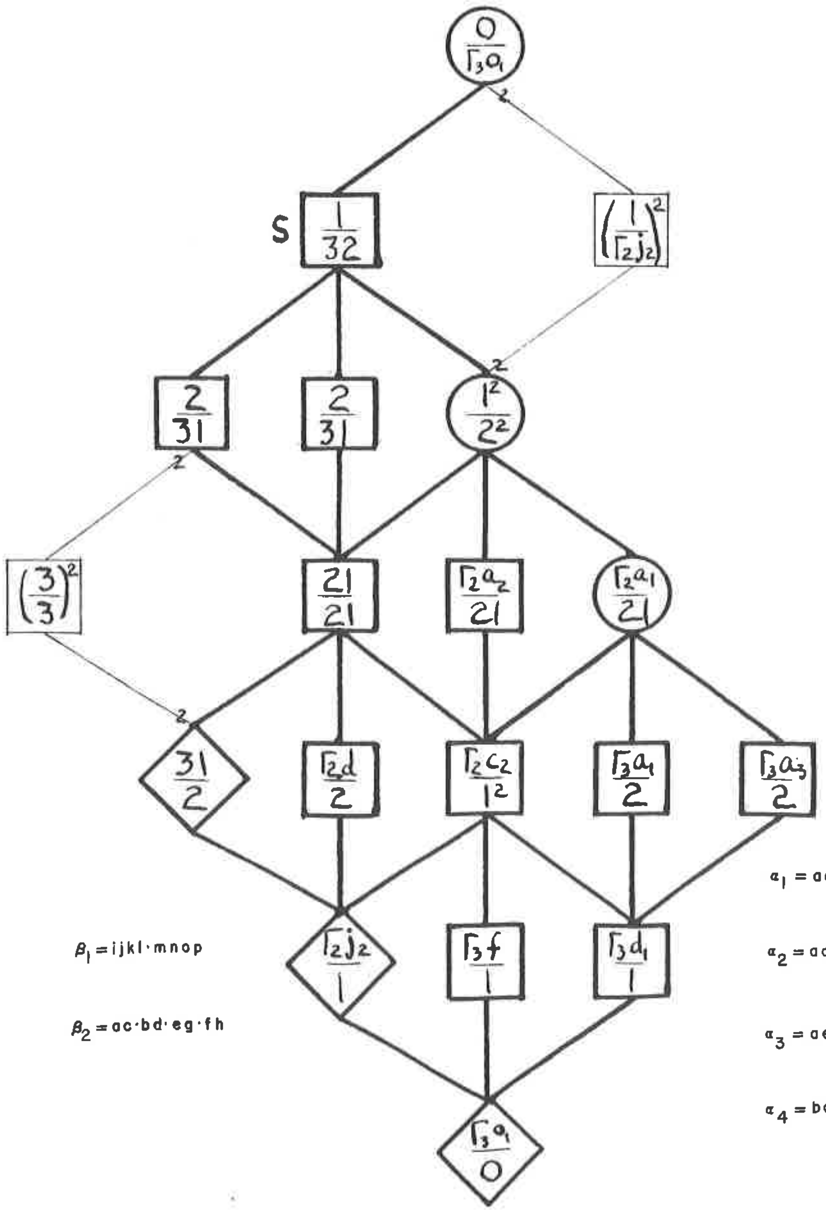
$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh \cdot imjnkolp$

$\alpha_4 = aecg \cdot bhdf$

$\beta_1 = ac \cdot bd \cdot eg \cdot fh \cdot ijkl \cdot mnop$

$\beta_2 = ac \cdot bd \cdot eg \cdot fh$



$\beta_1 = ijkl \cdot mnop$

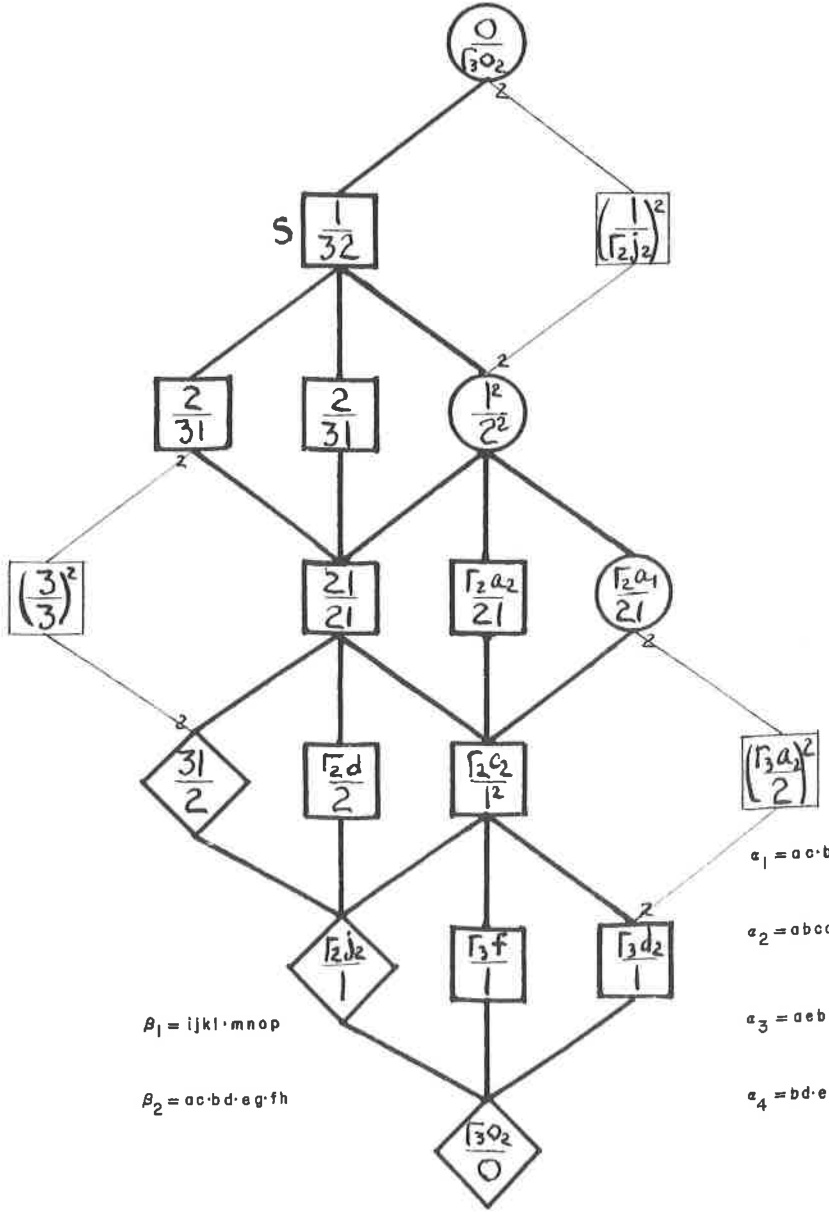
$\beta_2 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = adcb \cdot ehgf$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot eh \cdot fg \cdot imjnkolp$



$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

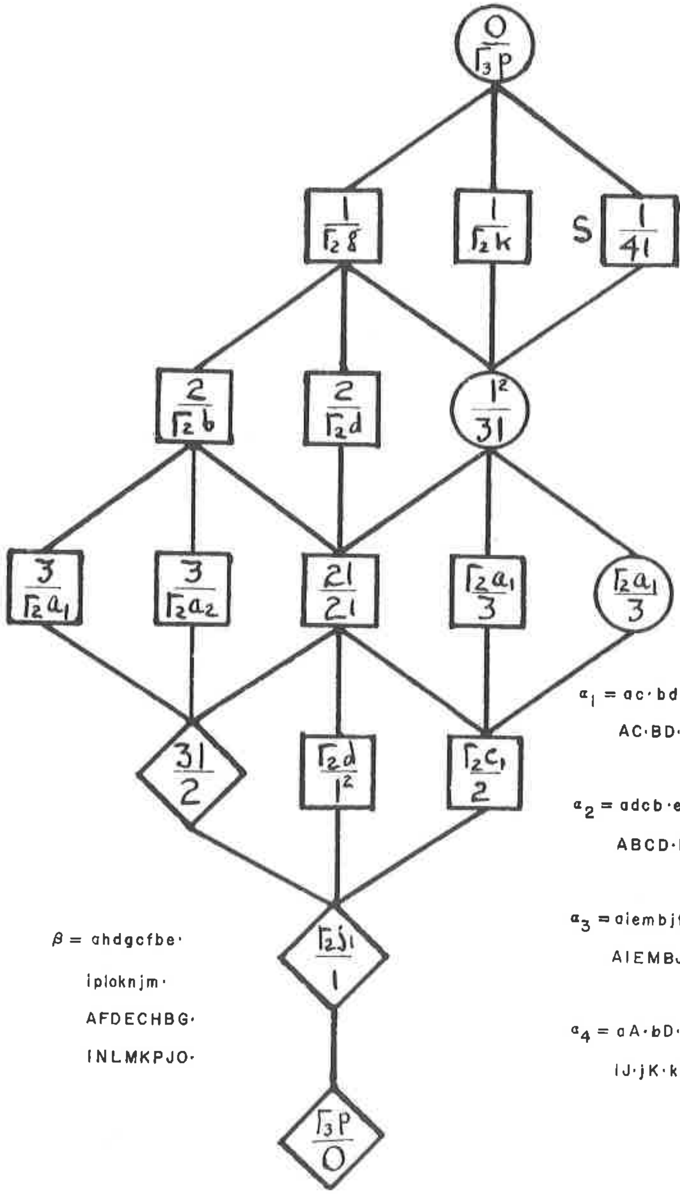
$\alpha_2 = abcd \cdot efgh$

$\alpha_3 = aebfcgdh$

$\alpha_4 = bd \cdot ef \cdot gh \cdot imjnkolp$

$\beta_1 = ijkl \cdot mnop$

$\beta_2 = ac \cdot bd \cdot eg \cdot fh$



α_1 = ac·bd·eg·fh·lk·jl·mo·np

AC·BD·EG·FH·IK·JL·MO·NP

α_2 = adcb·ehgf·ilkj·mpon

ABCD·EFGH·IJKL·MNOP

α_3 = aiembjfnckgodlhp·

AIEMBJFNCKGODLHP·

α_4 = aA·bD·cC·dB·eG·fH·gE·hF

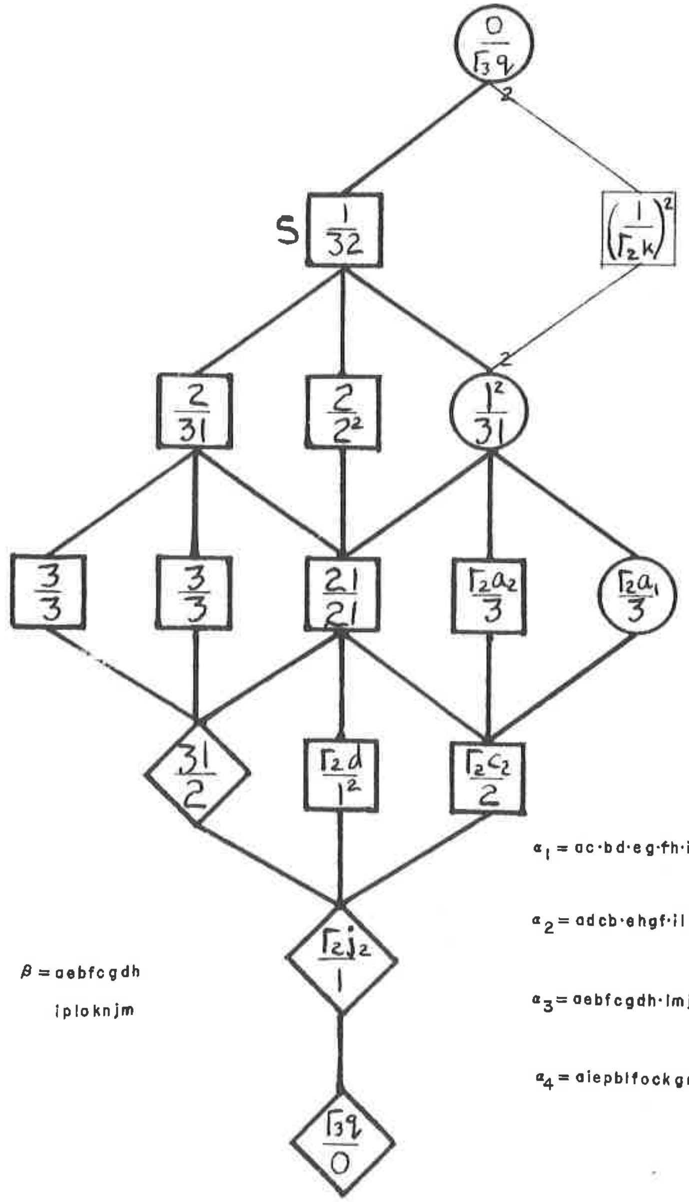
lJ·jK·kL·lI·mP·nM·oN·pO

β = ahdgcfbe·

iplokn jm·

AFDECHBG·

INLMKPJO·



α_1 = ac·bd·eg·fh·ik·jl·mo·np

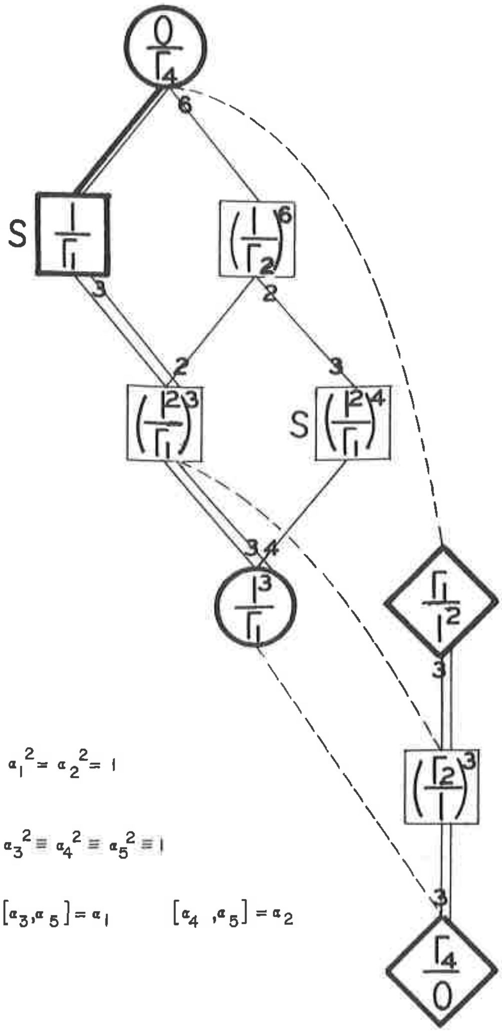
α_2 = adcb·ehgf·ilkj·mpon

α_3 = aebfcgdh·lmjnkolp

α_4 = aiepblfockgndjhm·

β = aebfcgdh

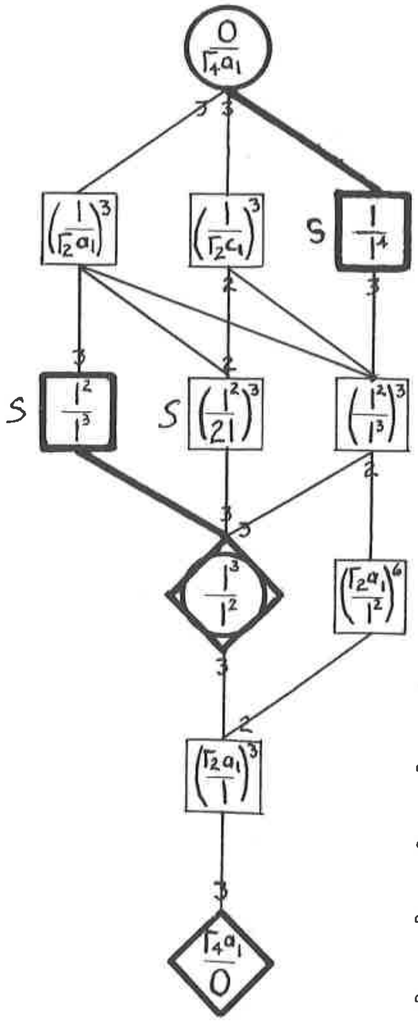
iploknjm



$\alpha_1{}^2 = \alpha_2{}^2 = 1$

$\alpha_3{}^2 \equiv \alpha_4{}^2 \equiv \alpha_5{}^2 \equiv 1$

$[\alpha_3, \alpha_5] = \alpha_1$ $\qquad$ $[\alpha_4, \alpha_5] = \alpha_2$



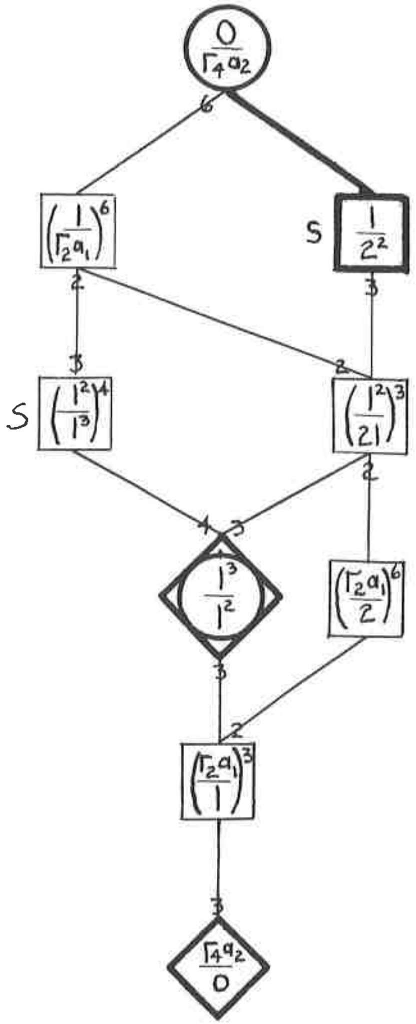
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ab \cdot cd$

$\alpha_4 = ef \cdot gh$

$\alpha_5 = bd \cdot fh$



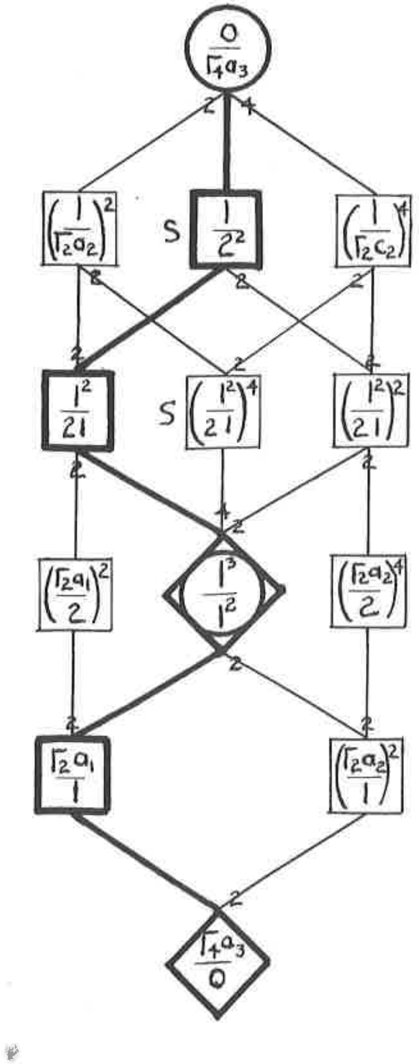
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = abcd$

$\alpha_4 = efgh$

$\alpha_5 = bd \cdot fh$



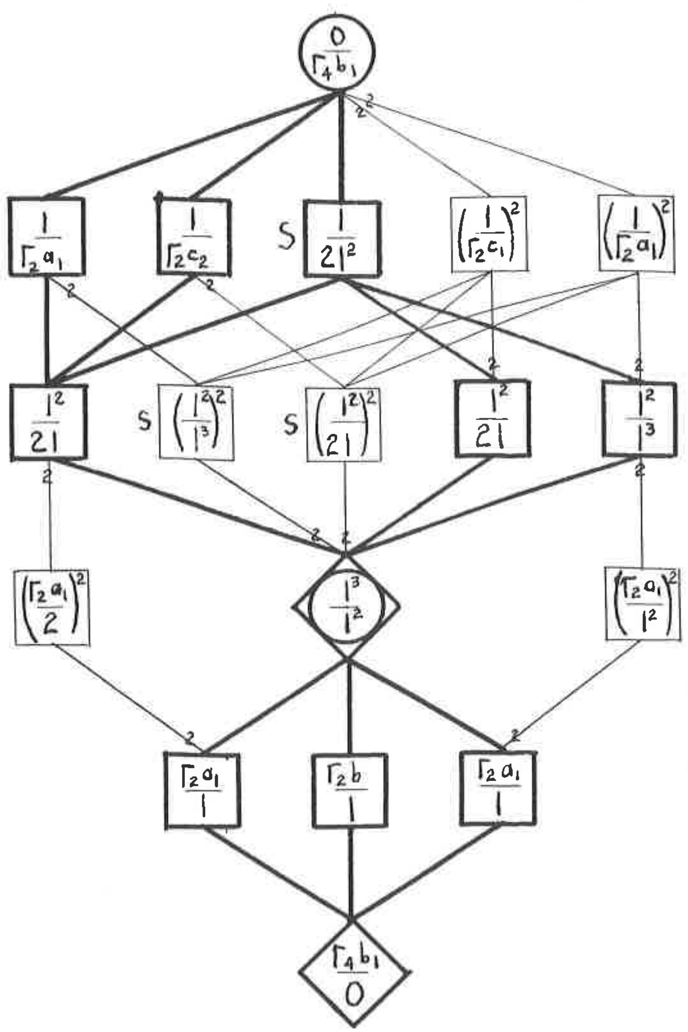
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = abcd \cdot efgh$

$\alpha_4 = ijkl$

$\alpha_5 = aecg \cdot bhdf \cdot jl$



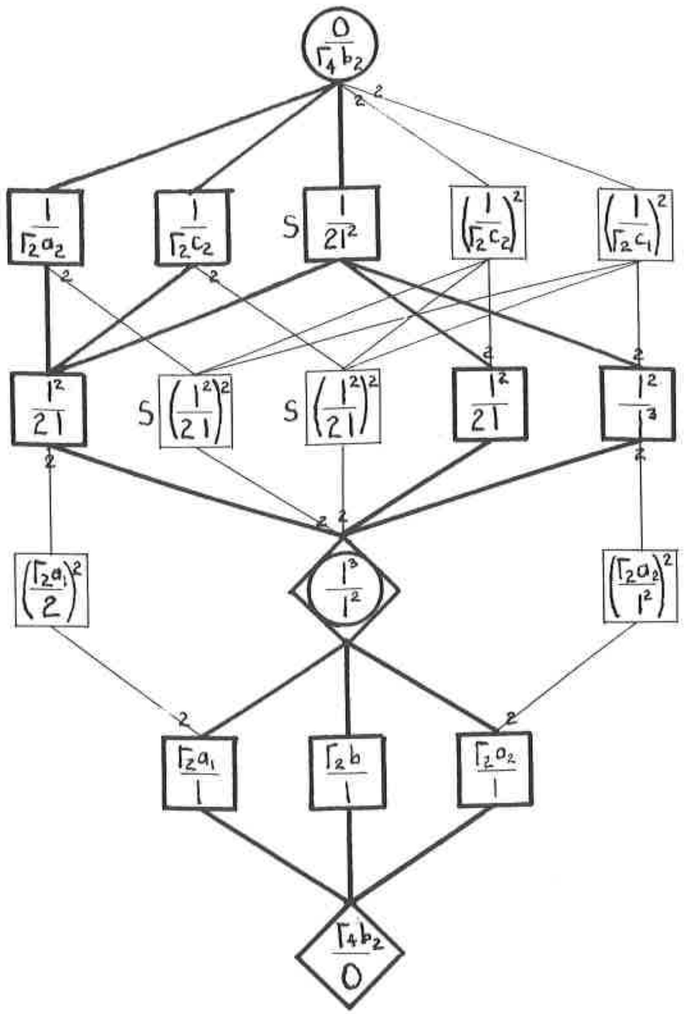
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = abcd$

$\alpha_4 = ef \cdot gh$

$\alpha_5 = bd \cdot fh$



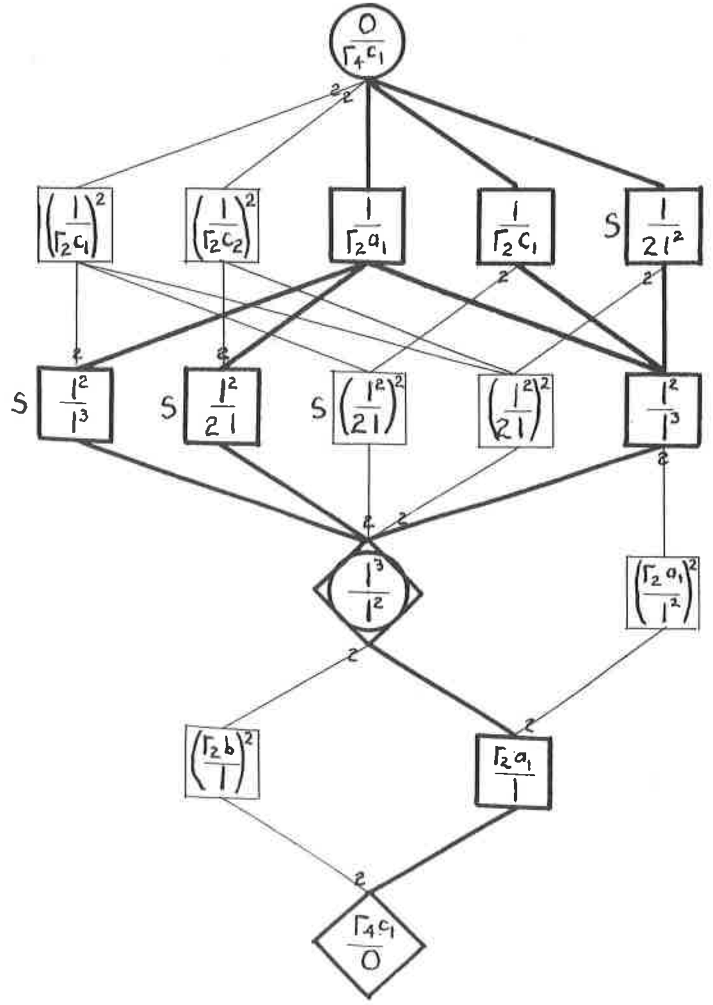
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh \cdot$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = abcd \cdot efgh$

$\alpha_4 = ij \cdot kl$

$\alpha_5 = aecg \cdot bhdf \cdot jl$



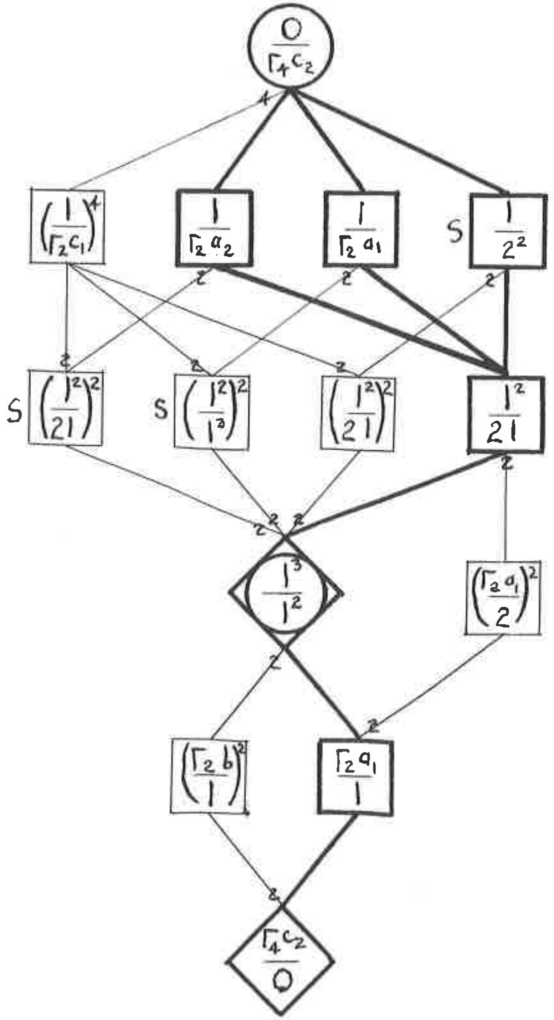
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = eg \cdot fh$

$\alpha_4 = abcd \cdot efgh \cdot jl$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot ij \cdot kl$



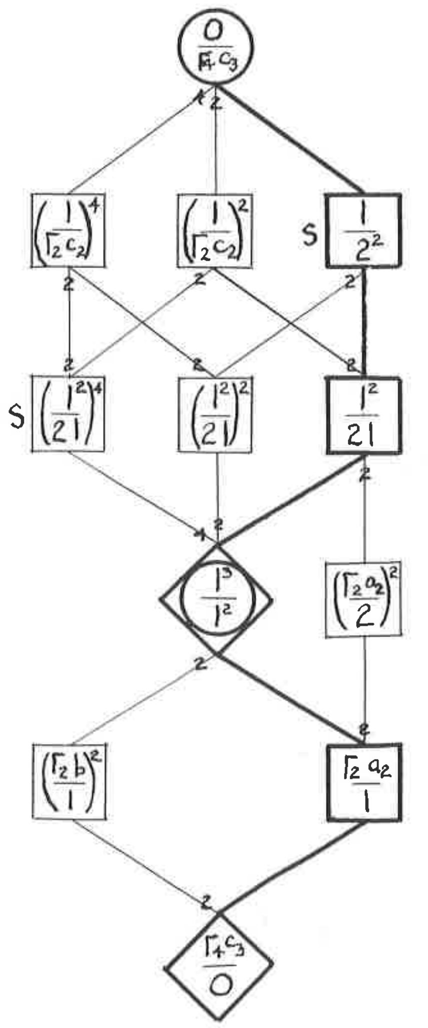
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = abcd \cdot ehgf$

$\alpha_4 = abcd \cdot efgh \cdot ljkl$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot jl$



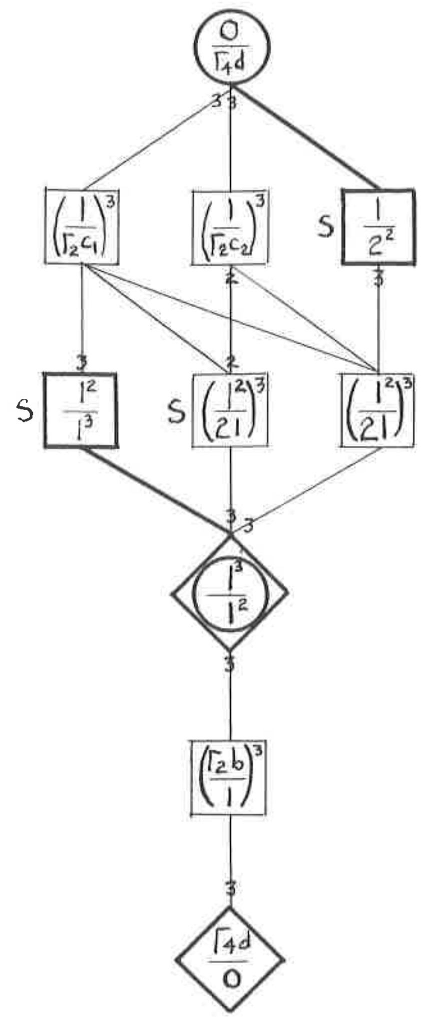
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl \cdot mo \cdot np$

$\alpha_3 = abcd \cdot ehgf$

$\alpha_4 = abcd \cdot efgh \cdot ijkl \cdot mnop$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot imko \cdot jpln$



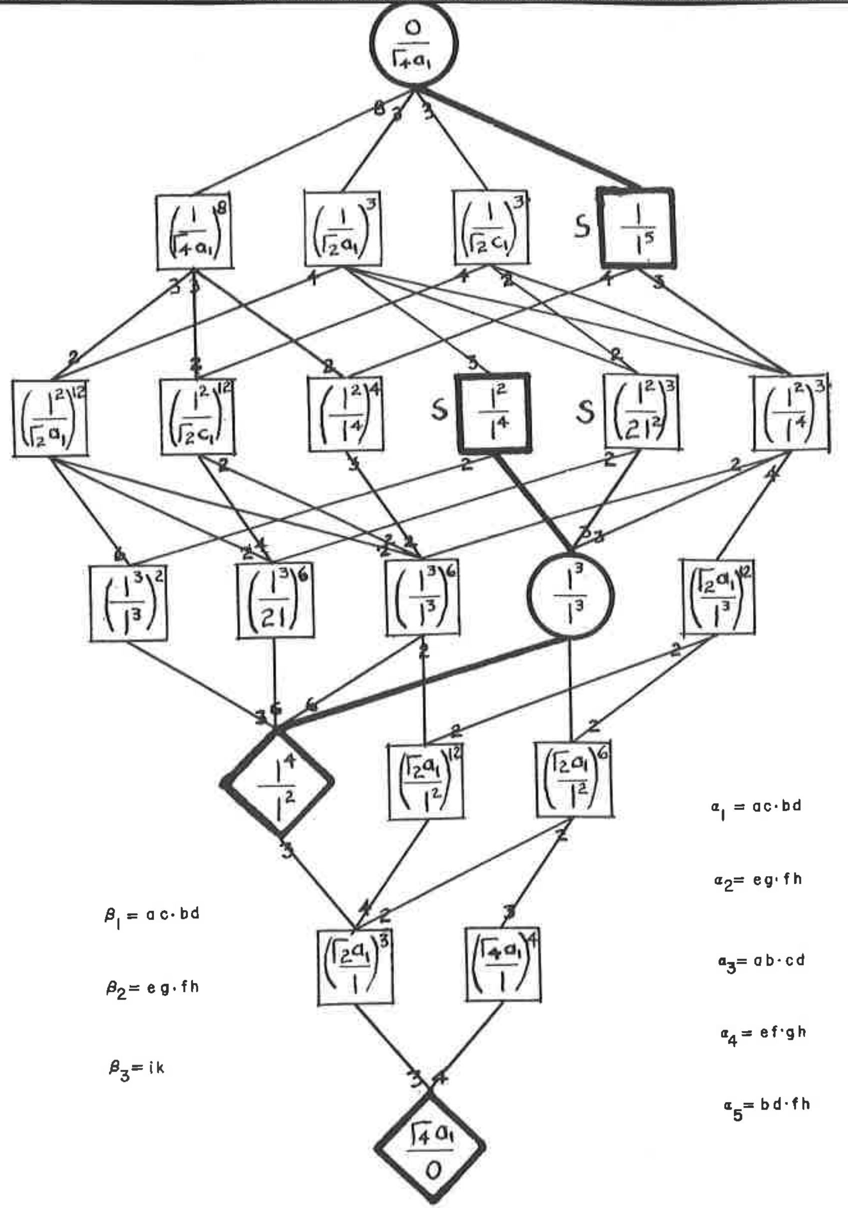
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl \cdot mo \cdot np$

$\alpha_3 = eg\ fh \cdot ijkl \cdot mnop$

$\alpha_4 = abcd \cdot efgh \cdot ijkl \cdot mpon$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot im \cdot jn \cdot ko \cdot lp \cdot$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = ik$

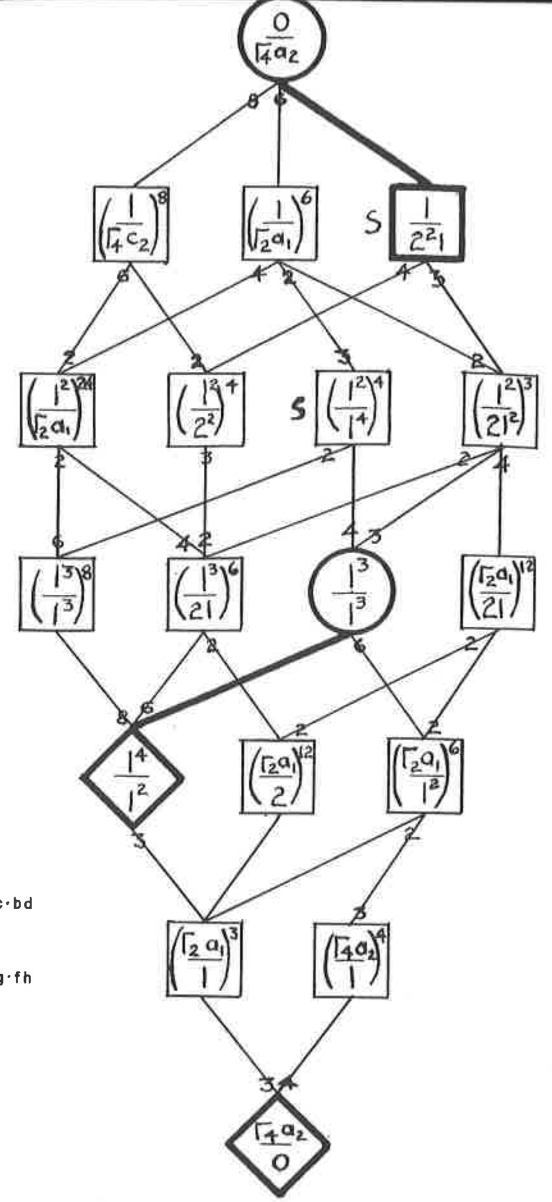
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ab \cdot cd$

$\alpha_4 = ef \cdot gh$

$\alpha_5 = bd \cdot fh$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = ik$

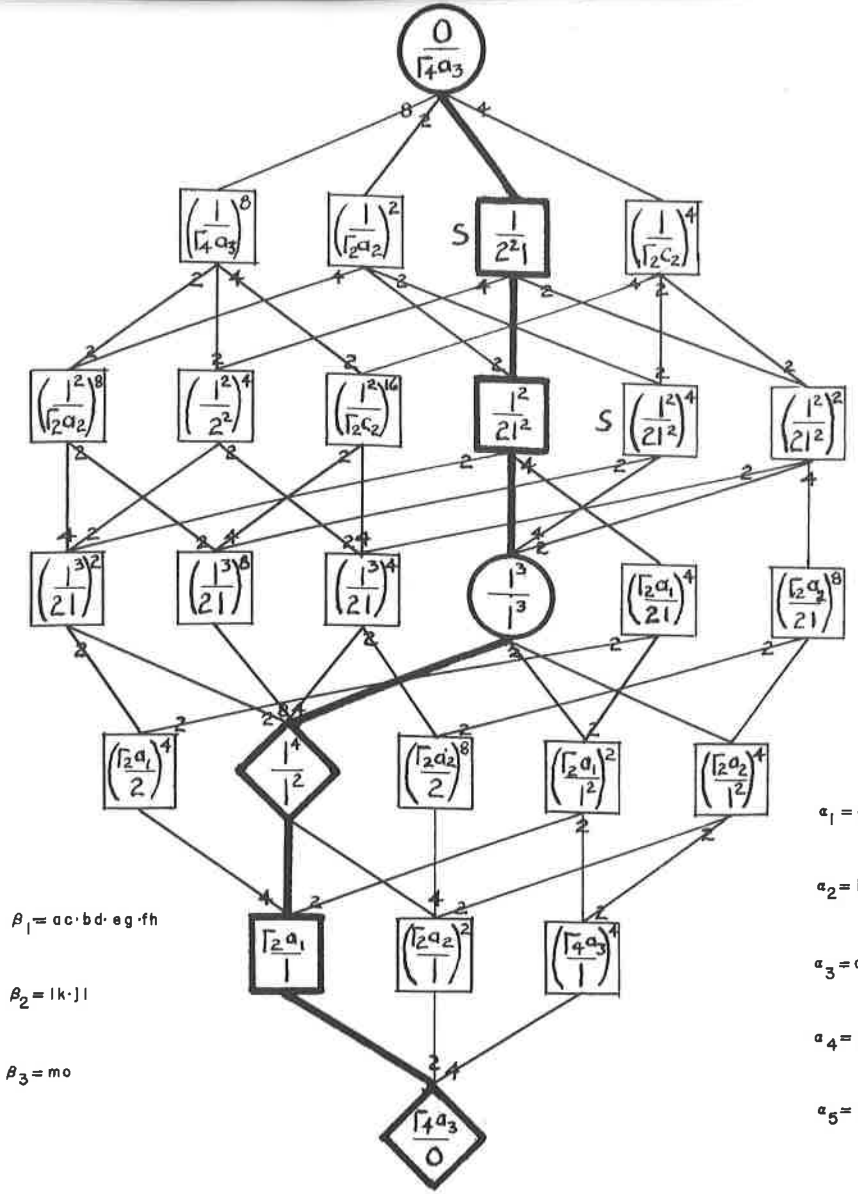
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ab\ cd$

$\alpha_4 = efgh$

$\alpha_5 = bd \cdot fh$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

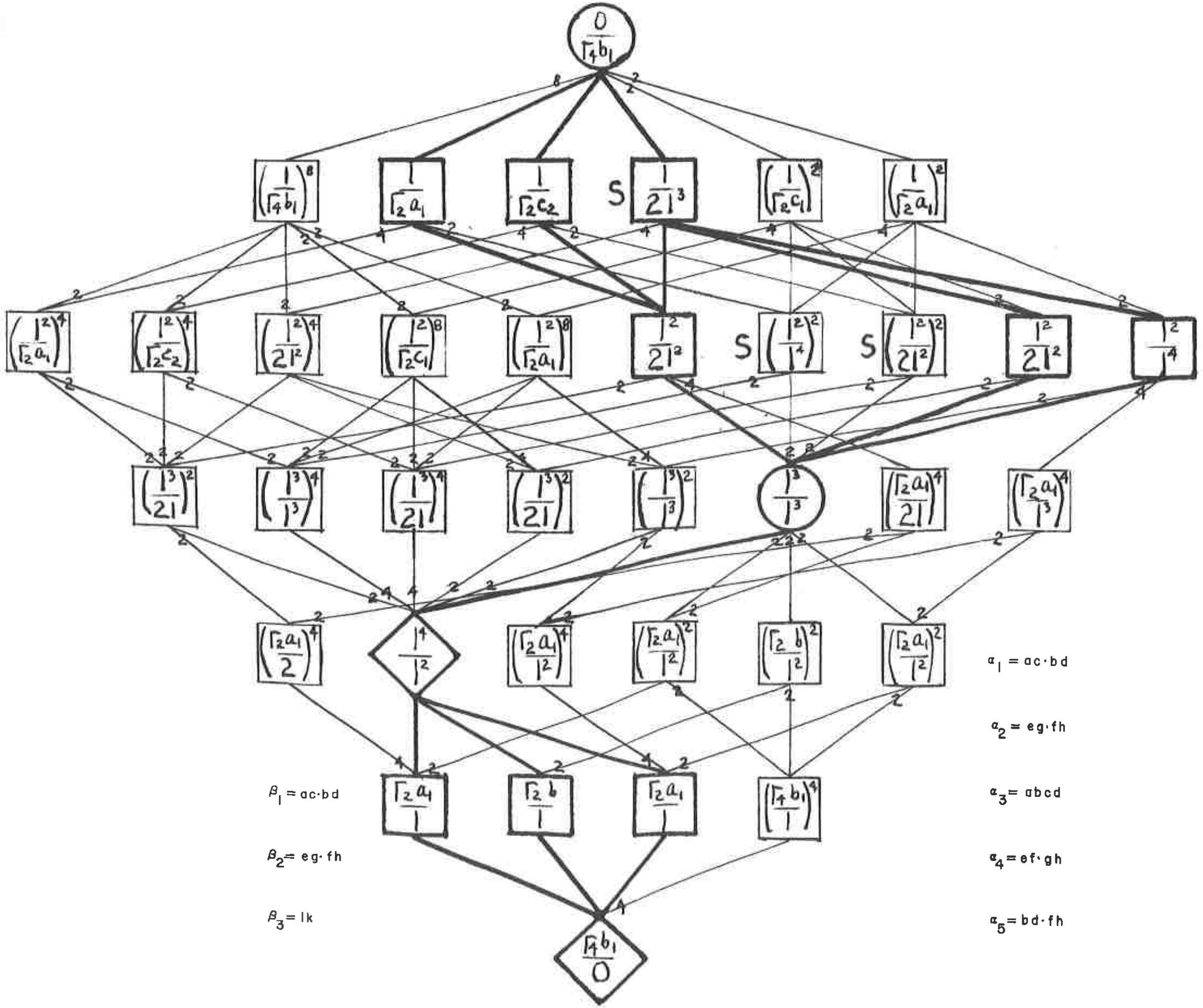
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = abcd \cdot efgh$

$\alpha_4 = ijkl$

$\alpha_5 = aecg \cdot bhdf \cdot jl$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = lk$

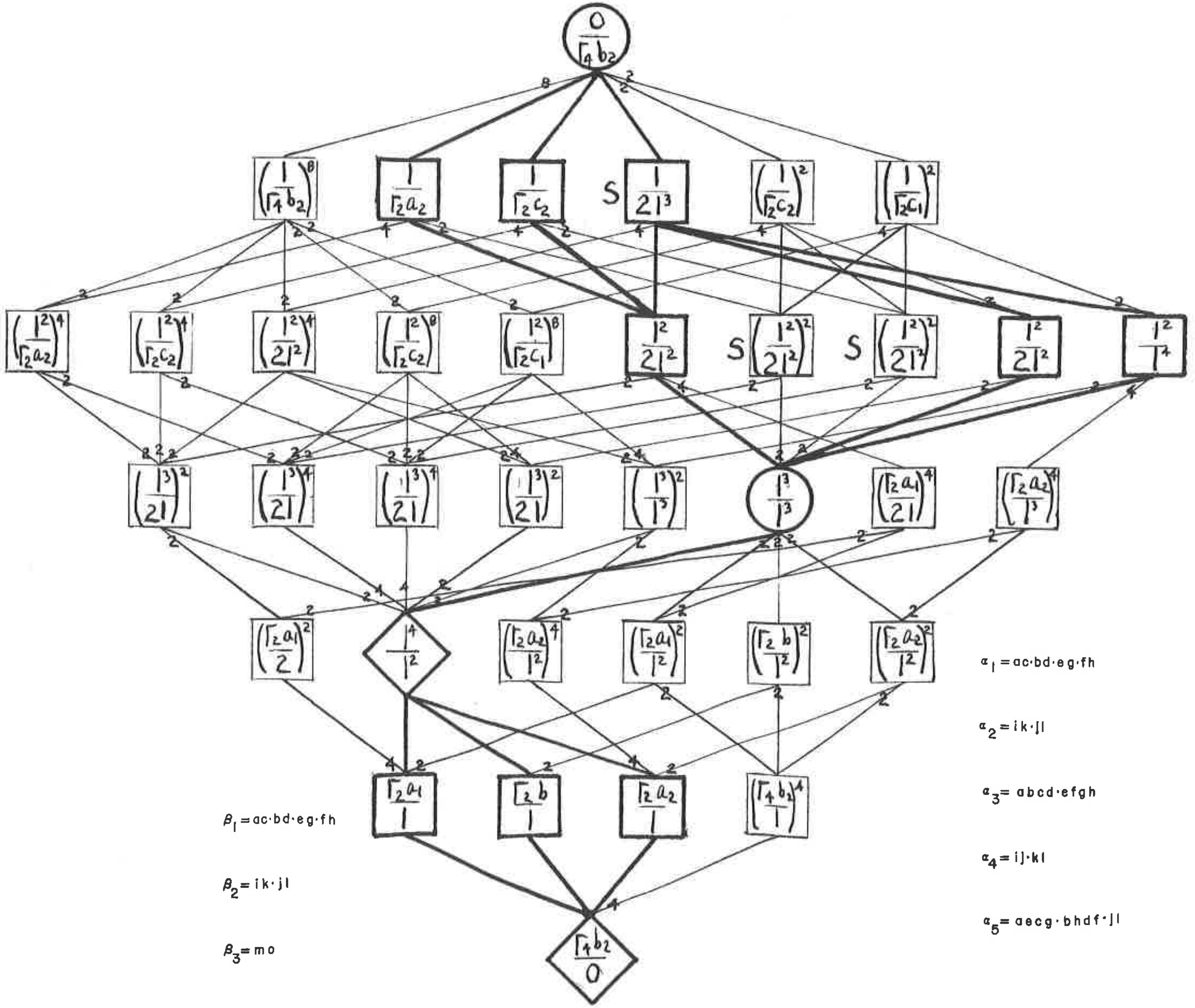
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = abcd$

$\alpha_4 = ef \cdot gh$

$\alpha_5 = bd \cdot fh$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

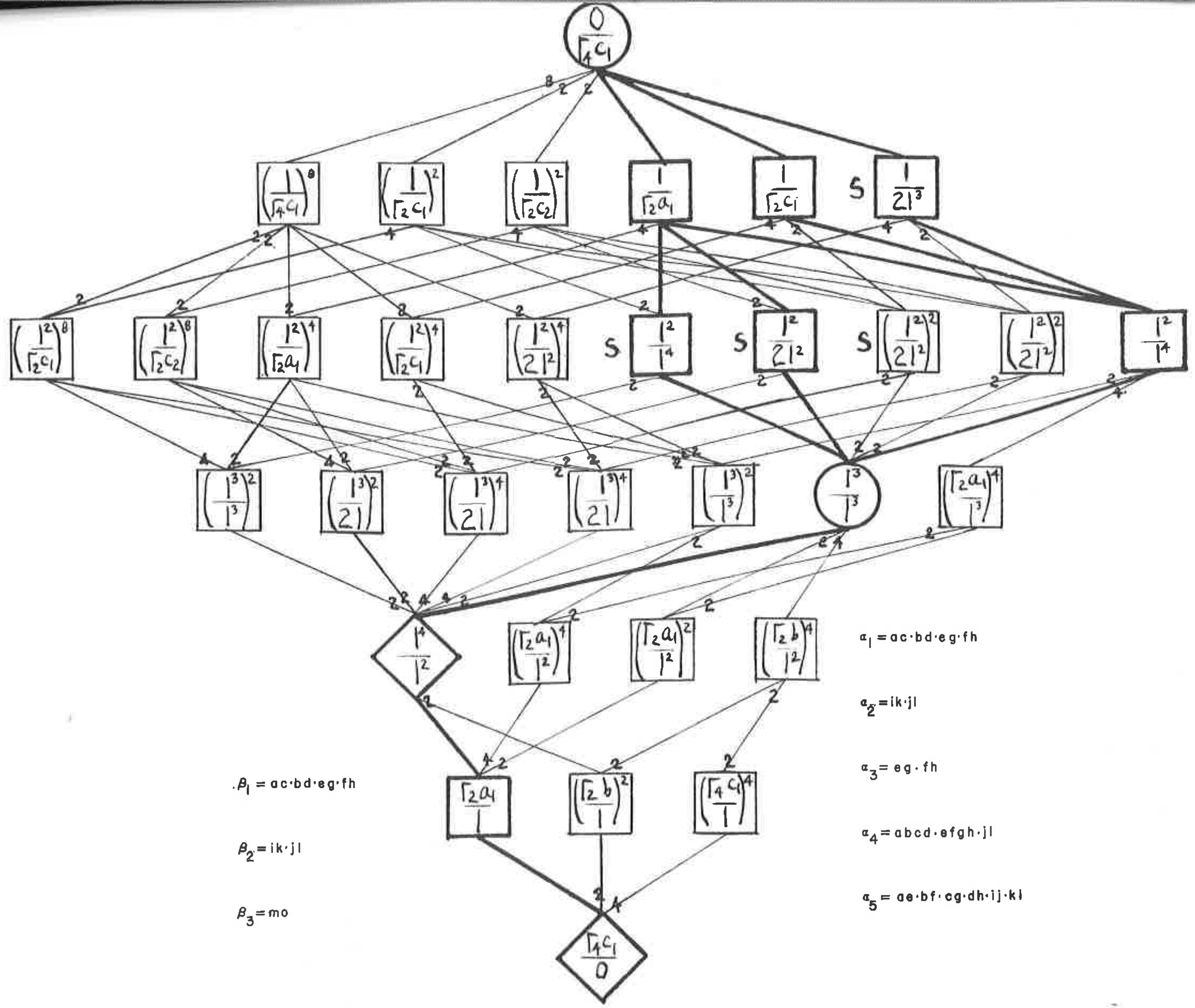
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = abcd \cdot efgh$

$\alpha_4 = ij \cdot kl$

$\alpha_5 = aecg \cdot bhdf \cdot jl$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

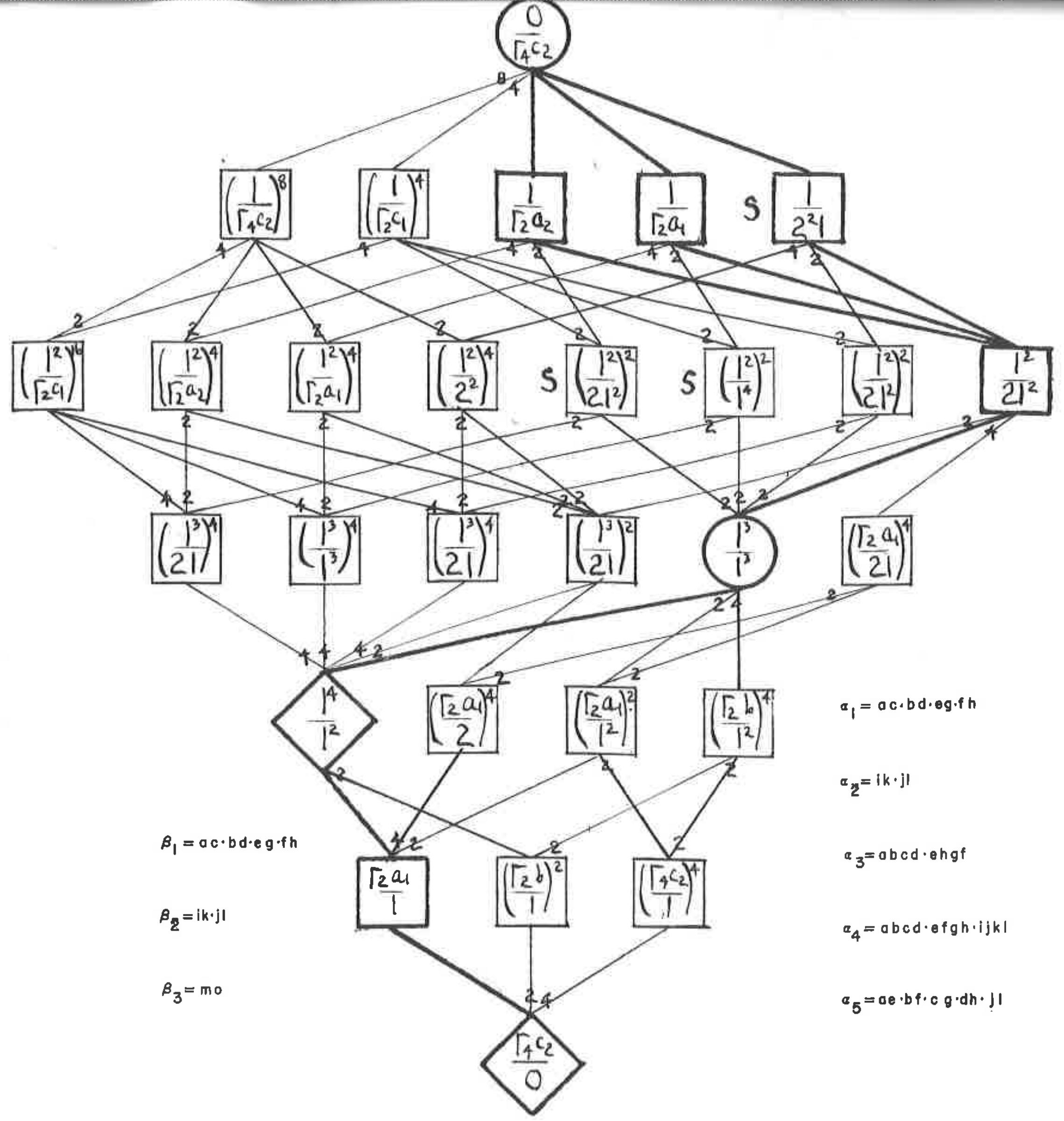
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = eg \cdot fh$

$\alpha_4 = abcd \cdot efgh \cdot jl$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot ij \cdot kl$



$\beta_1 = ac \cdot bd \cdot eg \cdot fh$

$\beta_2 = ik \cdot jl$

$\beta_3 = mo$

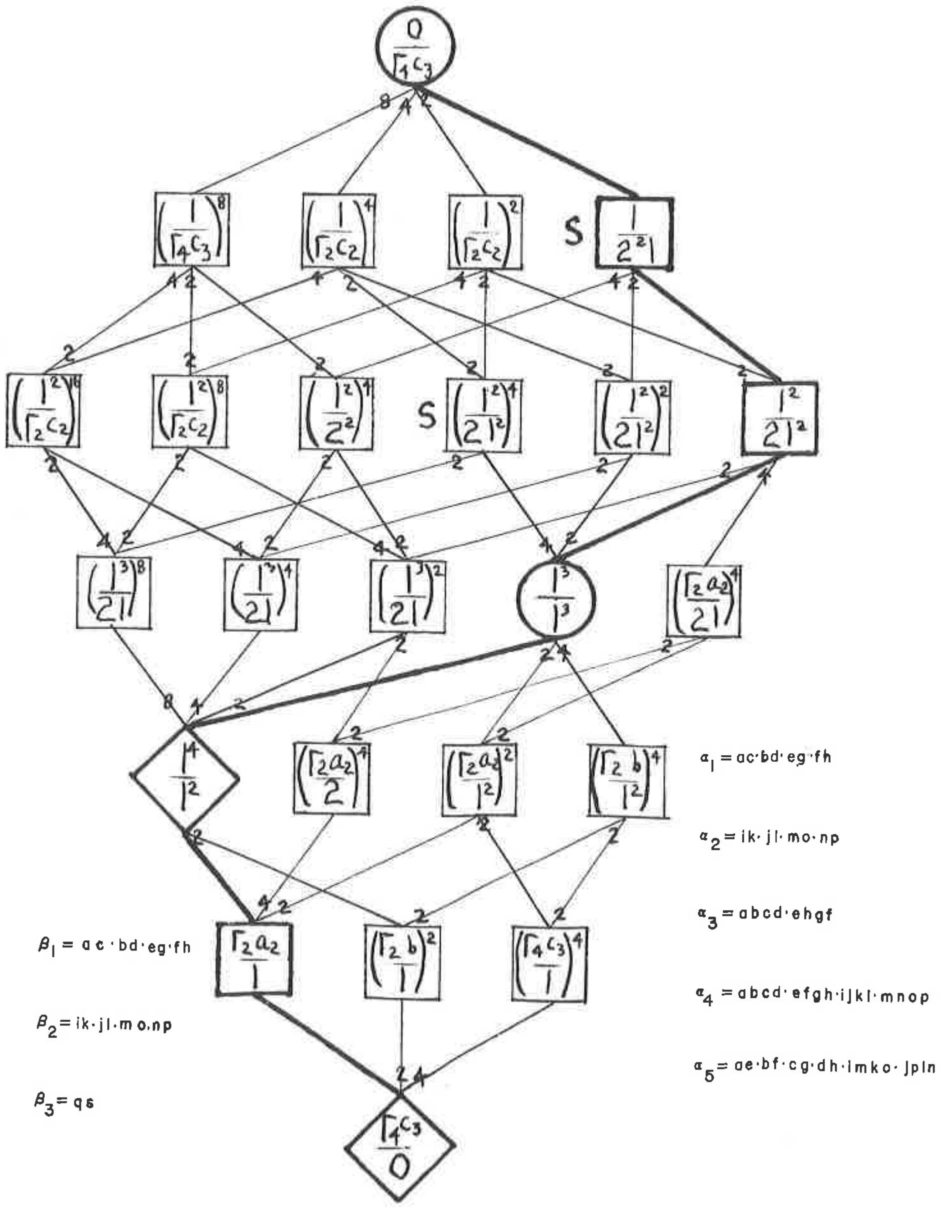
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

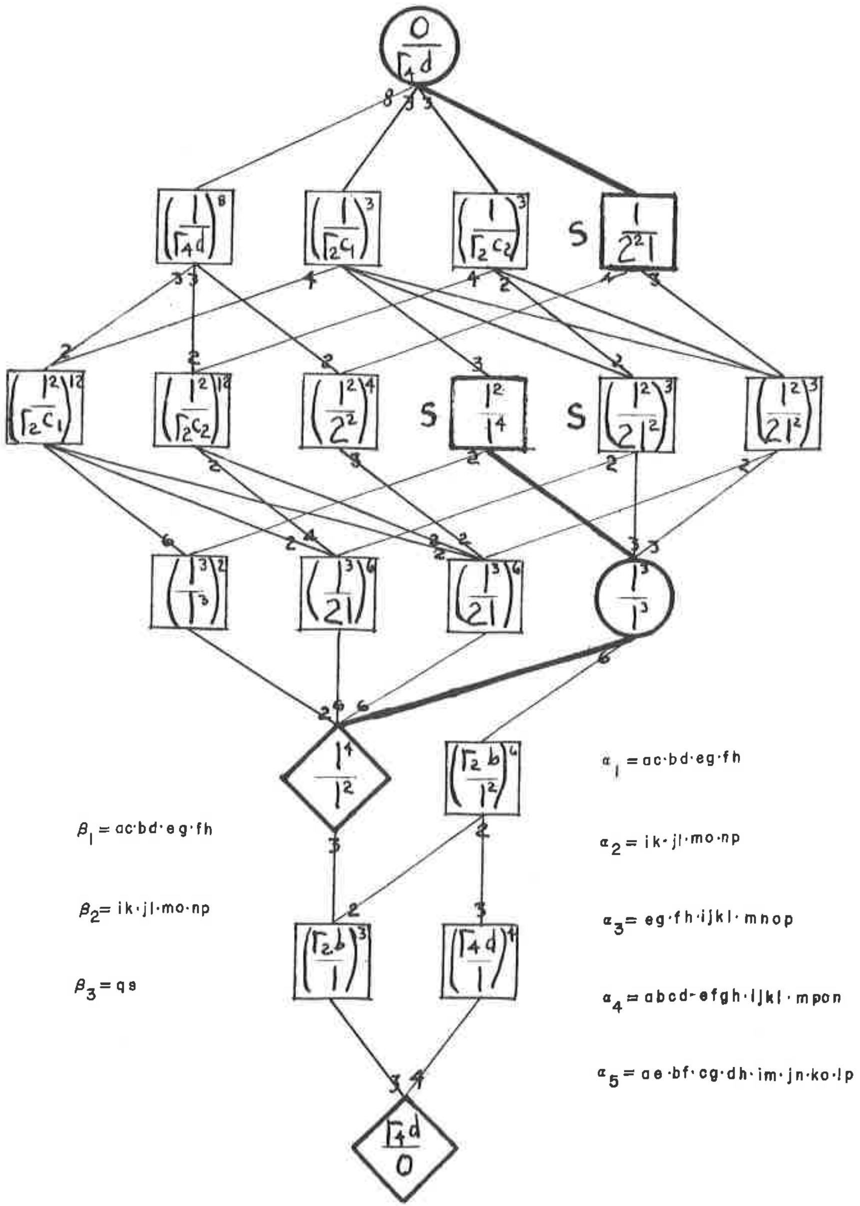
$\alpha_2 = ik \cdot jl$

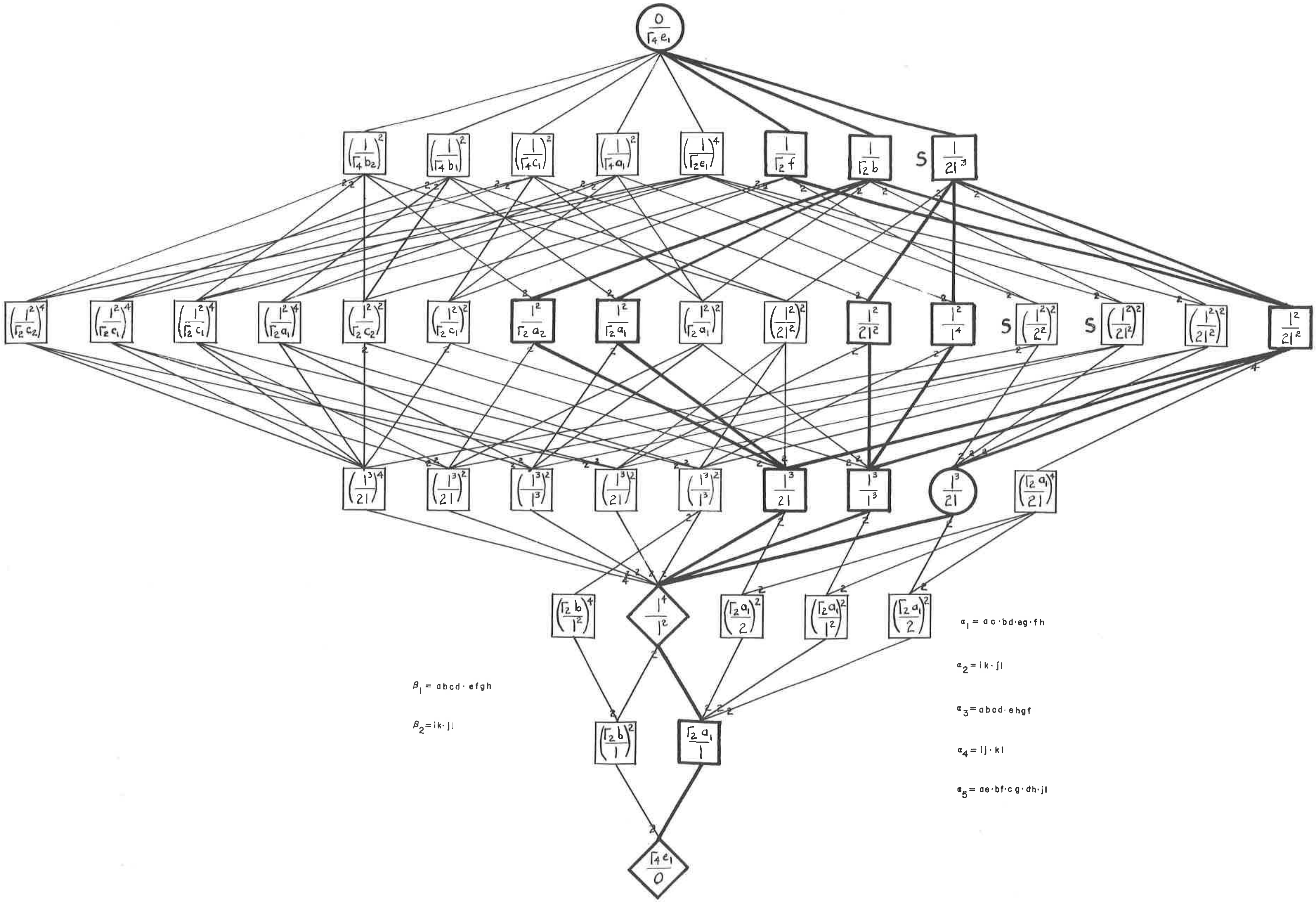
$\alpha_3 = abcd \cdot ehgf$

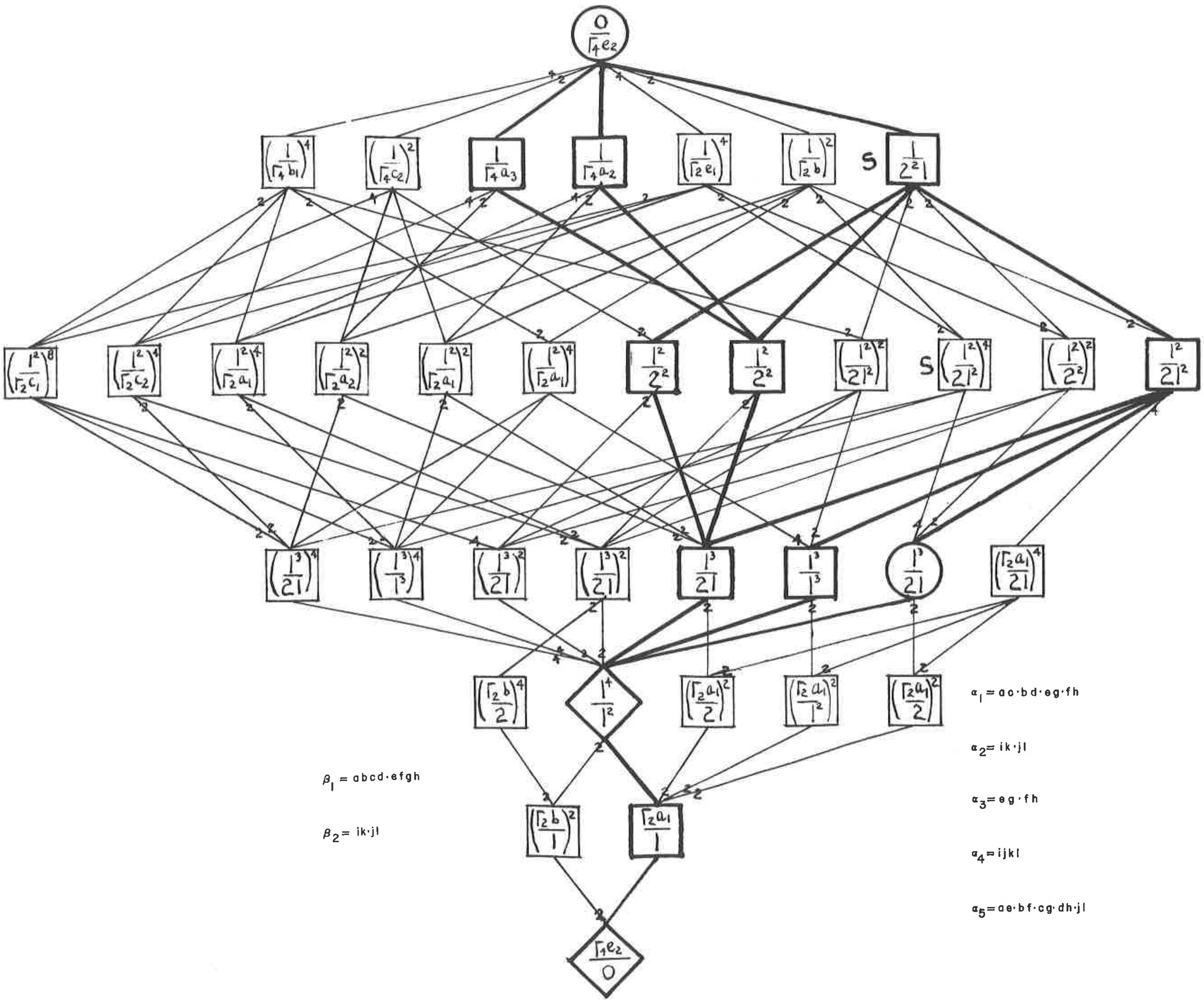
$\alpha_4 = abcd \cdot efgh \cdot ijkl$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot jl$


S
S
α_1 = ac·bd·eg·fh
α_2 = ik·jl·mo·np
α_3 = abcd·ehgf
α_4 = abcd·efgh·ijkl·mnop
α_5 = ae·bf·cg·dh·imko·jpln
β_1 = ac·bd·eg·fh
β_2 = ik·jl·mo.np
β_3 = qs


S
S
S
β_1 = ac·bd·eg·fh
β_2 = ik·jl·mo·np
β_3 = qs
α_1 = ac·bd·eg·fh
α_2 = ik·jl·mo·np
α_3 = eg·fh·ijkl·mnop
α_4 = abcd·efgh·ijkl·mpon
α_5 = ae·bf·cg·dh·im·jn·ko·lp


β1 = abcd · efgh
β2 = ik · jl
α1 = ac · bd · eg · fh
α2 = ik · jl
α3 = abcd · ehgf
α4 = ij · kl
α5 = ae · bf · cg · dh · jl



$\beta_1 = abcd \cdot efgh$

$\beta_2 = ik \cdot jl$

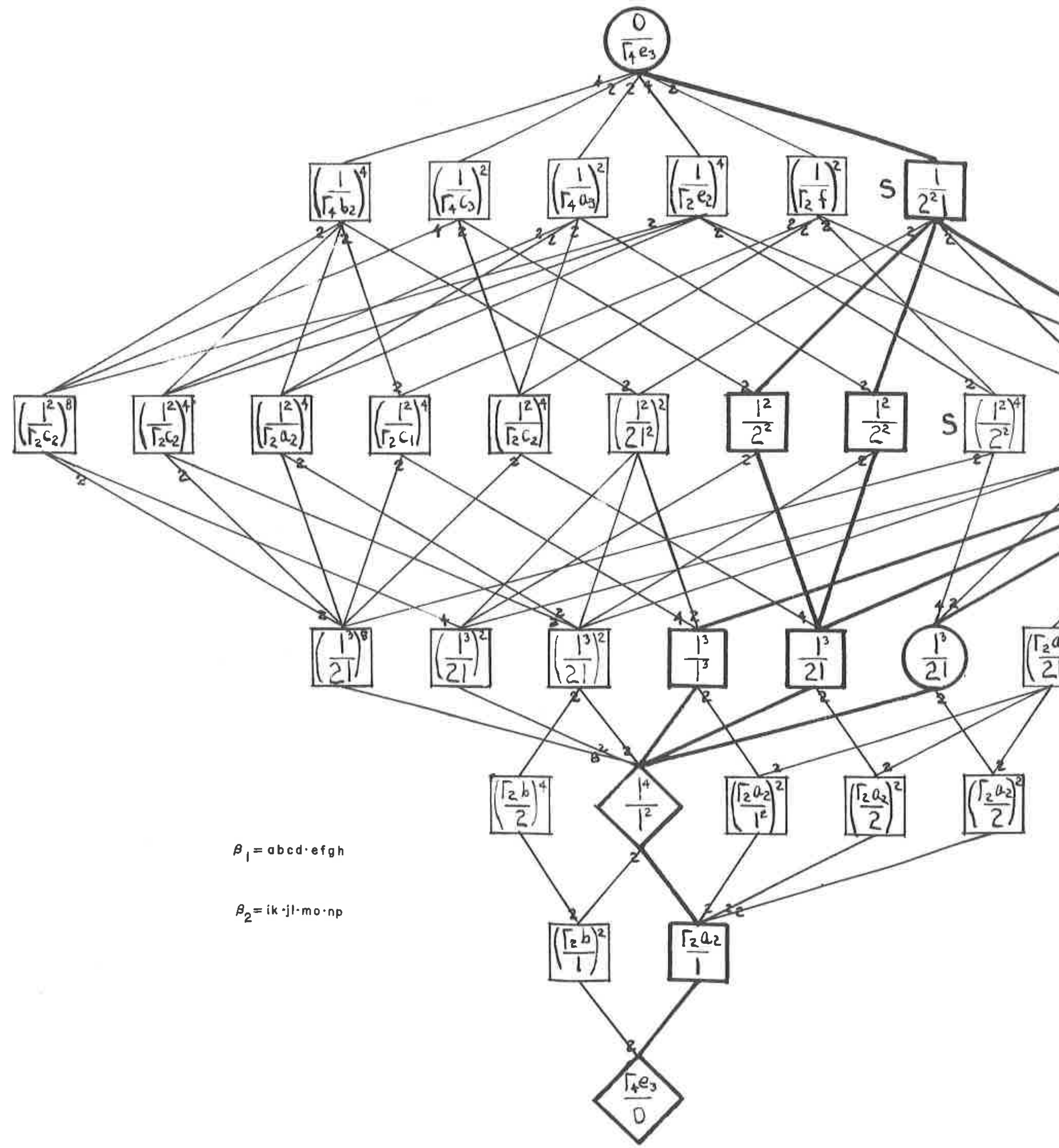
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ik \cdot jl$

$\alpha_3 = eg \cdot fh$

$\alpha_4 = ijkl$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot jl$



β_1 = abcd·efgh

β_2 = ik·jl·mo·np

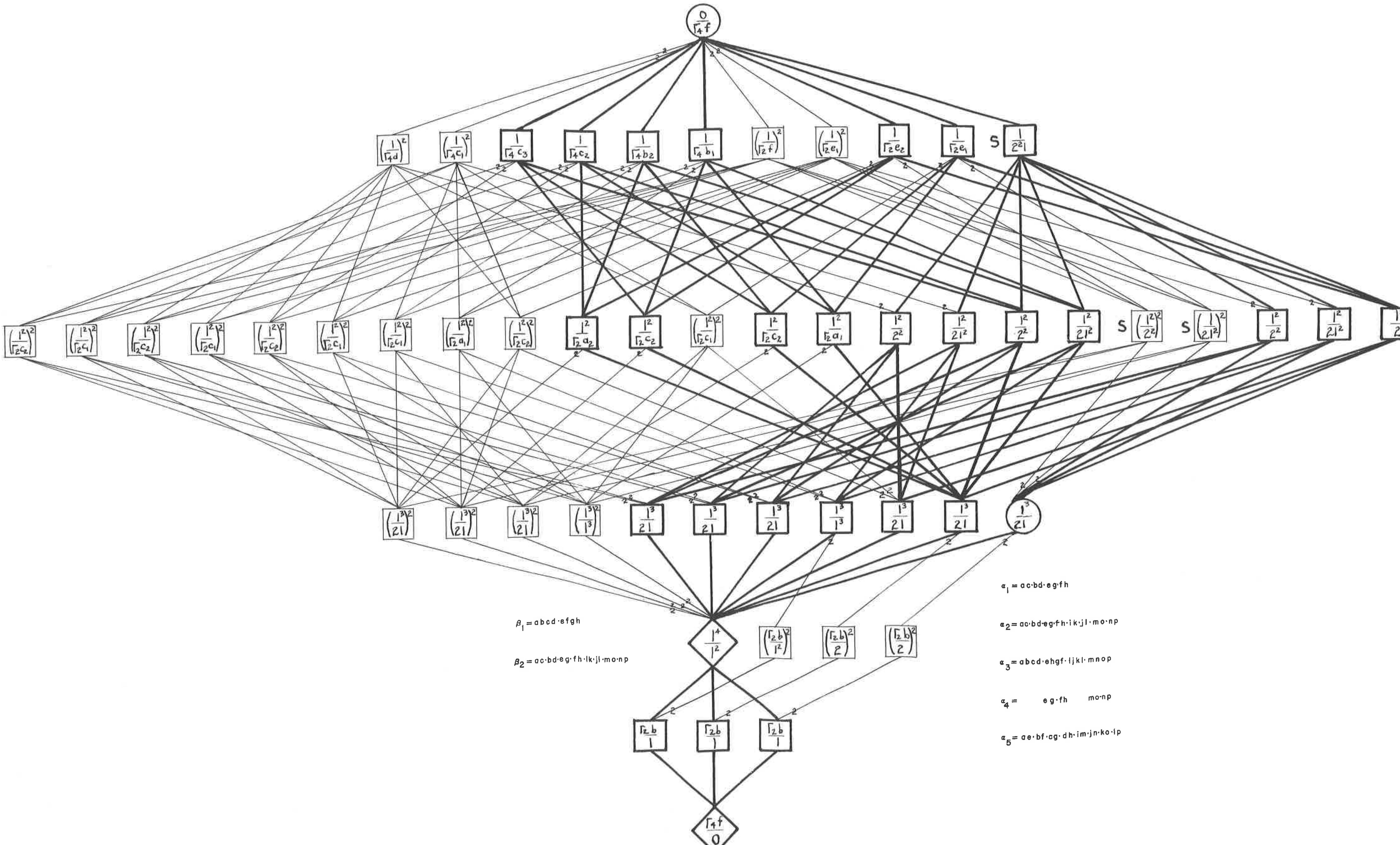
α_1 = ac·bd·eg·fh

α_2 = ik·jl·mo·np

α_3 = eg·fh

α_4 = ijkl·mnop

α_5 = ae·bf·cg·dh·imko·jpln



$\beta_1 = abcd \cdot efgh$

$\beta_2 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

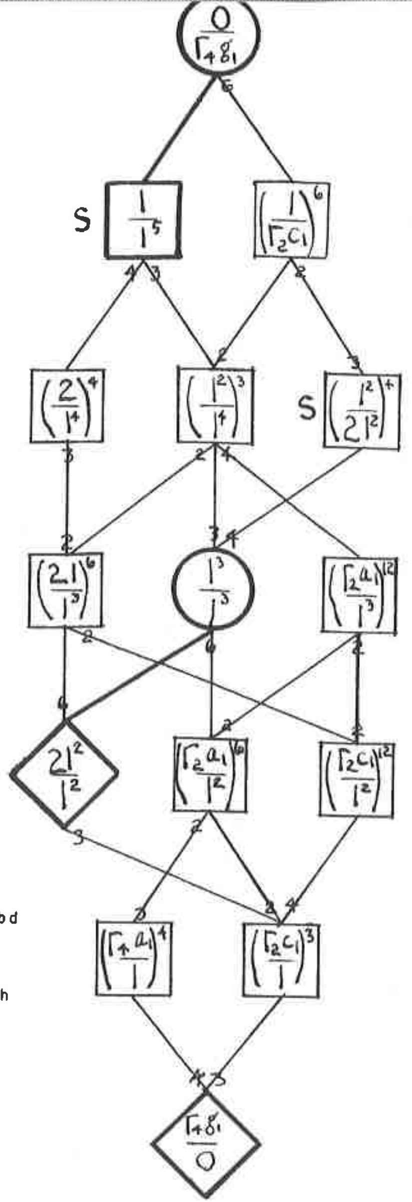
$\alpha_1 = ac \cdot bd \cdot eg \cdot fh$

$\alpha_2 = ac \cdot bd \cdot eg \cdot fh \cdot ik \cdot jl \cdot mo \cdot np$

$\alpha_3 = abcd \cdot ehgf \cdot ijkl \cdot mnop$

$\alpha_4 = \quad eg \cdot fh \quad mo \cdot np$

$\alpha_5 = ae \cdot bf \cdot cg \cdot dh \cdot im \cdot jn \cdot ko \cdot lp$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = ik \cdot jl$

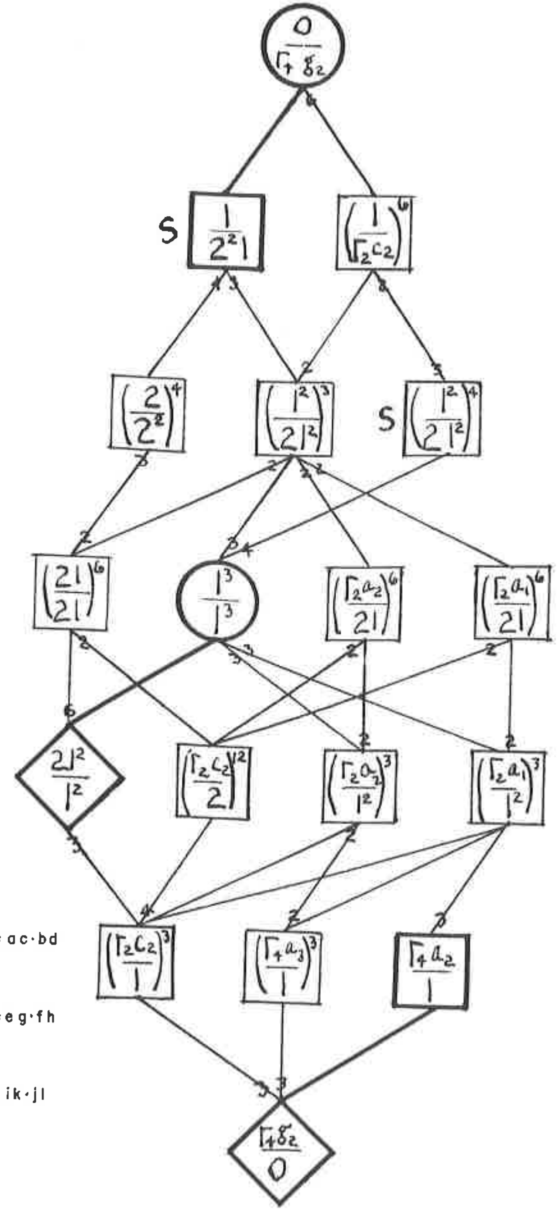
$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

$\alpha_3 = ab \cdot cd$

$\alpha_4 = ef \cdot gh$

$\alpha_5 = bd \cdot fh \cdot ijkl$



$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$

$\beta_3 = ik \cdot jl$

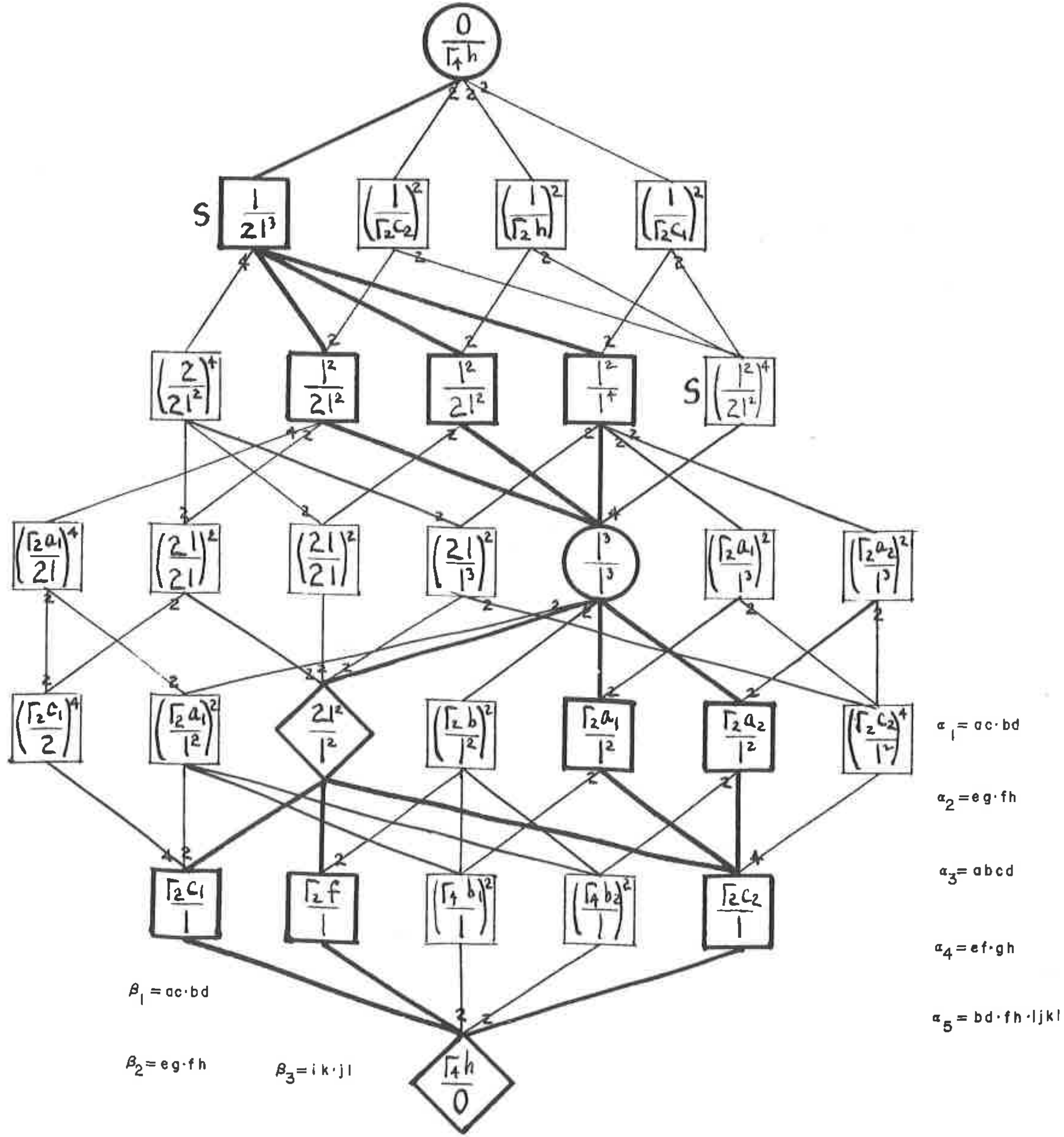
$\alpha_1 = ac \cdot bd$

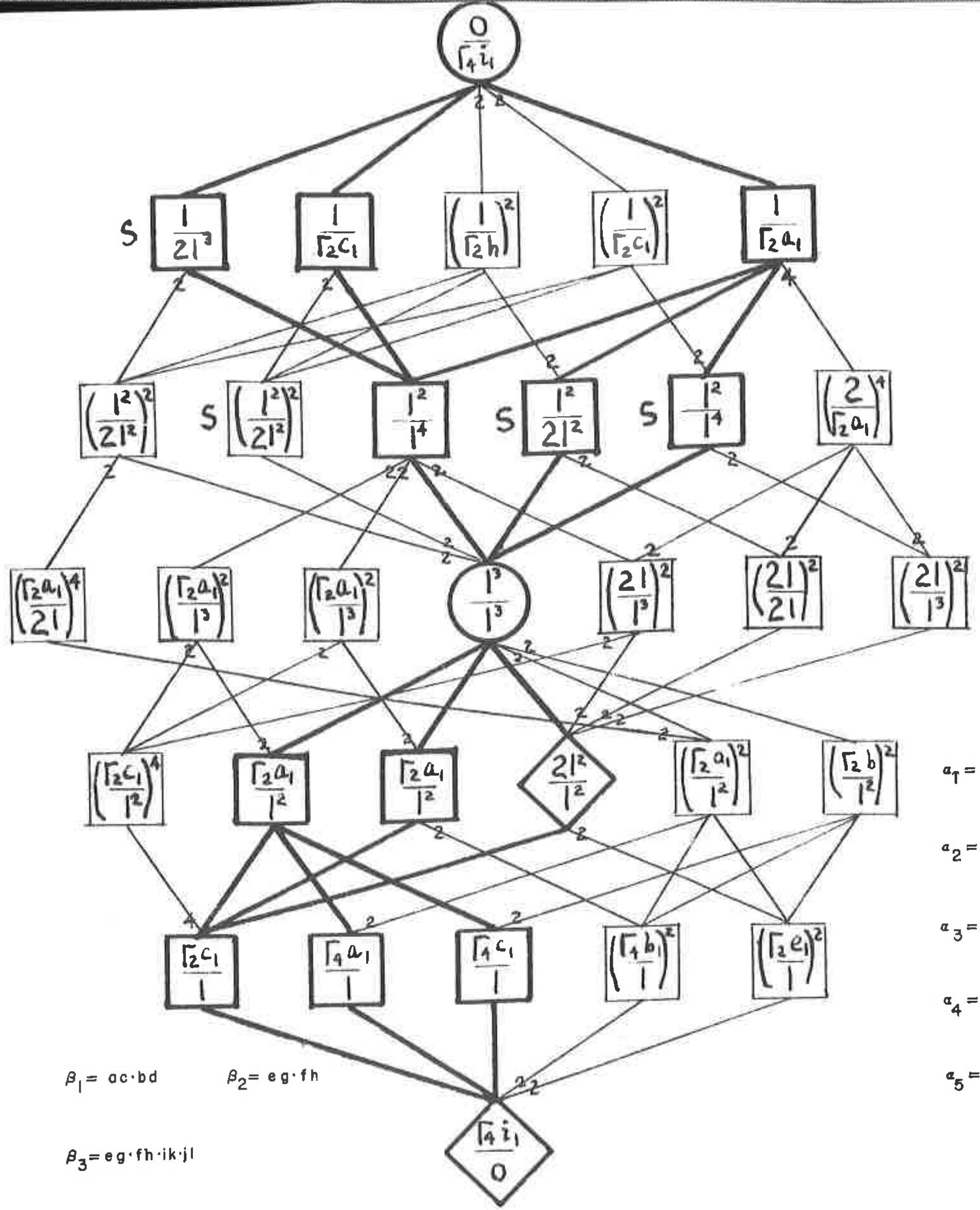
$\alpha_2 = eg \cdot fh$

$\alpha_3 = abcd$

$\alpha_4 = efgh$

$\alpha_5 = bd \cdot fh \cdot ijkl$


$\alpha_1 = ac \cdot bd$
$\alpha_2 = eg \cdot fh$
$\alpha_3 = abcd$
$\alpha_4 = ef \cdot gh$
$\alpha_5 = bd \cdot fh \cdot ljkl$
$\beta_1 = ac \cdot bd$
$\beta_2 = eg \cdot fh$
$\beta_3 = ik \cdot jl$



$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

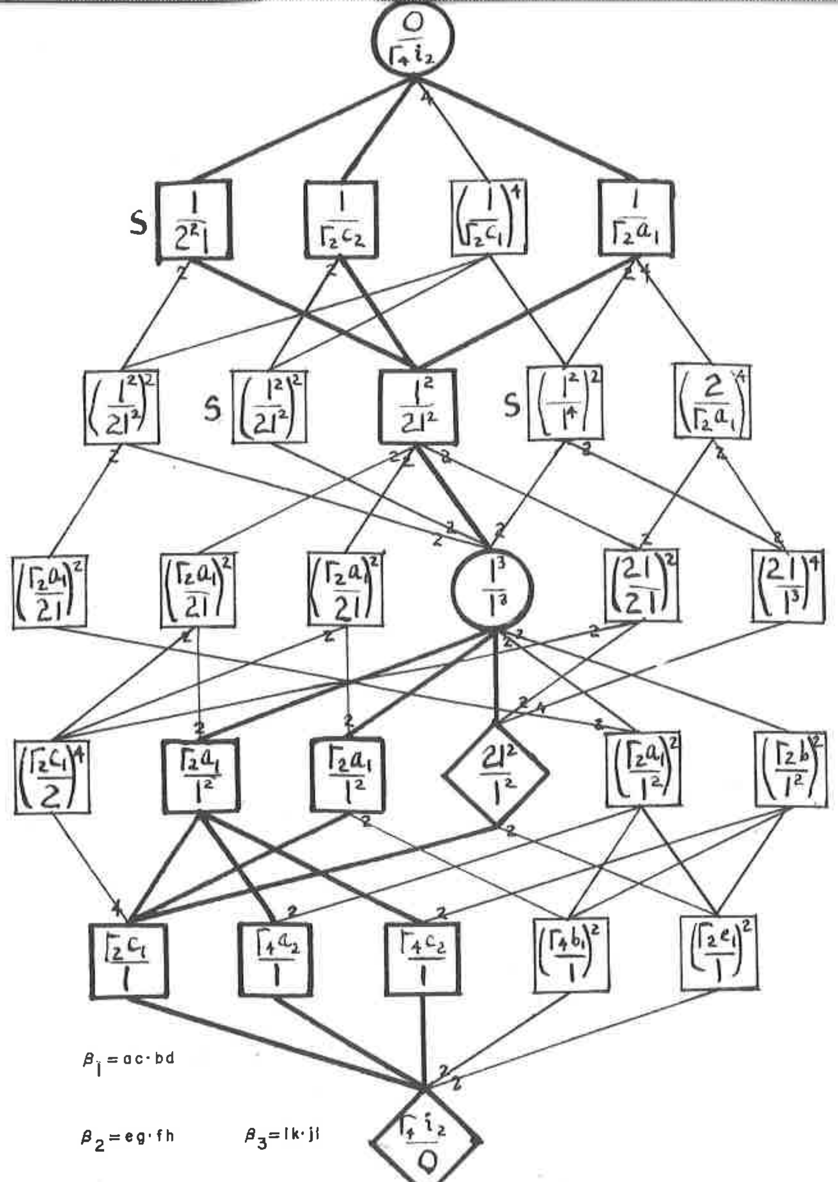
$\alpha_3 = ab \cdot cd$

$\alpha_4 = ef \cdot gh \cdot ijkl$

$\alpha_5 = bd \cdot fh$

$\beta_1 = ac \cdot bd$ $\beta_2 = eg \cdot fh$

$\beta_3 = eg \cdot fh \cdot ik \cdot jl$



$\alpha_1 = ac \cdot bd$

$\alpha_2 = eg \cdot fh$

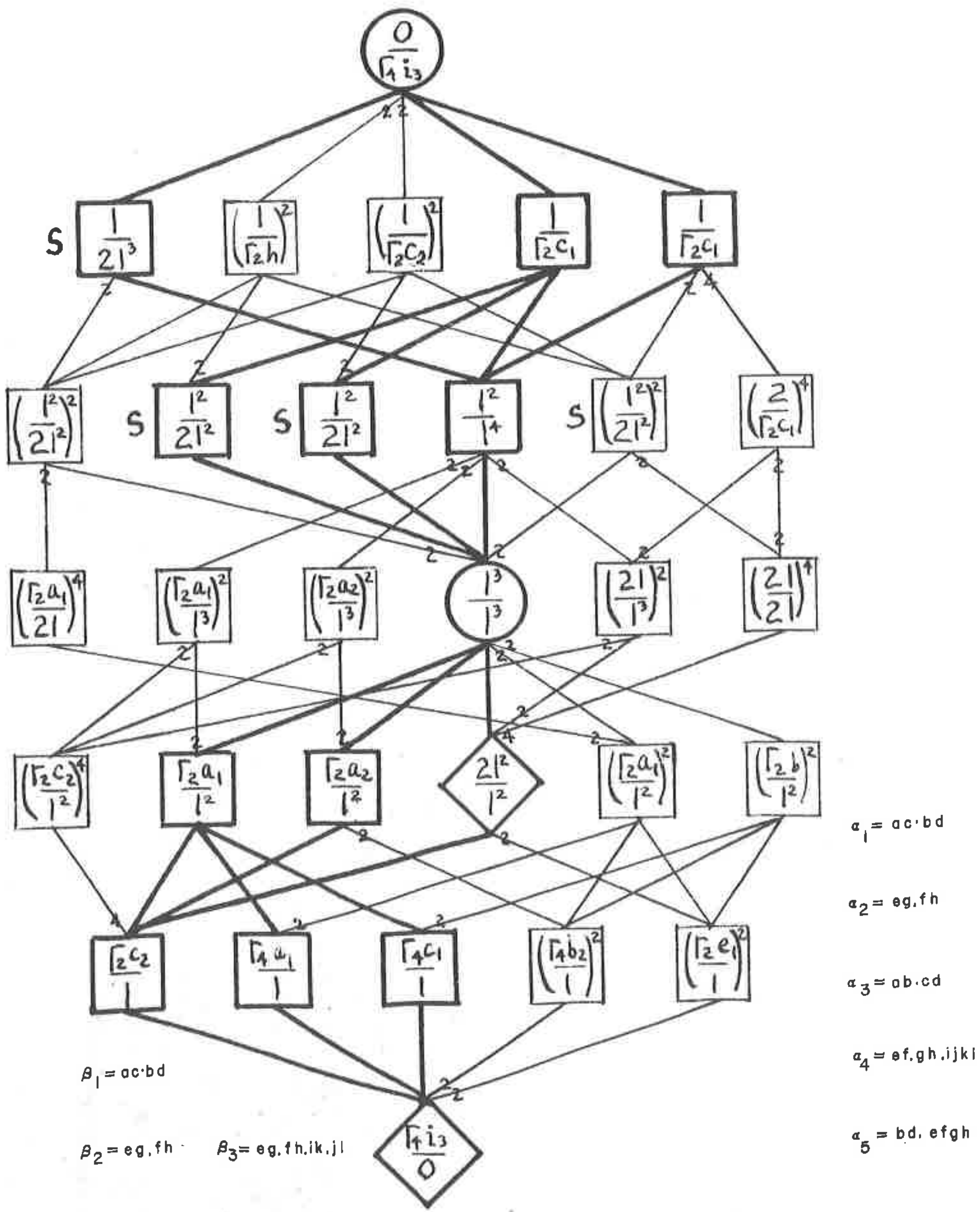
$\alpha_3 = abcd$

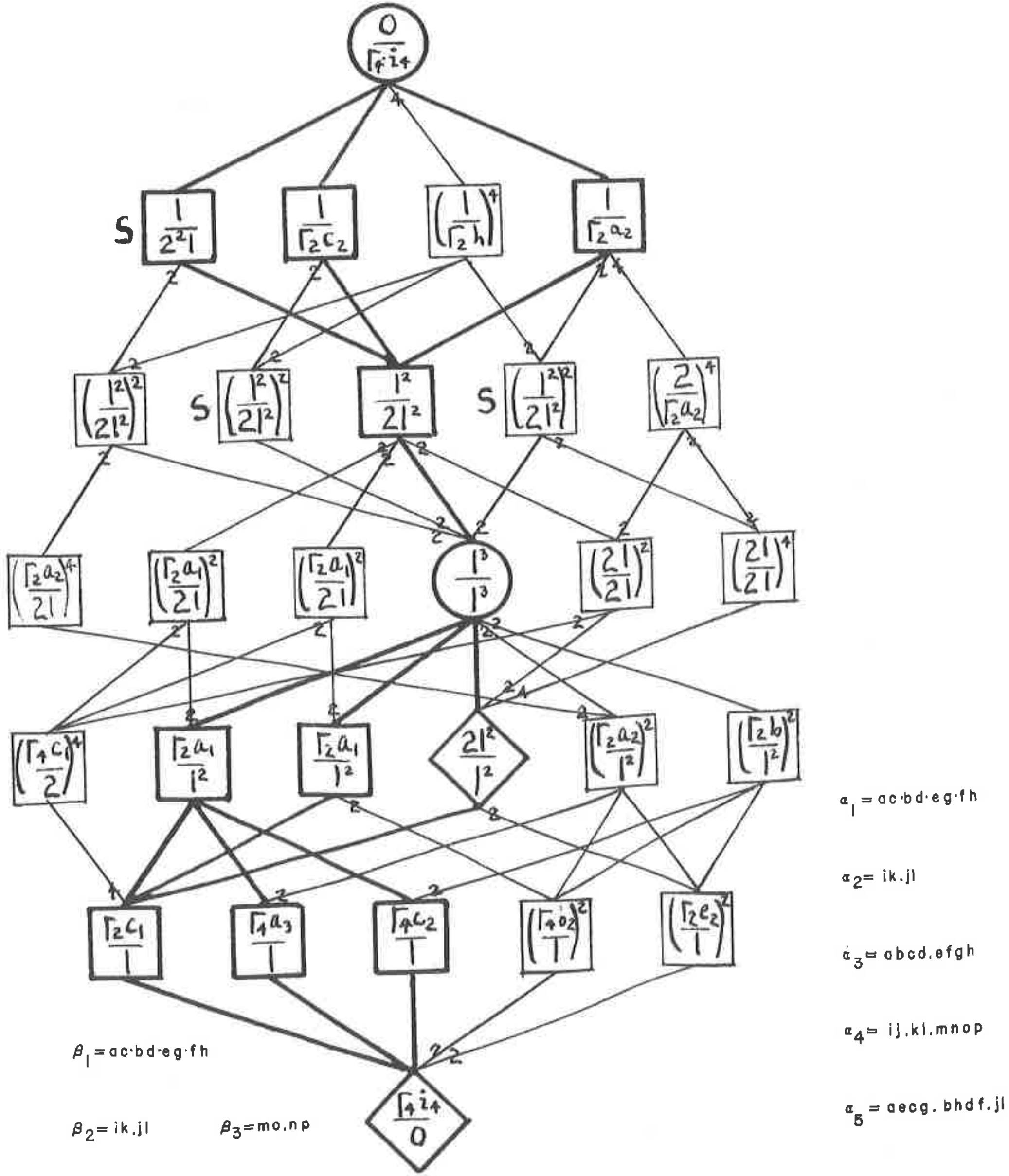
$\alpha_4 = ef \cdot gh \cdot ijkl$

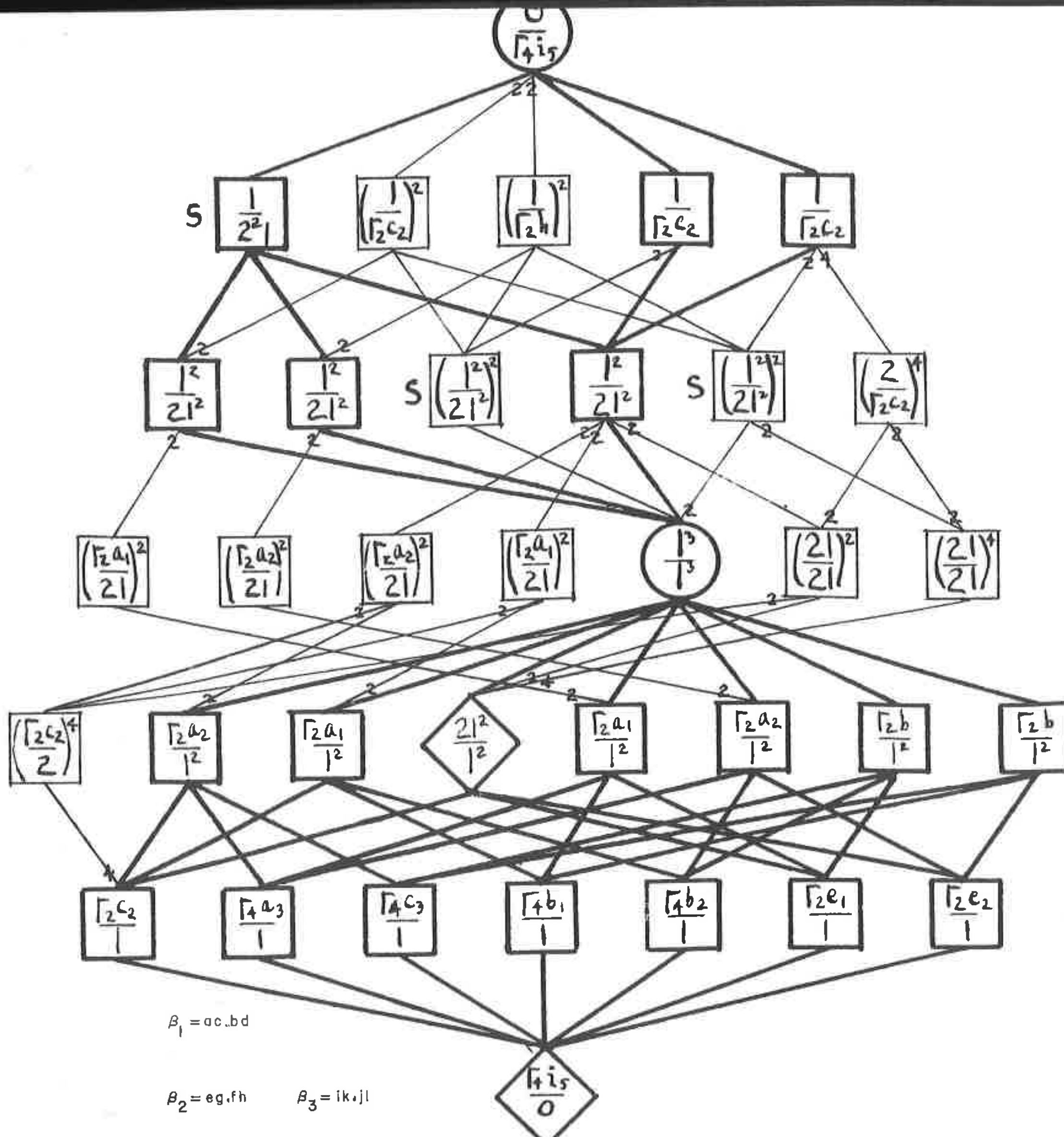
$\alpha_5 = bd \cdot fh$

$\beta_1 = ac \cdot bd$

$\beta_2 = eg \cdot fh$ $\beta_3 = lk \cdot jl$


α1 = ac·bd
α2 = eg.fh
α3 = ab·cd
α4 = ef.gh.ijkl
α5 = bd. efgh
β1 = ac·bd
β2 = eg.fh
β3 = eg.fh.ik.jl


α1 = ac·bd·eg·fh
α2 = ik.jl
α3 = abcd.efgh
α4 = ij.kl.mnop
α5 = aecg. bhdf.jl
β1 = ac·bd·eg·fh
β2 = ik.jl
β3 = mo.np



$\alpha_1 = ac.bd$

$\alpha_2 = eg.fh$

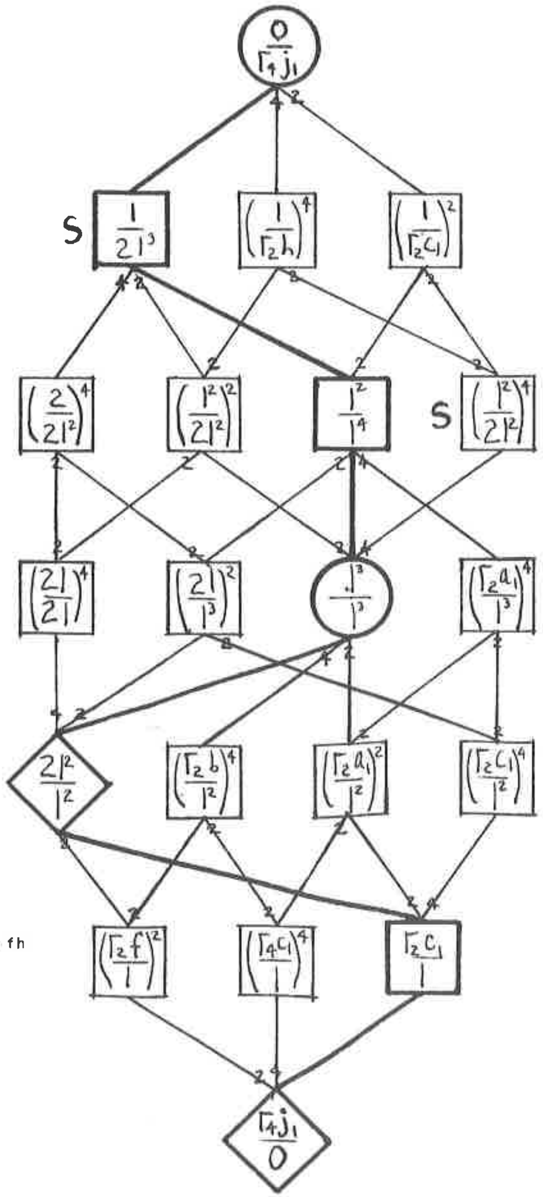
$\alpha_3 = abcd$

$\alpha_4 = ef.gh.ij kl$

$\alpha_5 = bd.efgh$

$\beta_1 = ac.bd$

$\beta_2 = eg.fh$ $\beta_3 = ik.jl$



β_1 = ac. bd. eg. fh

β_2 = Ik.jl

β_3 = mo.np

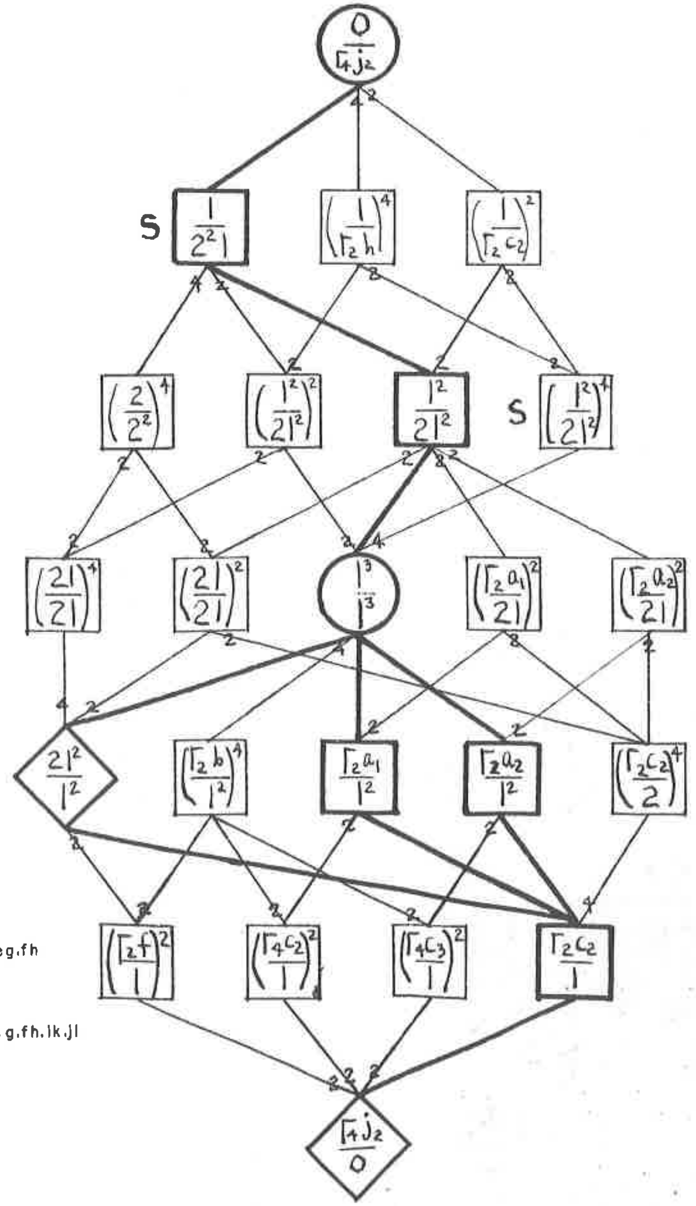
α_1 = ac.bd.eg.fh

α_2 = ik.jl

α_3 = eg.fh

α_4 = abcd.efgh.ik

α_5 = ae.bf.cg.dh.ij.kl.mn.op



β_1 = ac. bd.eg.fh

β_2 = ac.bd.eg.fh.ik.jl

β_3 = mo.np

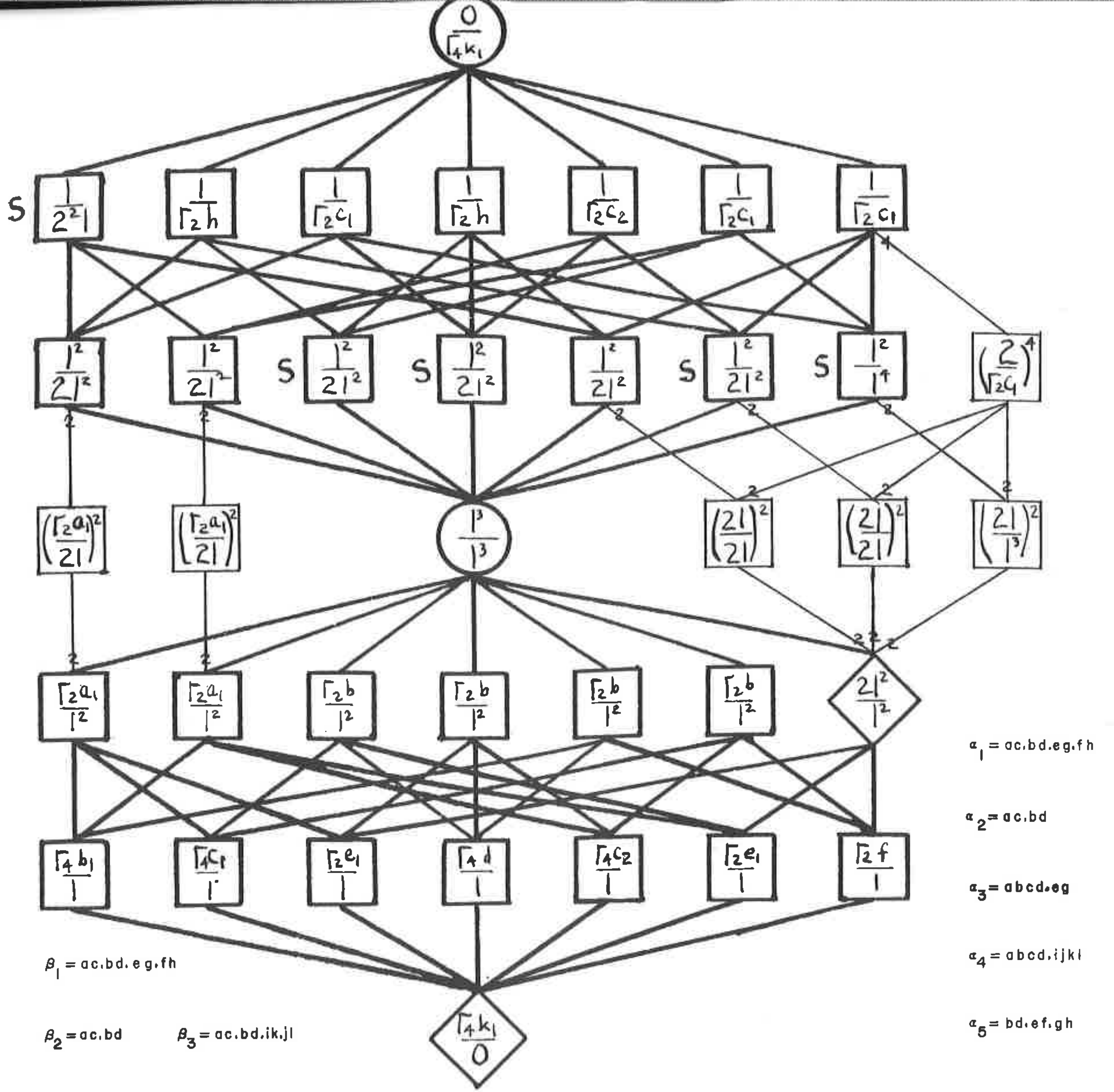
α_1 = ac.bd.eg.fh

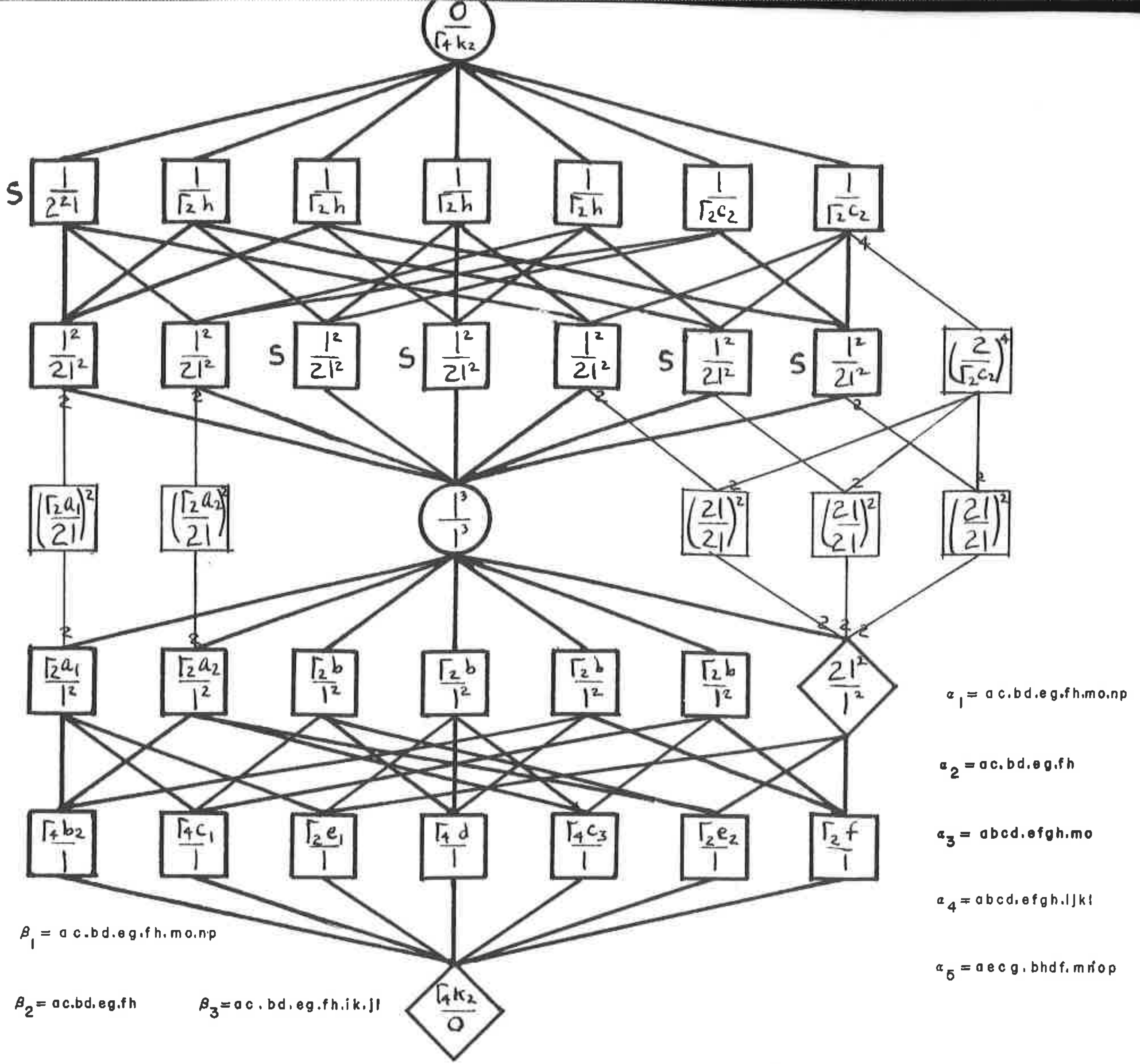
α_2 = ik.jl

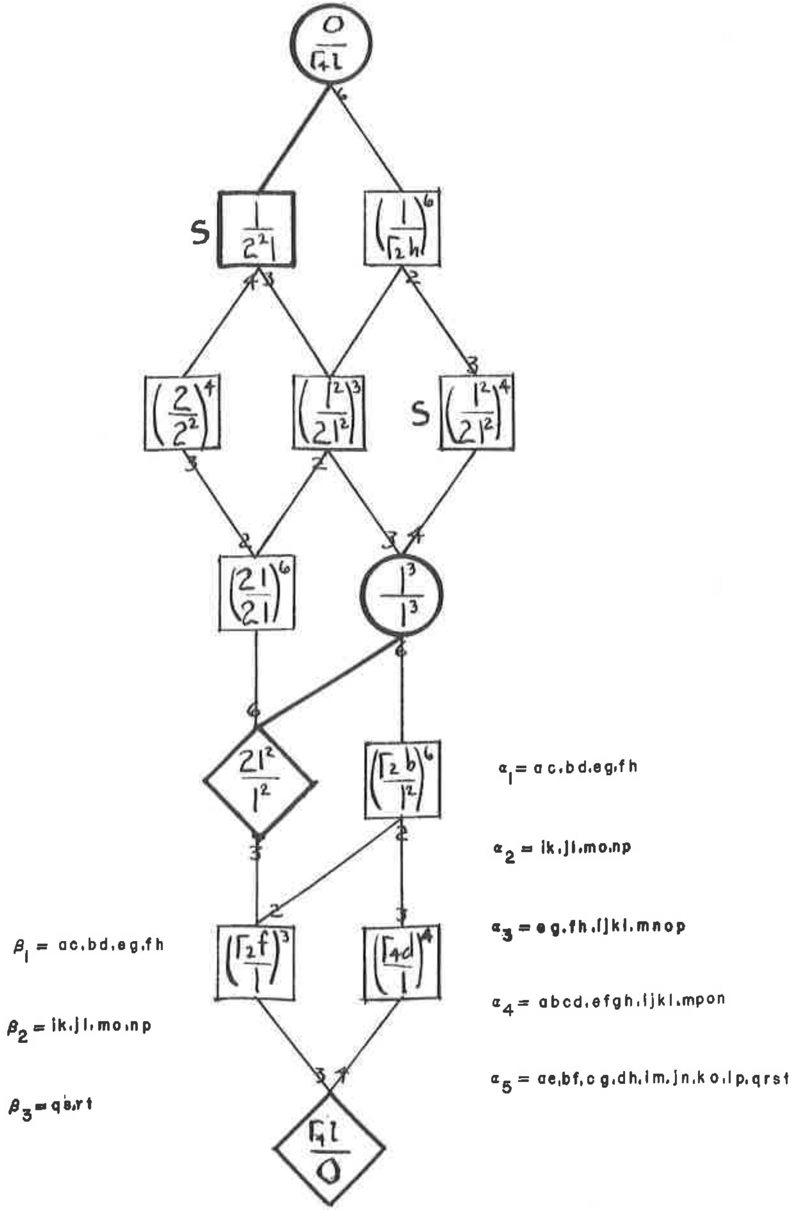
α_3 = abcd.ehgf

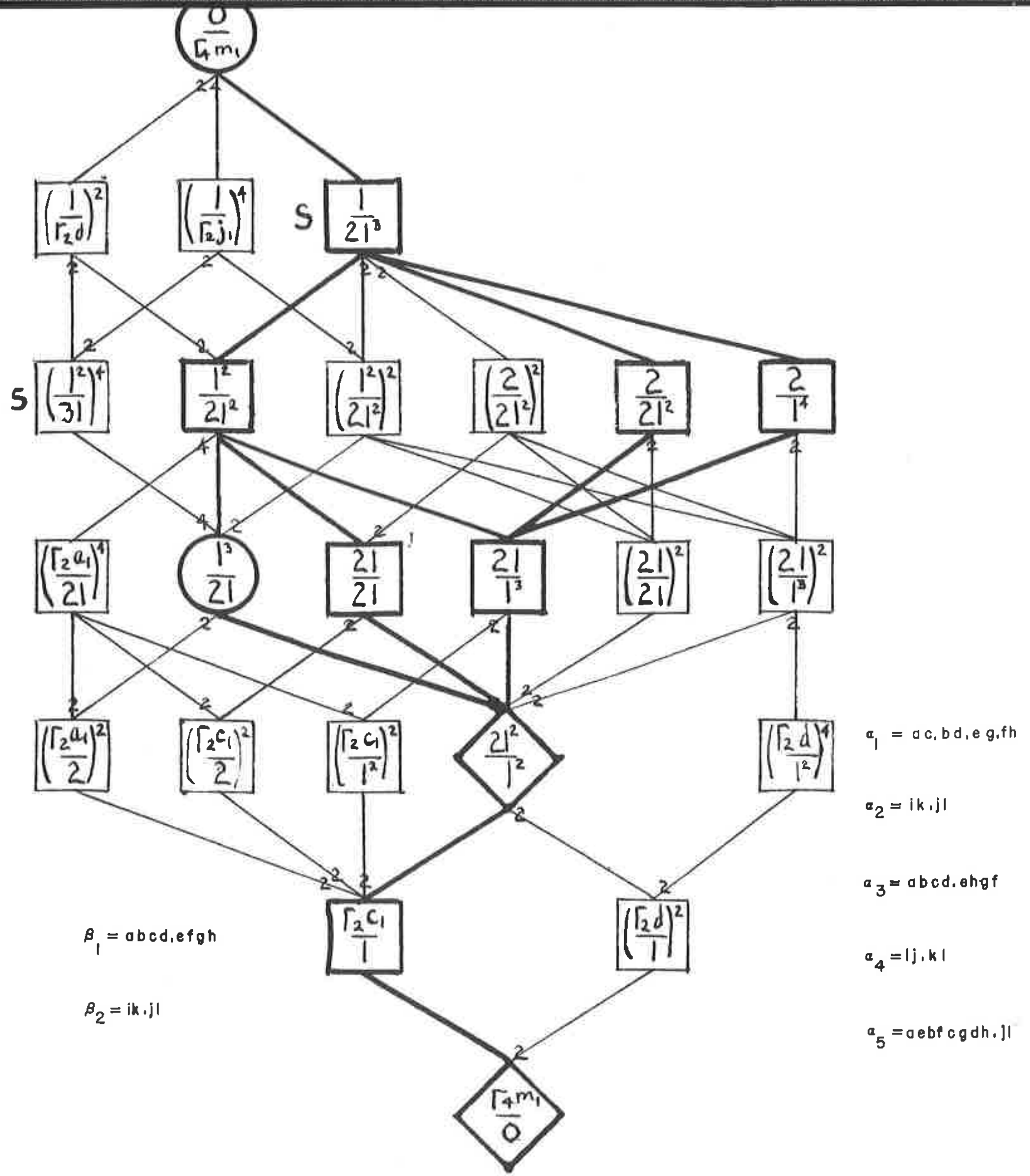
α_4 = eg.fh.ijkl

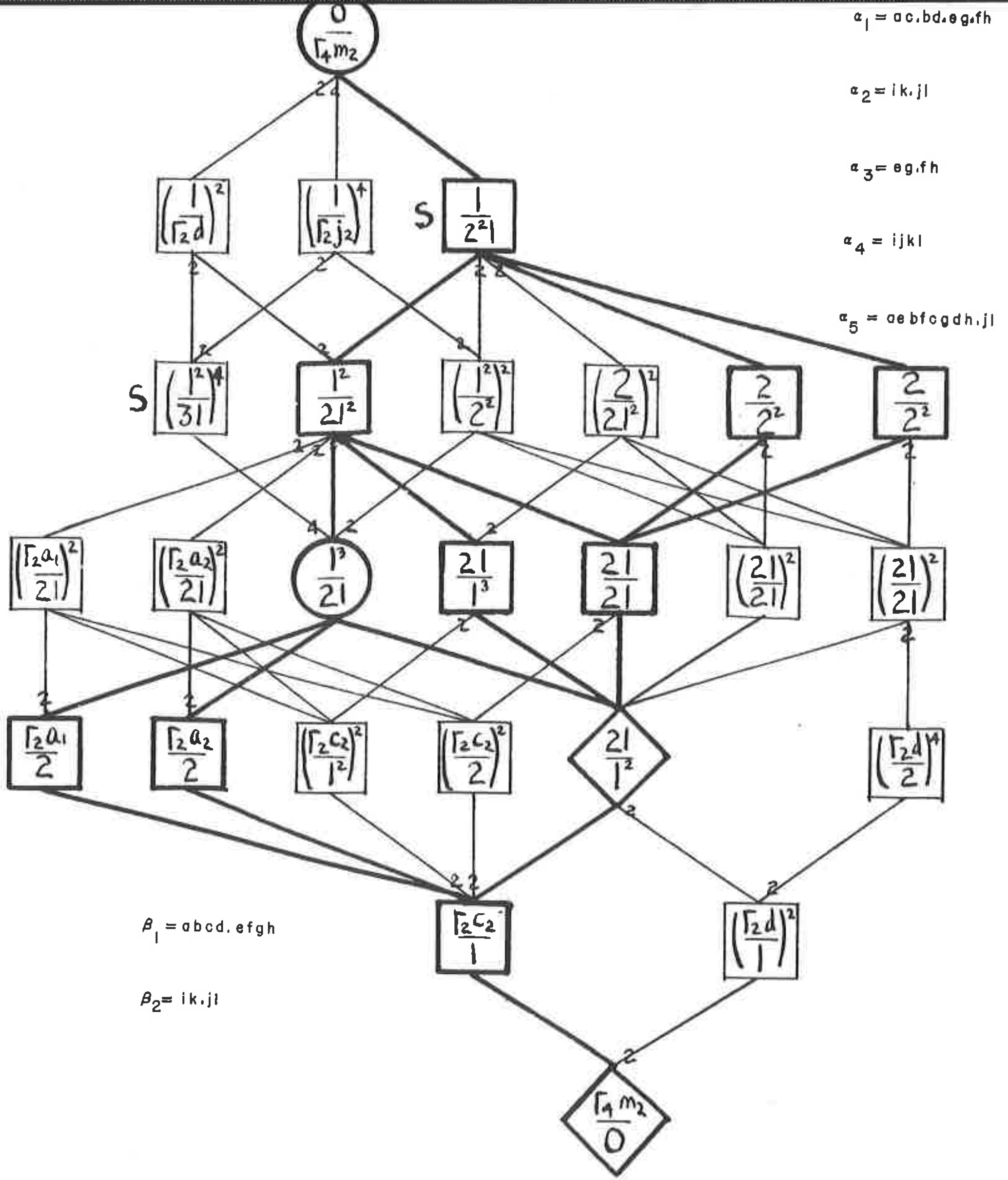
α_5 = ae.bf.cg.dh.jl.mnop

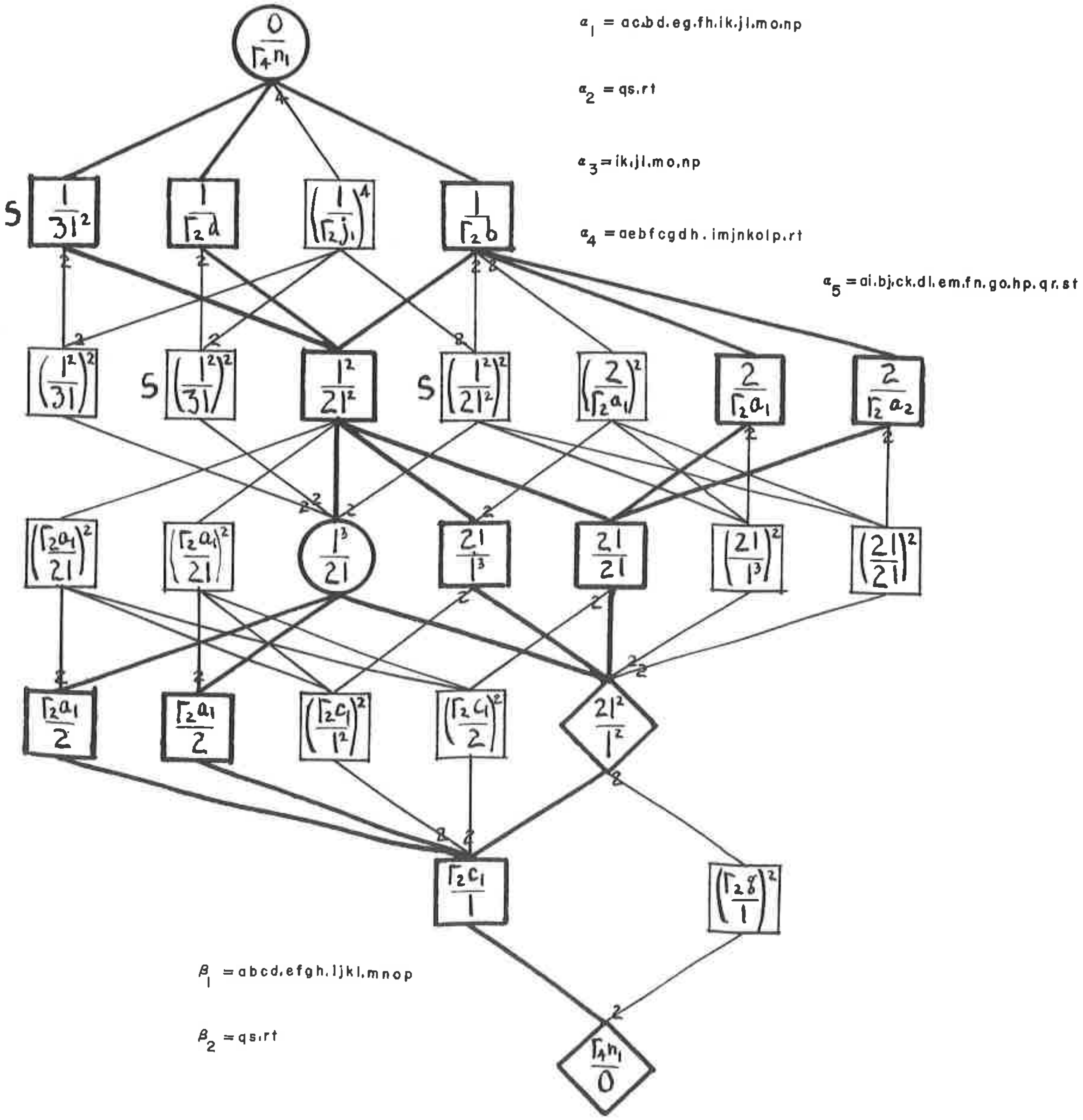

α1 = ac.bd.eg.fh
α2 = ac.bd
α3 = abcd.eg
α4 = abcd.ijkl
α5 = bd.ef.gh
β1 = ac.bd.eg.fh
β2 = ac.bd
β3 = ac.bd.ik.jl

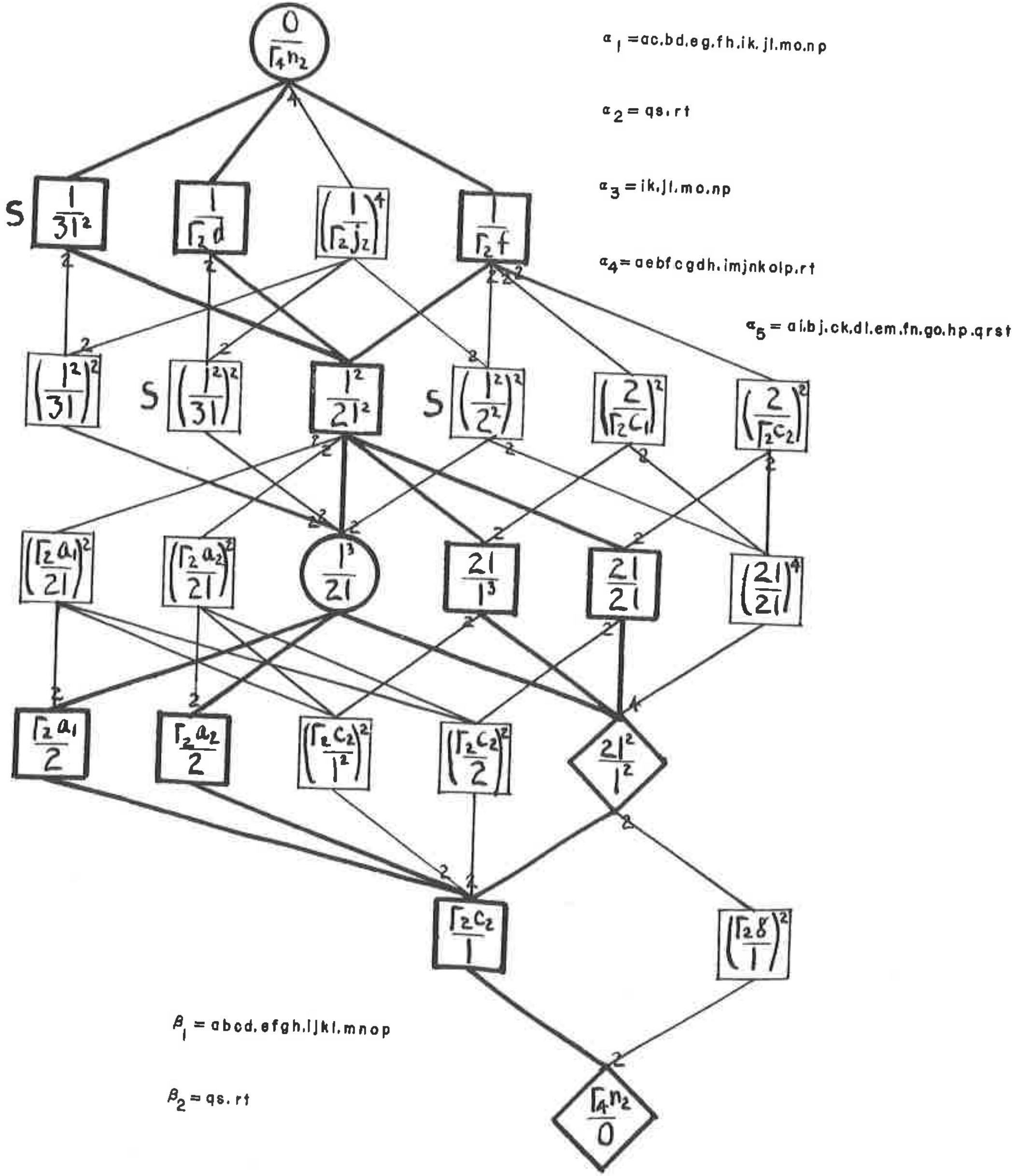

α1 = ac.bd.eg.fh.mo.np
α2 = ac.bd.eg.fh
α3 = abcd.efgh.mo
α4 = abcd.efgh.ijkl
α5 = aecg.bhdf.mnop
β1 = ac.bd.eg.fh.mo.np
β2 = ac.bd.eg.fh
β3 = ac.bd.eg.fh.ik.jl

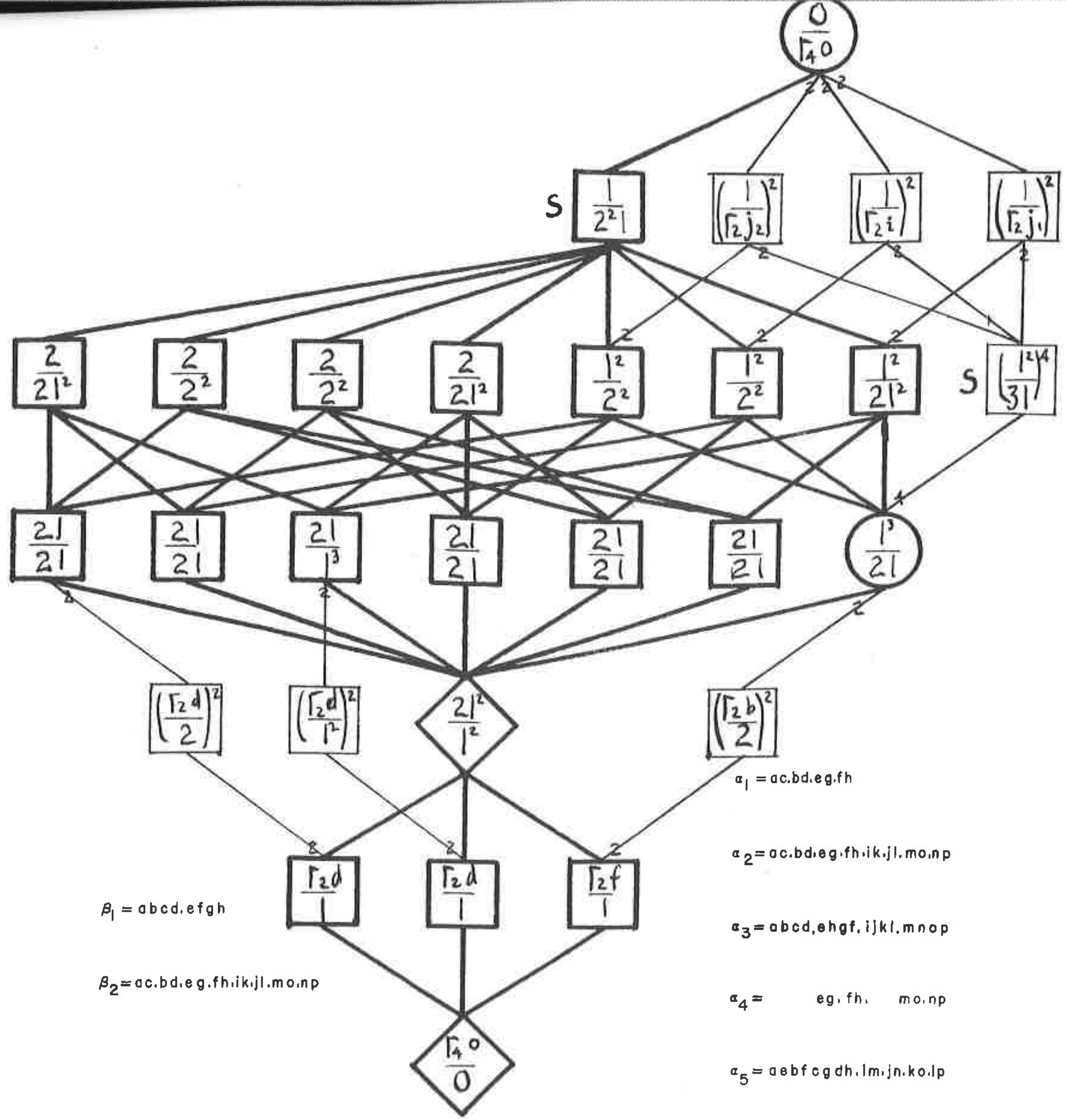

α1 = ac.bd.eg.fh
α2 = ik.jl.mo.np
α3 = eg.fh.ijkl.mnop
α4 = abcd.efgh.ijkl.mpon
α5 = ae.bf.cg.dh.im.jn.ko.lp.qrst
β1 = ac.bd.eg.fh
β2 = ik.jl.mo.np
β3 = qs.rt

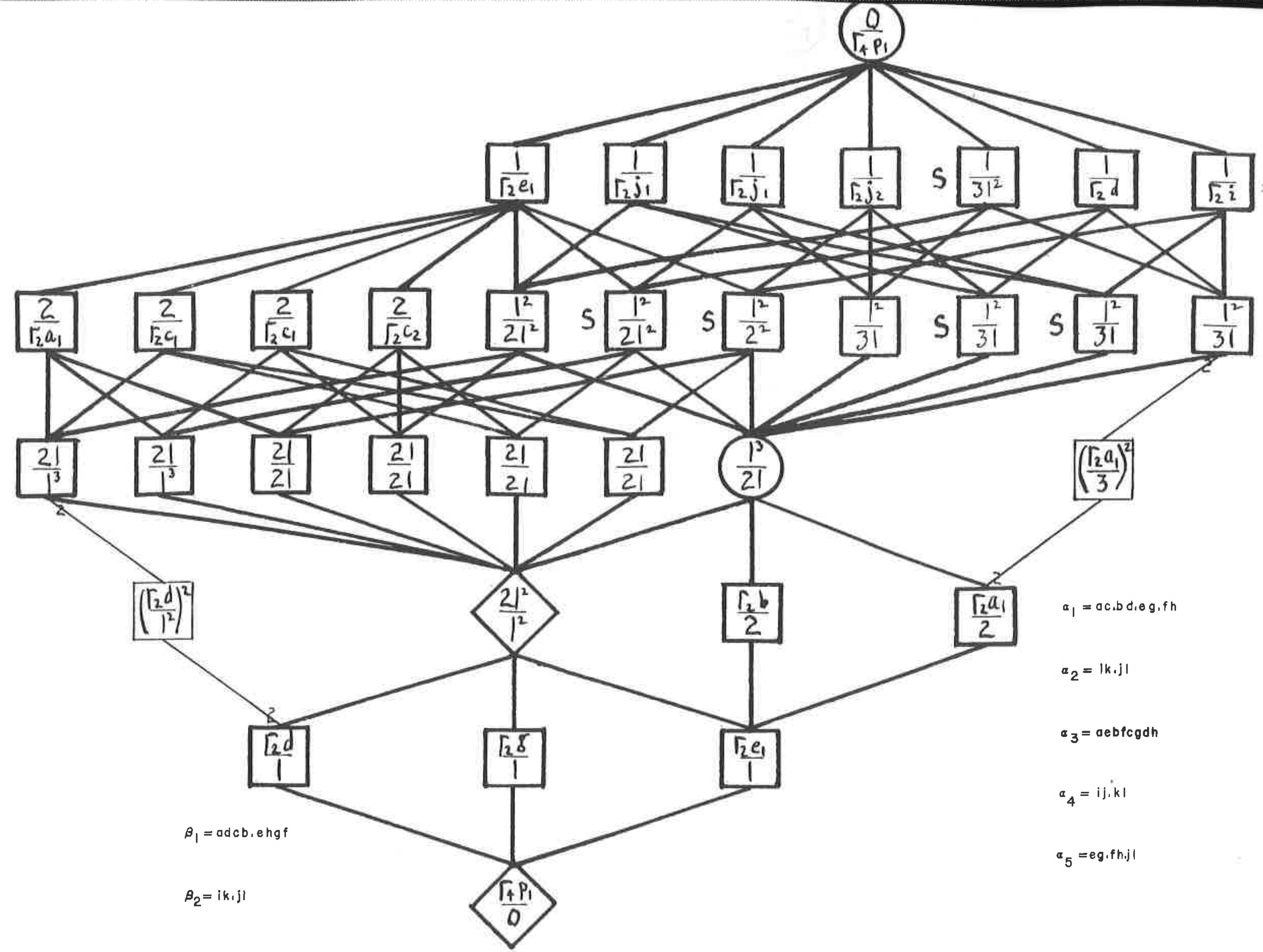

α1 = ac, bd, eg, fh
α2 = ik, jl
α3 = abcd, ehgf
α4 = lj, kl
α5 = aebfcgdh, jl
β1 = abcd, efgh
β2 = ik, jl

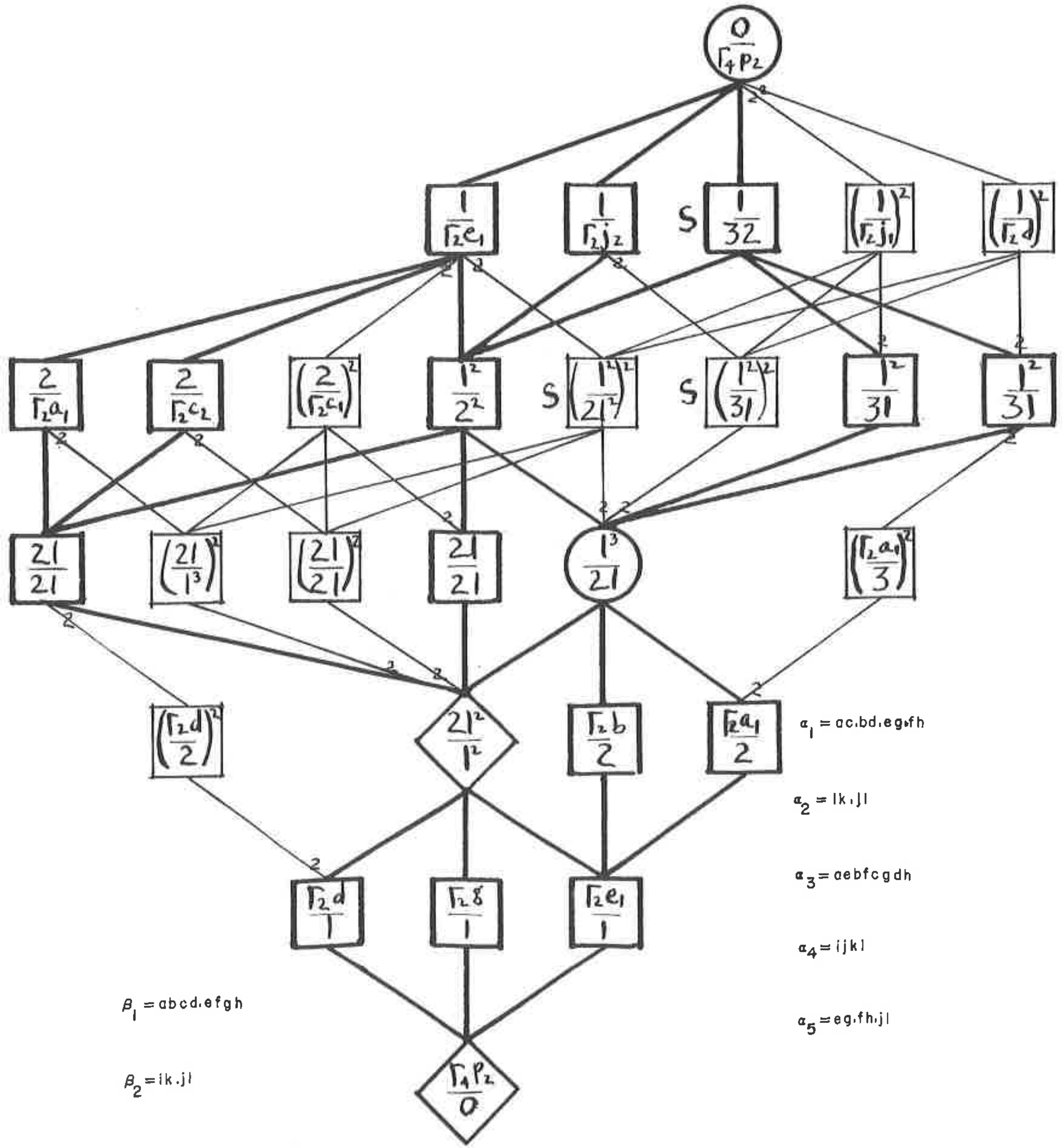

α1 = ac, bd, eg, fh
α2 = ik, jl
α3 = eg, fh
α4 = ijkl
α5 = aebfcgdh, jl
β1 = abcd, efgh
β2 = ik, jl


α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = qs.rt
α_3 = ik.jl.mo.np
α_4 = aebfcgdh.imjnkolp.rt
α_5 = ai.bj.ck.dl.em.fn.go.hp.qr.st
β_1 = abcd.efgh.ijkl.mnop
β_2 = qs.rt


α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = qs.rt
α_3 = ik.jl.mo.np
α_4 = aebfcgdh.imjnkolp.rt
α_5 = ai.bj.ck.dl.em.fn.go.hp.qrst
β_1 = abcd.efgh.ijkl.mnop
β_2 = qs.rt


α_1 = ac.bd.eg.fh
α_2 = ac.bd.eg.fh.ik.jl.mo.np
α_3 = abcd.ehgf.ijkl.mnop
α_4 = eg.fh. mo.np
α_5 = aebfcgdh.lm.jn.ko.lp
β_1 = abcd.efgh
β_2 = ac.bd.eg.fh.ik.jl.mo.np


α_1 = ac.bd.eg.fh
α_2 = lk.jl
α_3 = aebfcgdh
α_4 = ij.kl
α_5 = eg.fh.jl
β_1 = adcb.ehgf
β_2 = ik.jl



α_1 = ac.bd.eg.fh

α_2 = ik.jl

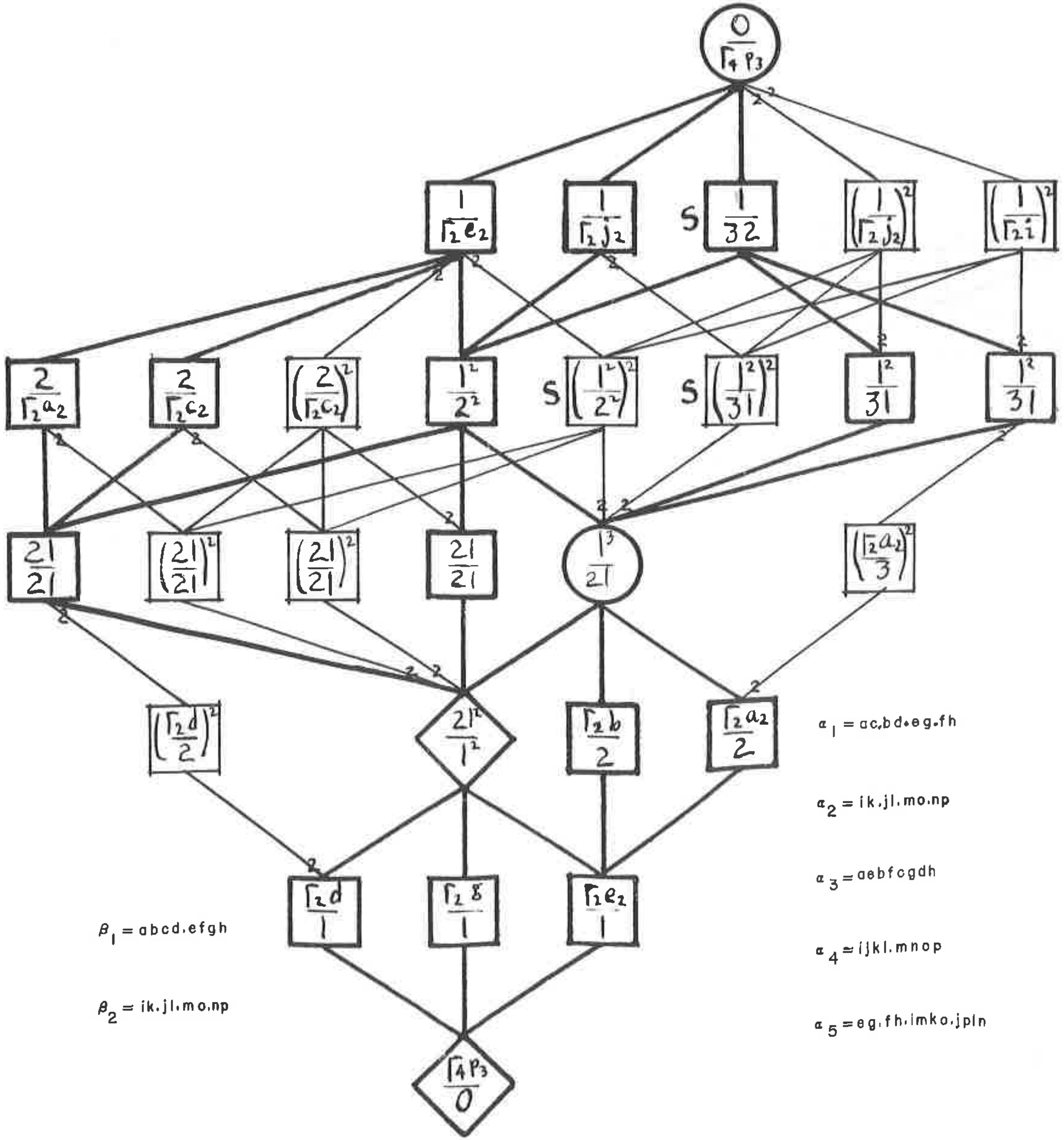
α_3 = aebfcgdh

α_4 = ijkl

α_5 = eg.fh.jl

β_1 = abcd.efgh

β_2 = ik.jl



α_1 = ac.bd.eg.fh

α_2 = ik.jl.mo.np

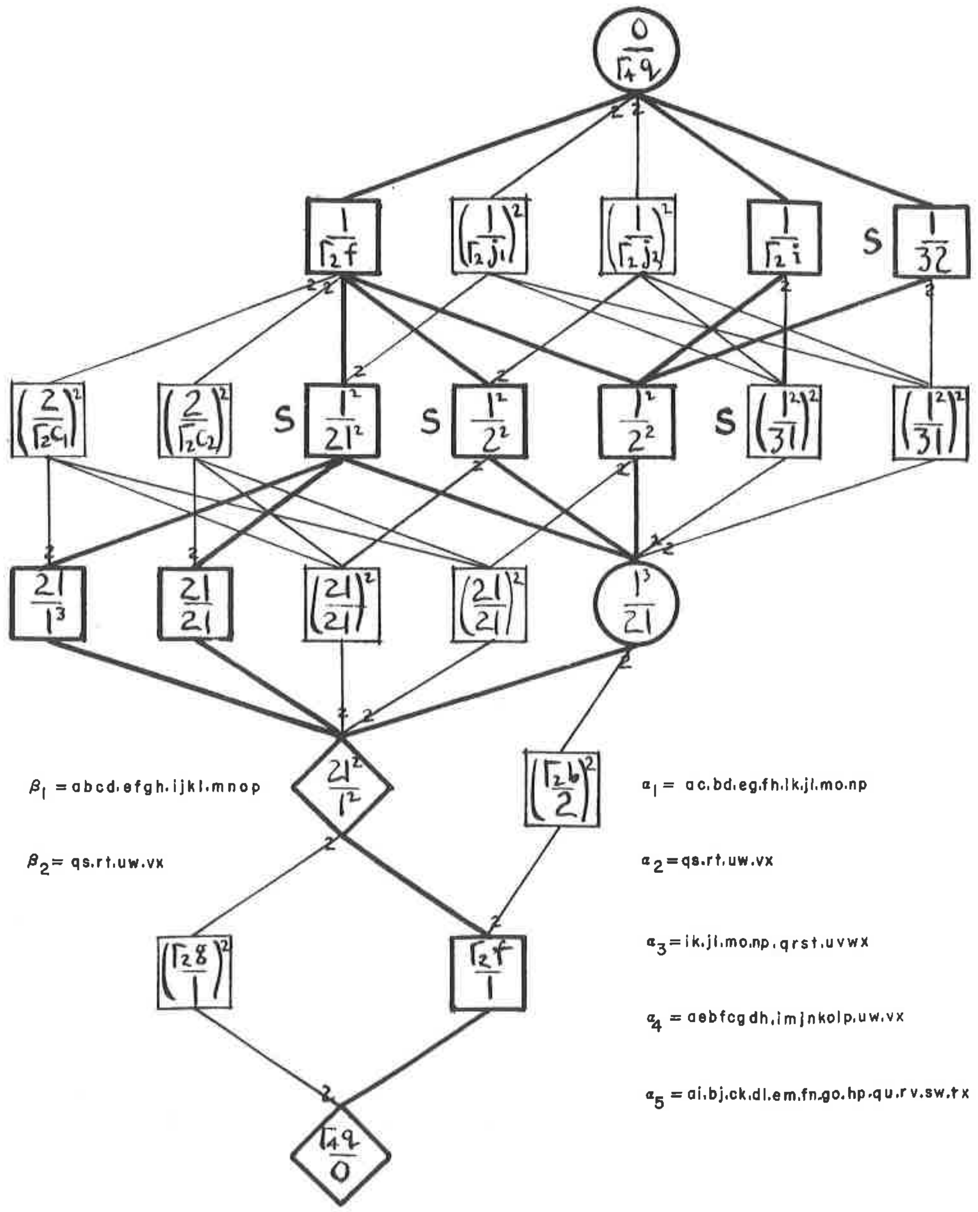
α_3 = aebfcgdh

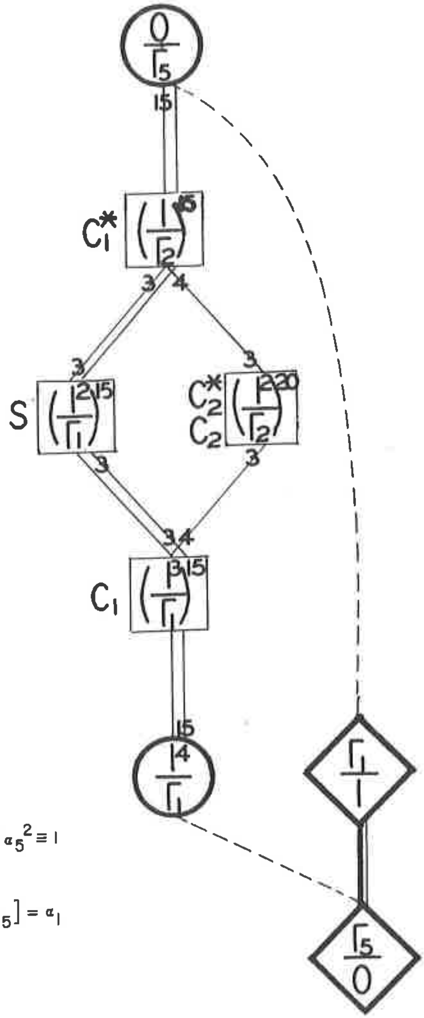
α_4 = ijkl.mnop

α_5 = eg.fh.lmko.jpln

β_1 = abcd.efgh

β_2 = ik.jl.mo.np

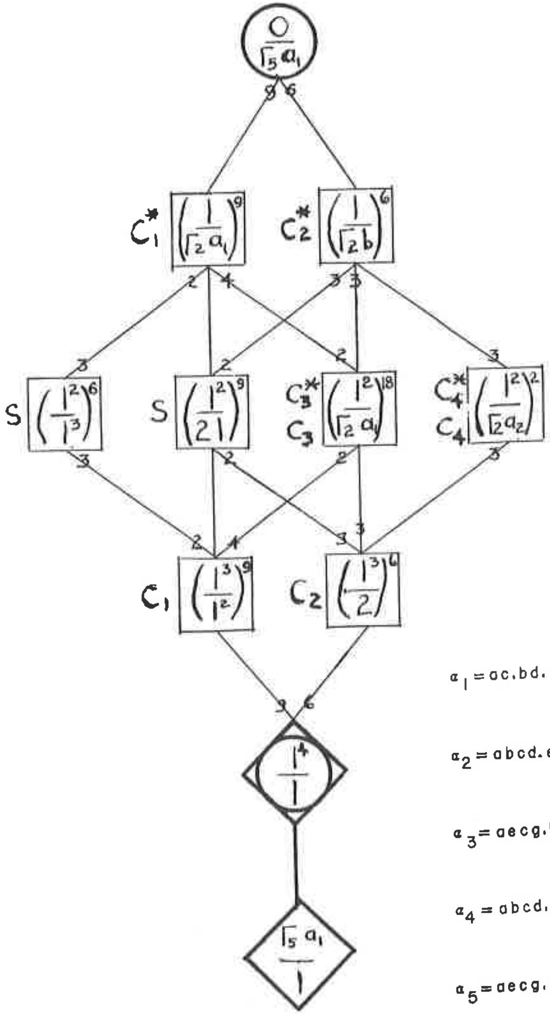

β1 = abcd.efgh.ijkl.mnop
β2 = qs.rt.uw.vx
α1 = ac.bd.eg.fh.ik.jl.mo.np
α2 = qs.rt.uw.vx
α3 = ik.jl.mo.np.qrst.uvwx
α4 = aebfcgdh.imjnkolp.uw.vx
α5 = ai.bj.ck.dl.em.fn.go.hp.qu.rv.sw.tx



$\alpha_1{}^2 = 1$

$\alpha_2{}^2 \equiv \alpha_3{}^2 \equiv \alpha_4{}^2 \equiv \alpha_5{}^2 \equiv 1$

$[\alpha_3, \alpha_4] = [\alpha_2, \alpha_5] = \alpha_1$



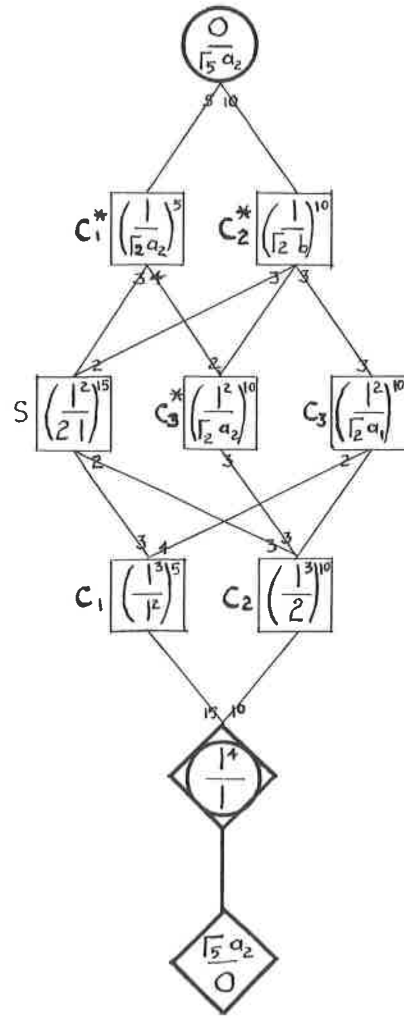
$\alpha_1 = ac.bd.eg.fh$

$\alpha_2 = abcd.efgh$

$\alpha_3 = aecg.bfdh$

$\alpha_4 = abcd.ehgf$

$\alpha_5 = aecg.bhdf$



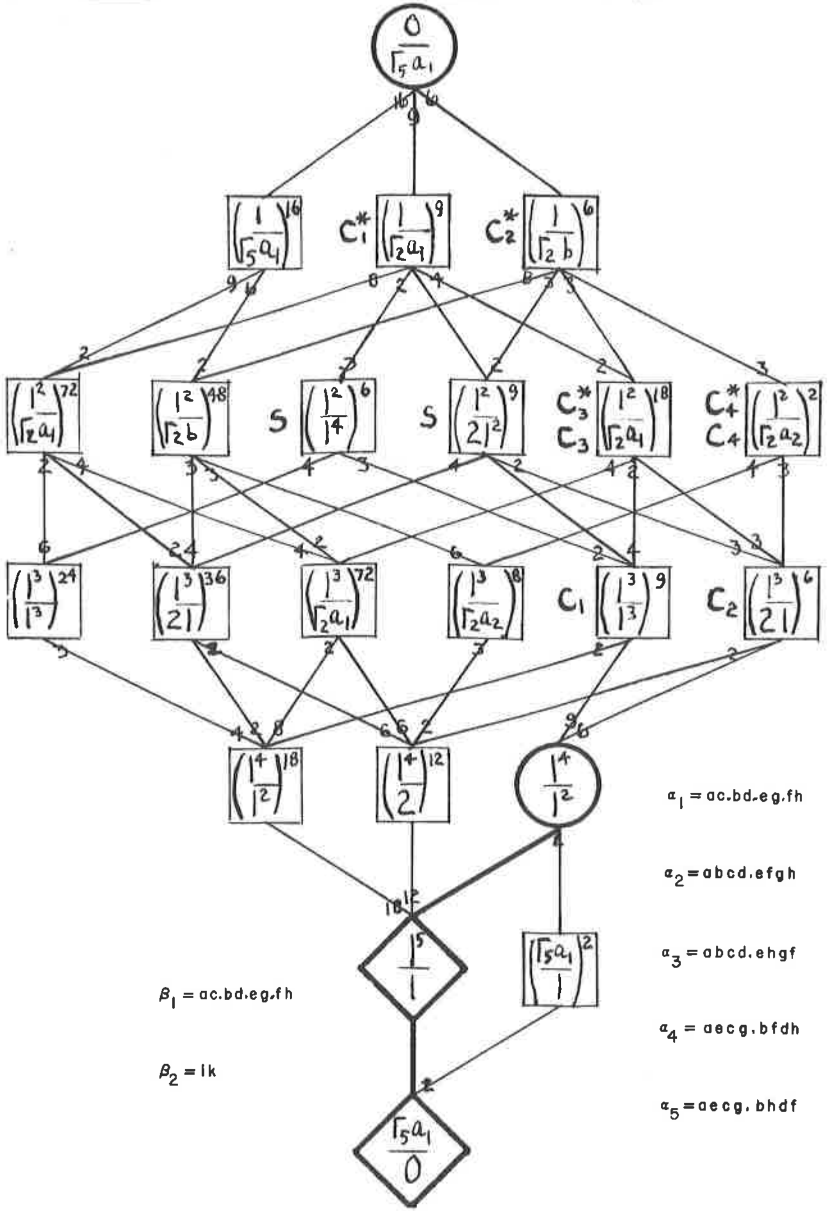
$\alpha_1 = ac.bd.eg.fh.ik.jl.mo.np$

$\alpha_2 = abcd.efgh.ijkl.mnop$

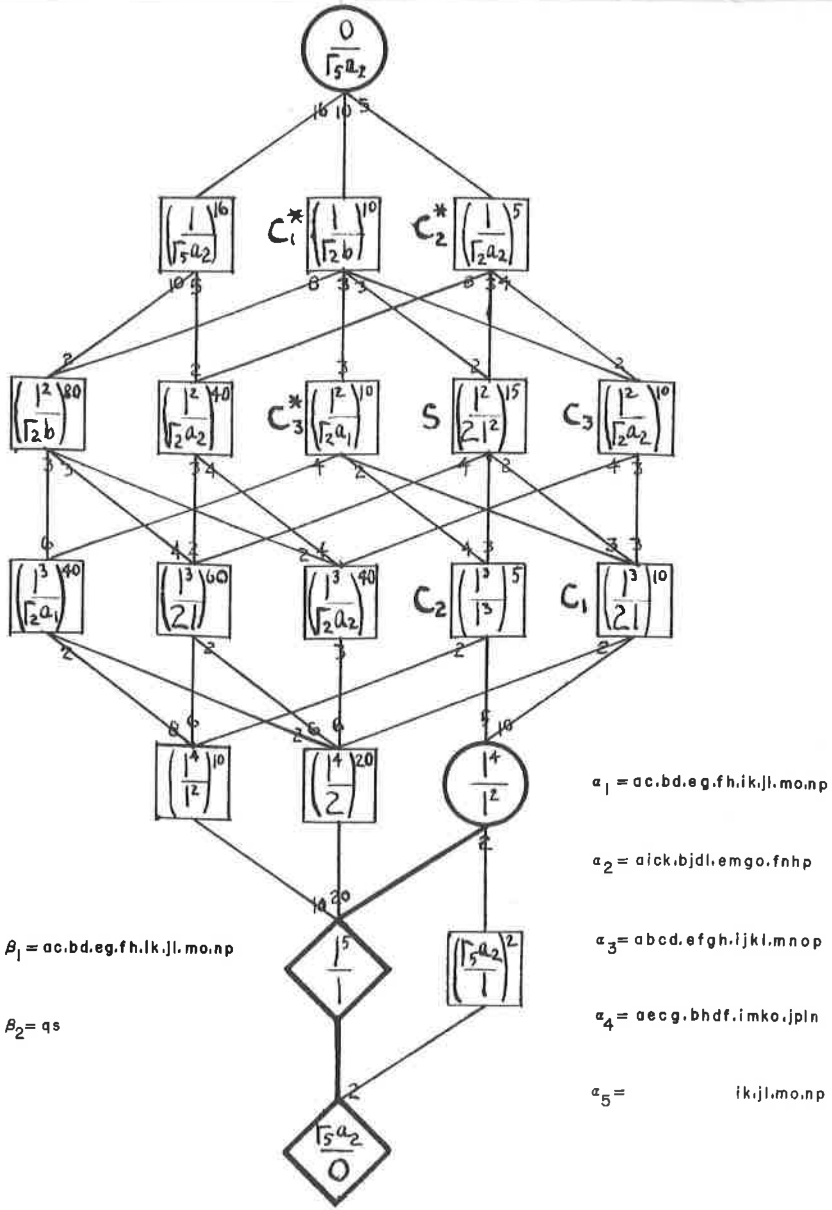
$\alpha_3 = aecg.bfdh.lmko.jnlp$

$\alpha_4 = aick.bjdl.eogm.fphn$

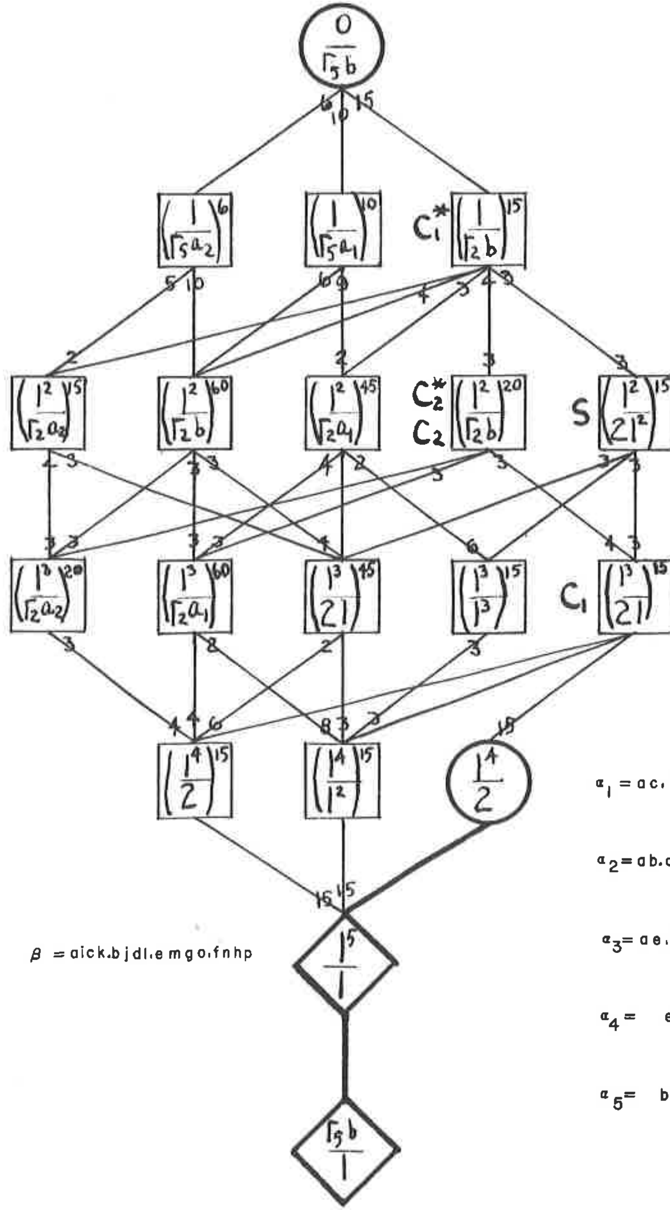
$\alpha_5 = bd.jl.fh.np$



α_1 = ac.bd.eg.fh
α_2 = abcd.efgh
α_3 = abcd.ehgf
α_4 = aecg.bfdh
α_5 = aecg.bhdf
β_1 = ac.bd.eg.fh
β_2 = ik



β_1 = ac.bd.eg.fh.ik.jl.mo.np
β_2 = qs
α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = aick.bjdl.emgo.fnhp
α_3 = abcd.efgh.ijkl.mnop
α_4 = aecg.bhdf.imko.jpln
α_5 = ik.jl.mo.np



β = aick.bjdl.emgo.fnhp

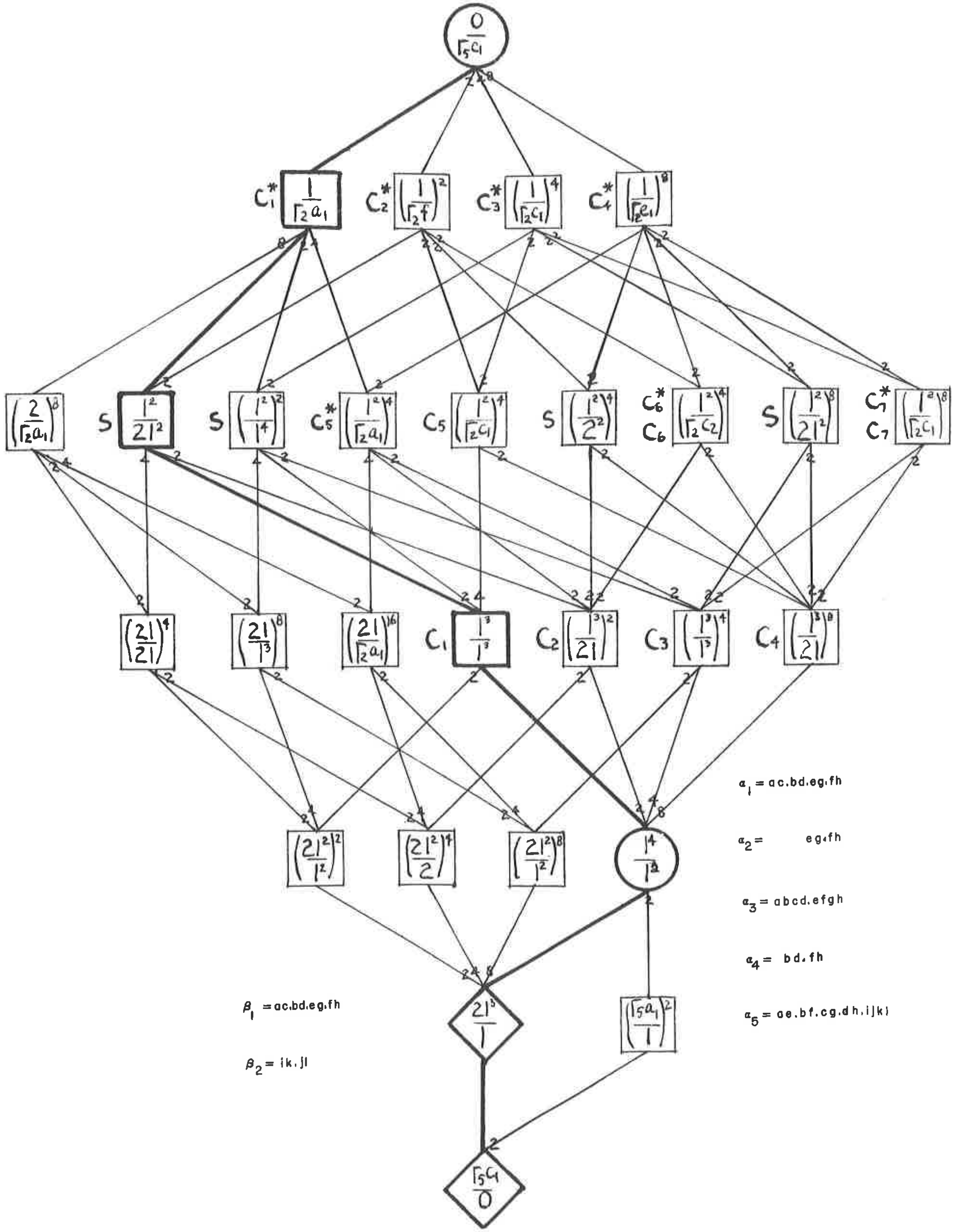
α_1 = ac, bd.eg.fh.ik.jl.mo.np

α_2 = ab.cd.ef.gh.ij.kl.mn.op

α_3 = ae.bf.cg.dh.im.jn.ko.lp

α_4 = eg.fh. mo.np

α_5 = bd.fh. jl.np



β_1 = ac.bd.eg.fh

β_2 = ik.jl

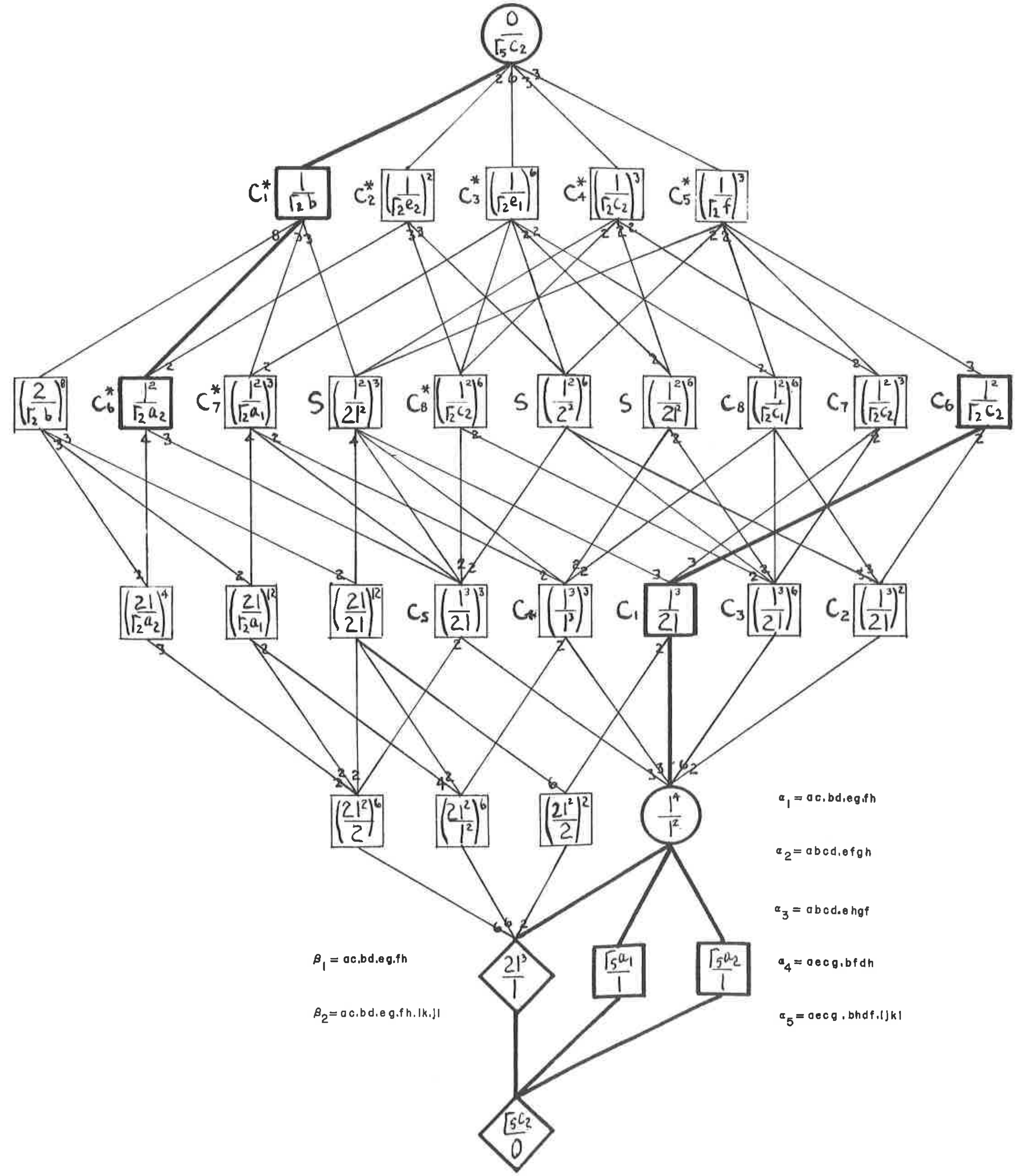
α_1 = ac.bd.eg.fh

α_2 = eg.fh

α_3 = abcd.efgh

α_4 = bd.fh

α_5 = ae.bf.cg.dh.ijkl



β_1 = ac.bd.eg.fh

β_2 = ac.bd.eg.fh.ik.jl

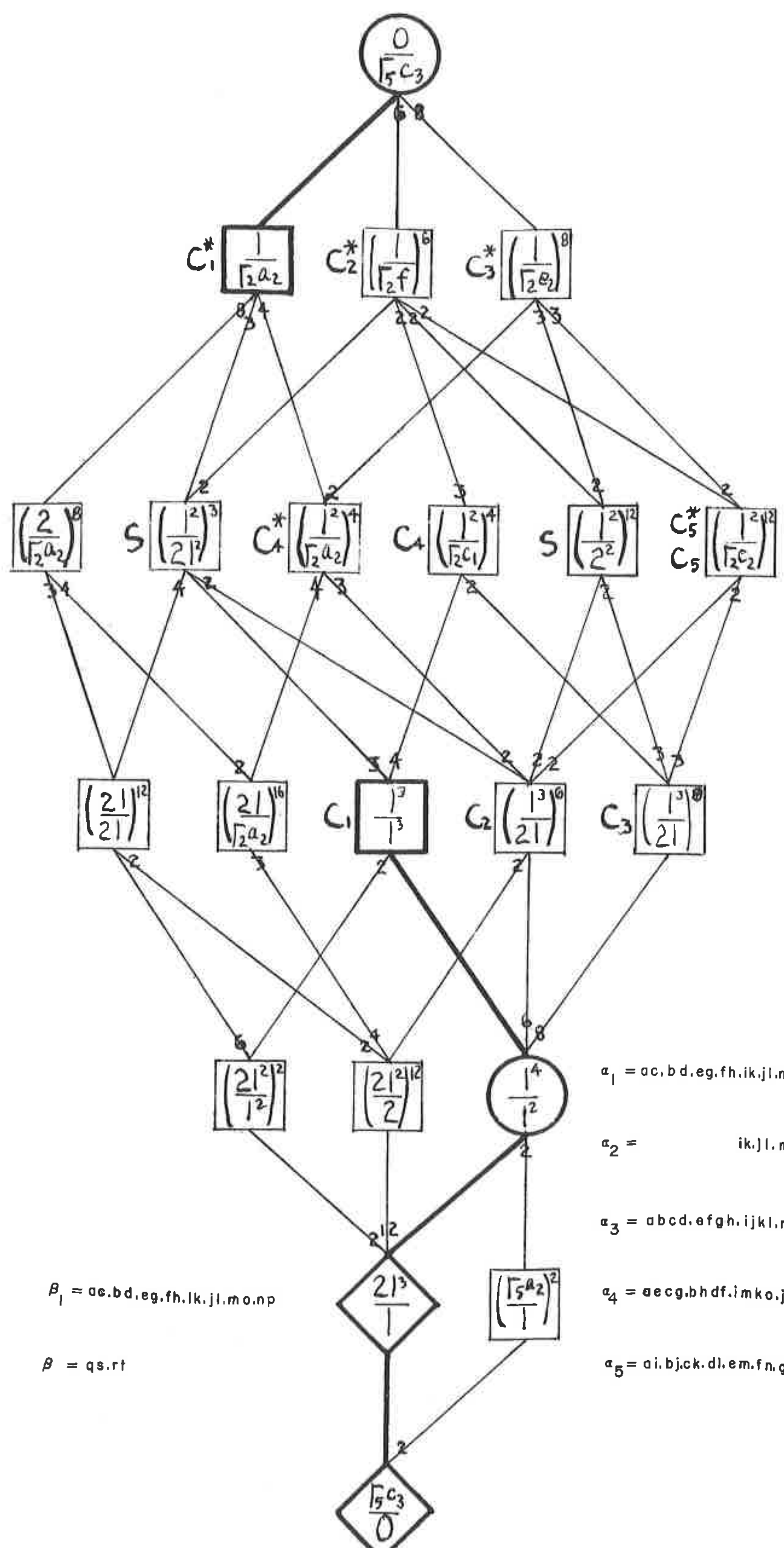
α_1 = ac.bd.eg.fh

α_2 = abcd.efgh

α_3 = abcd.ehgf

α_4 = aecg.bfdh

α_5 = aecg.bhdf.ijkl



β_1 = ac,bd,eg,fh,ik,jl,mo,np

β = qs,rt

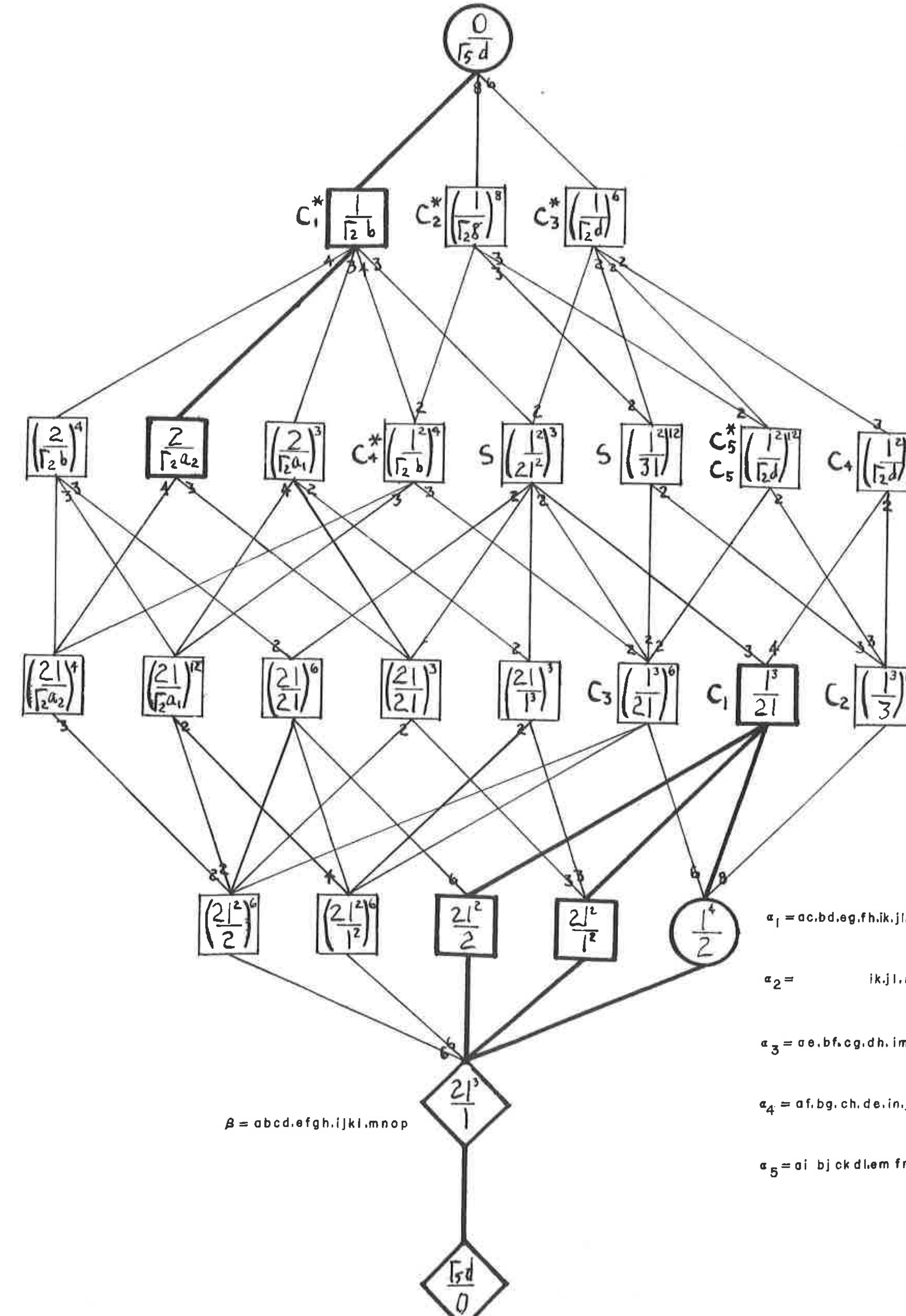
α_1 = ac,bd,eg,fh,ik,jl,mo,np

α_2 = ik,jl,mo,np

α_3 = abcd,efgh,ijkl,mnop

α_4 = aecg,bhdf,imko,jpln

α_5 = ai,bj,ck,dl,em,fn,go,hp,qrst



β = abcd,efgh,ijkl,mnop

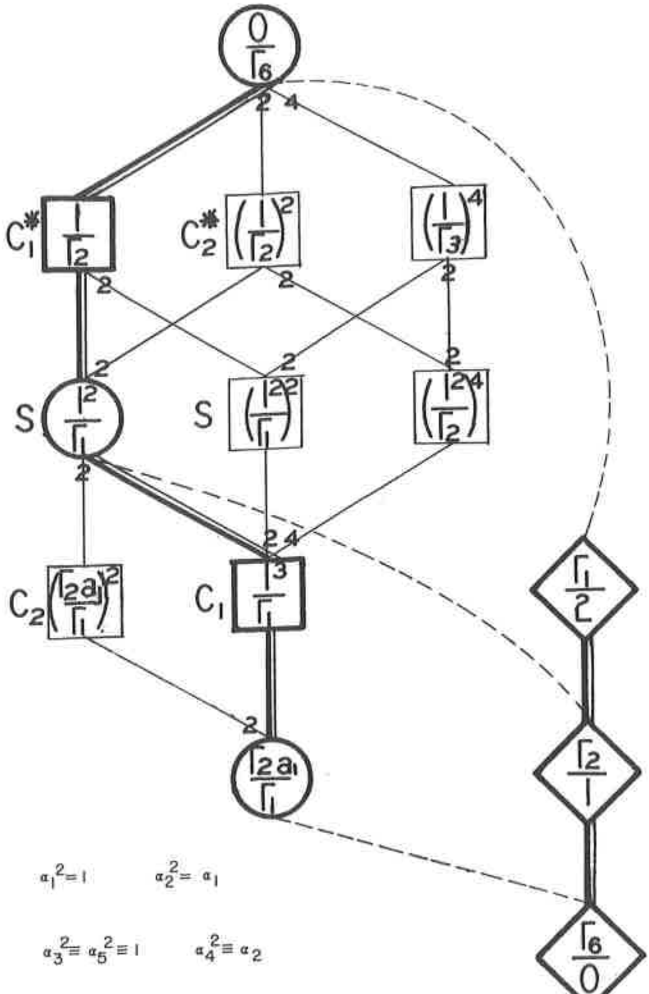
α_1 = ac,bd,eg,fh,ik,jl,mo,np

α_2 = ik,jl,mo,np

α_3 = ae,bf,cg,dh,im,jn,ko,lp

α_4 = af,bg,ch,de,in,jo,kp,lm

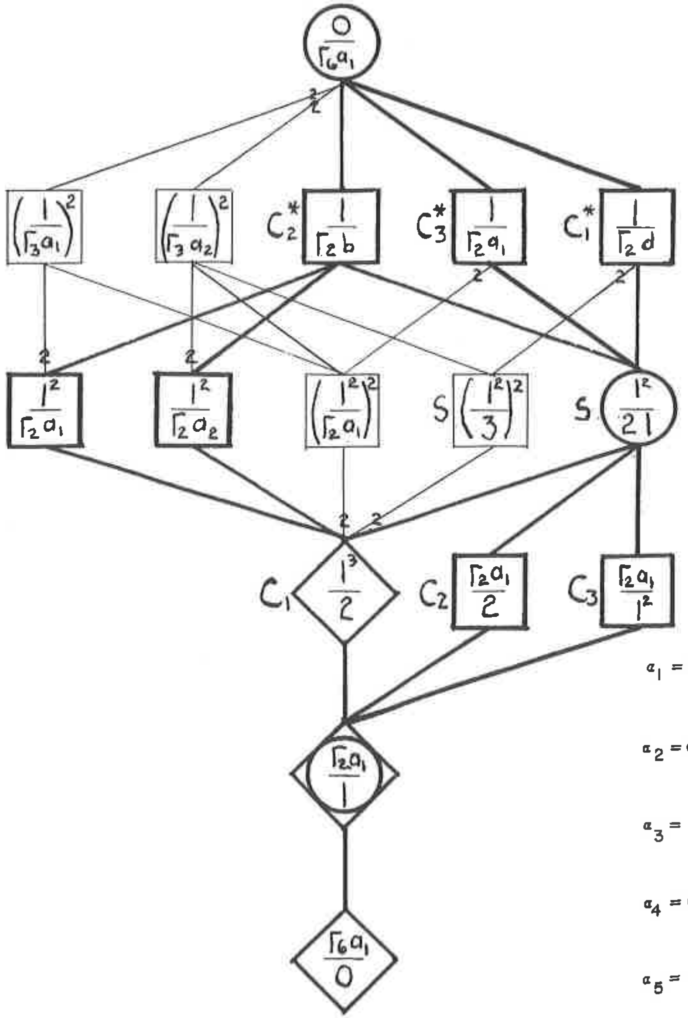
α_5 = ai bj ck dl,em fn go hp



$\alpha_1{}^2 = 1 \qquad \alpha_2^2 = \alpha_1$

$\alpha_3{}^2 \equiv \alpha_5{}^2 \equiv 1 \qquad \alpha_4^2 \equiv \alpha_2$

$[\alpha_3, \alpha_4] = [\alpha_2, \alpha_5] = \alpha_1 . \qquad [\alpha_4, \alpha_5] = \alpha_2$



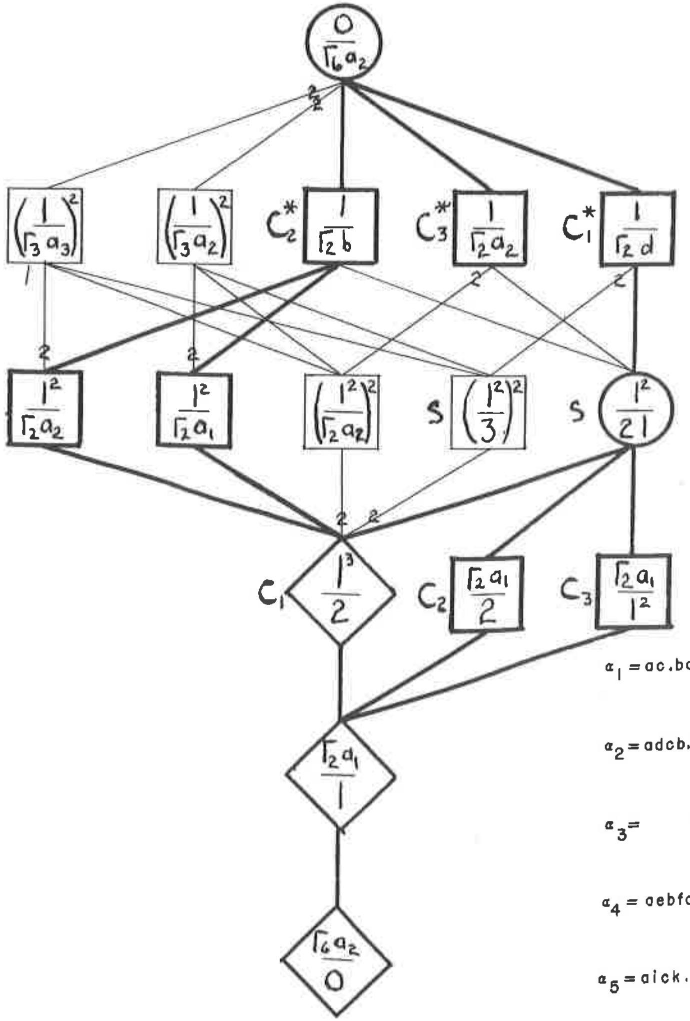
α_1 = ac.bd.eg.fh

α_2 = adcb.ehgf

α_3 = eg.fh

α_4 = aebfcgdh

α_5 = bd.eh.fg



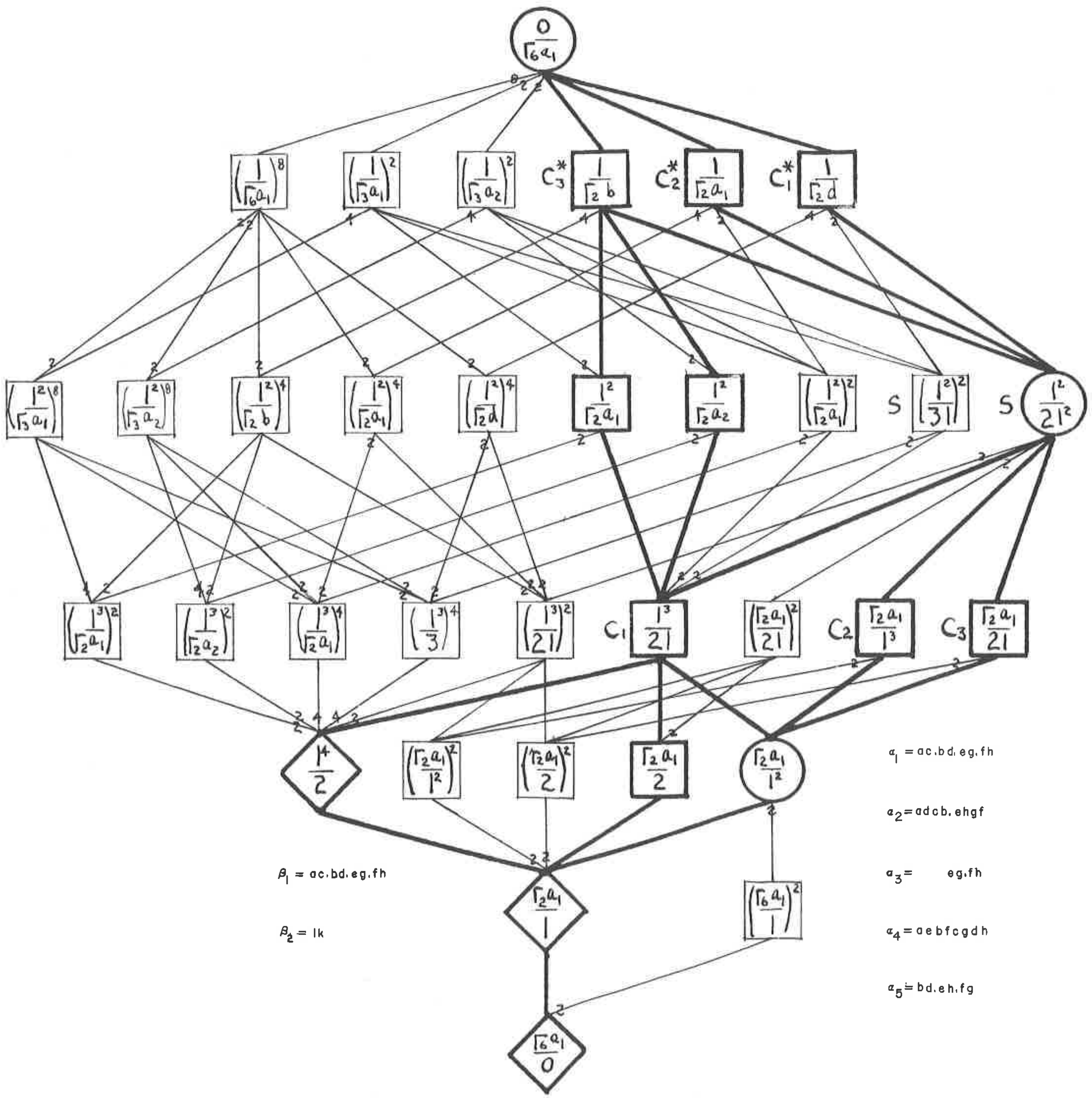
α_1 = ac.bd.eg.fh.lk.jl.mo.np

α_2 = adcb.ehgf.ilkj.mp on

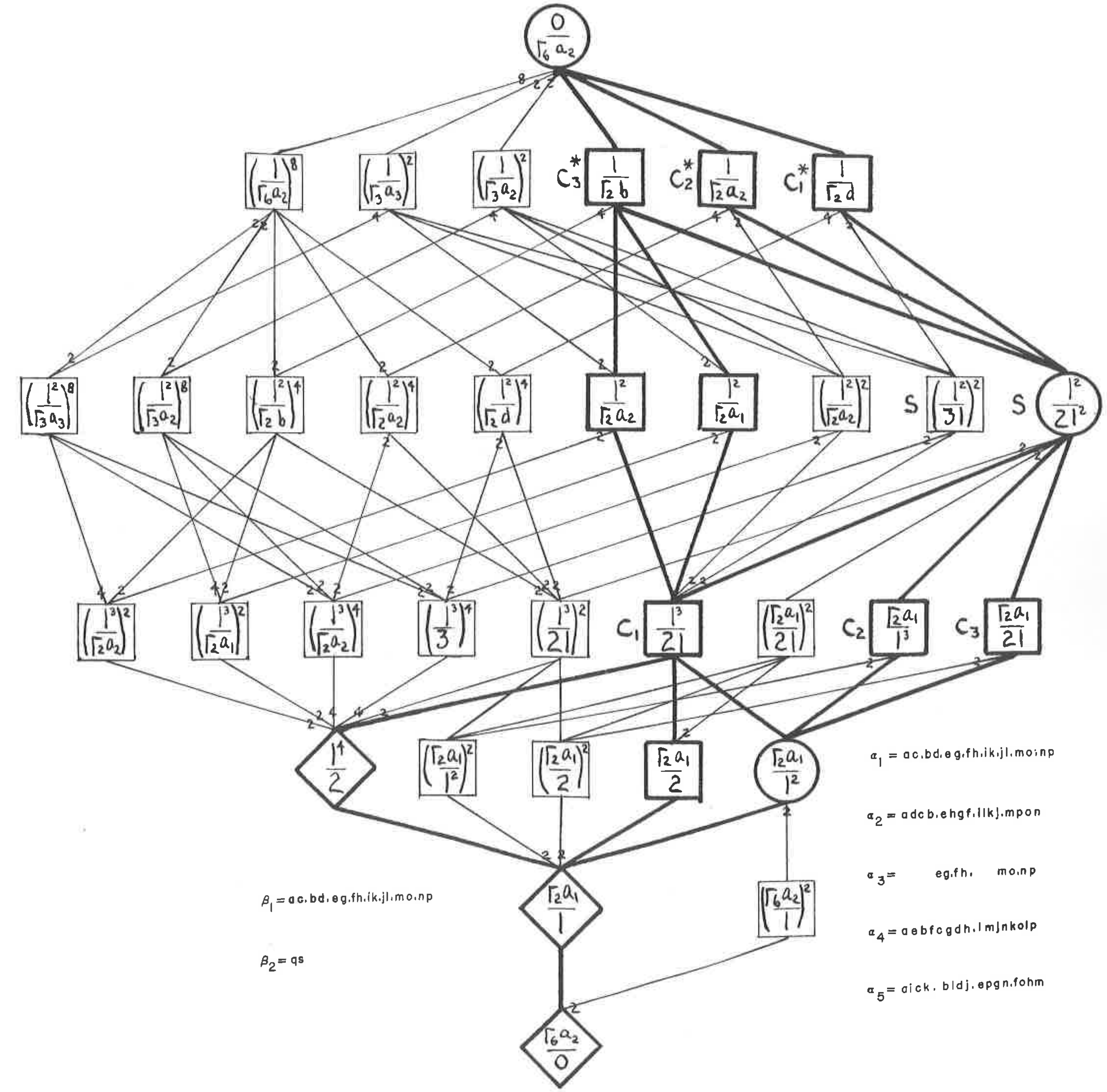
α_3 = eg.fh. mo.np

α_4 = aebfcgdh.imjnkolp

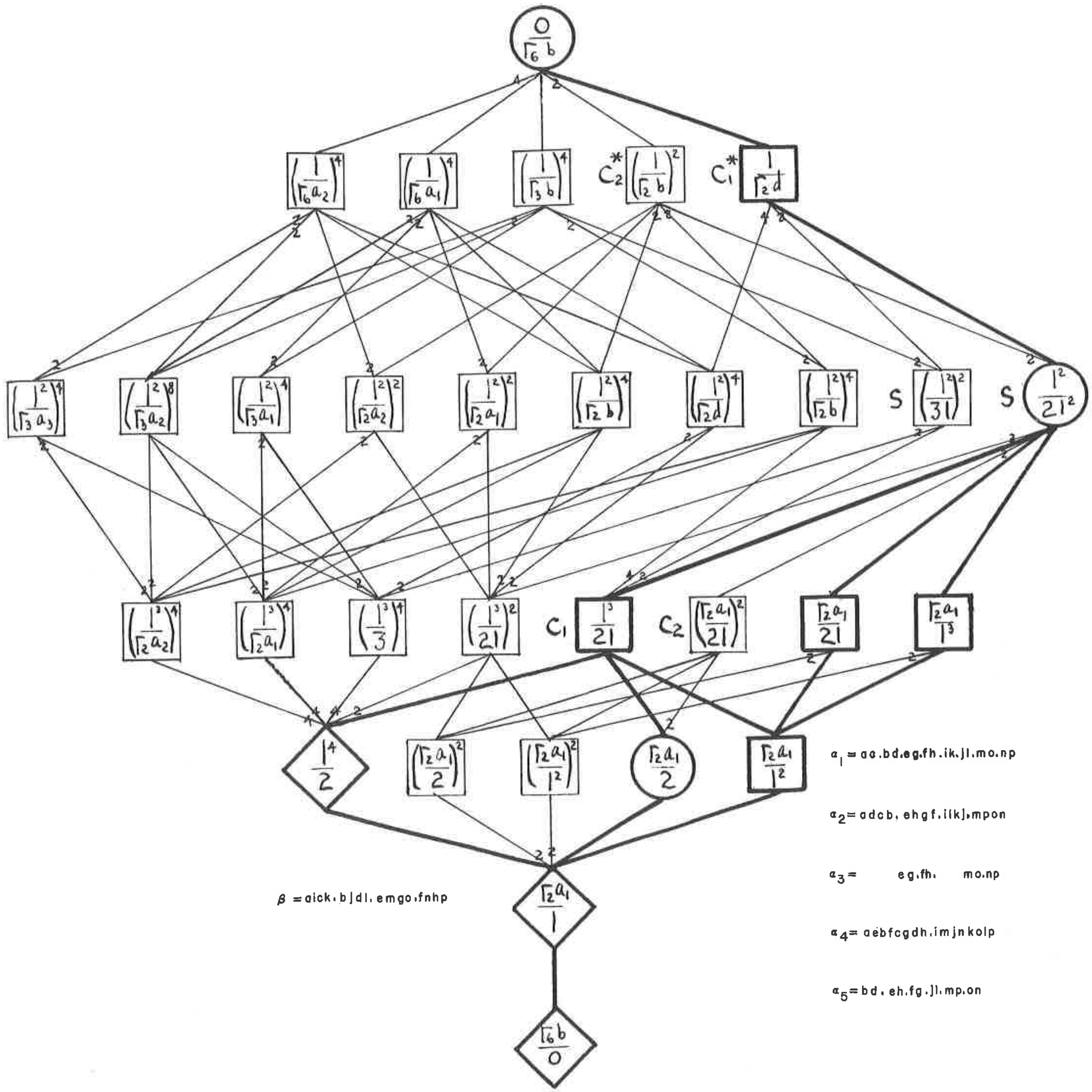
α_5 = aick.bldj.epgn.fohm

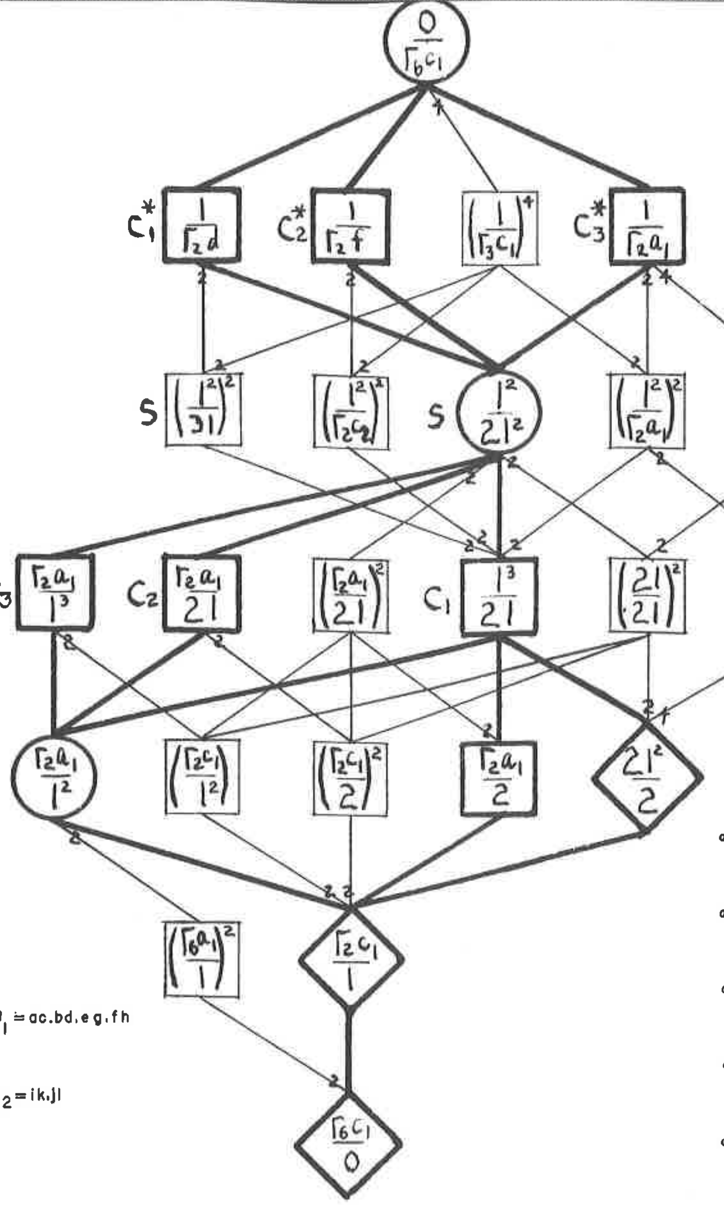


β_1 = ac.bd.eg.fh
β_2 = lk
α_1 = ac.bd.eg.fh
α_2 = adcb.ehgf
α_3 = eg.fh
α_4 = aebfcgdh
α_5 = bd.eh.fg



β_1 = ac.bd.eg.fh.ik.jl.mo.np
β_2 = qs
α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = adcb.ehgf.ilkj.mpon
α_3 = eg.fh, mo.np
α_4 = aebfcgdh.imjnkolp
α_5 = aick. bldj.epgn.fohm


C*2
C*1
S
S
C1
C2
β = aick, bjdl, emgo, fnhp
α1 = ac.bd.eg.fh.ik.jl.mo.np
α2 = adcb, ehgf, ilkj, mpon
α3 = eg, fh, mo, np
α4 = aebfcgdh, imjnkolp
α5 = bd, eh, fg, jl, mp, on



β_1 = ac.bd.eg.fh

β_2 = ik.jl

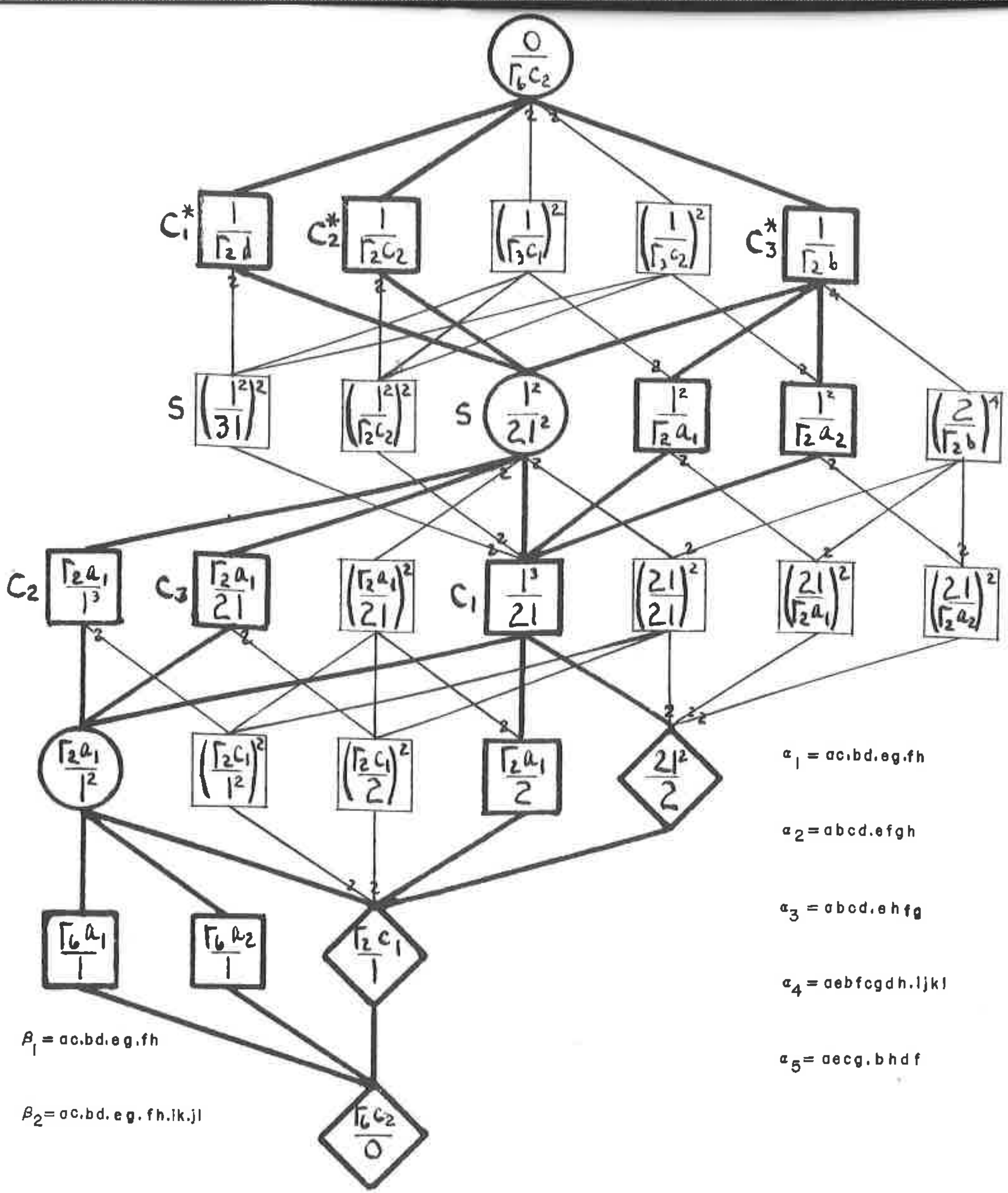
α_1 = ac.bd.eg.fh

α_2 = adcb.ehgf

α_3 = eg.fh

α_4 = aebfcgdh.ijkl

α_5 = bd.eh.fg



α_1 = ac.bd.eg.fh

α_2 = abcd.efgh

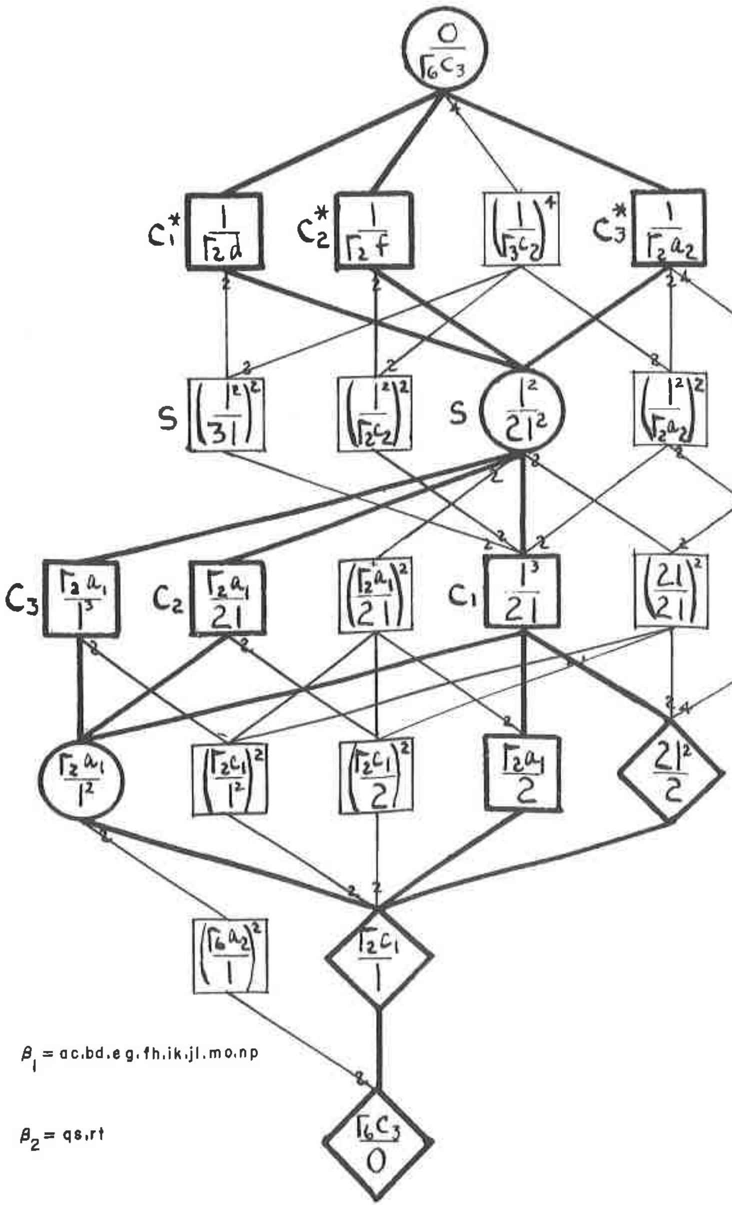
α_3 = abcd.ehfg

α_4 = aebfcgdh.ijkl

α_5 = aecg.bhdf

β_1 = ac.bd.eg.fh

β_2 = ac.bd.eg.fh.ik.jl



α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = adcb.ehgf.ilkj.mpon

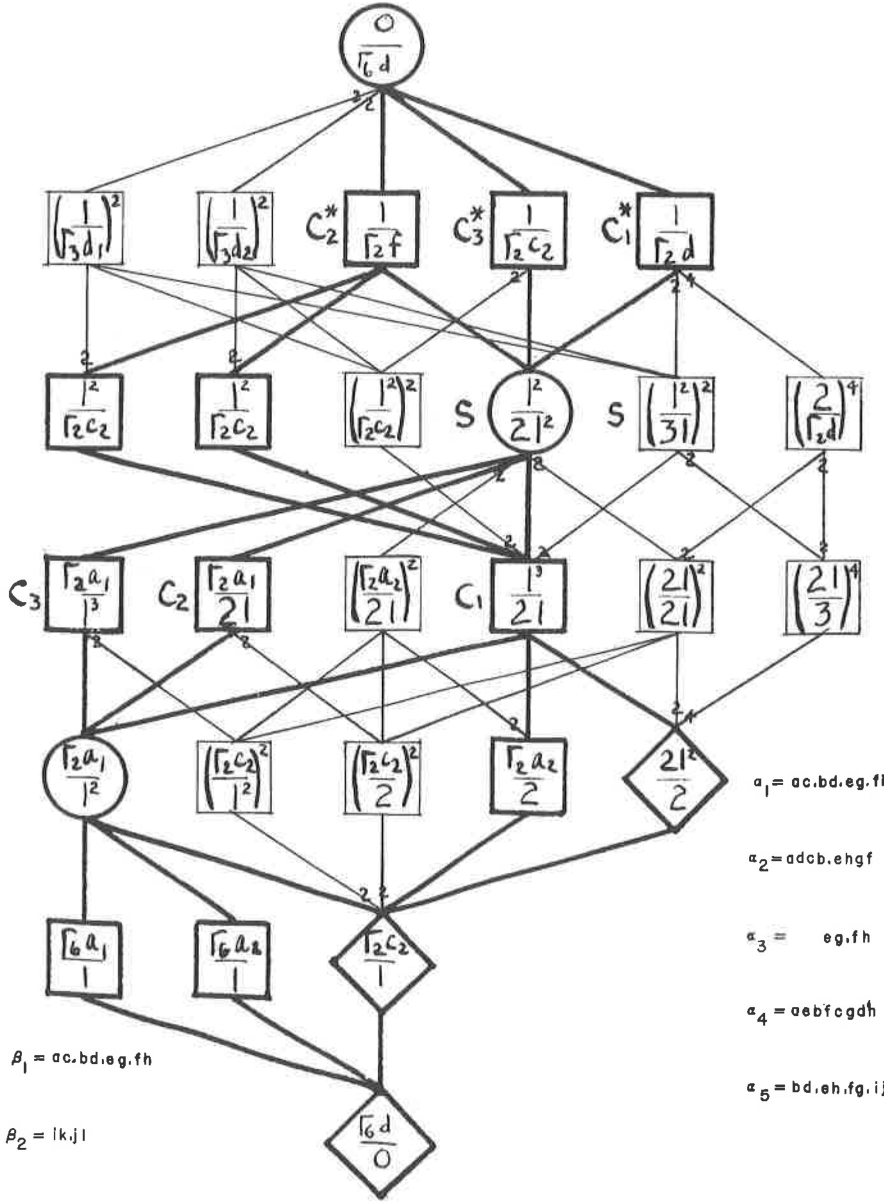
α_3 = eg.fh. mo.np

α_4 = aebfcgdh.lmjnkolp.qrst

α_5 = aick.bldj.epgn.fohm

β_1 = ac.bd.eg.fh.ik.jl.mo.np

β_2 = qs.rt



α_1 = ac.bd.eg.fh

α_2 = adcb.ehgf

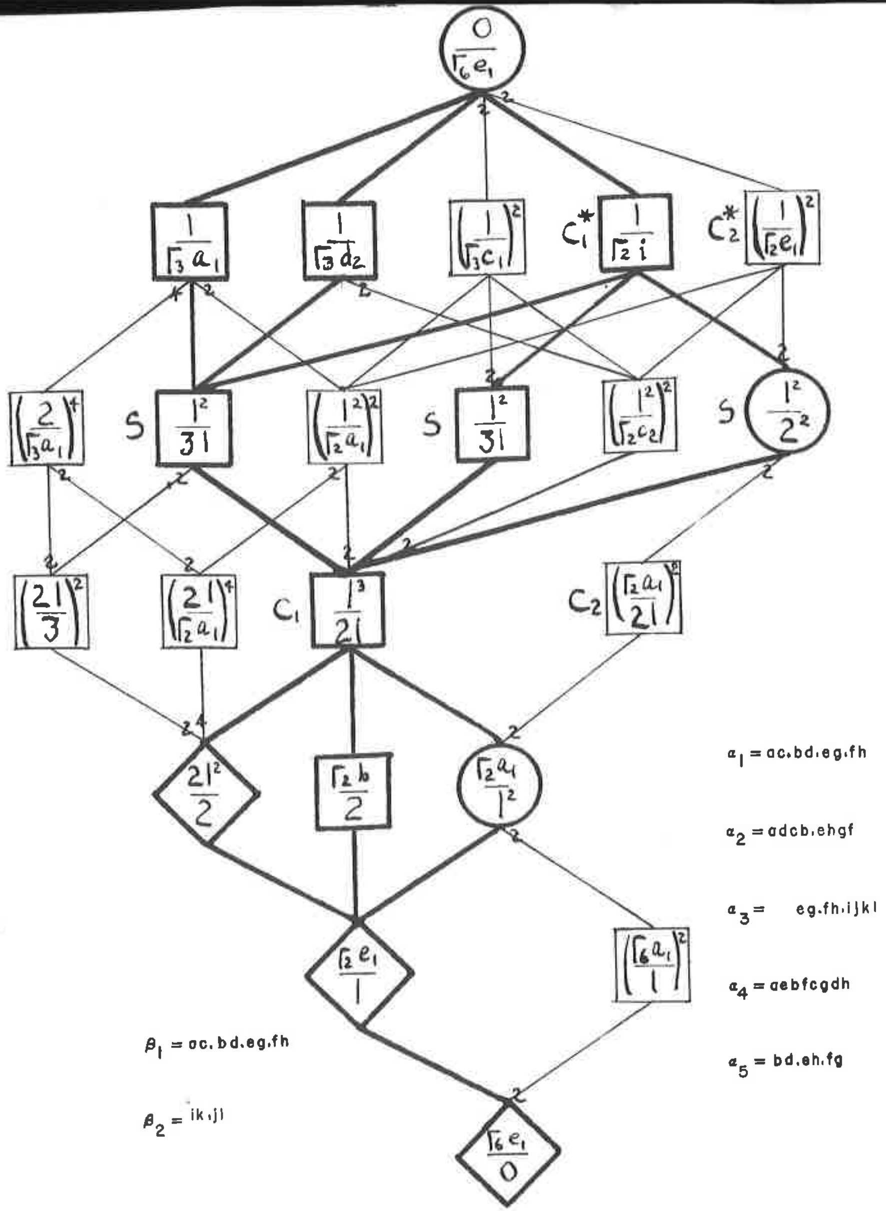
α_3 = eg.fh

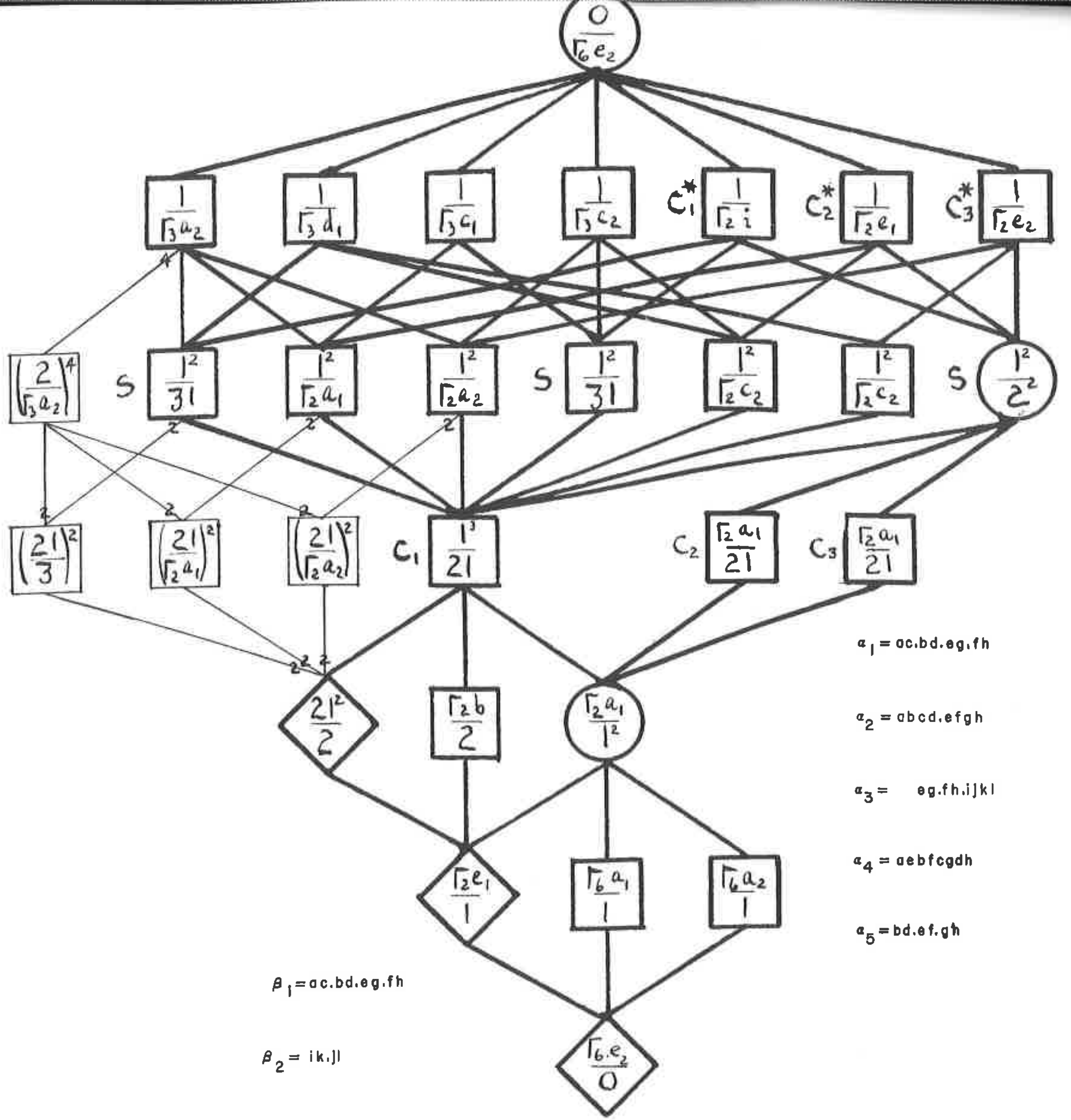
α_4 = aebfcgdh

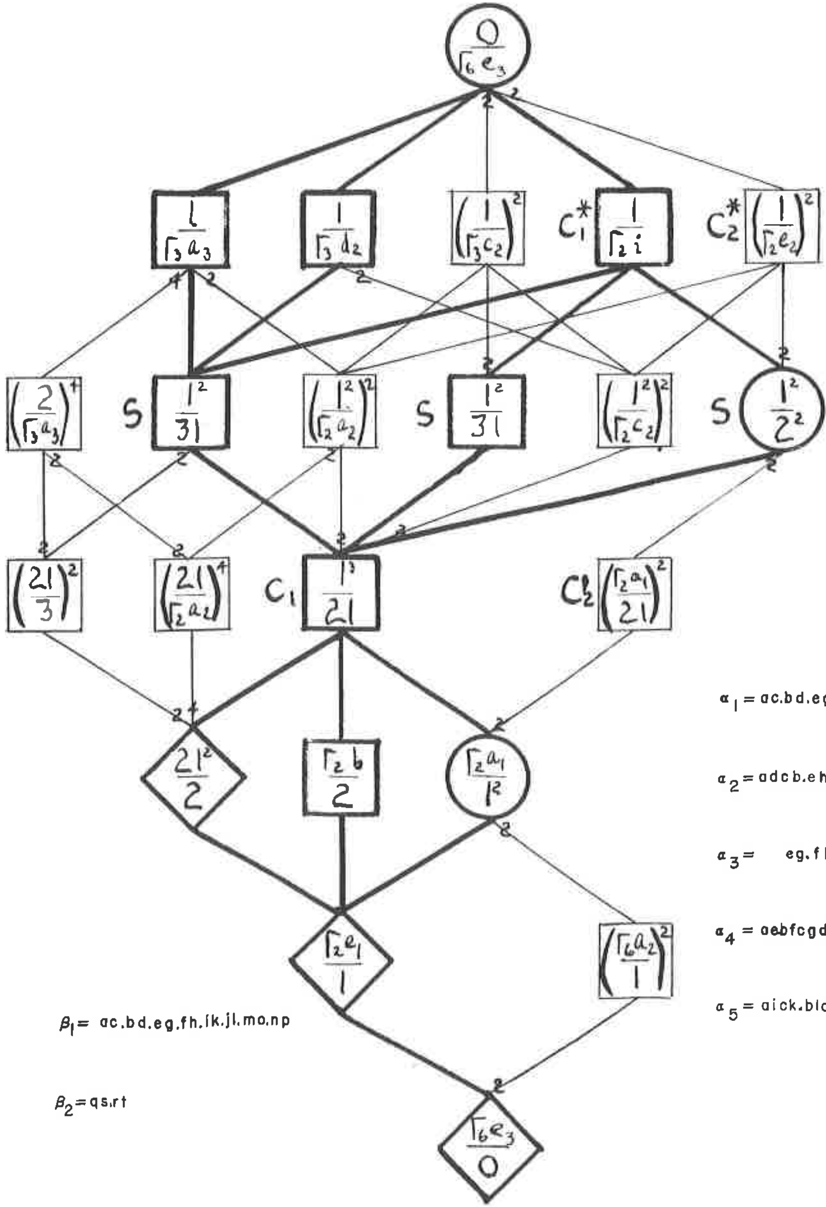
α_5 = bd.eh.fg.ijkl

β_1 = ac.bd.eg.fh

β_2 = ik.jl


α_1 = ac.bd.eg.fh
α_2 = adcb.ehgf
α_3 = eg.fh.ijkl
α_4 = aebfcgdh
α_5 = bd.eh.fg
β_1 = ac.bd.eg.fh
β_2 = ik.jl


α_1 = ac.bd.eg.fh
α_2 = abcd.efgh
α_3 = eg.fh.ijkl
α_4 = aebfcgdh
α_5 = bd.ef.gh
β_1 = ac.bd.eg.fh
β_2 = ik.jl



α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = adcb.ehgf.ilkj.mpon

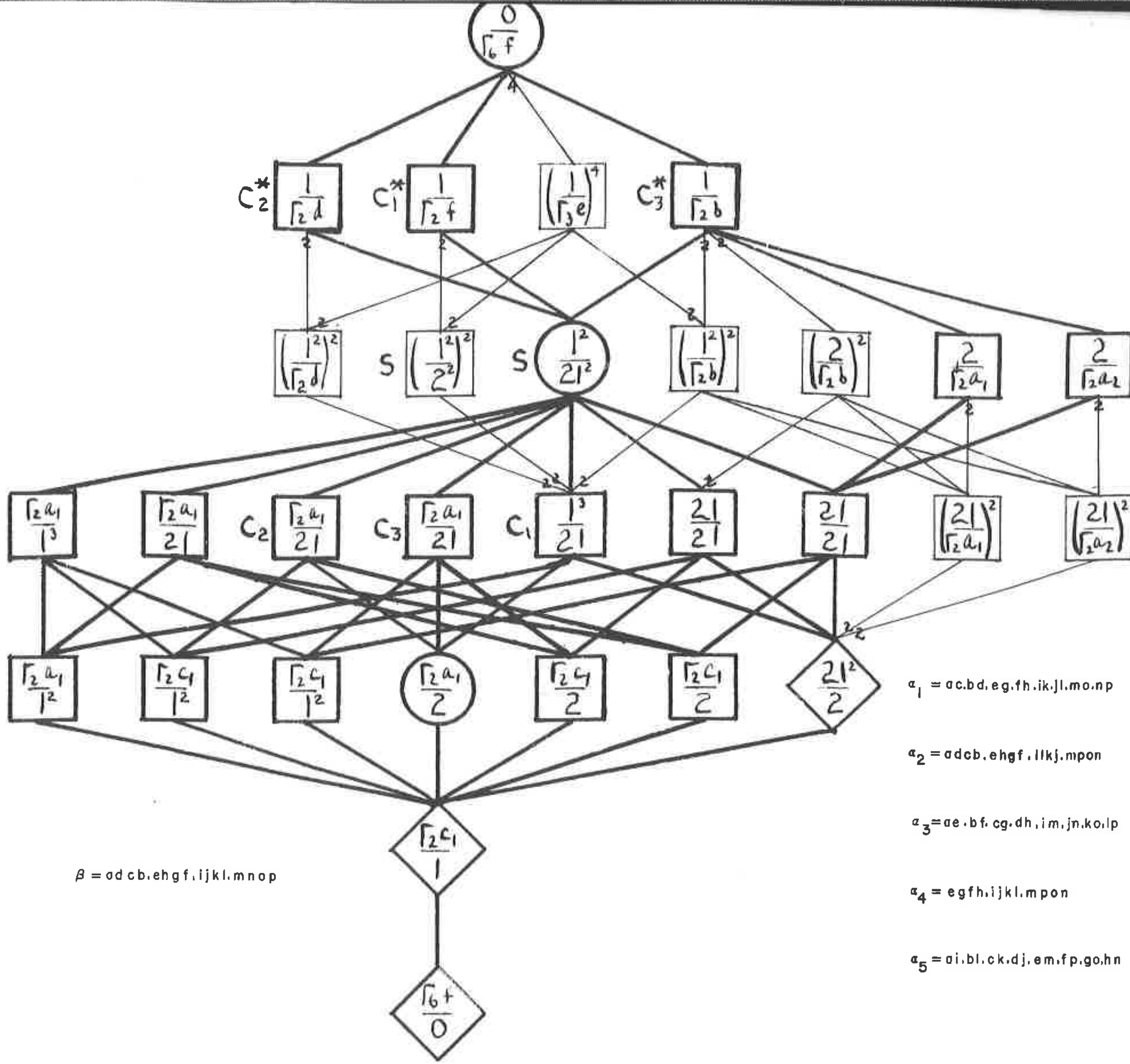
α_3 = eg.fh. mo.np.qrst

α_4 = aebfcgdh.imjnkolp

α_5 = aick.bldj.epgn.fohm

β_1 = ac.bd.eg.fh.ik.jl.mo.np

β_2 = qs.rt



α_1 = ac.bd.eg.fh.ik.jl.mo.np

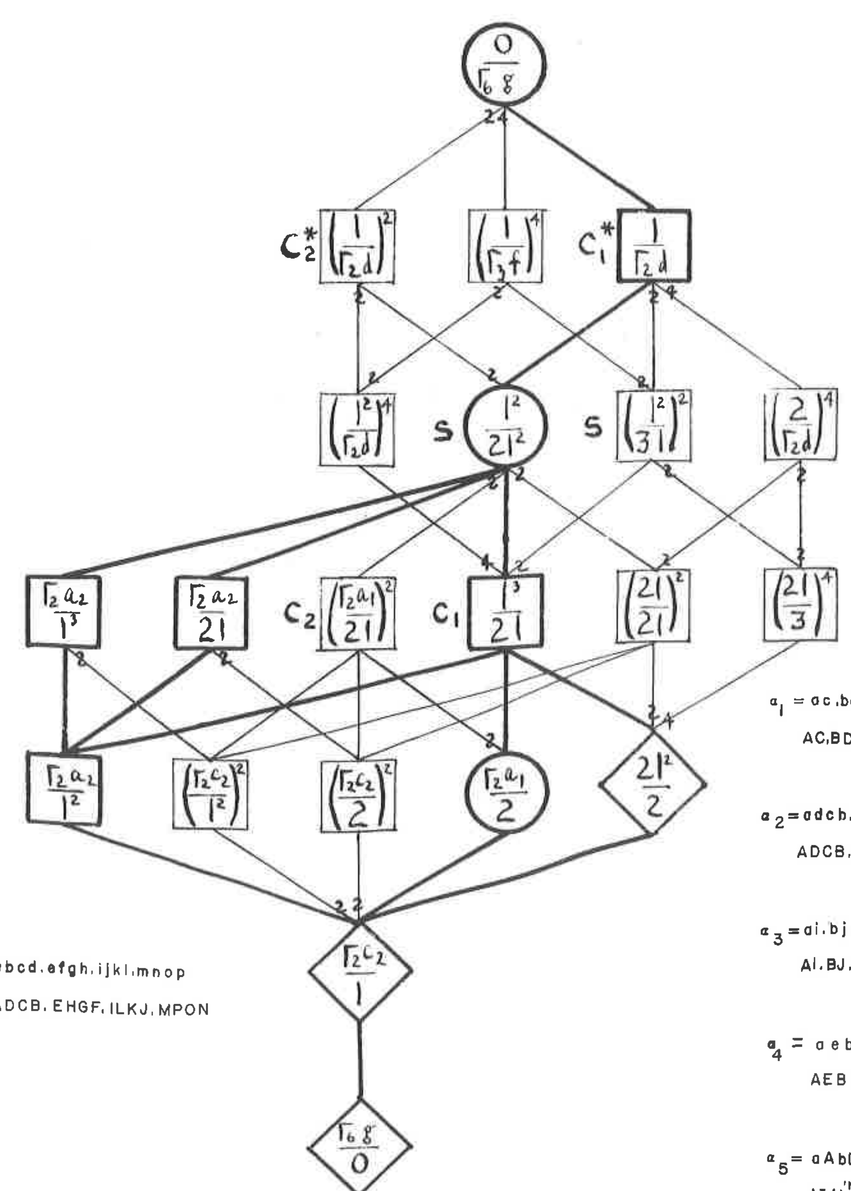
α_2 = adcb.ehgf.ilkj.mpon

α_3 = ae.bf.cg.dh.im.jn.ko.lp

α_4 = egfh.ijkl.mpon

α_5 = ai.bl.ck.dj.em.fp.go.hn

β = adcb.ehgf.ijkl.mnop



β = abcd.efgh.ijkl.mnop

ADCB.EHGF.ILKJ.MPON

α_1 = ac.bd.eg.fh.ik.jl.mo.np

AC.BD.EG.FH.IK.JL.MO.NP

α_2 = adcb.ehgf.ilkj.mpon

ADCB.EHGF.ILKJ.MPON

α_3 = ai.bj.ck.dl.eo.fp.gm.hn

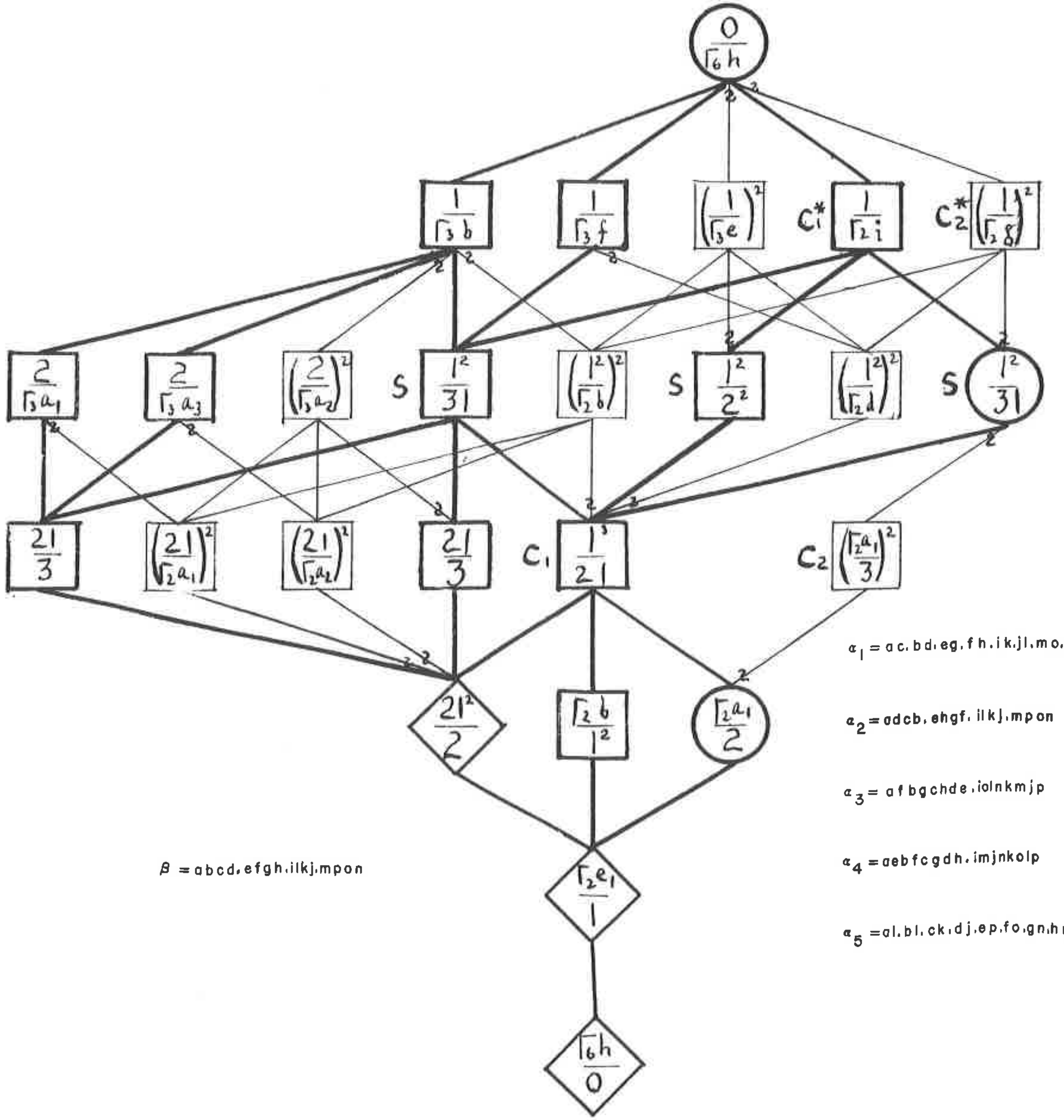
AI.BJ.CK.DL.EO.FP.GM.HN

α_4 = aebfcgdh.imjnkolp

AEBFCGDH.IMJNKOLP

α_5 = aAbDcCdB.eHfGgFhE.

iIjLkKlJ.mPnOoNpM



β = abcd.efgh.ilkj.mpon

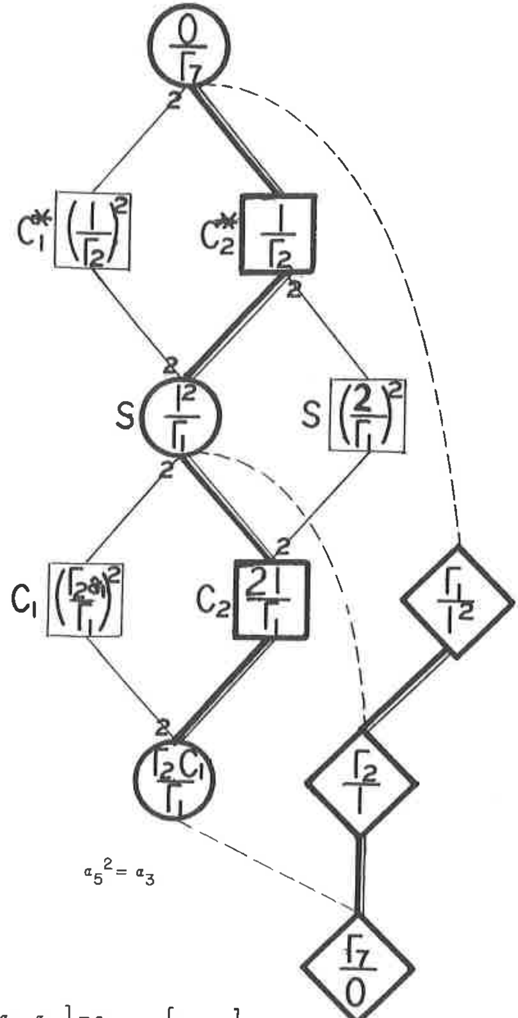
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = adcb.ehgf.ilkj.mpon

α_3 = afbgchde.iolnkmjp

α_4 = aebfcgdh.imjnkolp

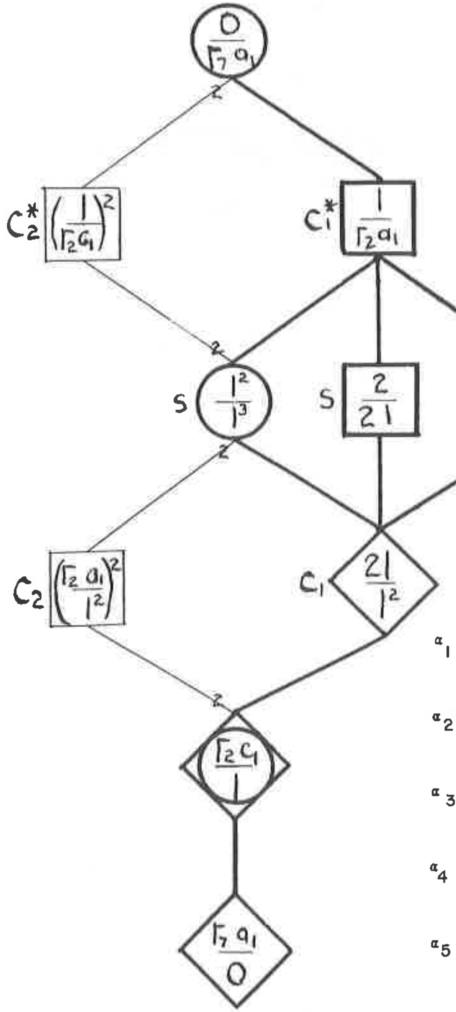
α_5 = al.bl.ck.dj.ep.fo.gn.hm



$\alpha_1^{\,2} = \alpha_2^{\,2} = 1$ $\qquad$ $\alpha_5^{\,2} = \alpha_3$

$\alpha_3^{\,2} \equiv \alpha_4^{\,2} \equiv 1$

$[\alpha_3, \alpha_4] = [\alpha_2, \alpha_5] = \alpha_1$ $\qquad$ $[\alpha_4, \alpha_5] = \alpha_2$



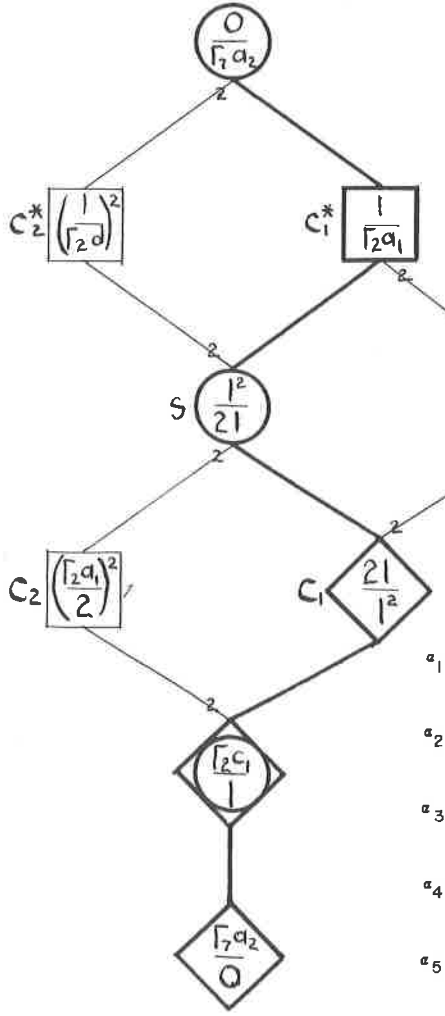
α_1 = ac, bd, eg, fh

α_2 = ac, bd

α_3 = ab, cd, ef, gh

α_4 = bd, fh

α_5 = aebf, cgdh



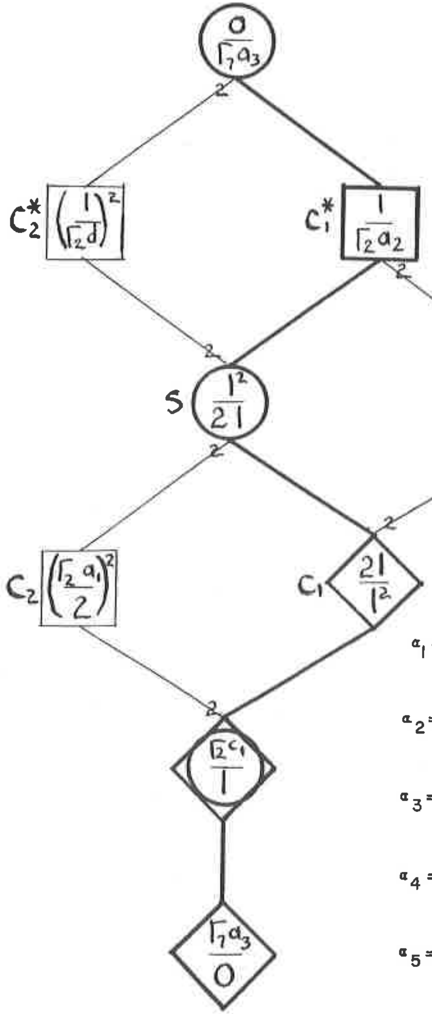
α_1 = ac, bd, eg, fh

α_2 = ac, bd

α_3 = abcd, efgh

α_4 = bd, fh

α_5 = aebfcgdh



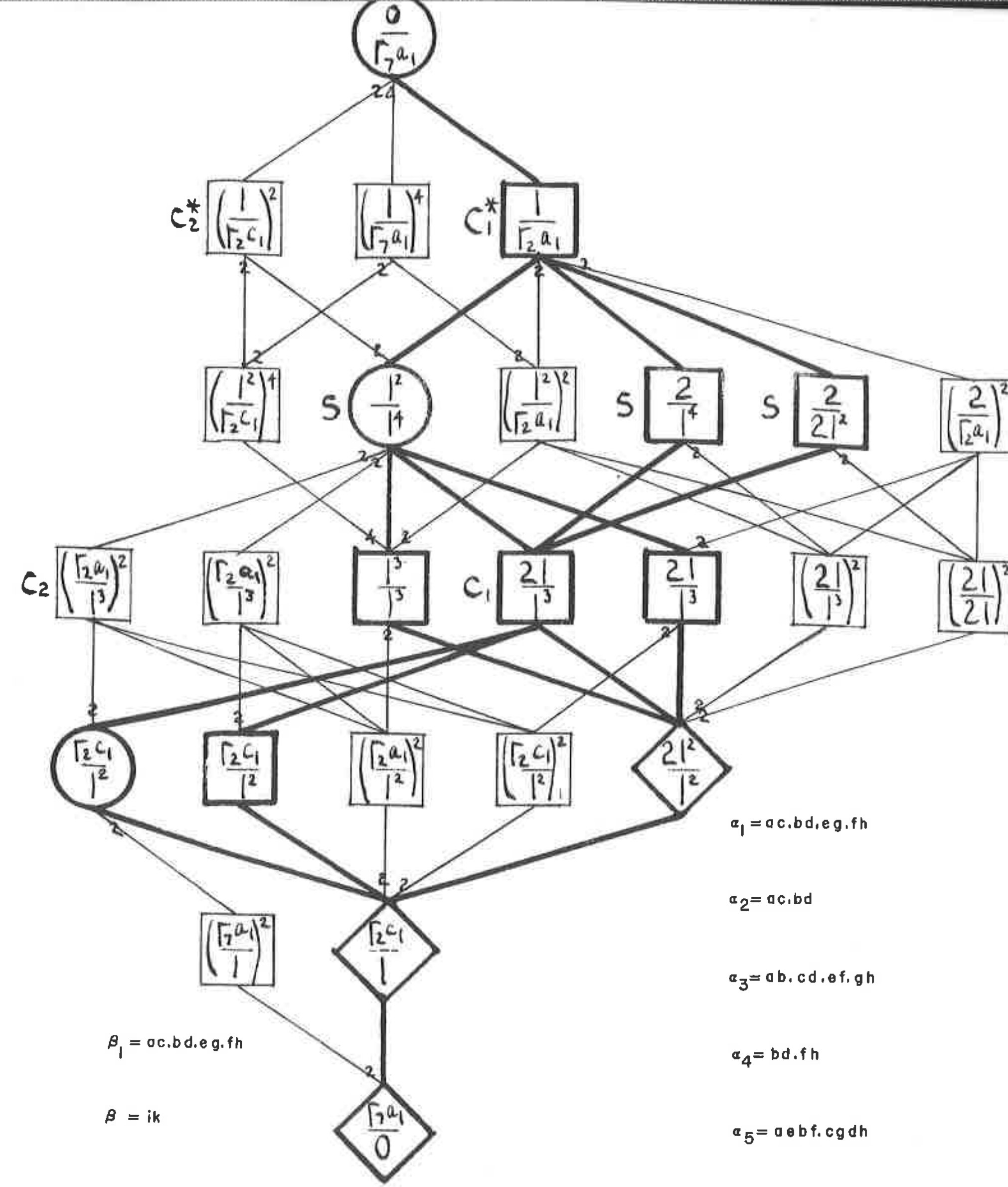
α_1 = ac, bd, eg, fh, ik, jl, mo, np

α_2 = ac, bd, ik, jl

α_3 = abcd, efgh, ijkl, mnop

α_4 = aick, bldj, emgo, fphn

α_5 = aebfcgdh, imjnkolp



α_1 = ac.bd.eg.fh

α_2 = ac.bd

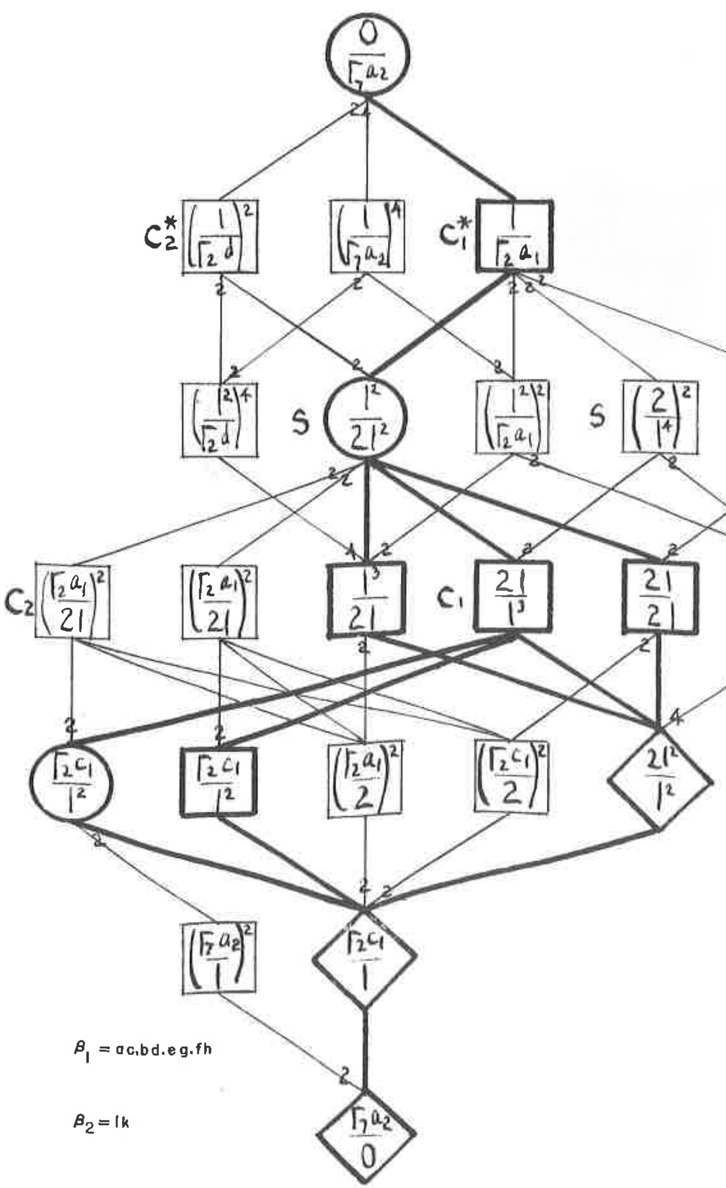
α_3 = ab.cd.ef.gh

α_4 = bd.fh

α_5 = aebf.cgdh

β_1 = ac.bd.eg.fh

β = ik



α_1 = ac.bd.eg.fh

α_2 = ac.bd

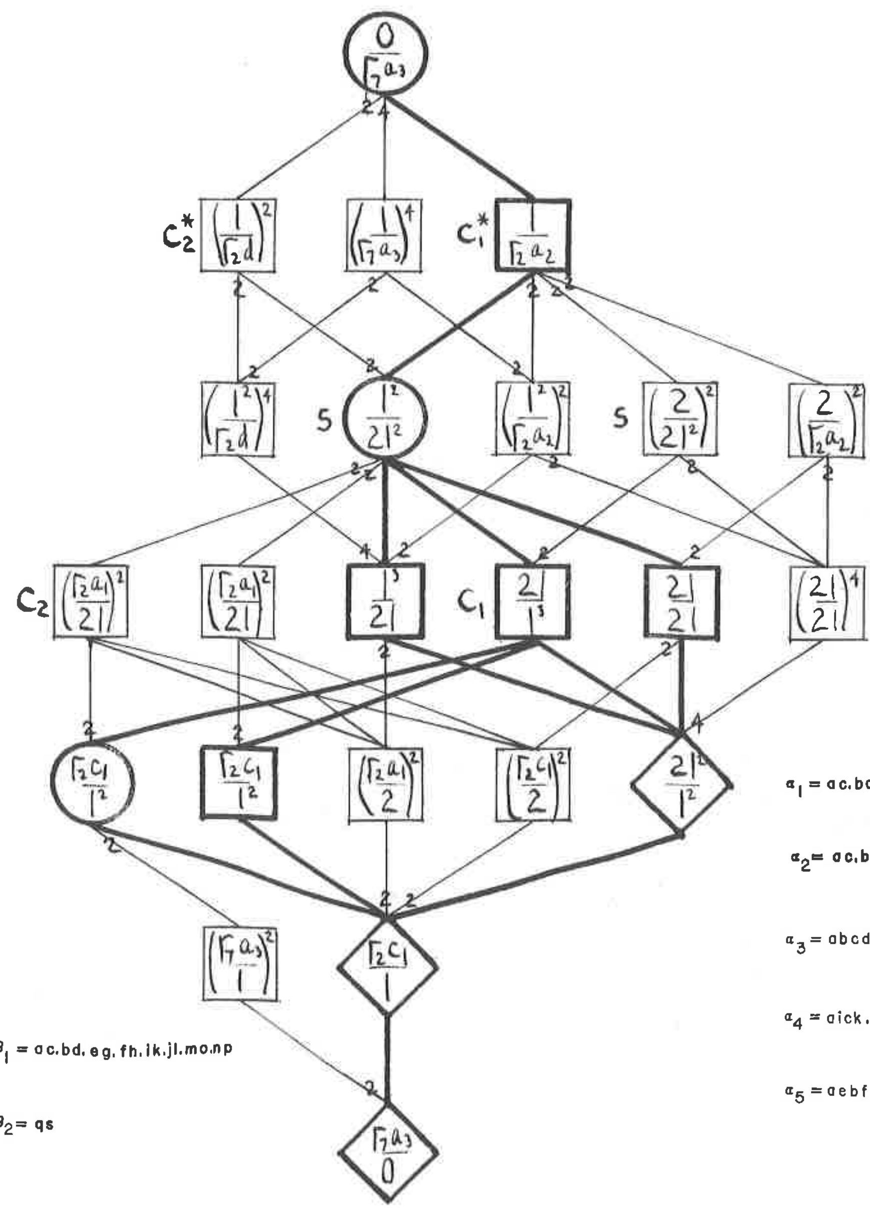
α_3 = abcd.efgh

α_4 = bd.fh

α_5 = aebfcgdh

β_1 = ac.bd.eg.fh

β_2 = lk



α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = ac.bd.ik.jl

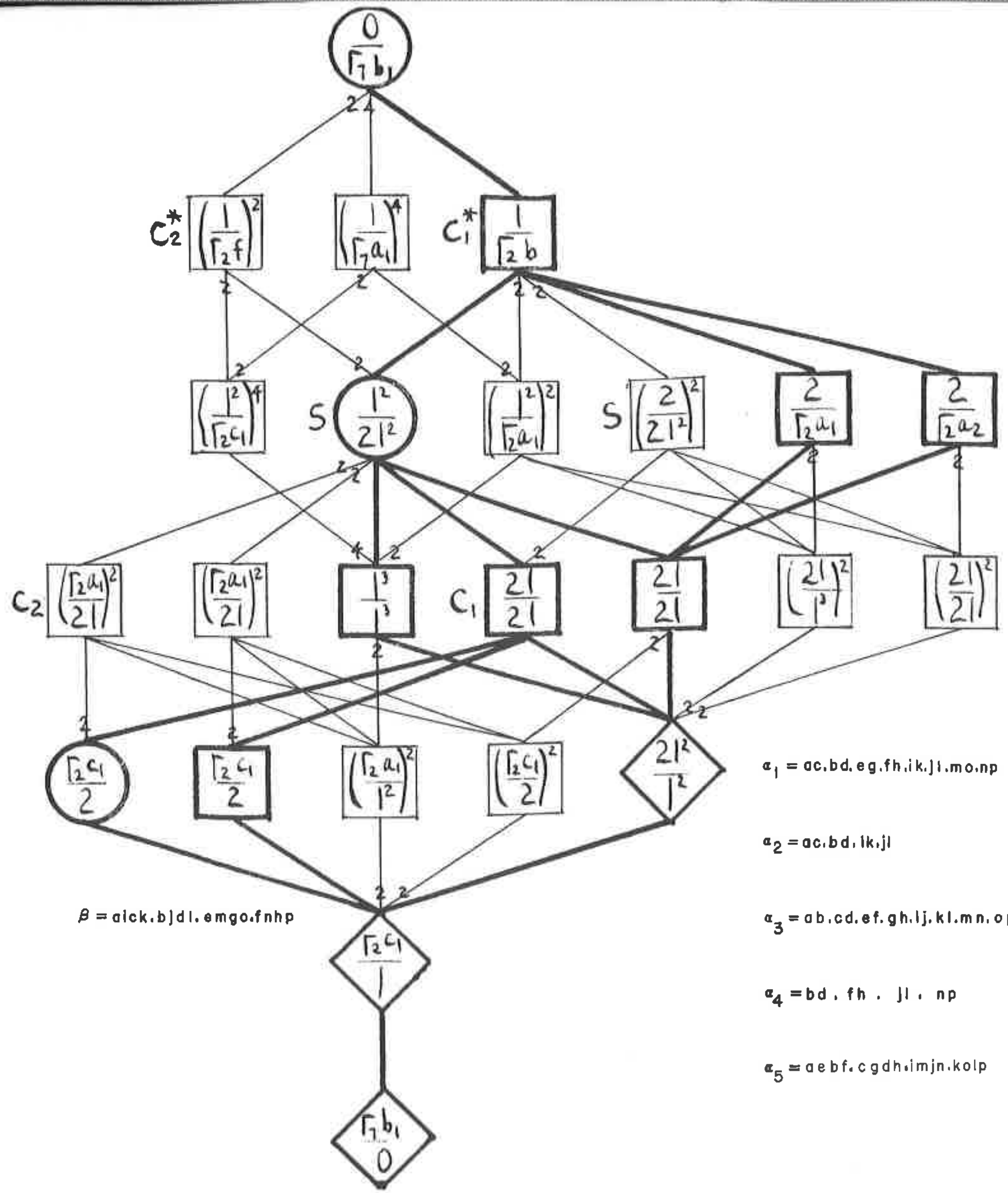
α_3 = abcd.efgh.ijkl.mnop

α_4 = aick.bldj.emgo.fphn

α_5 = aebfcgdh.imjnkolp

β_1 = ac.bd.eg.fh.ik.jl.mo.np

β_2 = qs



β = aick.bjdl.emgo.fnhp

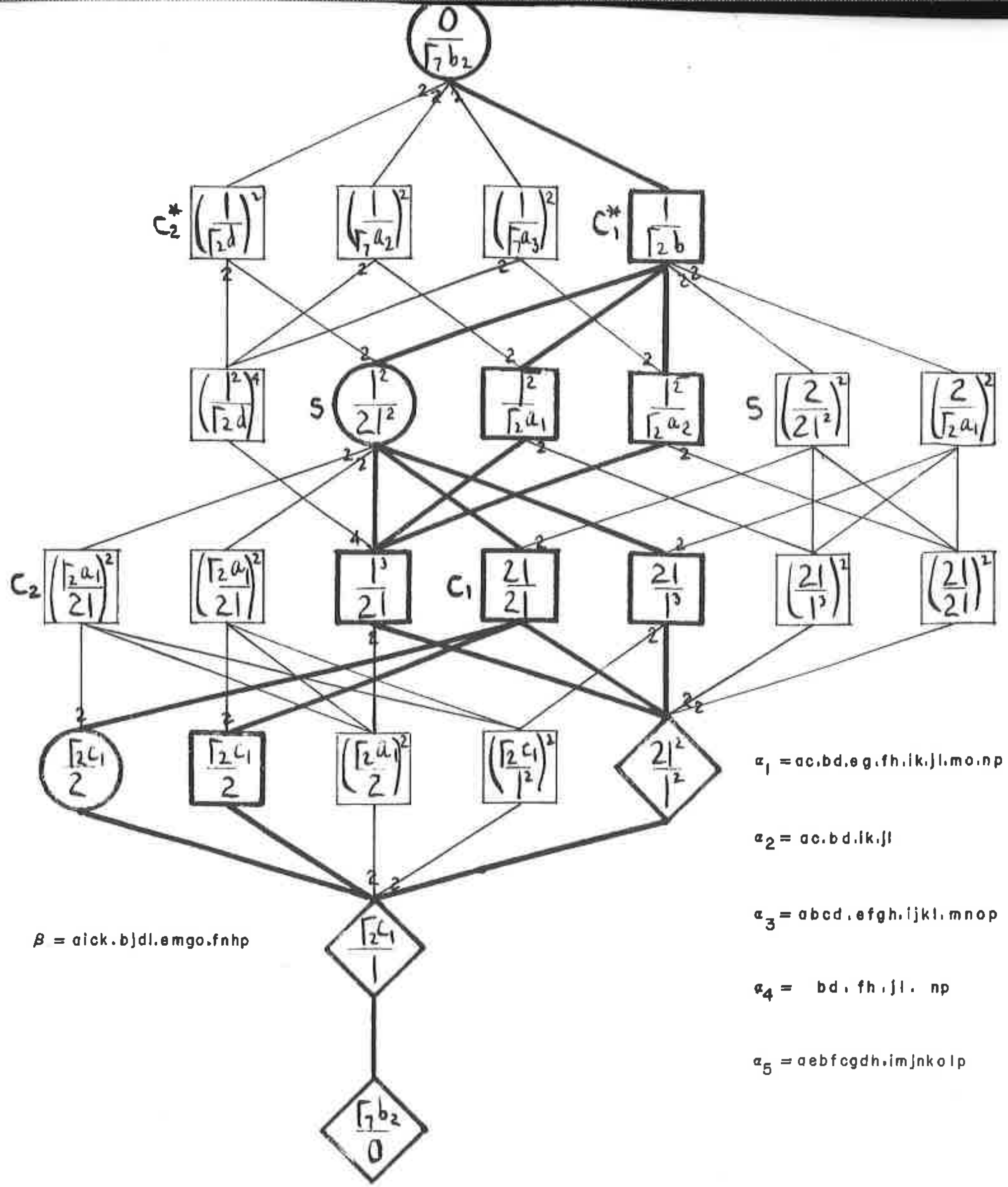
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = ac.bd.ik.jl

α_3 = ab.cd.ef.gh.ij.kl.mn.op

α_4 = bd . fh . jl . np

α_5 = aebf.cgdh.imjn.kolp



β = aick.bjdl.emgo.fnhp

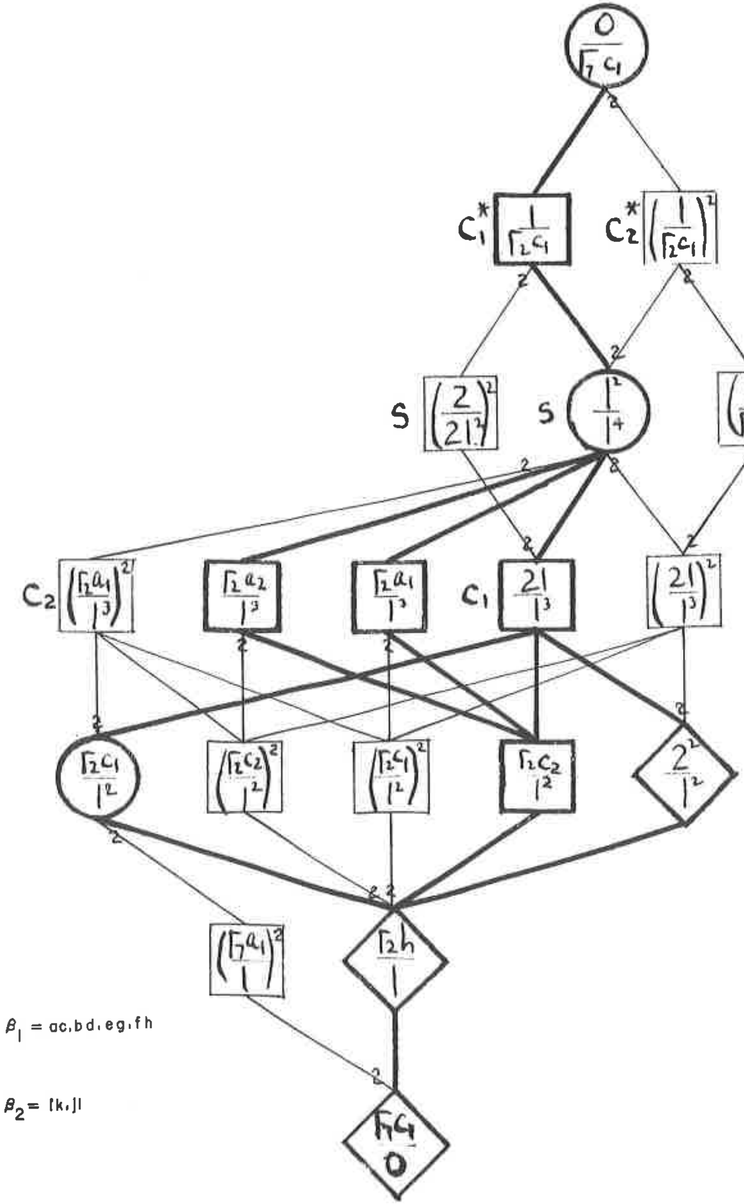
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = ac.bd.ik.jl

α_3 = abcd.efgh.ijkl.mnop

α_4 = bd.fh.jl. np

α_5 = aebfcgdh.imjnkolp



α_1 = ac.bd.eg.fh

α_2 = ac.bd

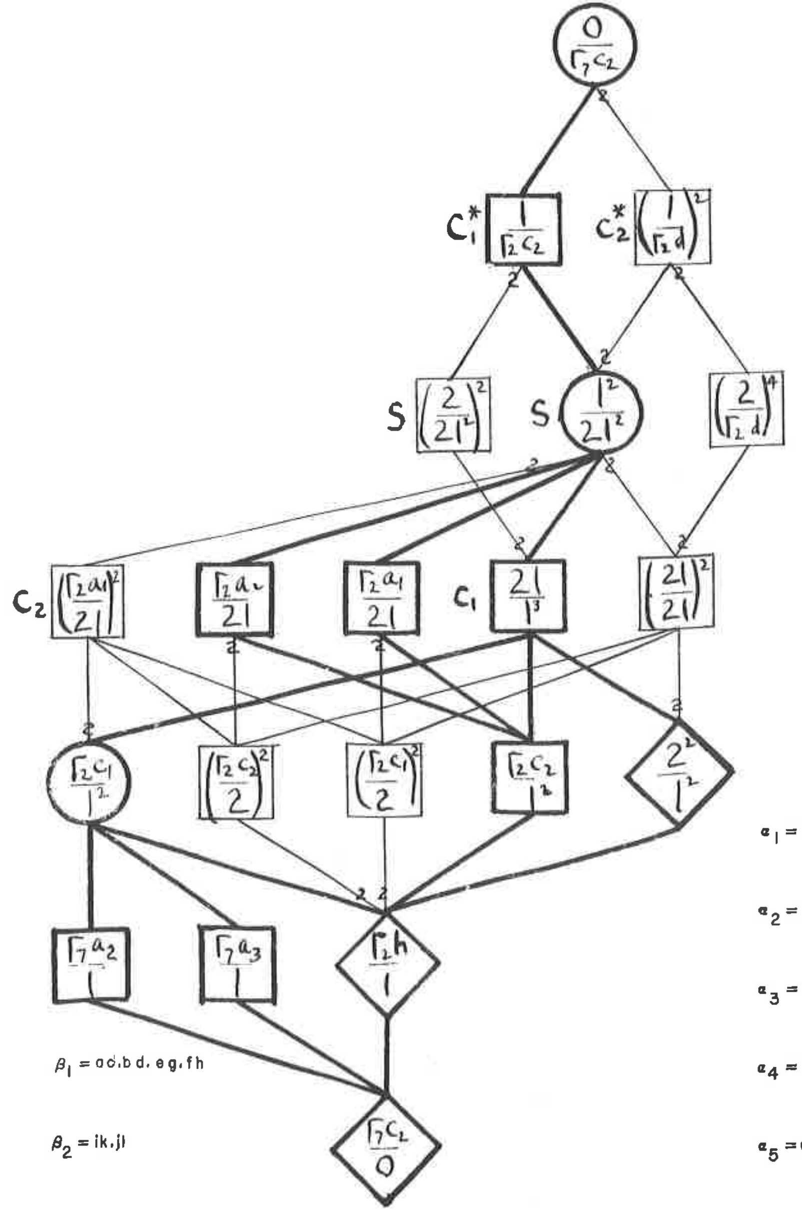
α_3 = ab.cd.ef.gh

α_4 = bd.fh.ljkl

α_5 = aebf.cgdh

β_1 = ac.bd.eg.fh

β_2 = lk.jl



α_1 = ac.bd.eg.fh

α_2 = ac.bd

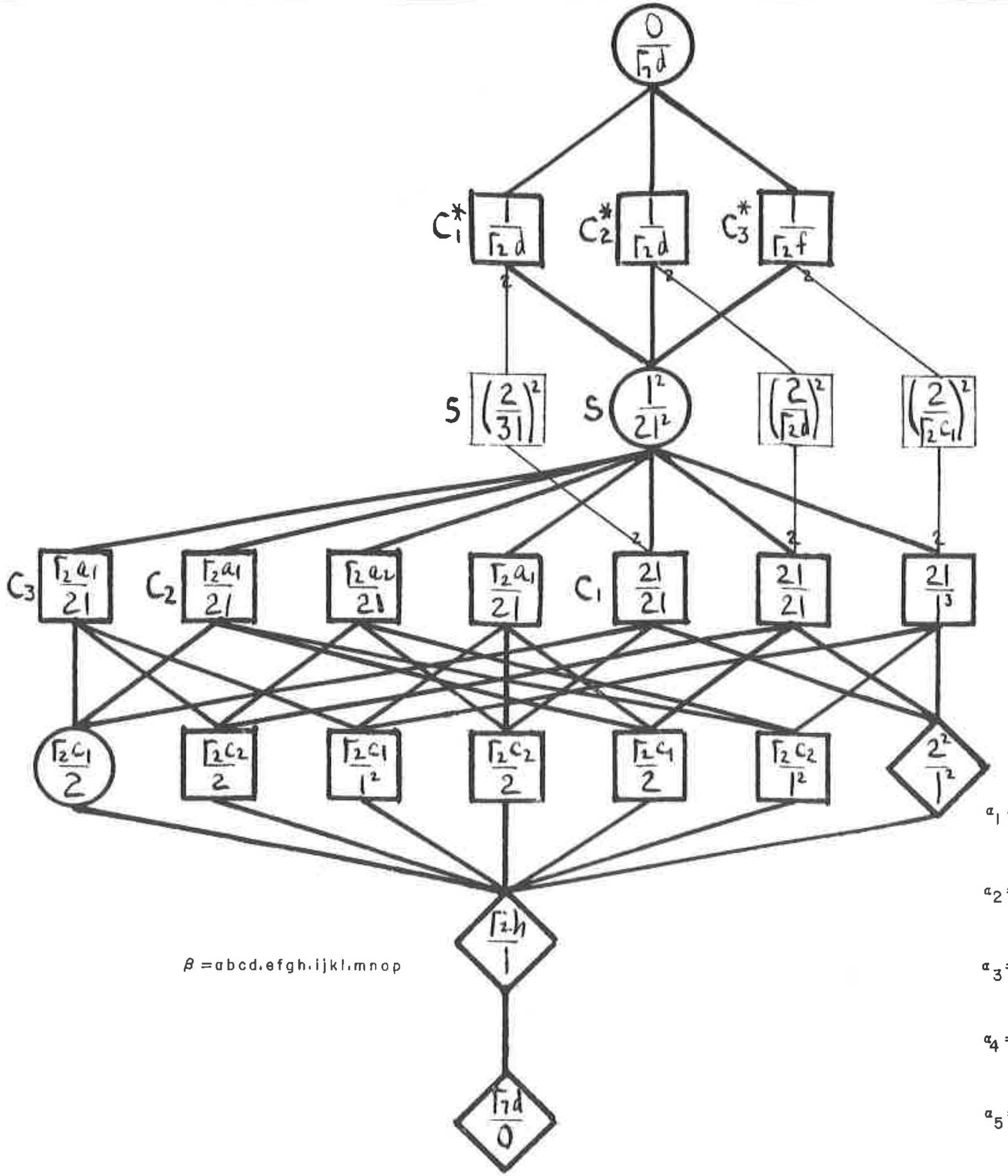
α_3 = abcd.efgh

α_4 = bd.fh.ljkl

α_5 = aebfcgdh

β_1 = ac.bd.eg.fh

β_2 = ik.jl



β = abcd.efgh.ijkl.mnop

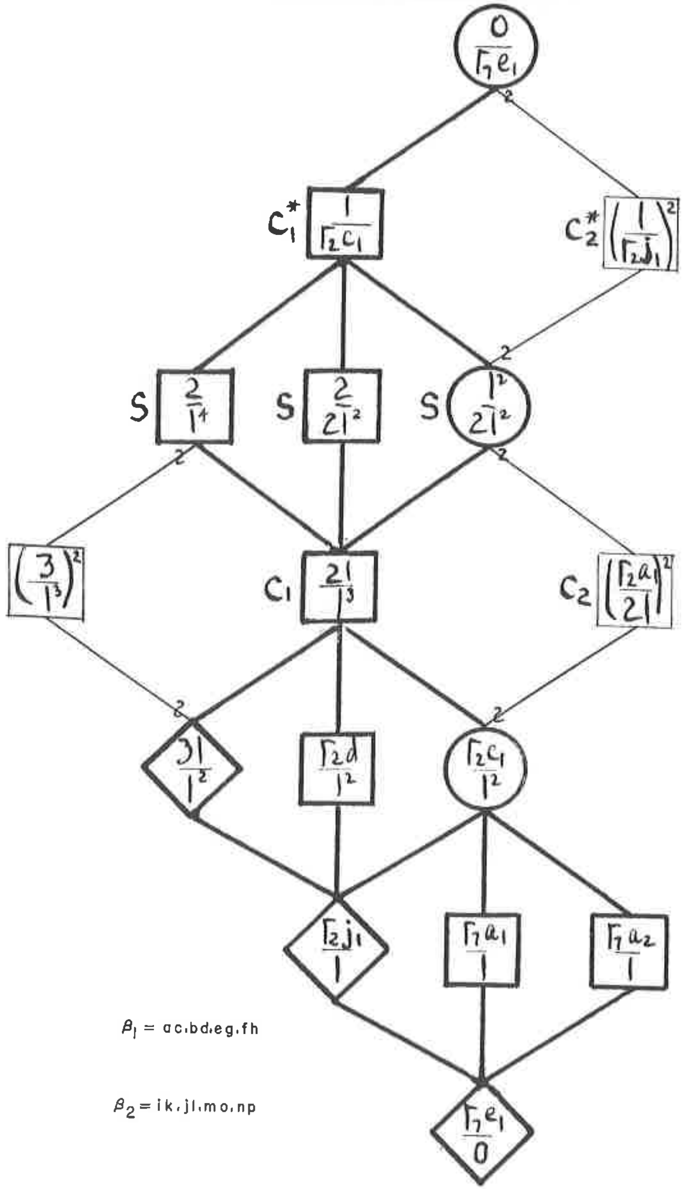
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = ai.bj.ck.dl.em.fn.go.hp

α_3 = eg.fh.mo.np

α_4 = aebfcgdh.imjnkolp

α_5 = ik.jl.emgo.fnhp



β_1 = ac.bd.eg.fh

β_2 = ik.jl.mo.np

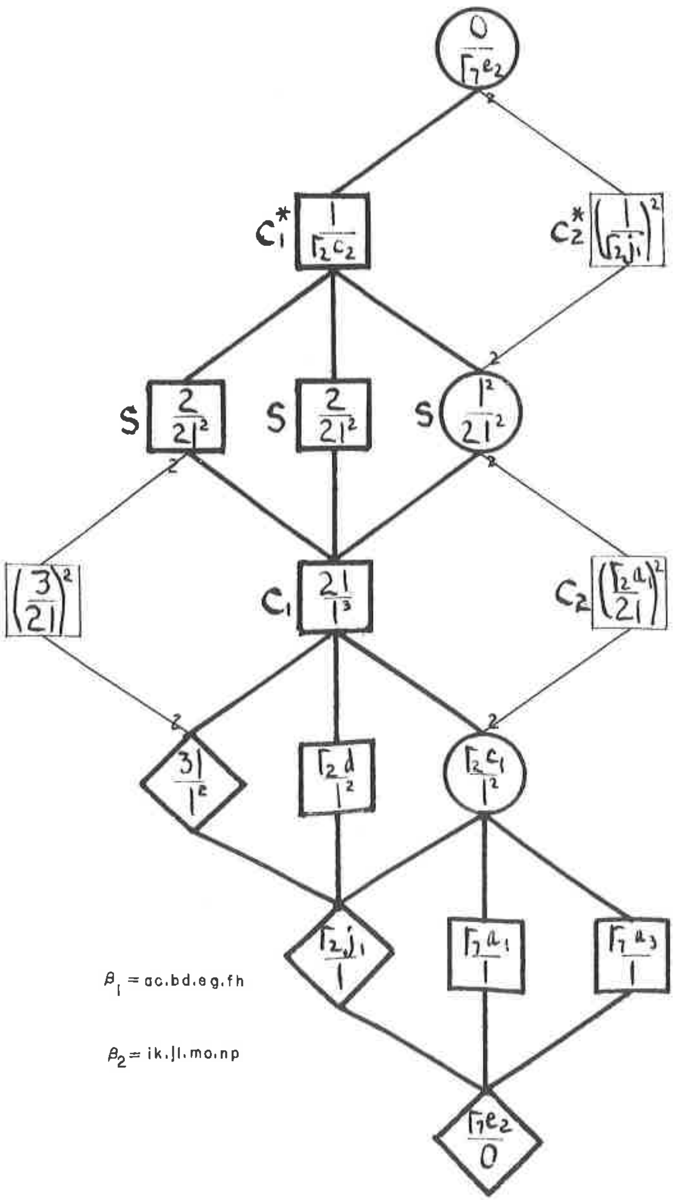
α_1 = ac.bd.eg.fh

α_2 = ac.bd

α_3 = ab.cd.ef.gh.ij kl.mn op

α_4 = bd . fh

α_5 = aebf.cgdh.imjnkolp



β_1 = ac.bd.eg.fh

β_2 = ik.jl.mo.np

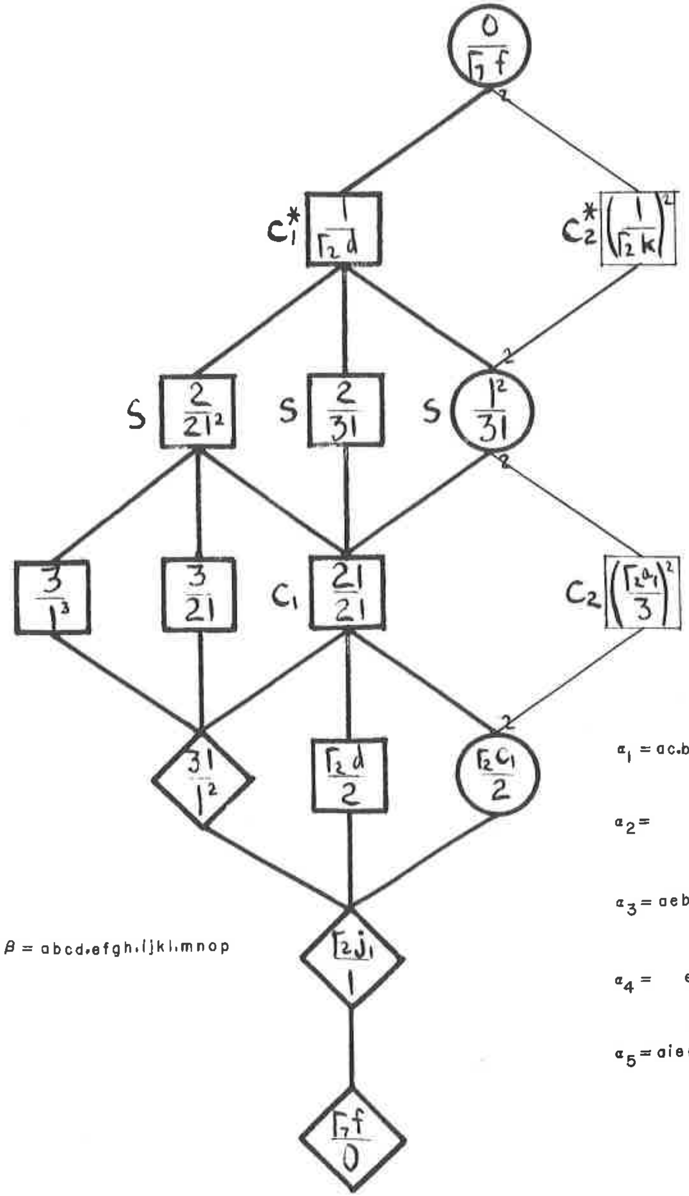
α_1 = ac.bd.eg.fh

α_2 = ac.bd

α_3 = ab.cd.ef.gh.ijkl.mnop

α_4 = ab.cd.efgh

α_5 = aebf.cgdh.imjnkolp



β = abcd.efgh.ijkl.mnop

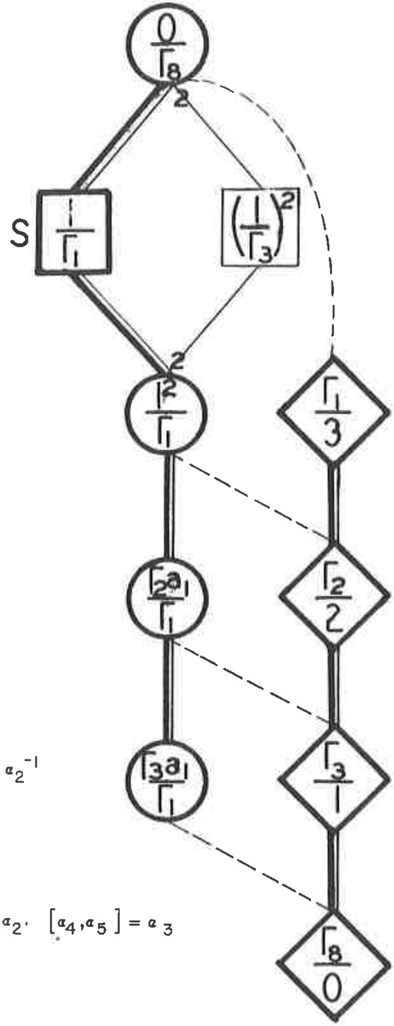
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = ikjl.mo.np

α_3 = aebfcgdh.imjnkolp

α_4 = eg.fh.ik.jl

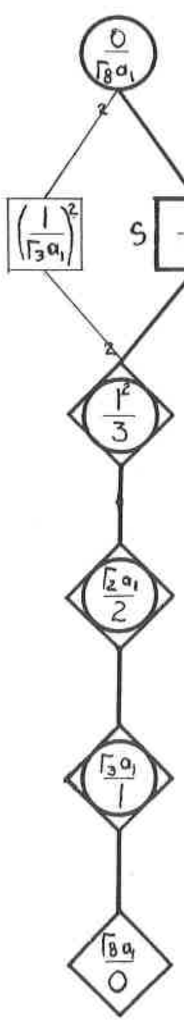
α_5 = aiembjfnckgodlhp



$\alpha_1^{\,2} = 1 \qquad \alpha_2^{\,2} = \alpha_1 \qquad \alpha_3^{\,2} = \alpha_2^{-1}$

$\alpha_4^{\,2} \equiv \alpha_3^{-1} \qquad \alpha_5^{\,2} \equiv 1$

$[\alpha_2, \alpha_5] = \alpha_1 \cdot \quad [\alpha_3, \alpha_5] = \alpha_2 \cdot \quad [\alpha_4, \alpha_5] = \alpha_3$



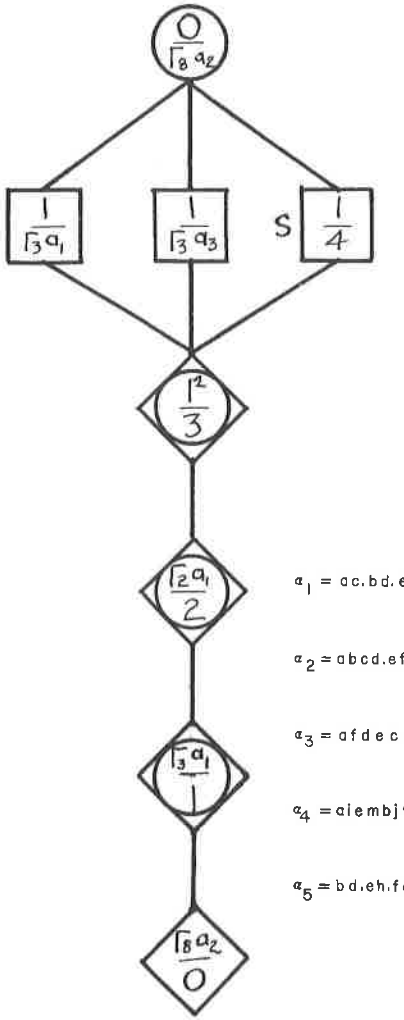
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = abcd.efgh.ijkl.mnop

α_3 = ahdgcfbe.iploknjm

α_4 = aiembjfnckgodlhp

α_5 = bd.eh.fg.ip.jo.kn.lm



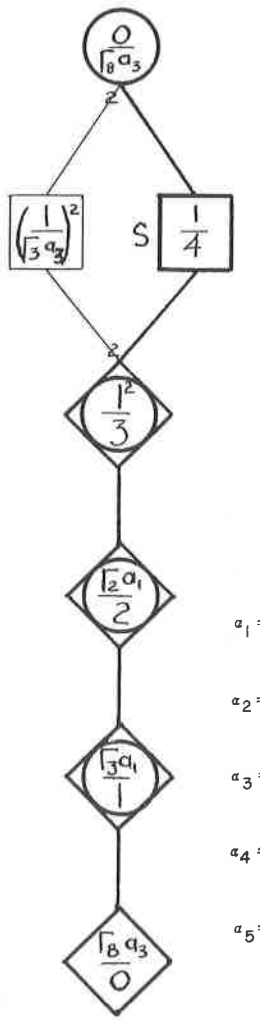
α_1 = ac.bd.eg.fh.ik.jl.mo.np

α_2 = abcd.efgh.ijkl.mnop

α_3 = afdechbg.inlmkpjo

α_4 = aiembjfnckgodlhp

α_5 = bd.eh.fg.in.lm.kp.jo



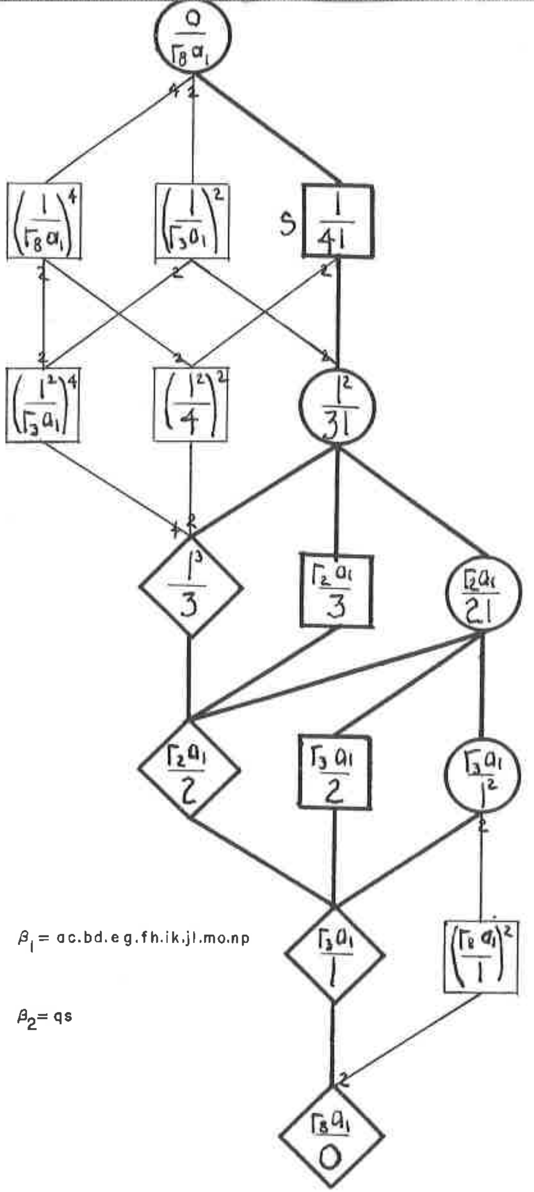
α_1 = ac.bd.eg.fh.ik.jl.mo.np.AC.BD.EG.FH.IK.JL.MO.NP

α_2 = abcd.efgh.ijkl.mnop.ABCD.EFGH.IJKL.MNOP

α_3 = ahdgcfbe.iploknjm.AHDGCFBE.IPLOKNJM

α_4 = aiembjfnckgodlhp.AIEMBJFNCKGODLHP

α_5 = aAcC.bDdB.eHgF.fGhE.iPkN.jOlM.mLoJ.nKpI



$\beta_1 =$ ac.bd.eg.fh.ik.jl.mo.np

$\beta_2 =$ qs

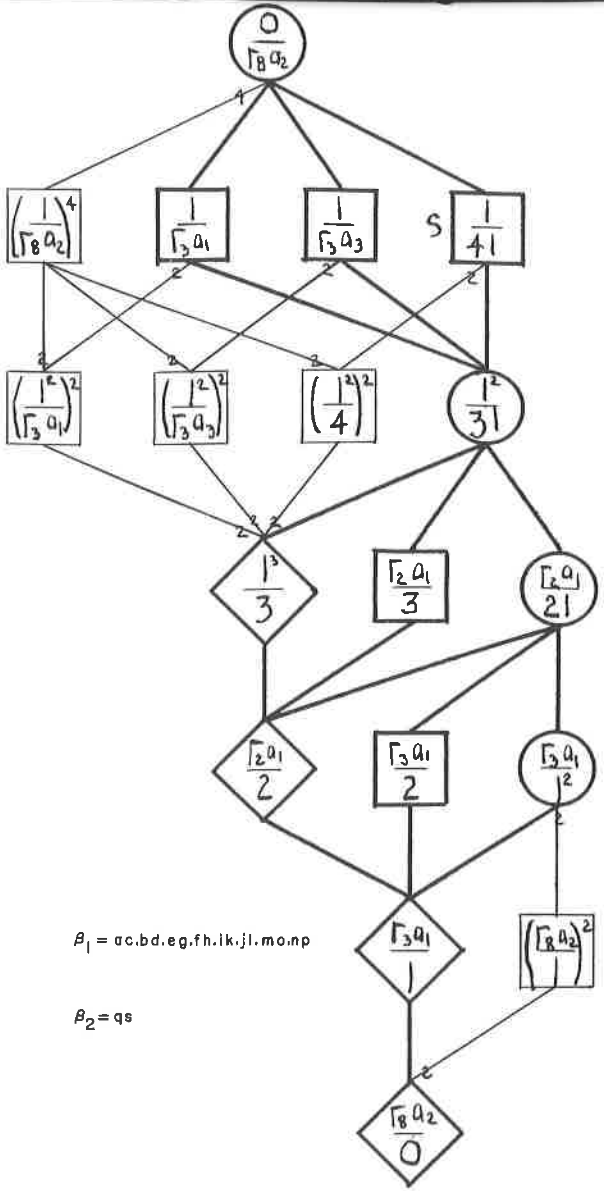
$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

$\alpha_2 =$ abcd.efgh.ijkl.mnop

$\alpha_3 =$ ahdgcfbe.iploknjm

$\alpha_4 =$ aiembjfnckgodlhp

$\alpha_5 =$ bd.eh.fg.ip.jo.kn.lm



$\beta_1 =$ ac.bd.eg.fh.ik.jl.mo.np

$\beta_2 =$ qs

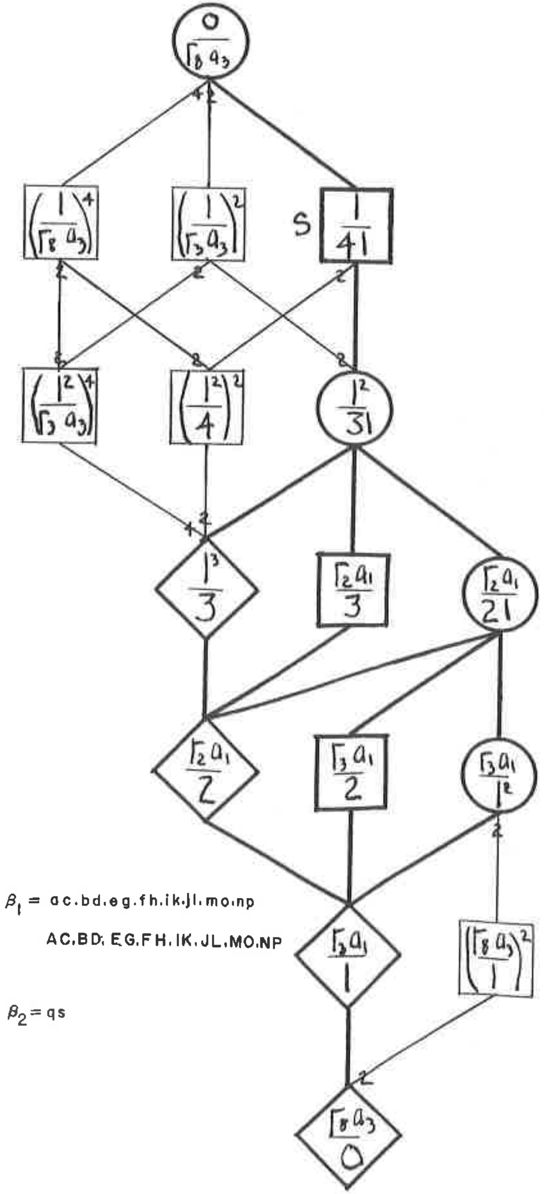
$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

$\alpha_2 =$ abcd.efgh.ijkl.mnop

$\alpha_3 =$ afdechbg.inlmkpjo

$\alpha_4 =$ aiembjfnckgodlhp

$\alpha_5 =$ bd.eh.fg.in.jm.kp.lo



β_1 = ac.bd.eg.fh.ik.jl.mo.np

AC.BD.EG.FH.IK.JL.MO.NP

β_2 = qs

α_1 = ac.bd.eg.fh.ik.jl.mo.np

AC.BD.EG.FH.IK.JL.MO.NP

α_2 = abcd.efgh.ijkl.mnop

ABCD.EFGH.IJKL.MNOP

α_3 = ahdgcfbe.iploknjm

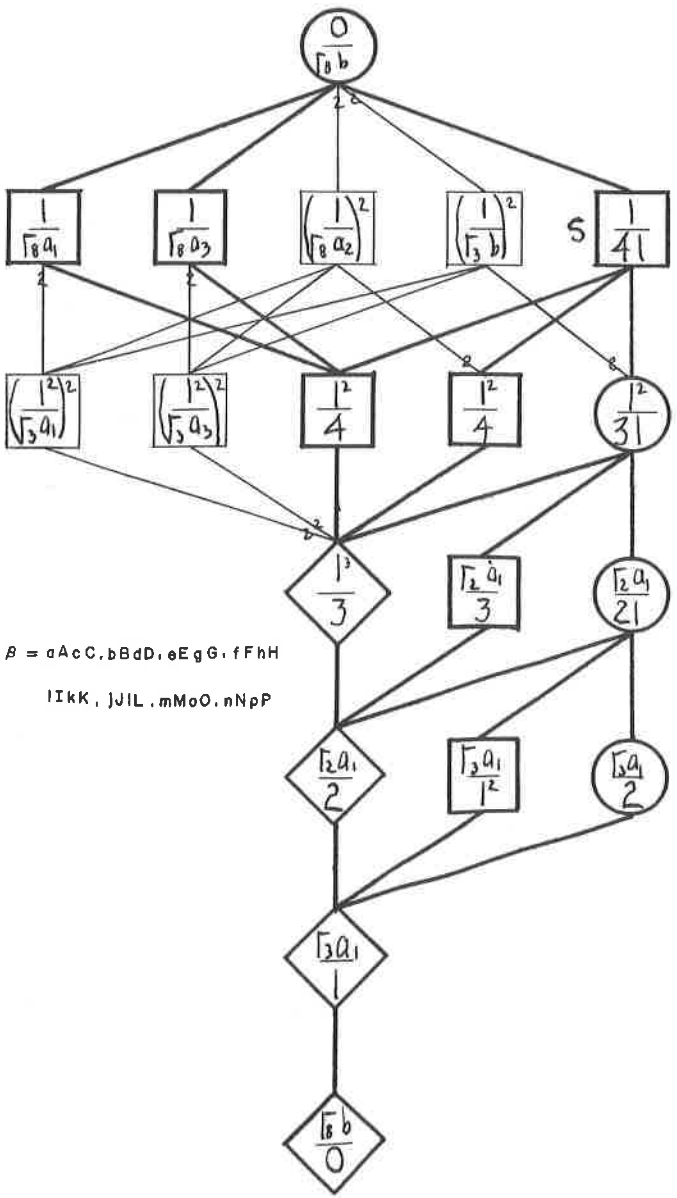
AHDGCFBE.IPLOKNJM

α_4 = aiembjfnckgodlhp

AIEMBJFNCKGODLHP

α_5 = aAcC.bDdB.eHgF.fGhE

iPkN.jOIM.mLoJ.nKpI



β = aAcC.bBdD.eEgG.fFhH

IIkK.JJIL.mMoO.nNpP

α_1 = ac.bd.eg.fh.ik.jl.mo.np

AC.BD.EG.FH.IK.JL.MO.NP

α_2 = abcd.efgh.ijkl.mnop

ABCD.EFGH.IJKL.MNOP

α_3 = ahdgcfbe.iploknjm

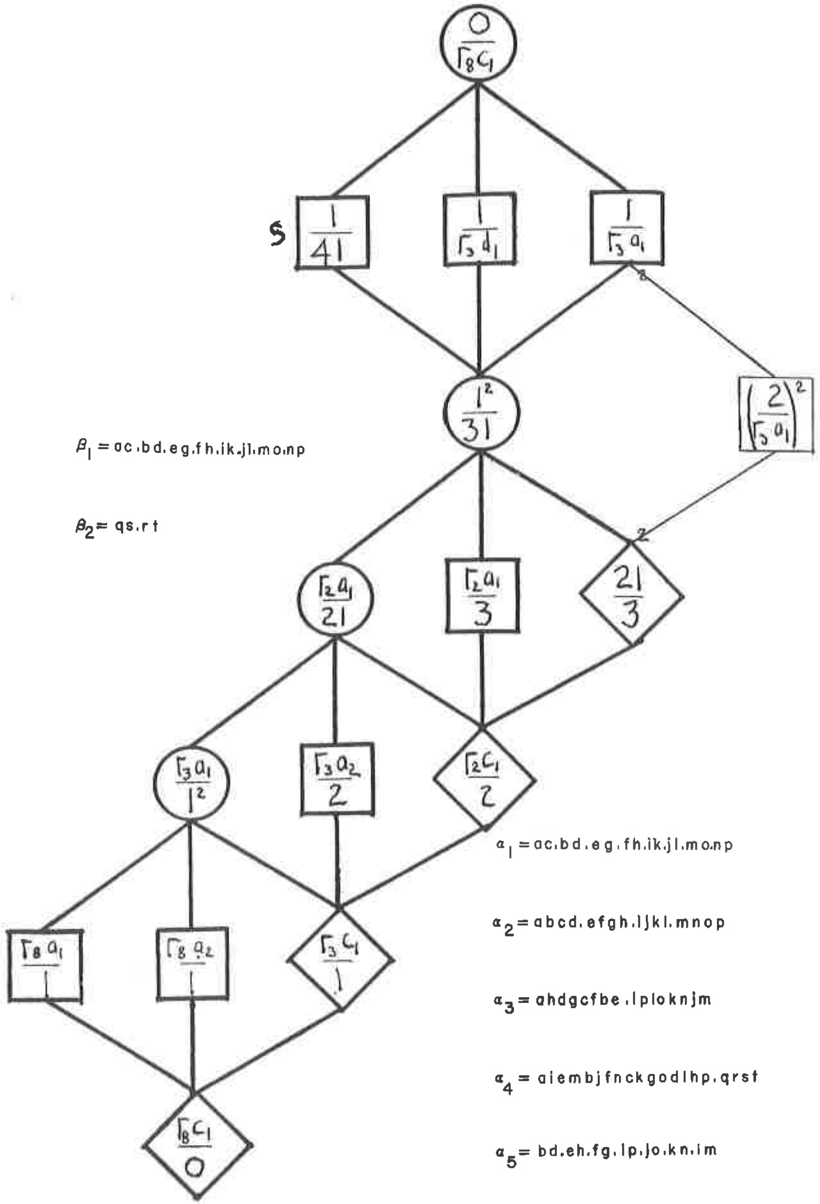
AHDGCFBE.IPLOKNJM

α_4 = aiembjfnckgodlhp

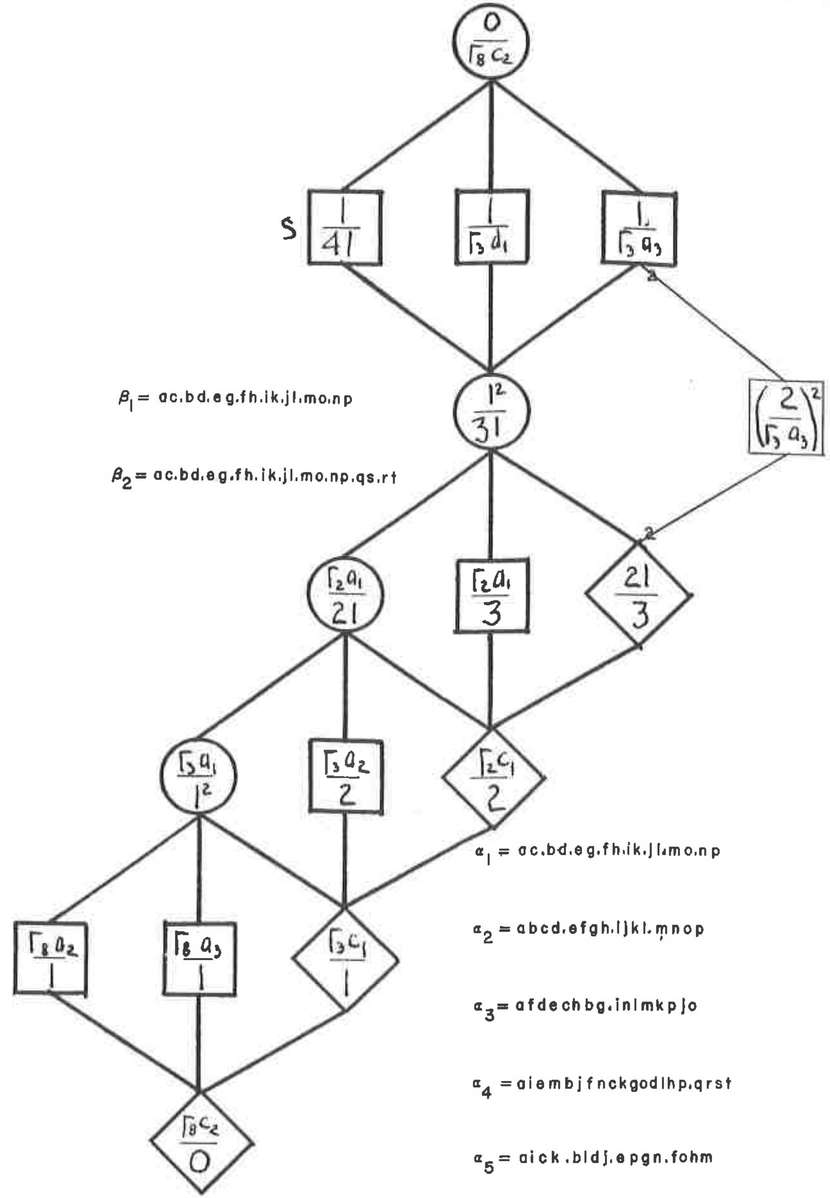
AIEMBJFNCKGODLHP

α_5 = bd.eh.fg.ip.jo.kn.lm

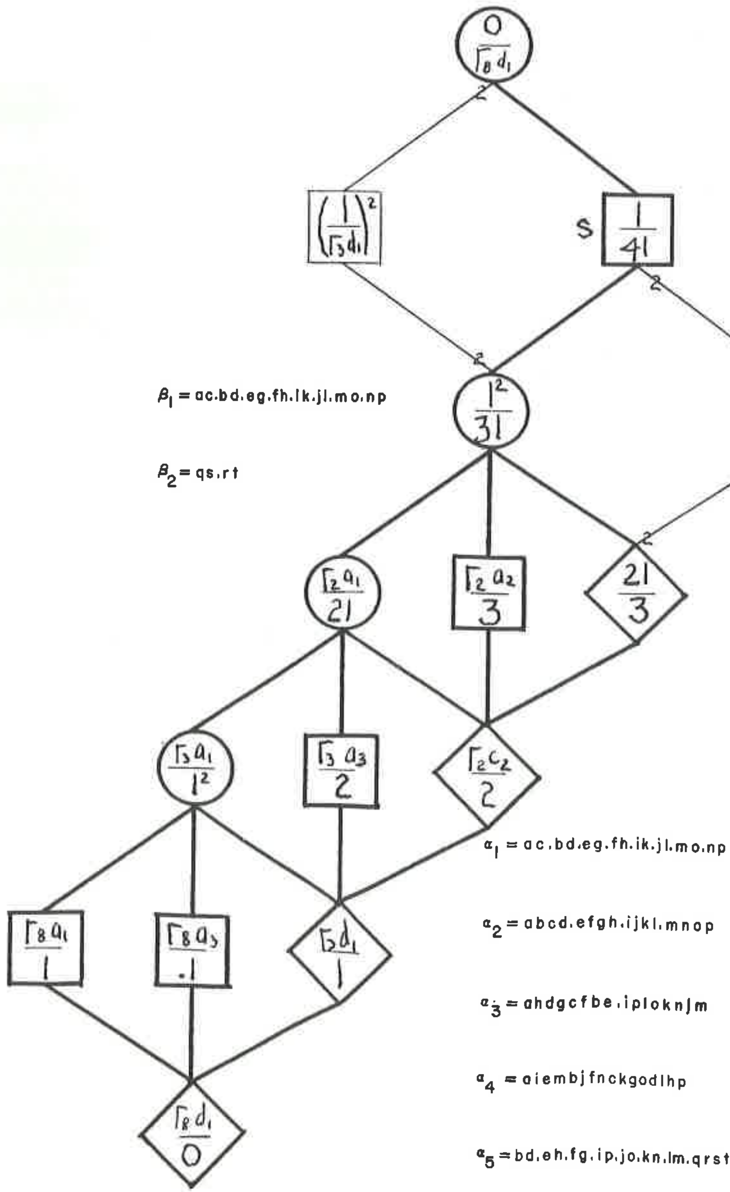
BD.EH.FG.IP.JO.KN.LM

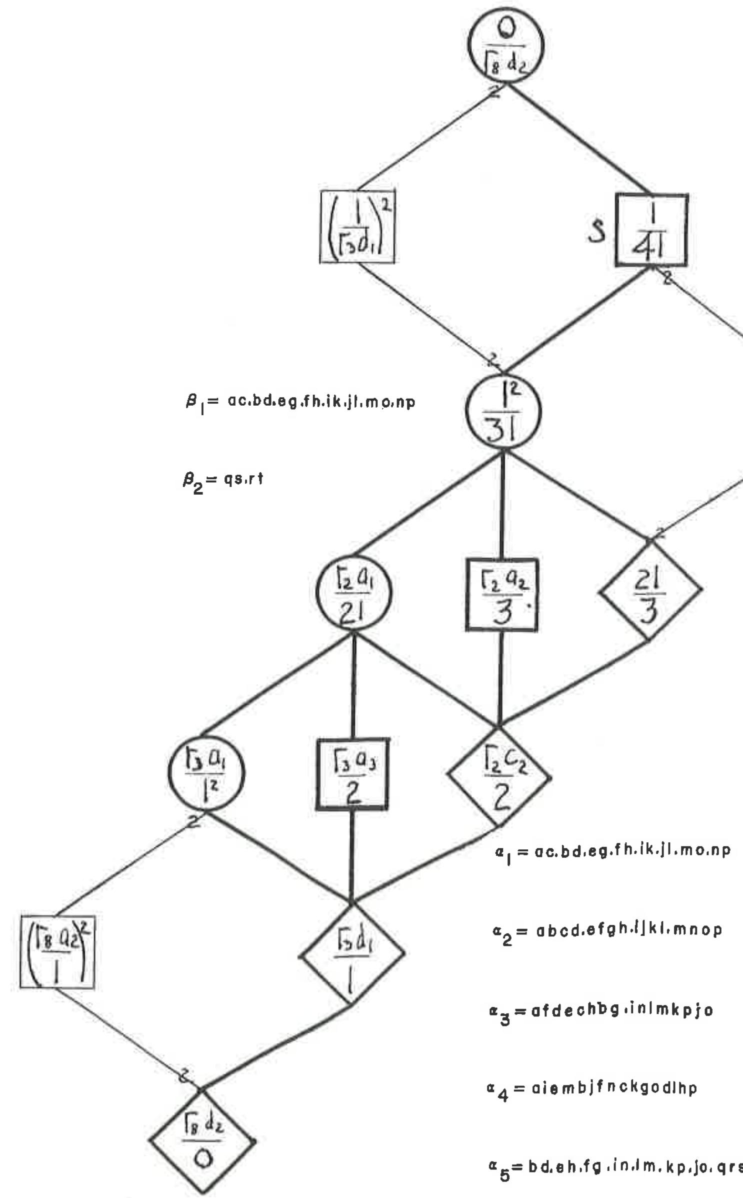


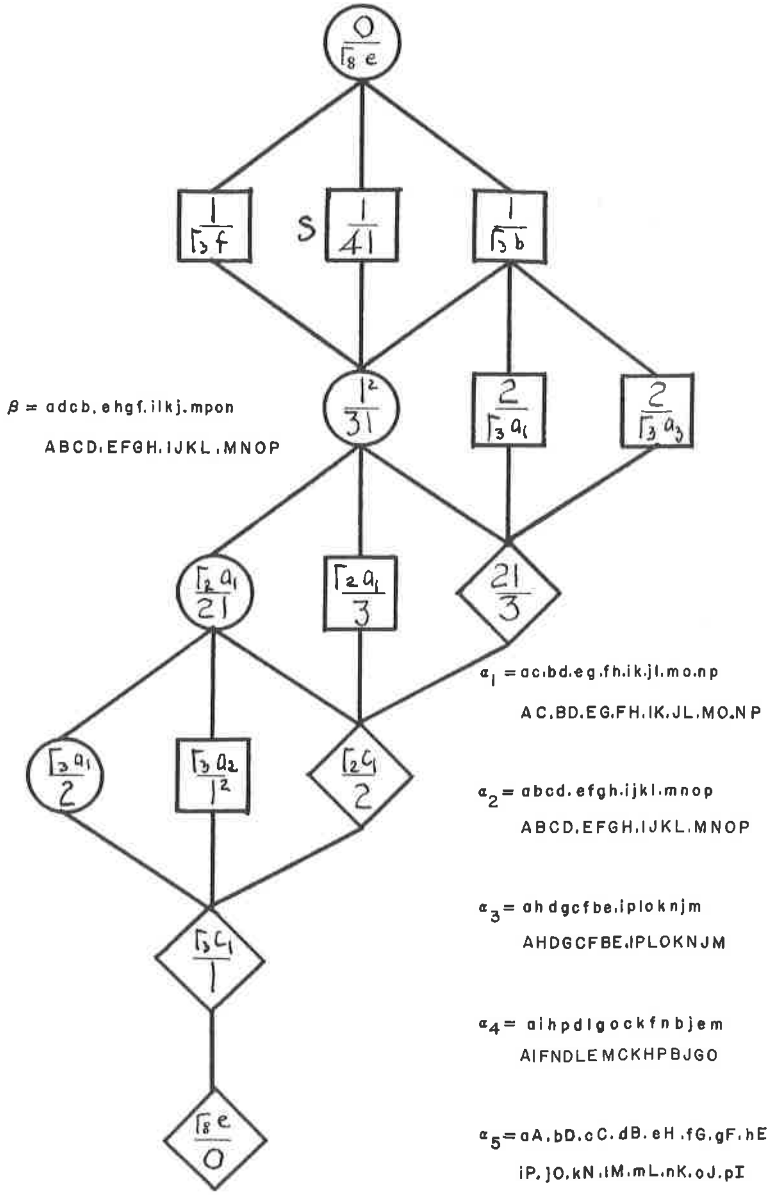
S
β_1 = ac.bd.eg.fh.ik.jl.mo.np
β_2 = qs.rt
α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = abcd.efgh.ljkl.mnop
α_3 = ahdgcfbe.lploknjm
α_4 = aiembjfnckgodlhp.qrst
α_5 = bd.eh.fg.lp.jo.kn.lm



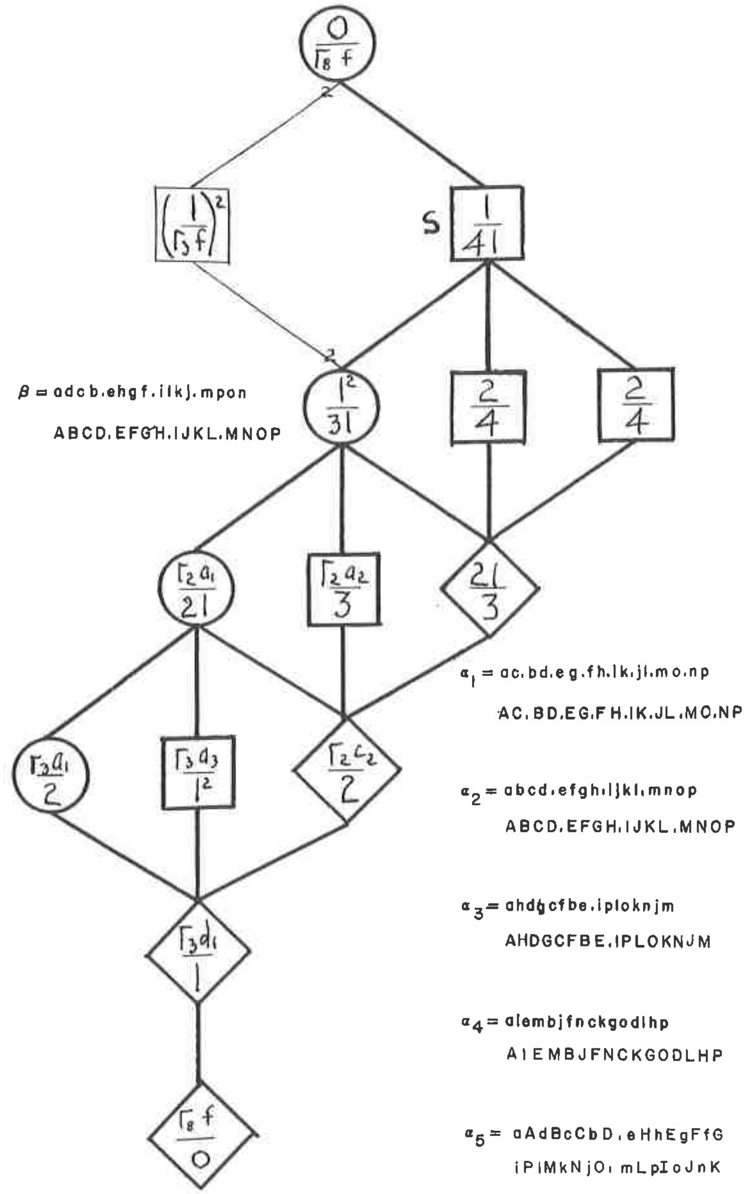
S
β_1 = ac.bd.eg.fh.ik.jl.mo.np
β_2 = ac.bd.eg.fh.ik.jl.mo.np.qs.rt
α_1 = ac.bd.eg.fh.ik.jl.mo.np
α_2 = abcd.efgh.ljkl.mnop
α_3 = afdechbg.inlmkpjo
α_4 = aiembjfnckgodlhp.qrst
α_5 = aick.bldj.epgn.fohm


β1 = ac.bd.eg.fh.ik.jl.mo.np
β2 = qs.rt
α1 = ac.bd.eg.fh.ik.jl.mo.np
α2 = abcd.efgh.ijkl.mnop
α3 = ahdgcfbe.iploknjm
α4 = aiembjfnckgodlhp
α5 = bd.eh.fg.ip.jo.kn.lm.qrst

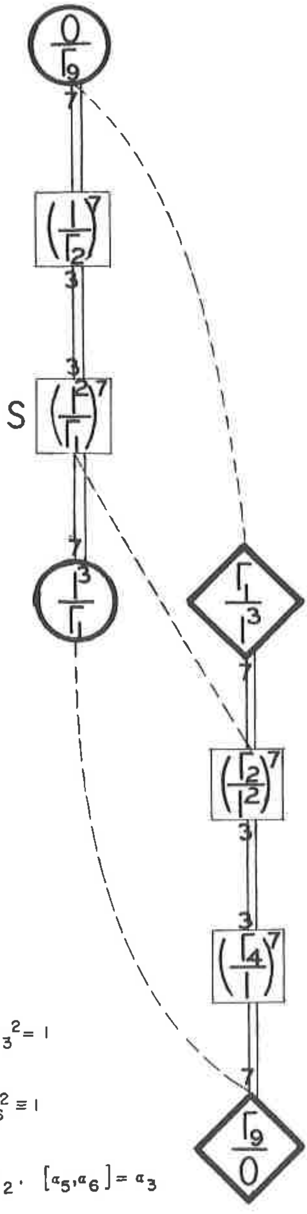

β1 = ac.bd.eg.fh.ik.jl.mo.np
β2 = qs.rt
α1 = ac.bd.eg.fh.ik.jl.mo.np
α2 = abcd.efgh.ijkl.mnop
α3 = afdechbg.inlmkpjo
α4 = aiembjfnckgodlhp
α5 = bd.eh.fg.in.lm.kp.jo.qrst



0 / Γ8 e
1 / Γ3 f
S 1 / 41
1 / Γ3 b
1² / 31
2 / Γ3 a1
2 / Γ3 a3
Γ2 a1 / 21
Γ2 a1 / 3
21 / 3
Γ3 a1 / 2
Γ3 a2 / 1²
Γ2 c1 / 2
Γ3 c1 / 1
Γ8 e / 0
β = adcb, ehgf. ilkj. mpon
ABCD, EFGH. IJKL. MNOP
α1 = ac, bd. eg. fh. ik. jl. mo. np
AC. BD. EG. FH. IK. JL. MO. NP
α2 = abcd. efgh. ijkl. mnop
ABCD. EFGH. IJKL. MNOP
α3 = ahdgcfbe. iploknjm
AHDGCFBE. IPLOKNJM
α4 = aihpdlgockfnbjem
AIFNDLEMCKHPBJGO
α5 = aA. bD. cC. dB. eH. fG. gF. hE
iP. jO. kN. lM. mL. nK. oJ. pI



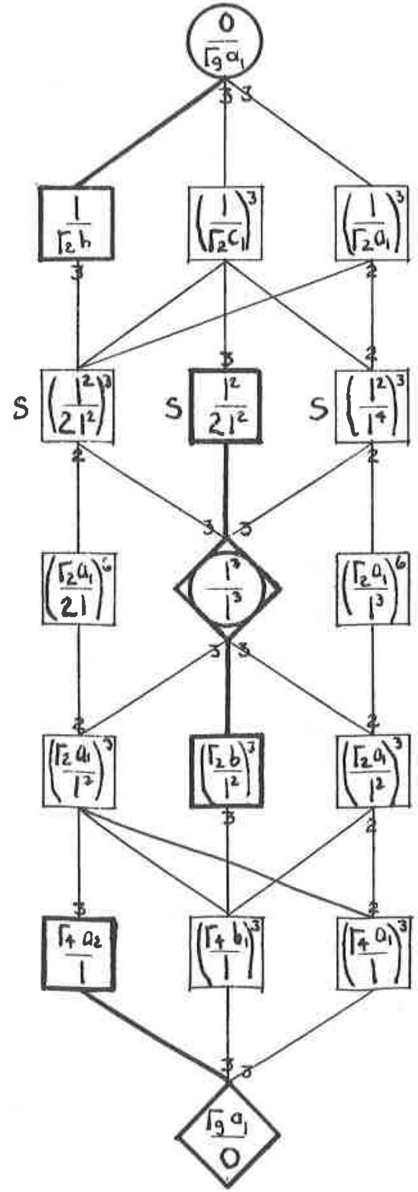
0 / Γ8 f
(1 / Γ3 f)²
S 1 / 41
1² / 31
2 / 4
2 / 4
Γ2 a1 / 21
Γ2 a2 / 3
21 / 3
Γ3 a1 / 2
Γ3 a3 / 1²
Γ2 c2 / 2
Γ3 d1 / 1
Γ8 f / 0
β = adcb. ehgf. ilkj. mpon
ABCD. EFGH. IJKL. MNOP
α1 = ac. bd. eg. fh. lk. jl. mo. np
AC. BD. EG. FH. IK. JL. MO. NP
α2 = abcd. efgh. ijkl. mnop
ABCD. EFGH. IJKL. MNOP
α3 = ahdgcfbe. iploknjm
AHDGCFBE. IPLOKNJM
α4 = aiembjfnckgodlhp
AIEMBJFNCKGODLHP
α5 = aAdBcCbD. eHhEgFfG
iPlMkNjO. mLpIoJnK



$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$

$\alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$

$[\alpha_4, \alpha_5] = \alpha_1, \quad [\alpha_4, \alpha_6] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3$



α_1 = ac, bd

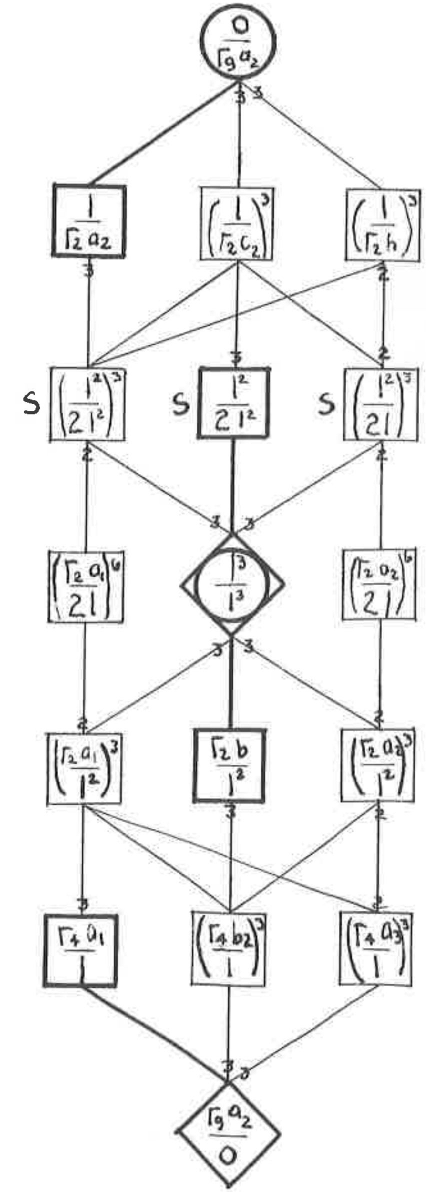
α_2 = eg, fh

α_3 = eg, fh, ik, jl

α_4 = ab, cd, fh

α_5 = bd, fh, jl

α_6 = ef, gh, ij, kl



α_1 = ac, bd, eg, fh

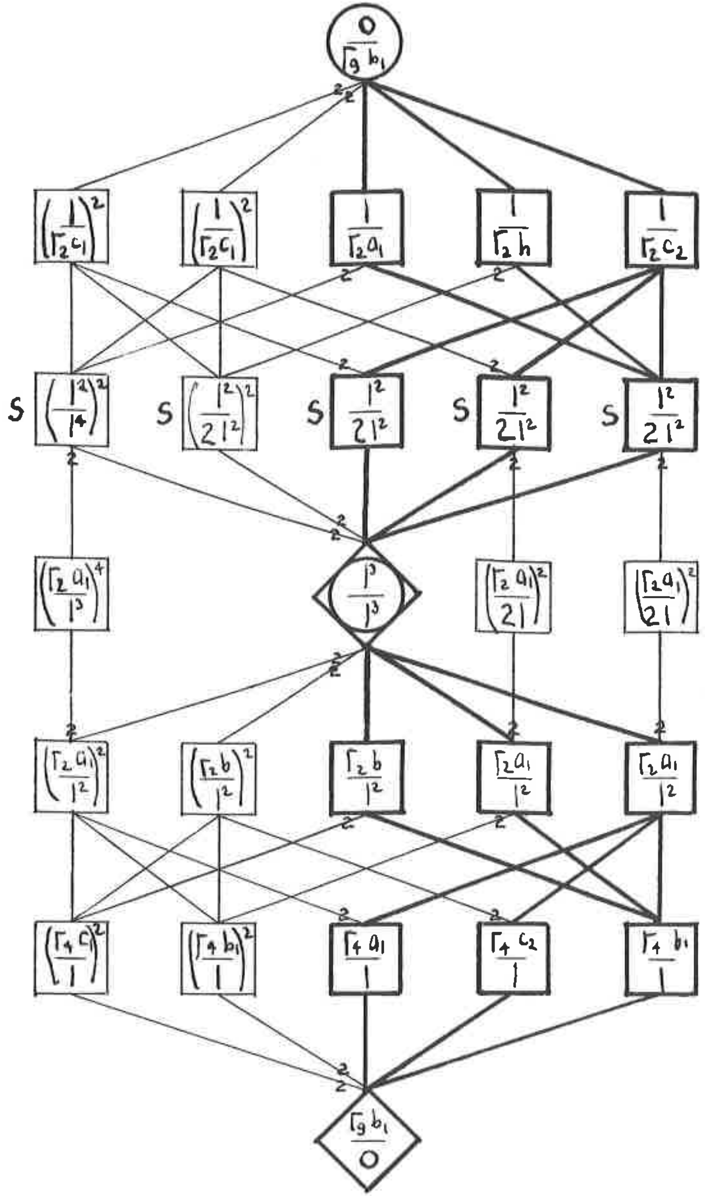
α_2 = ik, jl

α_3 = mo, np

α_4 = abcd, efgh, ij, kl

α_5 = aecg, bhdf, mn, op

α_6 = ijkl, mnop



α_1 = ac.bd

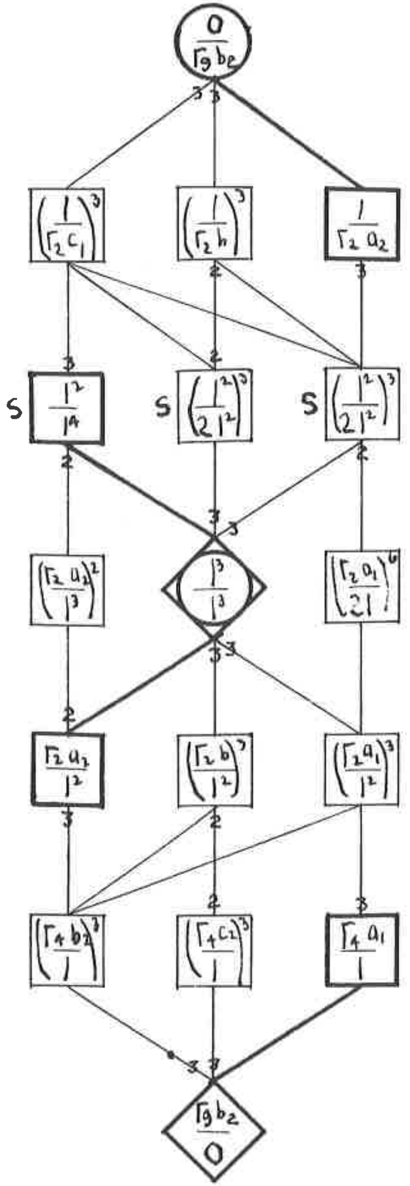
α_2 = eg.fh

α_3 = eg.fh.ik.jl

α_4 = bd.ef.gh

α_5 = ab.cd.ef.gh.ij.kl

α_6 = fh.ij.kl



α_1 = ac.bd.eg.fh

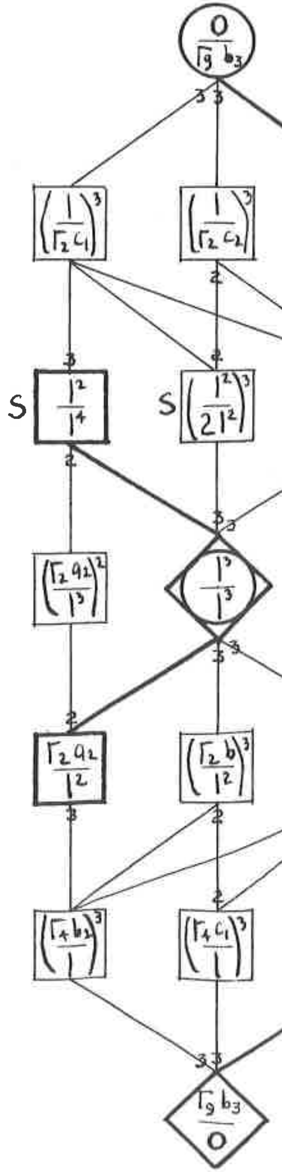
α_2 = ik.jl

α_3 = mo.np

α_4 = abcd.efgh.ij.kl

α_5 = aecg.bhdf.mn.op

α_6 = jl.np



α_1 = ac. bd. eg. fh

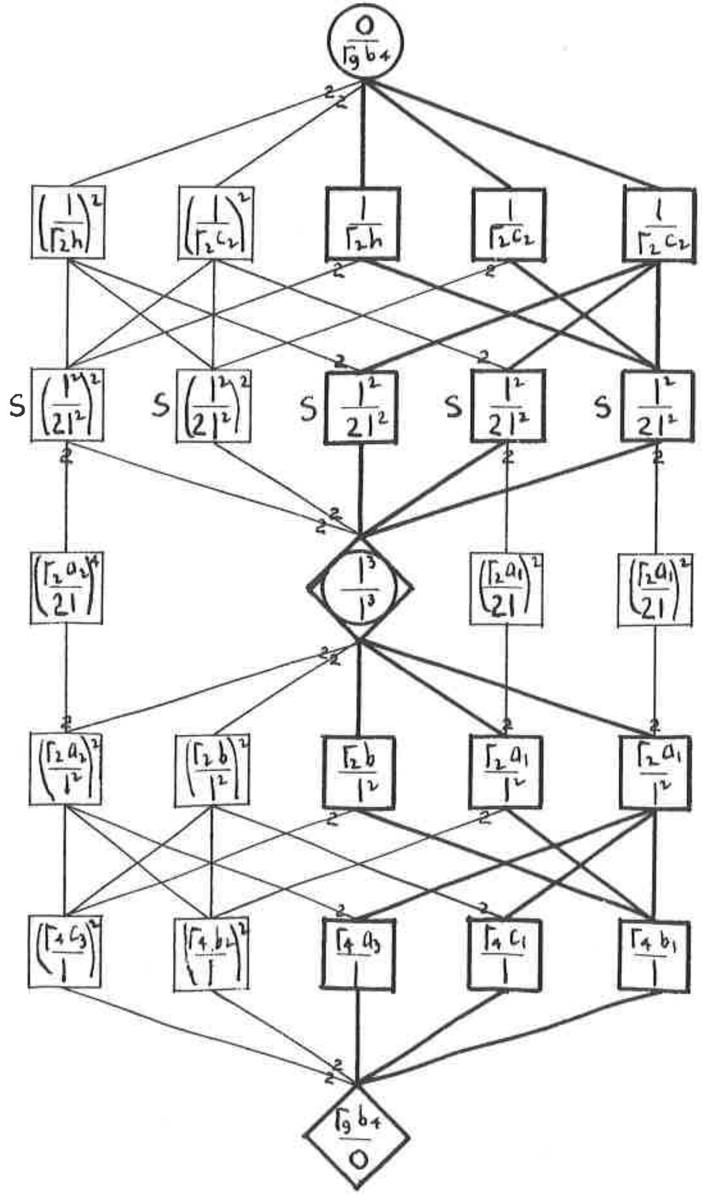
α_2 = ik. jl

α_3 = mo. np

α_4 = abcd. efgh. ijkl

α_5 = aecg. bhdf. mnop

α_6 = jl. np



α_1 = ac. bd

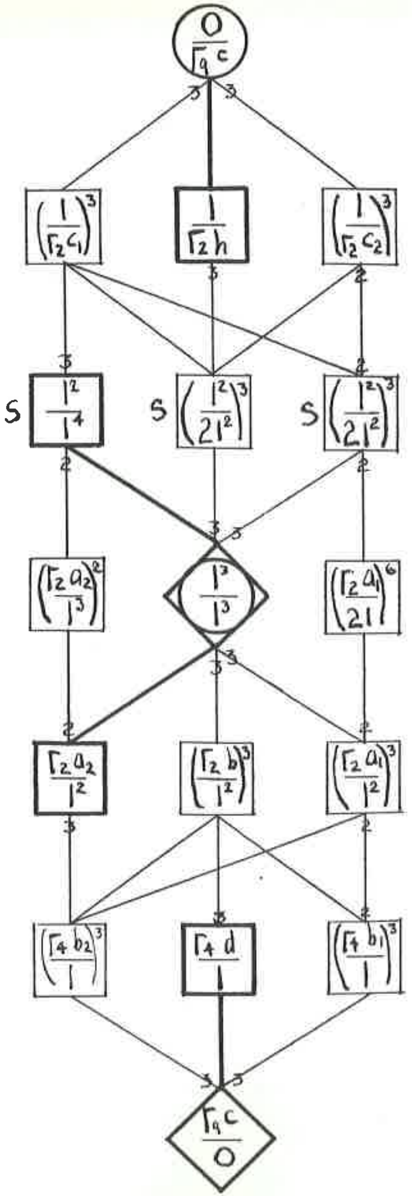
α_2 = eg. fh

α_3 = ik. jl. mo. np

α_4 = bd. efgh

α_5 = abcd. ijkl. mnop

α_6 = fh. imko. jpln



α_1 = ac.bd

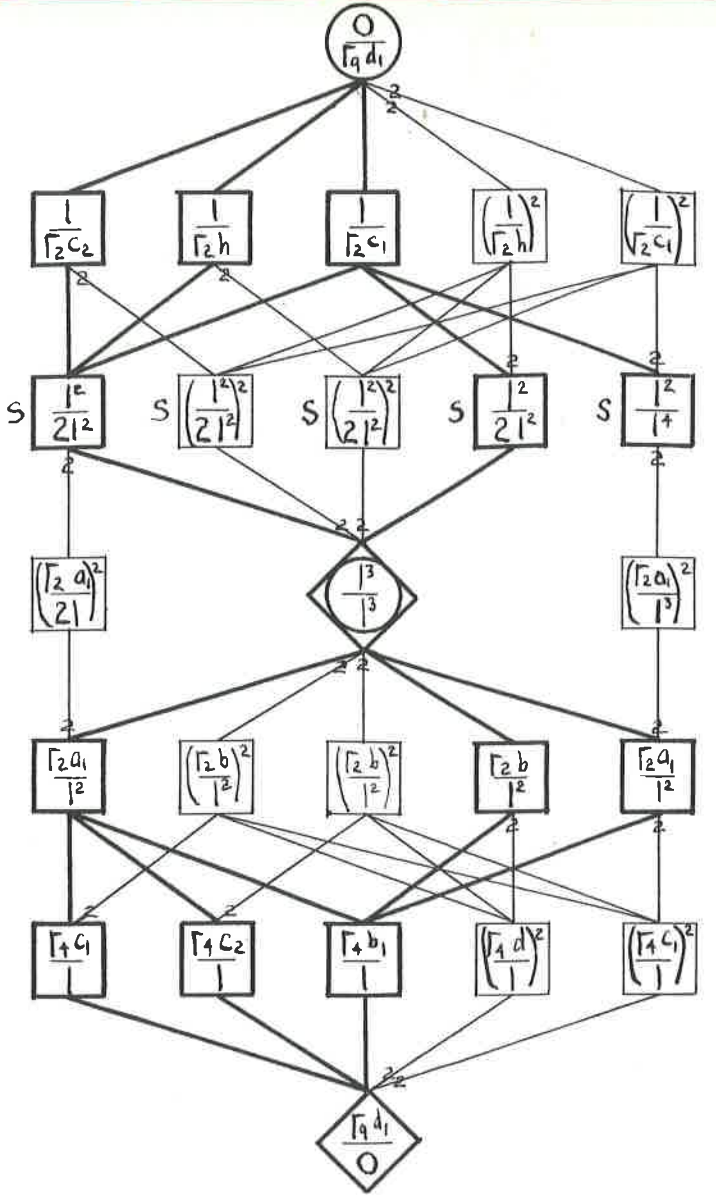
α_2 = eg.fh

α_3 = ik.jl

α_4 = abcd.ef.gh

α_5 = ab.cd. ijkl

α_6 = efgh ij.kl



α_1 = ac.bd

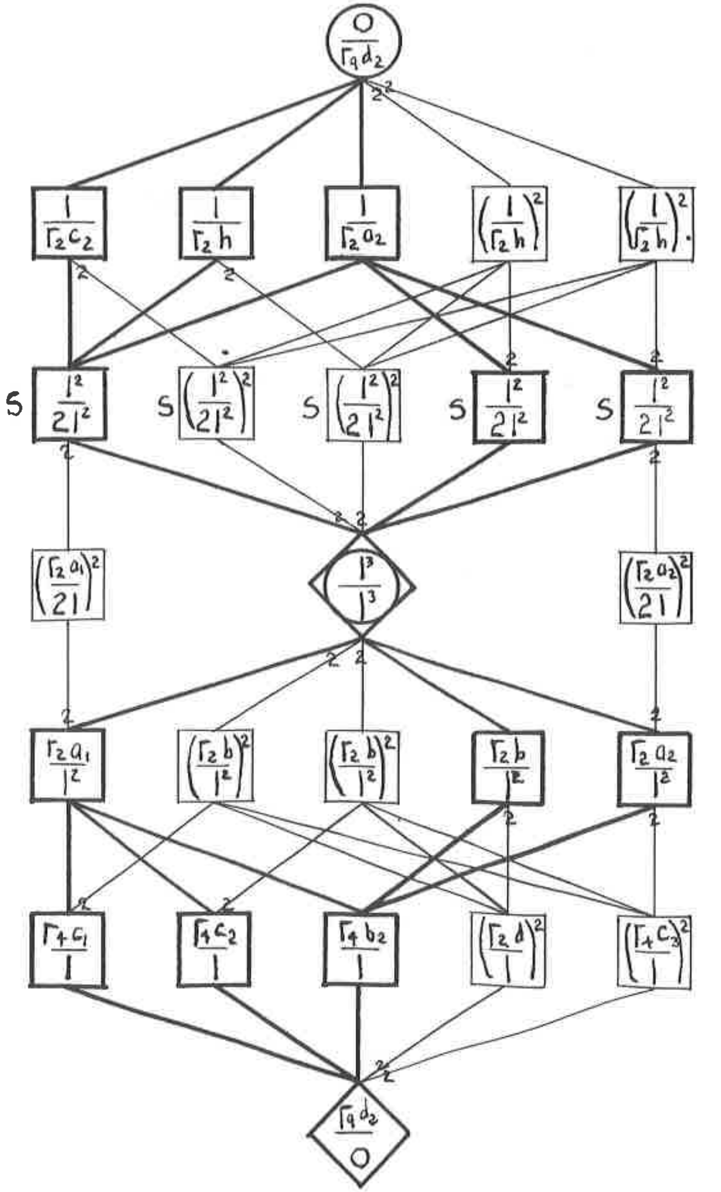
α_2 = ac.bd.ik.jl.mo.np

α_3 = eg.fh

α_4 = ab.cd.mo.np

α_5 = abcd.ef.gh.ijkl.mnop

α_6 = abcd.fh. im.jn.ko.lp



α_1 = ac.bd.eg.fh

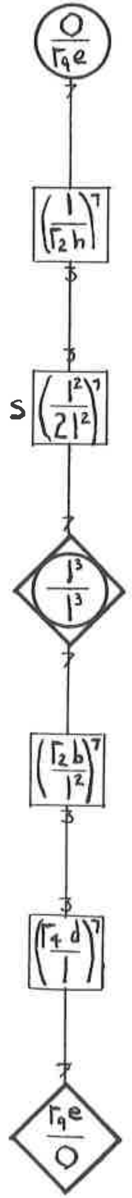
α_2 = ac.bd.eg.fh.ik.jl.mo.np

α_3 = ik.jl.mo.np.qs.rt

α_4 = abcd.efgh.ijkl.mpon

α_5 = aecg.bhdf. mo.np.qrst

α_6 = aecg.bhdf im.jn.kolp.rt



α_1 = ac.bd.eg.fh.ik.jl.mo.np

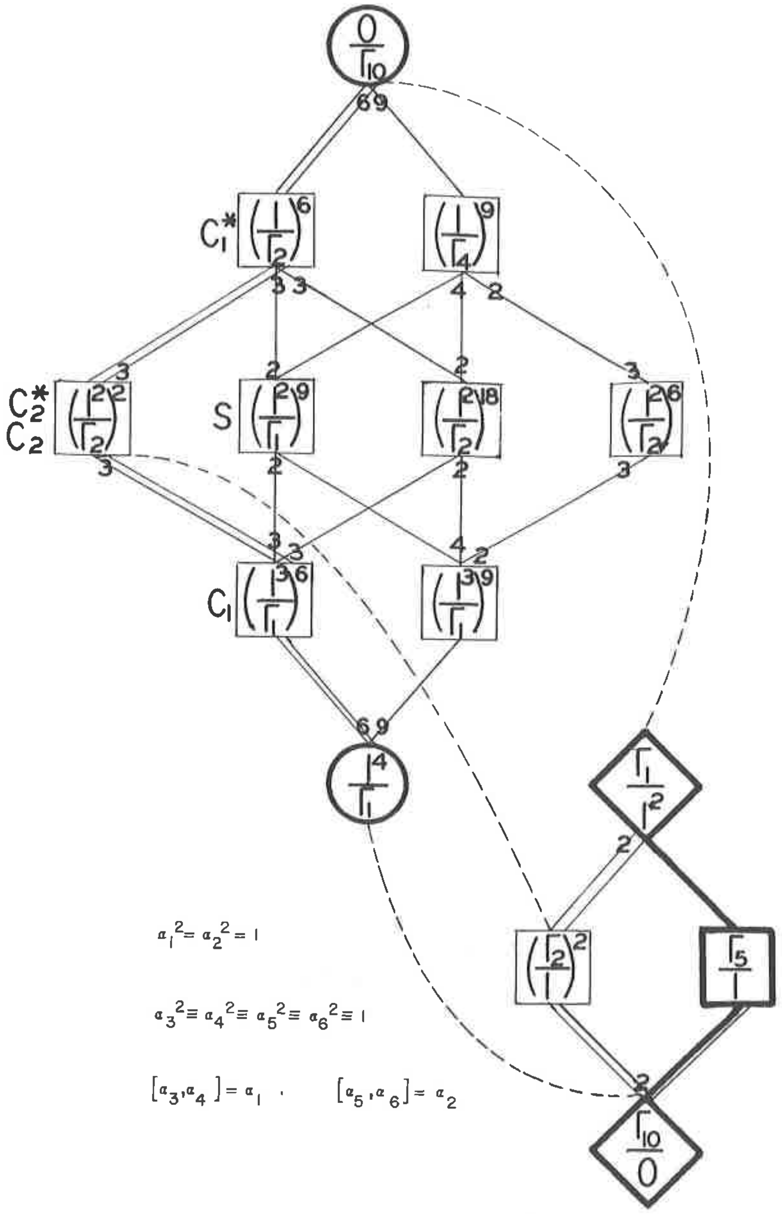
α_2 = ac.bd.eg.fh.ik.jl.mo.np.qs.rt.uw.vx

α_3 = ac.bd.eg.fh. qs.rt.uw.vx

α_4 = eg.fh. in kp.jolm.qvsx.rwtu

α_5 = af.bg.ch.de. imko. jnlp. qu.rv.sw.tx

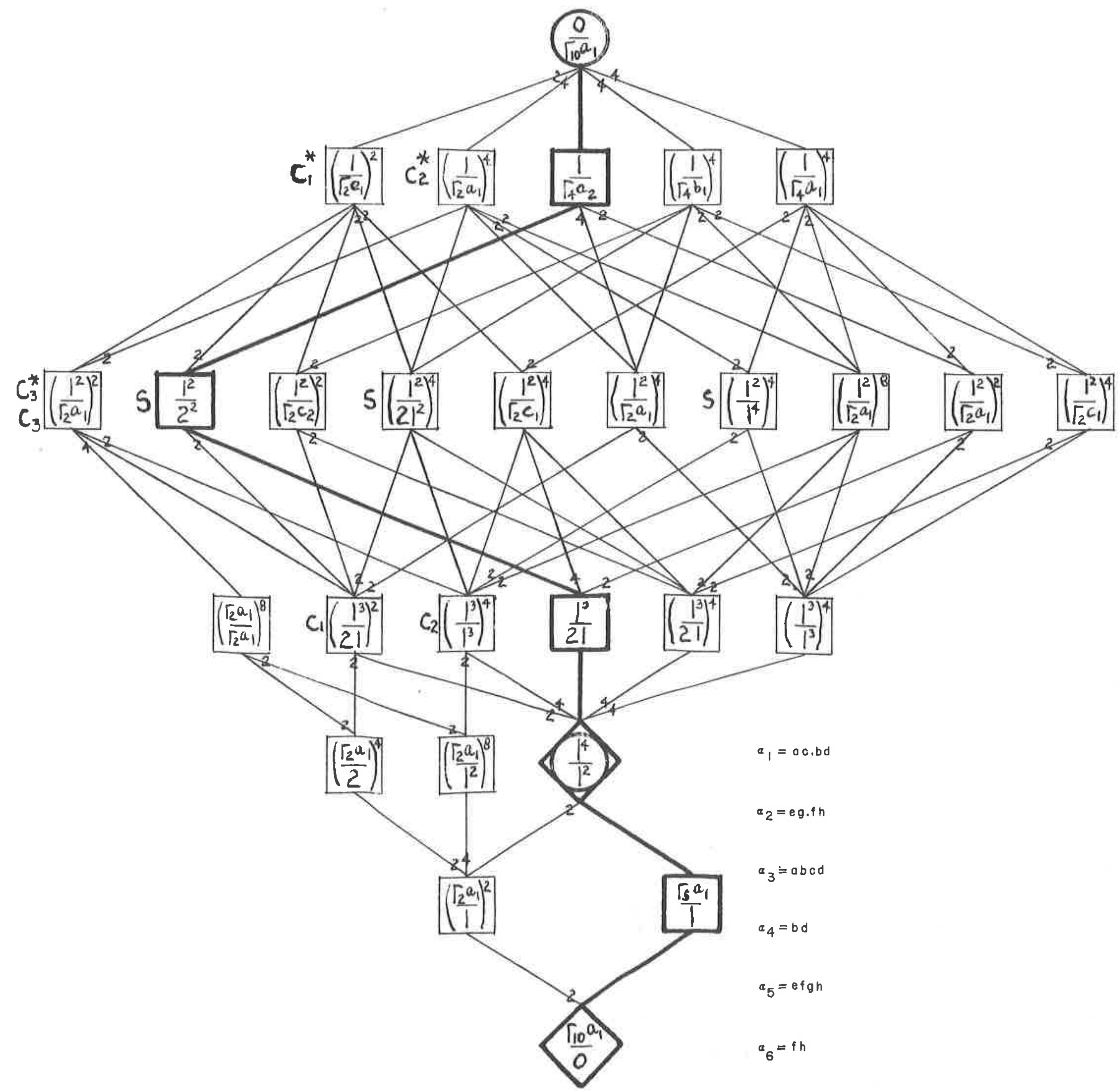
α_6 = ae.bf.cg.dh. in.jo.kp.lm. qusw.rvtx



$\alpha_1^2 = \alpha_2^2 = 1$

$\alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$

$[\alpha_3, \alpha_4] = \alpha_1$. $[\alpha_5, \alpha_6] = \alpha_2$



$\alpha_1 = ac.bd$

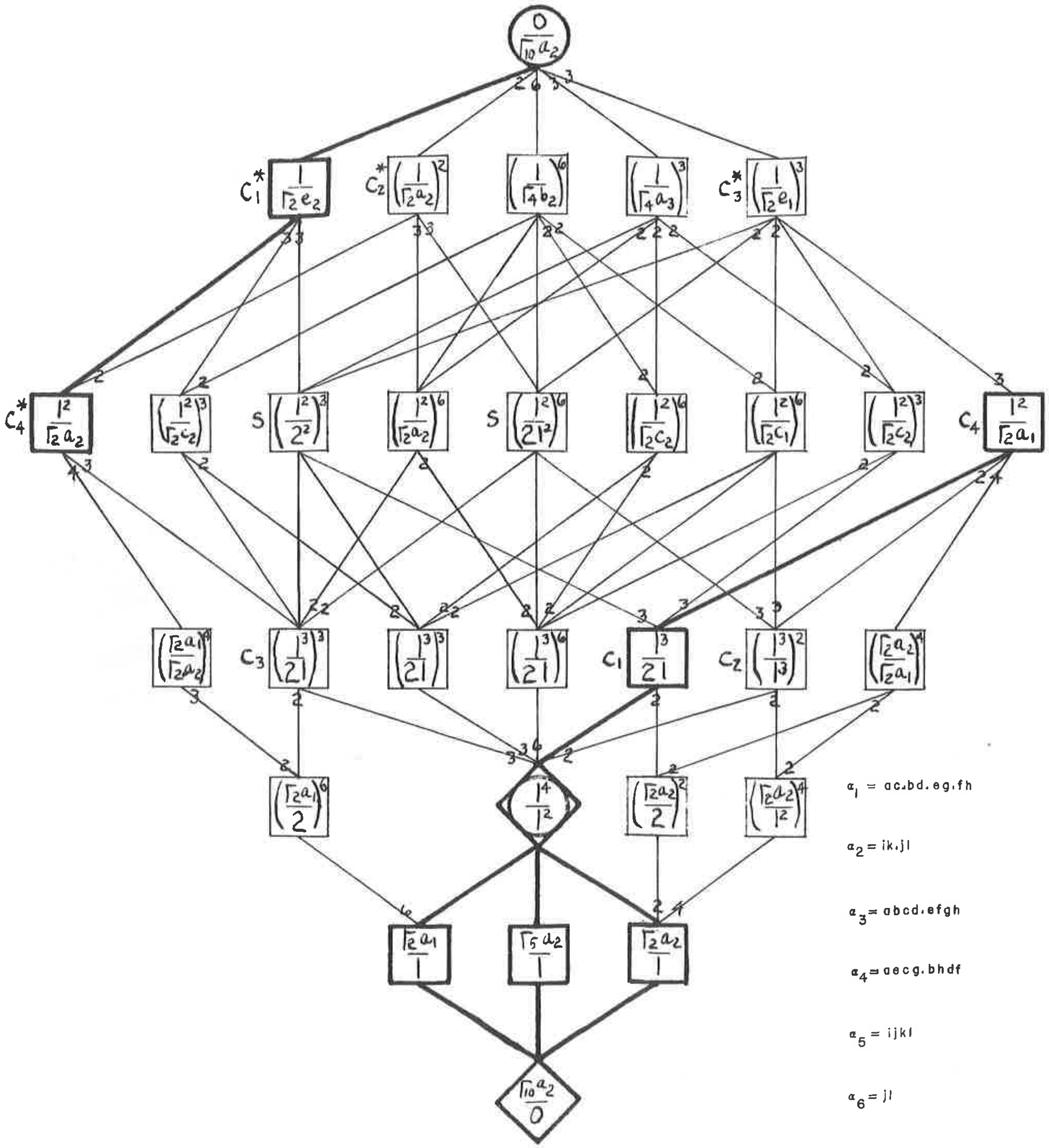
$\alpha_2 = eg.fh$

$\alpha_3 = abcd$

$\alpha_4 = bd$

$\alpha_5 = efgh$

$\alpha_6 = fh$



α_1 = ac.bd.eg.fh

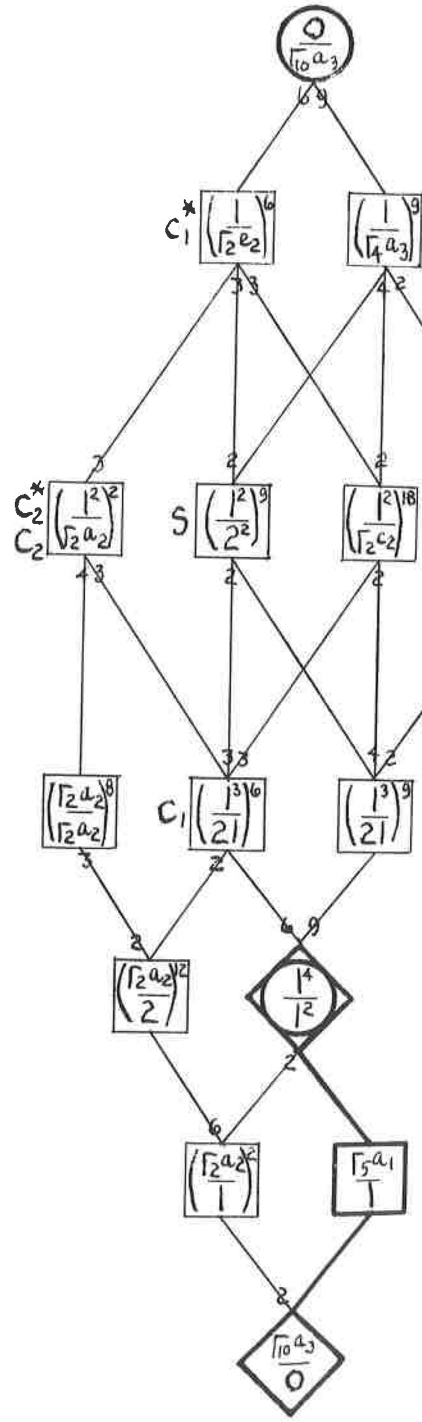
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = aecg.bhdf

α_5 = ijkl

α_6 = jl



α_1 = ac.bd.eg.fh

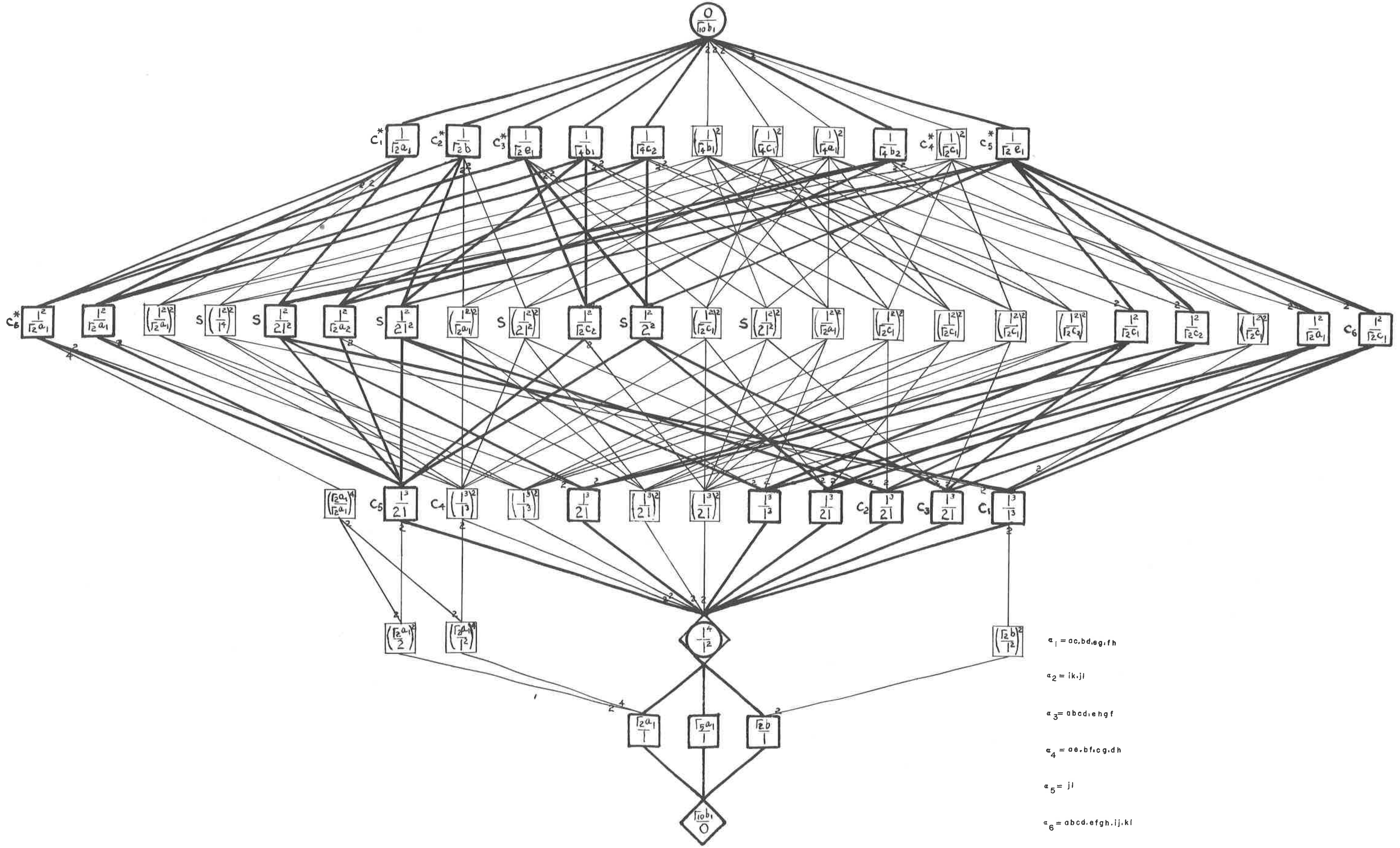
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = aecg.bhdf

α_5 = ijkl.mnop

α_6 = imko.jpln



$\alpha_1 = ac, bd, eg, fh$

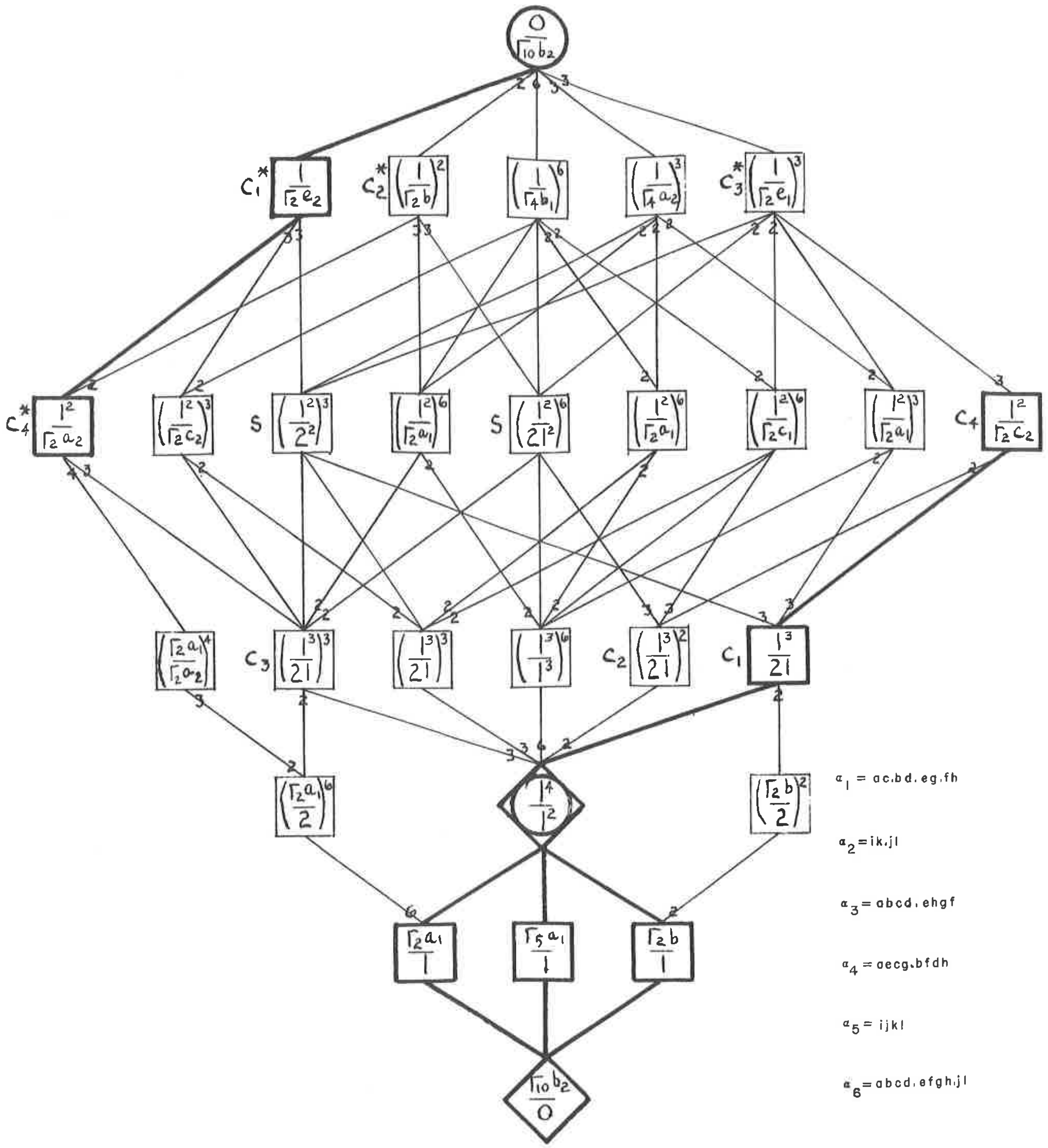
$\alpha_2 = ik, jl$

$\alpha_3 = abcd, ehgf$

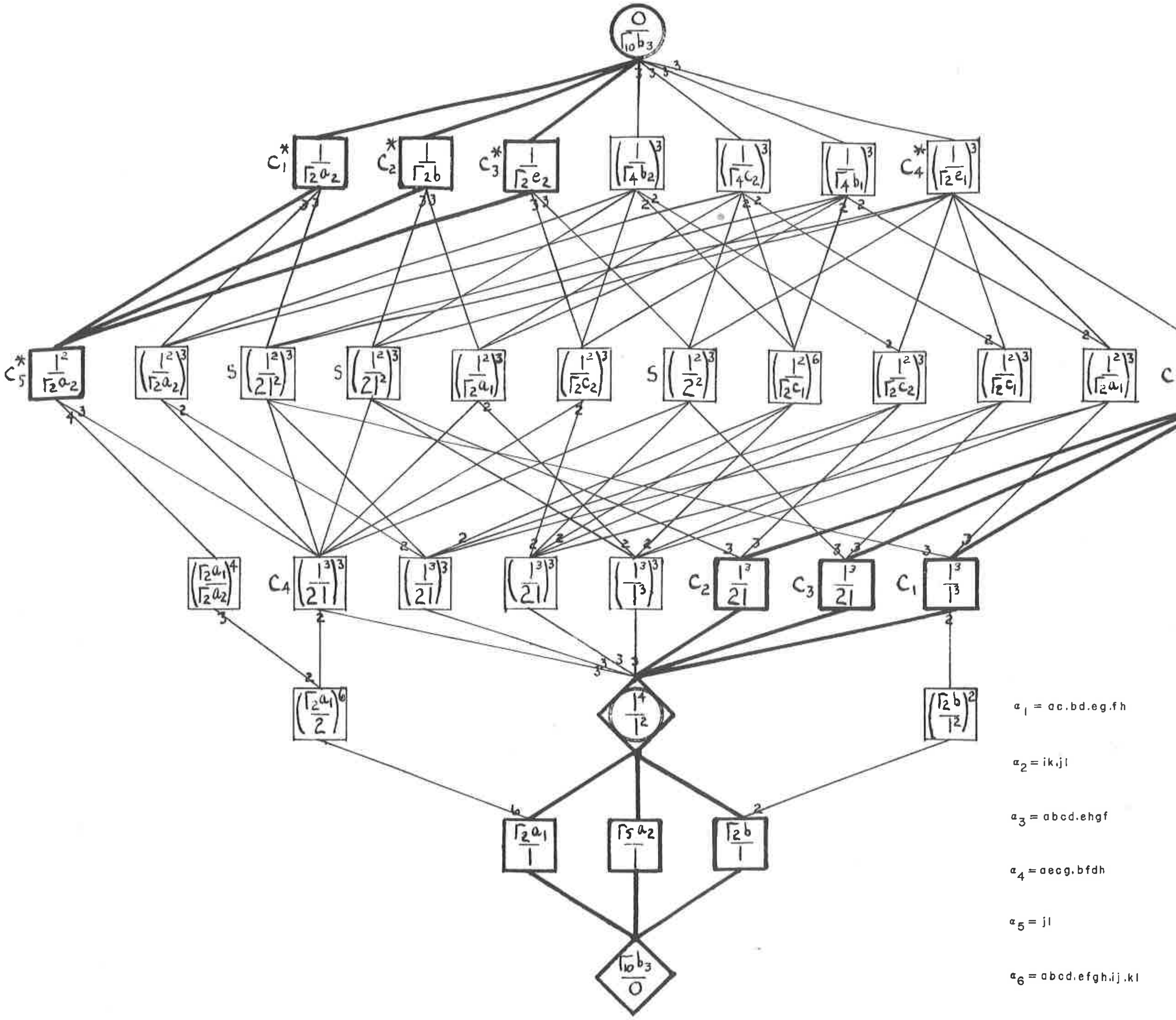
$\alpha_4 = ae, bf, cg, dh$

$\alpha_5 = jl$

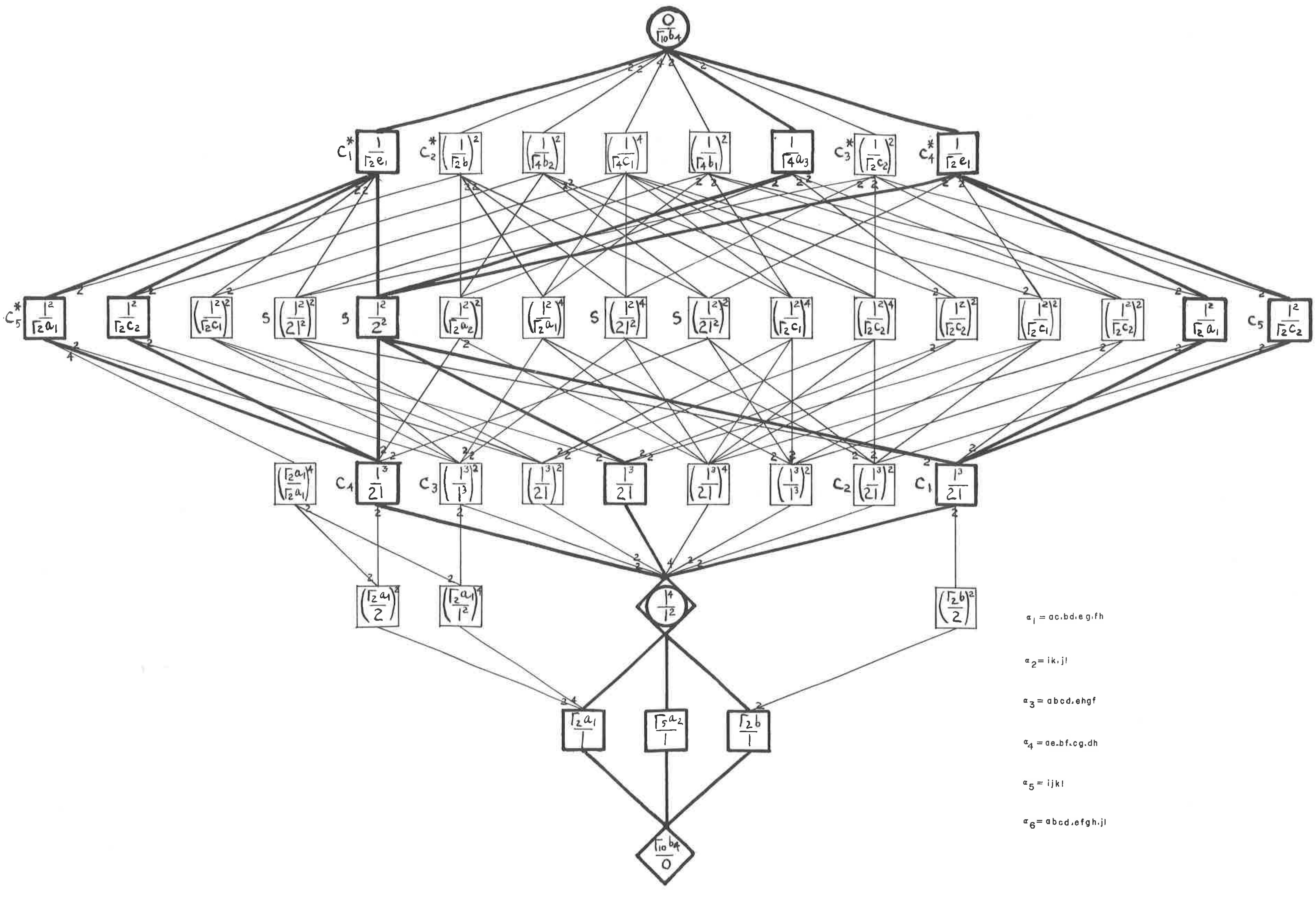
$\alpha_6 = abcd, efgh, ij, kl$



α1 = ac.bd.eg.fh
α2 = ik.jl
α3 = abcd.ehgf
α4 = aecg.bfdh
α5 = ijkl
α6 = abcd.efgh.jl



α1 = ac.bd.eg.fh
α2 = ik.jl
α3 = abcd.ehgf
α4 = aecg.bfdh
α5 = jl
α6 = abcd.efgh.ij.kl



$\alpha_1 = ac.bd.eg.fh$

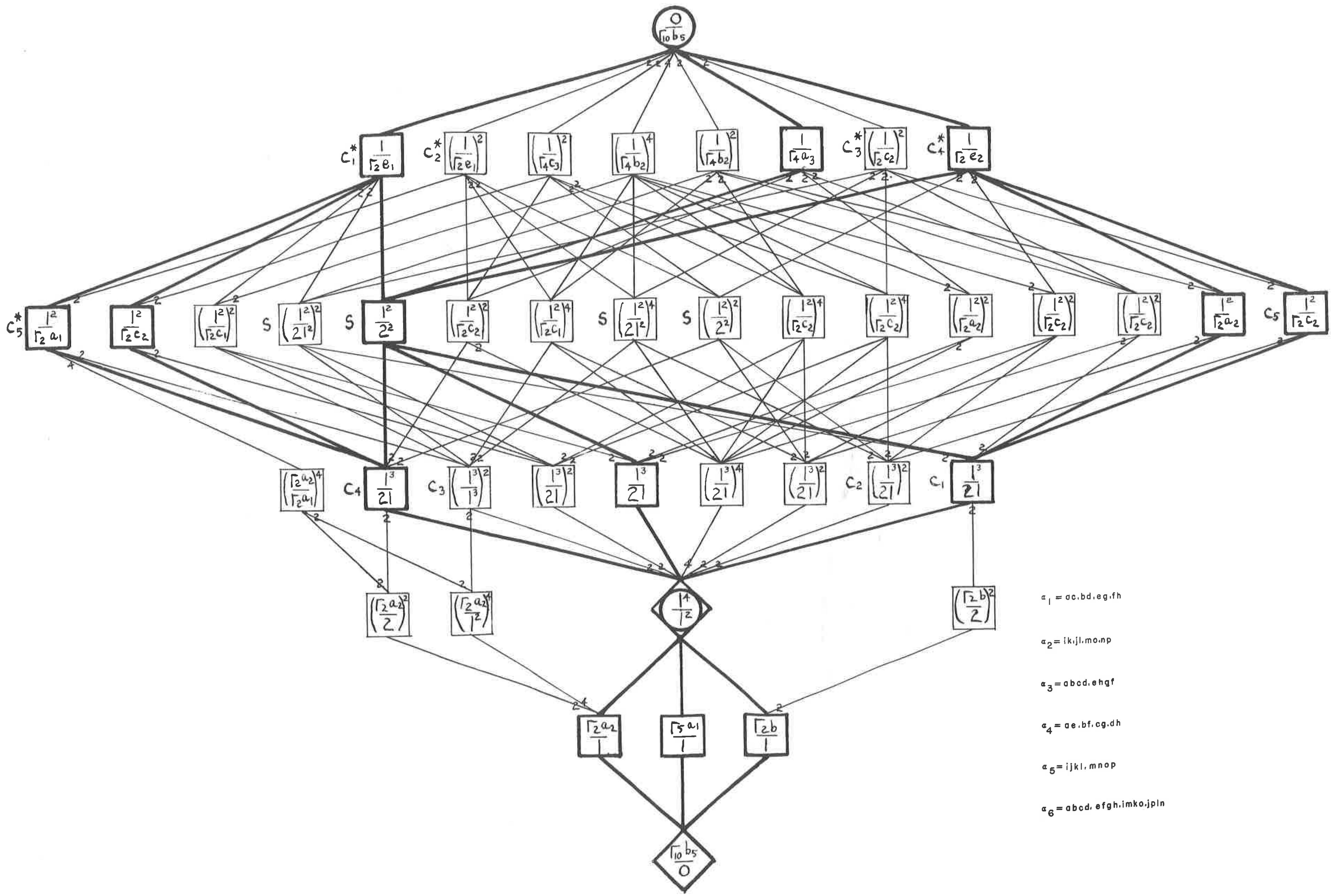
$\alpha_2 = ik.jl$

$\alpha_3 = abcd.ehgf$

$\alpha_4 = ae.bf.cg.dh$

$\alpha_5 = ijkl$

$\alpha_6 = abcd.efgh.jl$



α_1 = ac.bd.eg.fh

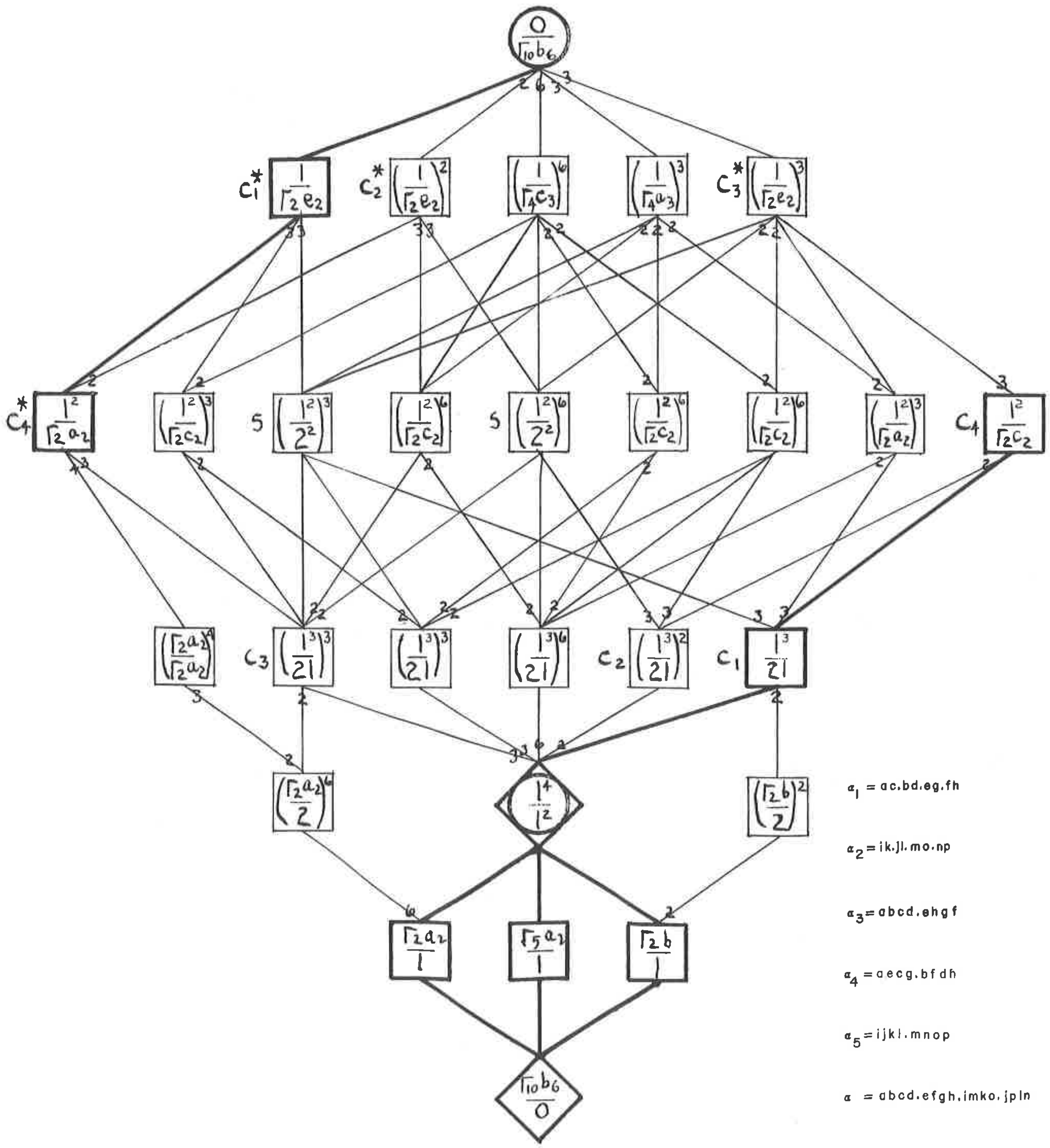
α_2 = ik.jl.mo.np

α_3 = abcd.ehgf

α_4 = ae.bf.cg.dh

α_5 = ijkl.mnop

α_6 = abcd.efgh.imko.jpln



α_1 = ac.bd.eg.fh

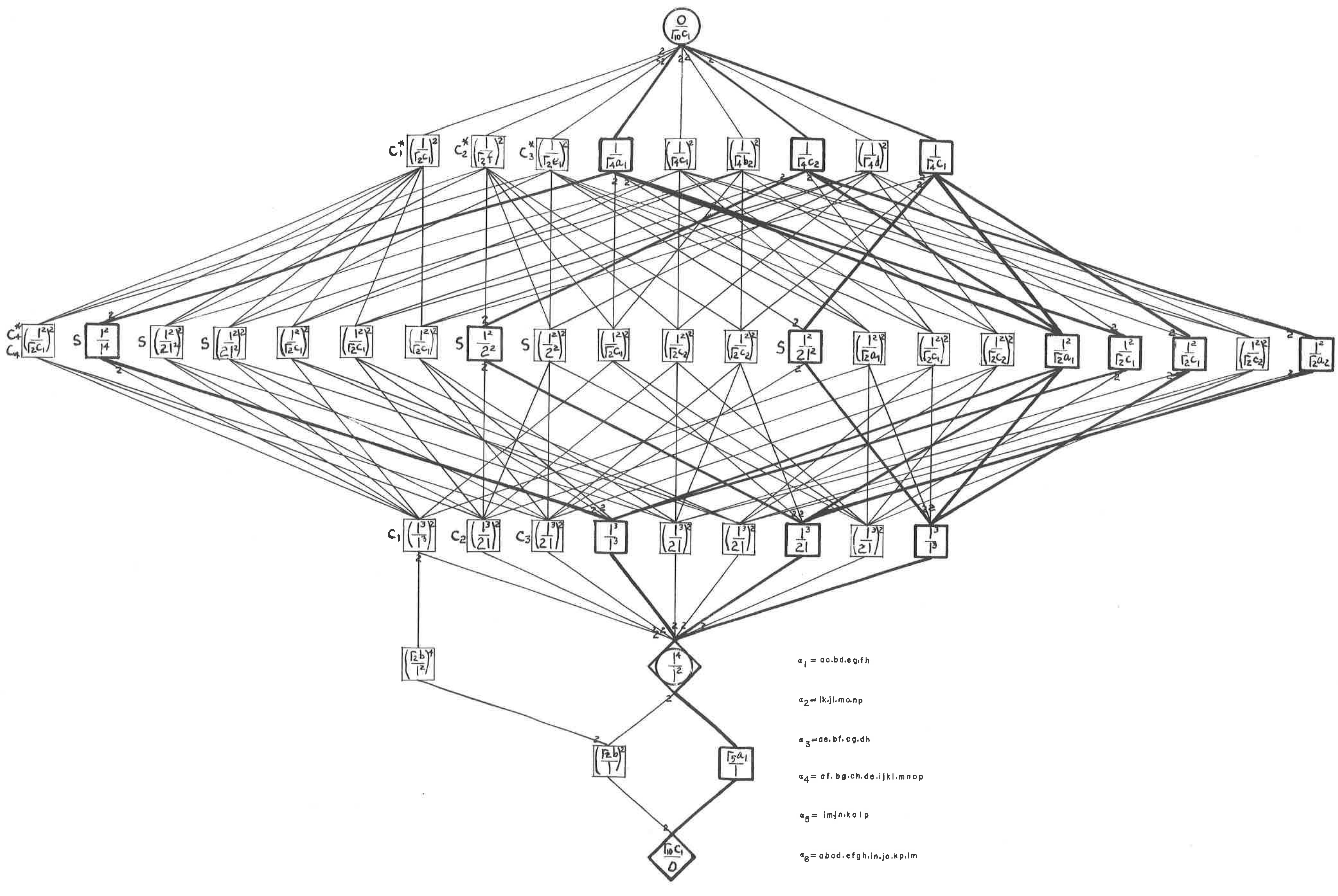
α_2 = ik.jl.mo.np

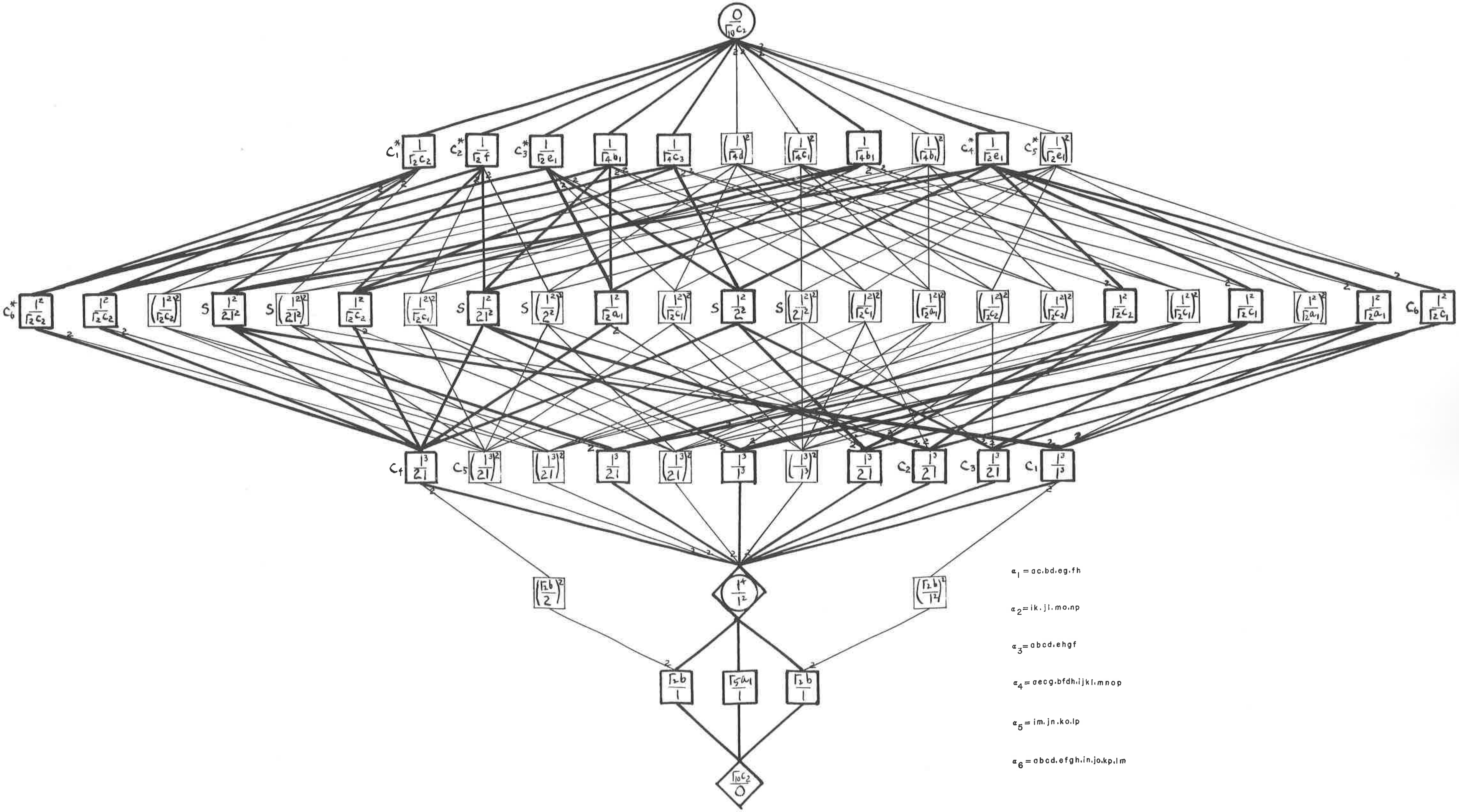
α_3 = abcd.ehgf

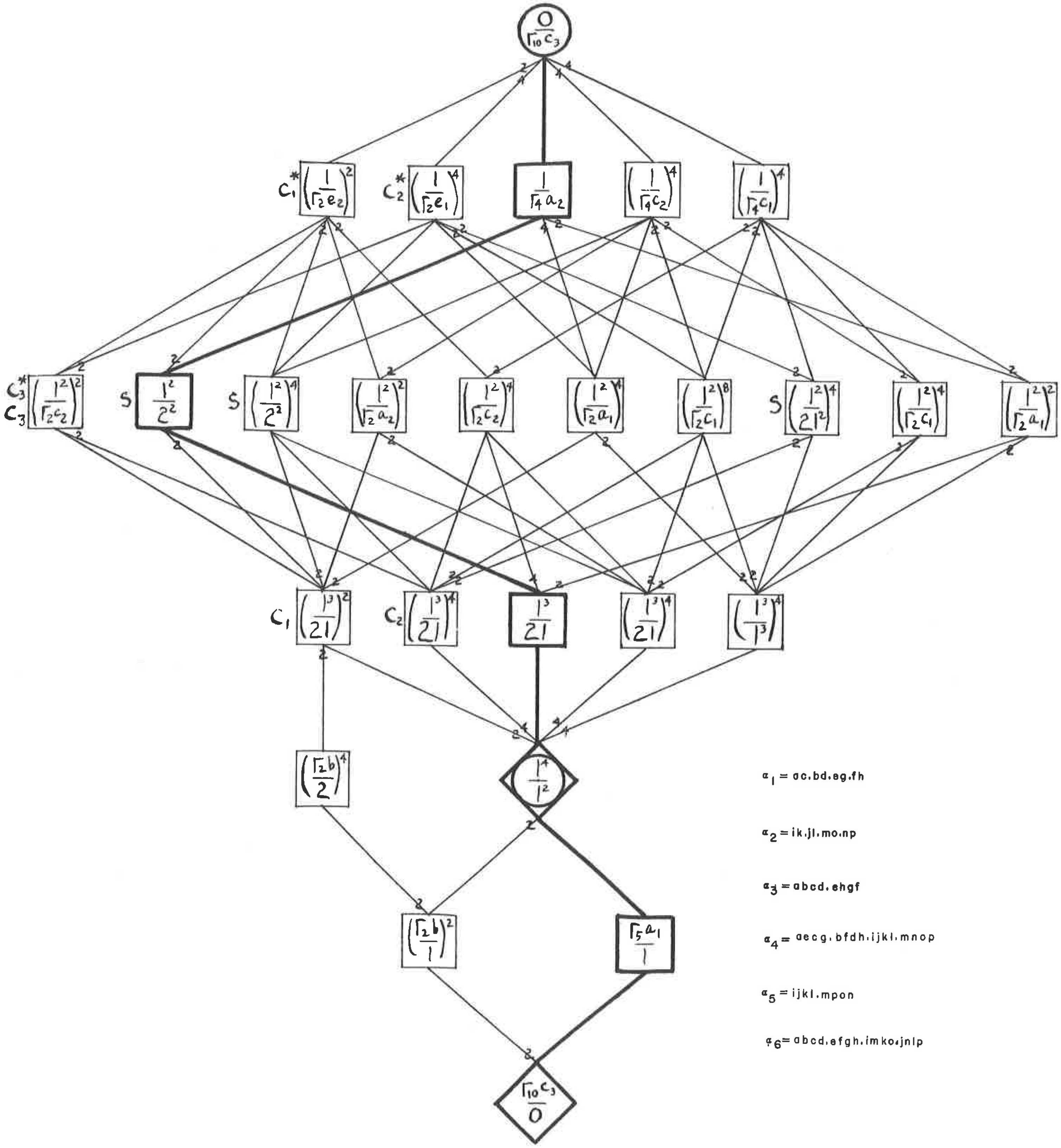
α_4 = aecg.bfdh

α_5 = ijkl.mnop

α = abcd.efgh.imko.jpln


α_1 = ac.bd.eg.fh
α_2 = ik.jl.mo.np
α_3 = ae.bf.cg.dh
α_4 = af. bg.ch.de.ijkl.mnop
α_5 = im.jn.ko lp
α_6 = abcd.efgh.in.jo.kp.lm


α1 = ac.bd.eg.fh
α2 = ik.jl.mo.np
α3 = abcd.ehgf
α4 = aecg.bfdh.ijkl.mnop
α5 = im.jn.ko.lp
α6 = abcd.efgh.in.jo.kp.lm



$\alpha_1 = ac.bd.eg.fh$

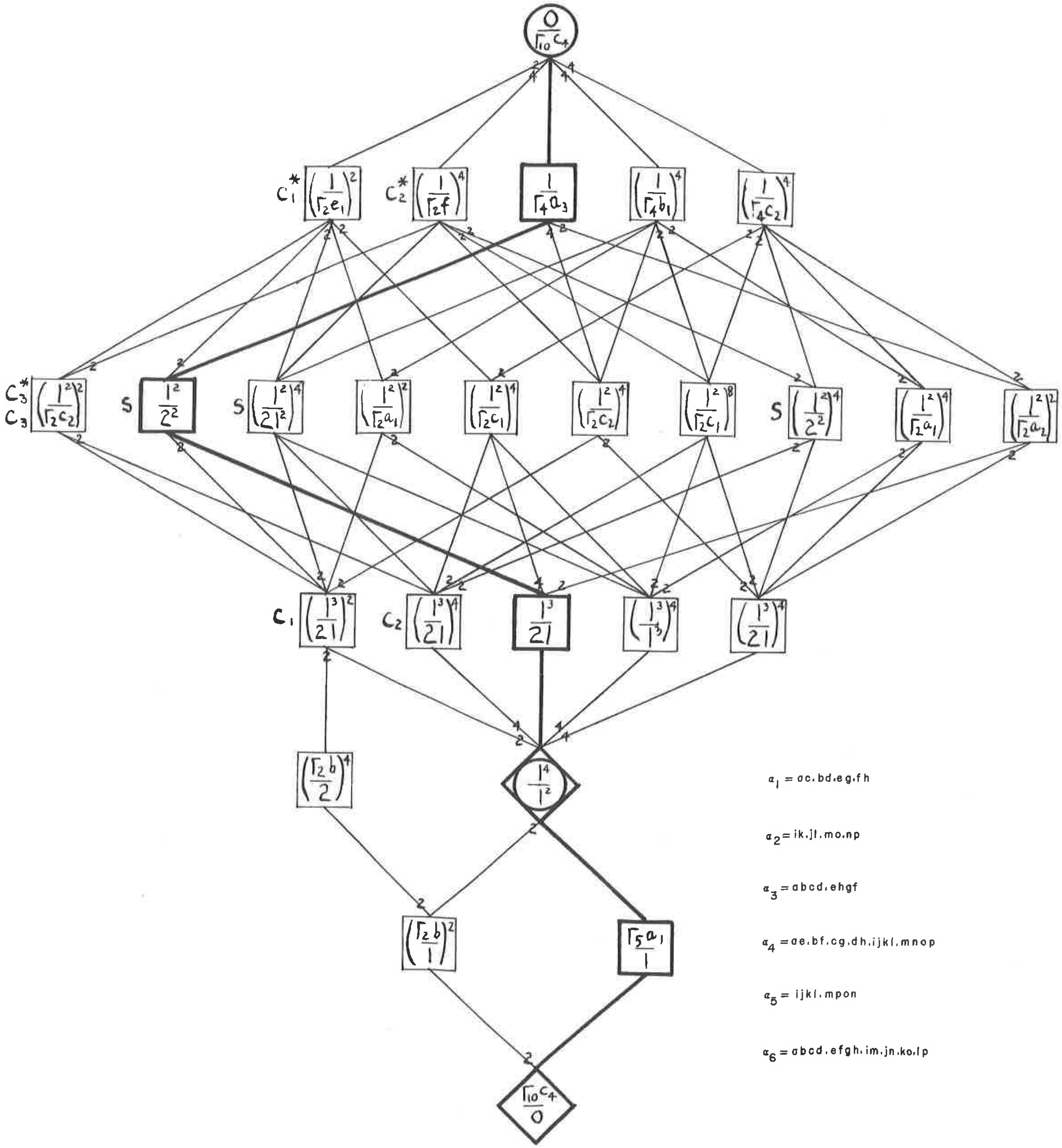
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = abcd.ehgf$

$\alpha_4 = aecg.bfdh.ijkl.mnop$

$\alpha_5 = ijkl.mpon$

$\alpha_6 = abcd.efgh.imko.jnlp$



$\alpha_1 = ac.bd.eg.fh$

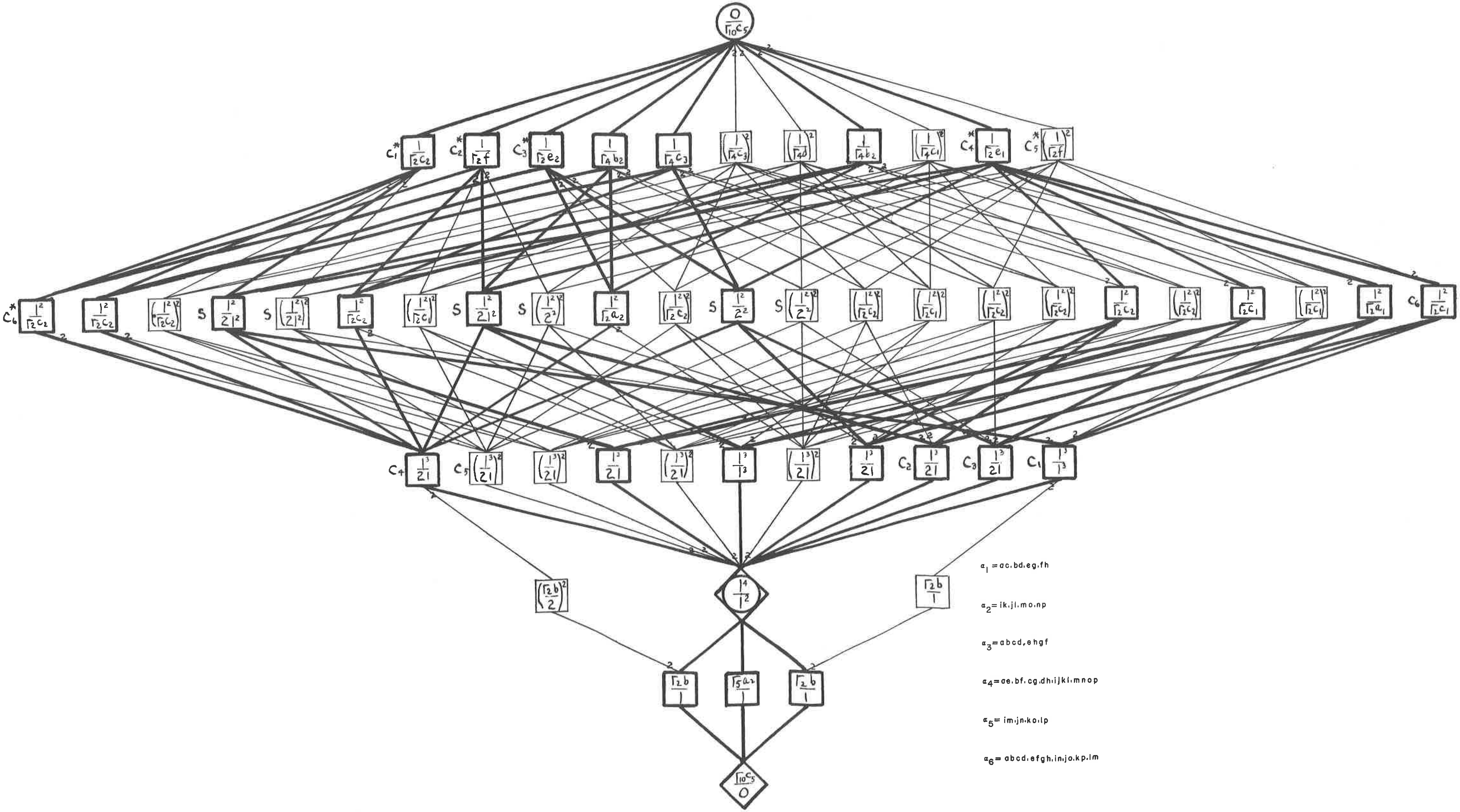
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = abcd.ehgf$

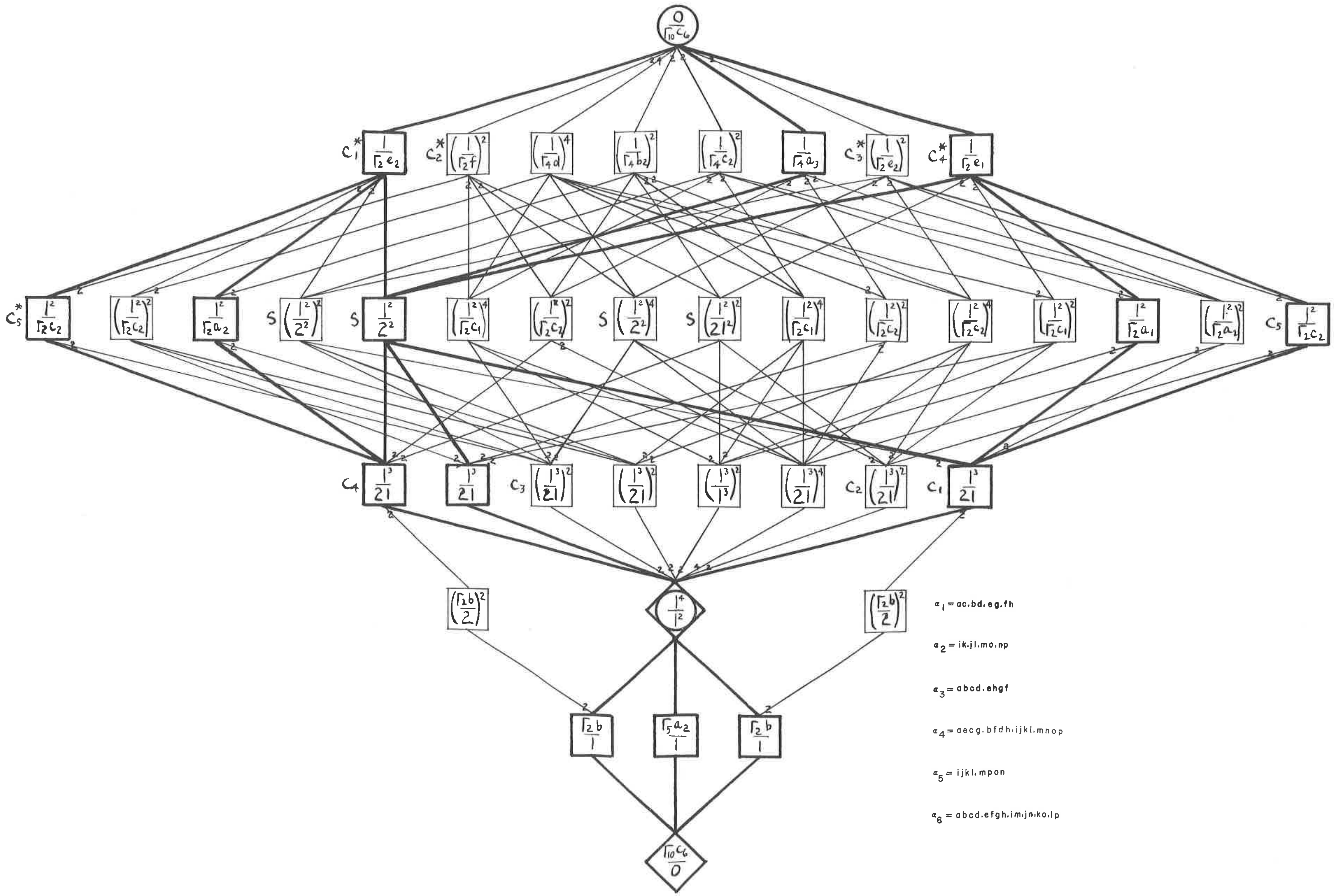
$\alpha_4 = ae.bf.cg.dh.ijkl.mnop$

$\alpha_5 = ijkl.mpon$

$\alpha_6 = abcd.efgh.im.jn.ko.lp$



α_1 = ac.bd.eg.fh
α_2 = ik.jl.mo.np
α_3 = abcd, ehgf
α_4 = ae.bf.cg.dh.ijkl.mnop
α_5 = im.jn.ko.lp
α_6 = abcd.efgh.in.jo.kp.lm



$\alpha_1 = ac.bd.eg.fh$

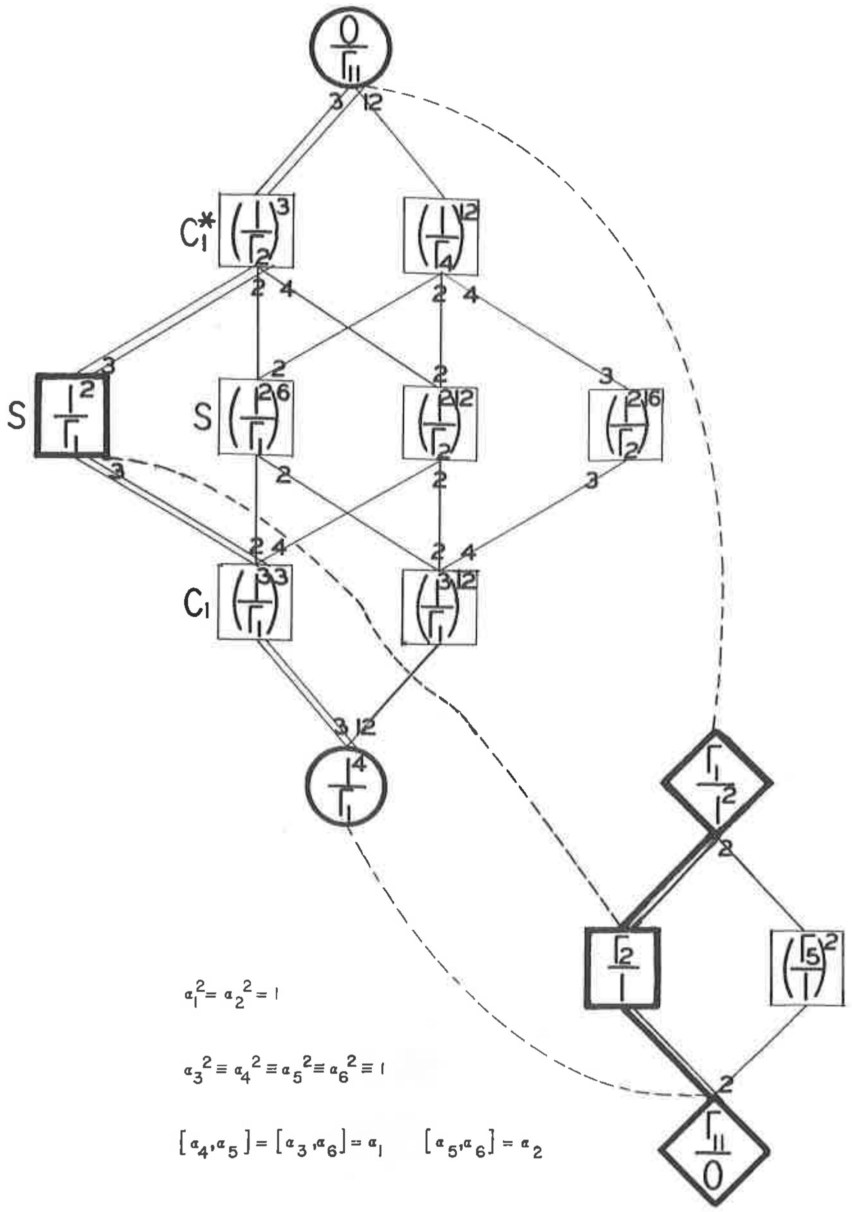
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = abcd.ehgf$

$\alpha_4 = aecg.bfdh.ijkl.mnop$

$\alpha_5 = ijkl.mpon$

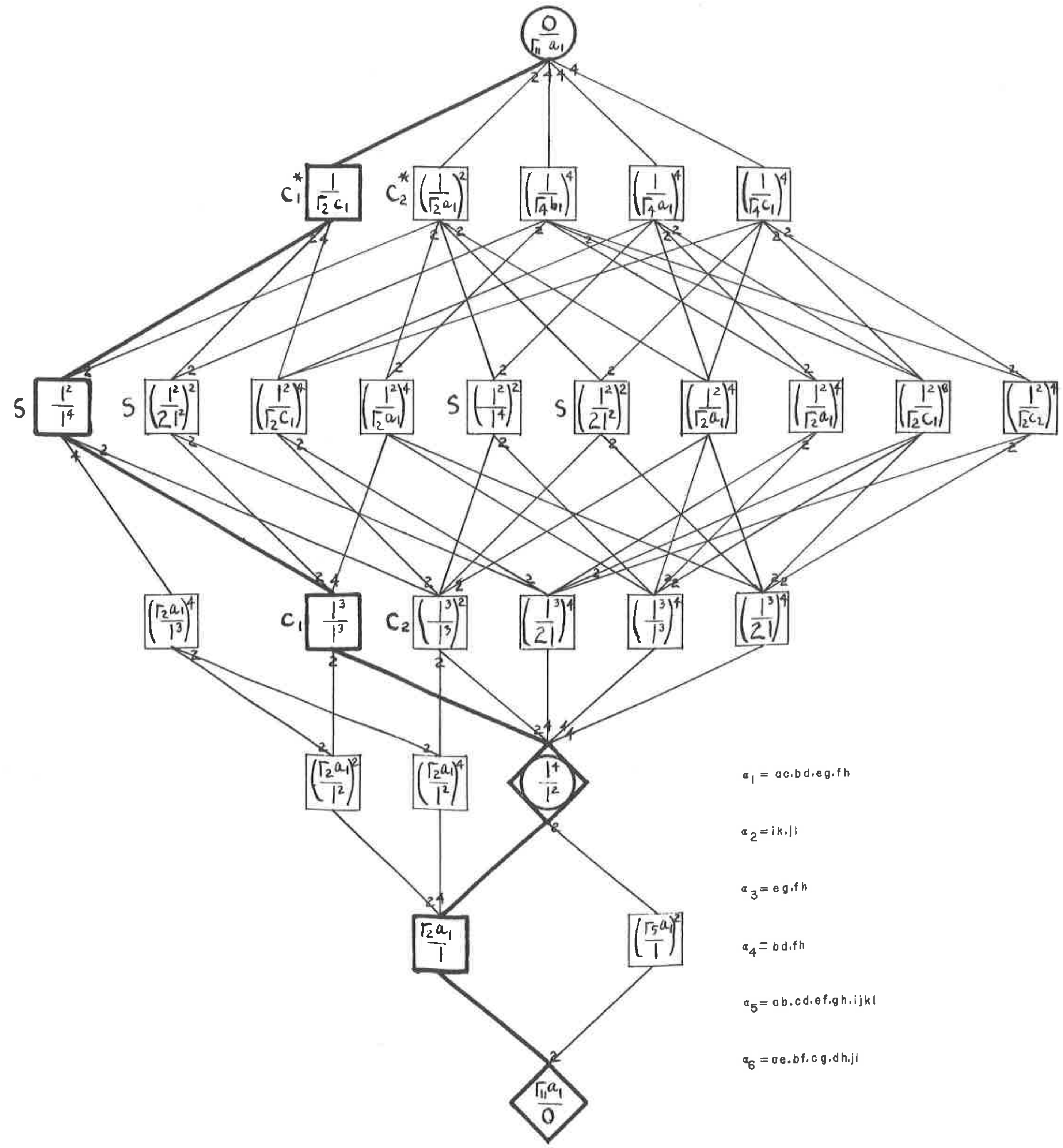
$\alpha_6 = abcd.efgh.im.jn.ko.lp$



$\alpha_1^2 = \alpha_2^2 = 1$

$\alpha_3^2 \equiv \alpha_4^2 \equiv \alpha_5^2 \equiv \alpha_6^2 \equiv 1$

$[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1 \qquad [\alpha_5, \alpha_6] = \alpha_2$



$\alpha_1 =$ ac.bd.eg.fh

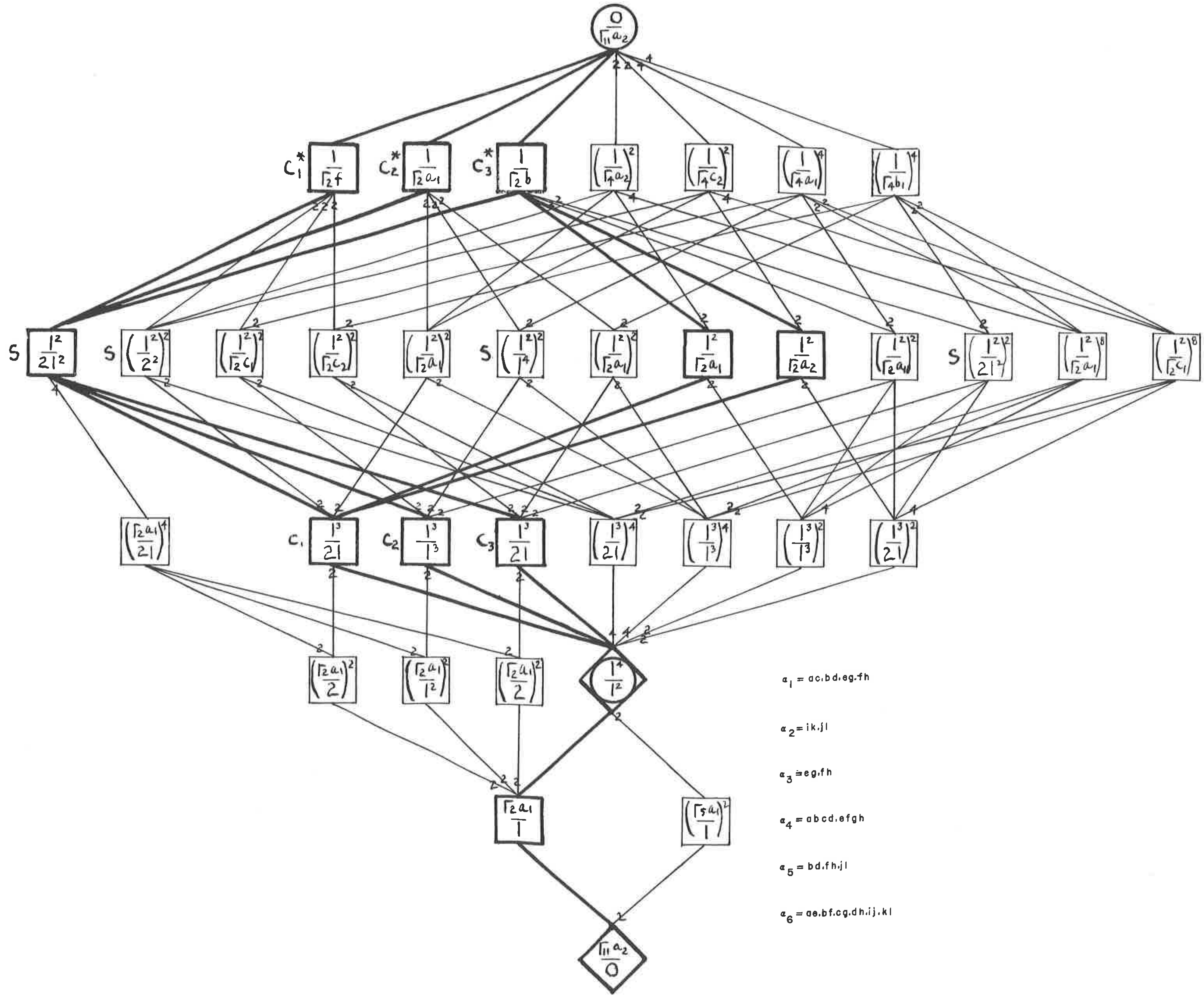
$\alpha_2 =$ ik.jl

$\alpha_3 =$ eg.fh

$\alpha_4 =$ bd.fh

$\alpha_5 =$ ab.cd.ef.gh.ijkl

$\alpha_6 =$ ae.bf.cg.dh.jl



$\alpha_1 = ac.bd.eg.fh$

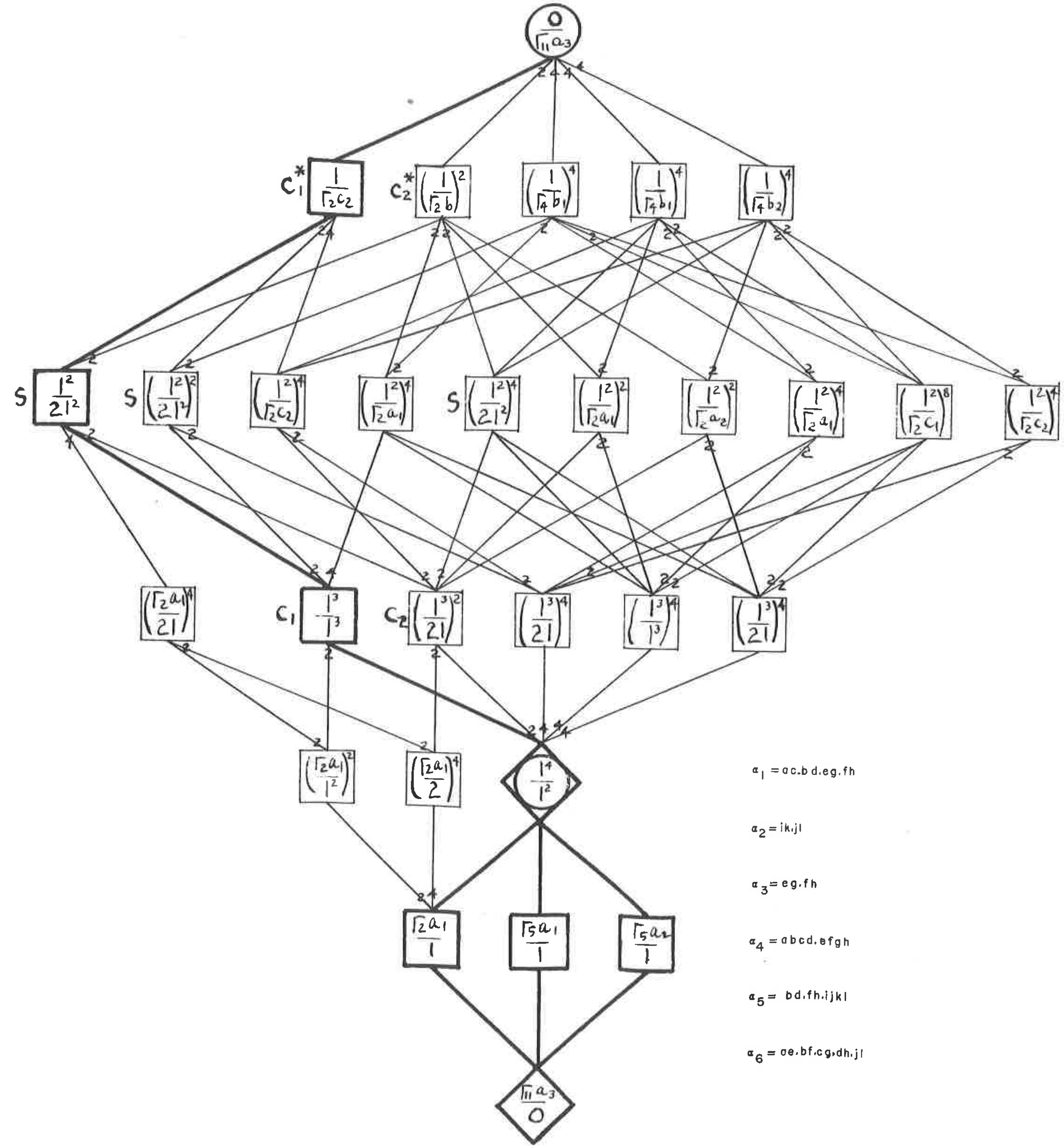
$\alpha_2 = ik.jl$

$\alpha_3 = eg.fh$

$\alpha_4 = abcd.efgh$

$\alpha_5 = bd.fh.jl$

$\alpha_6 = ae.bf.cg.dh.ij.kl$



α_1 = ac.bd.eg.fh

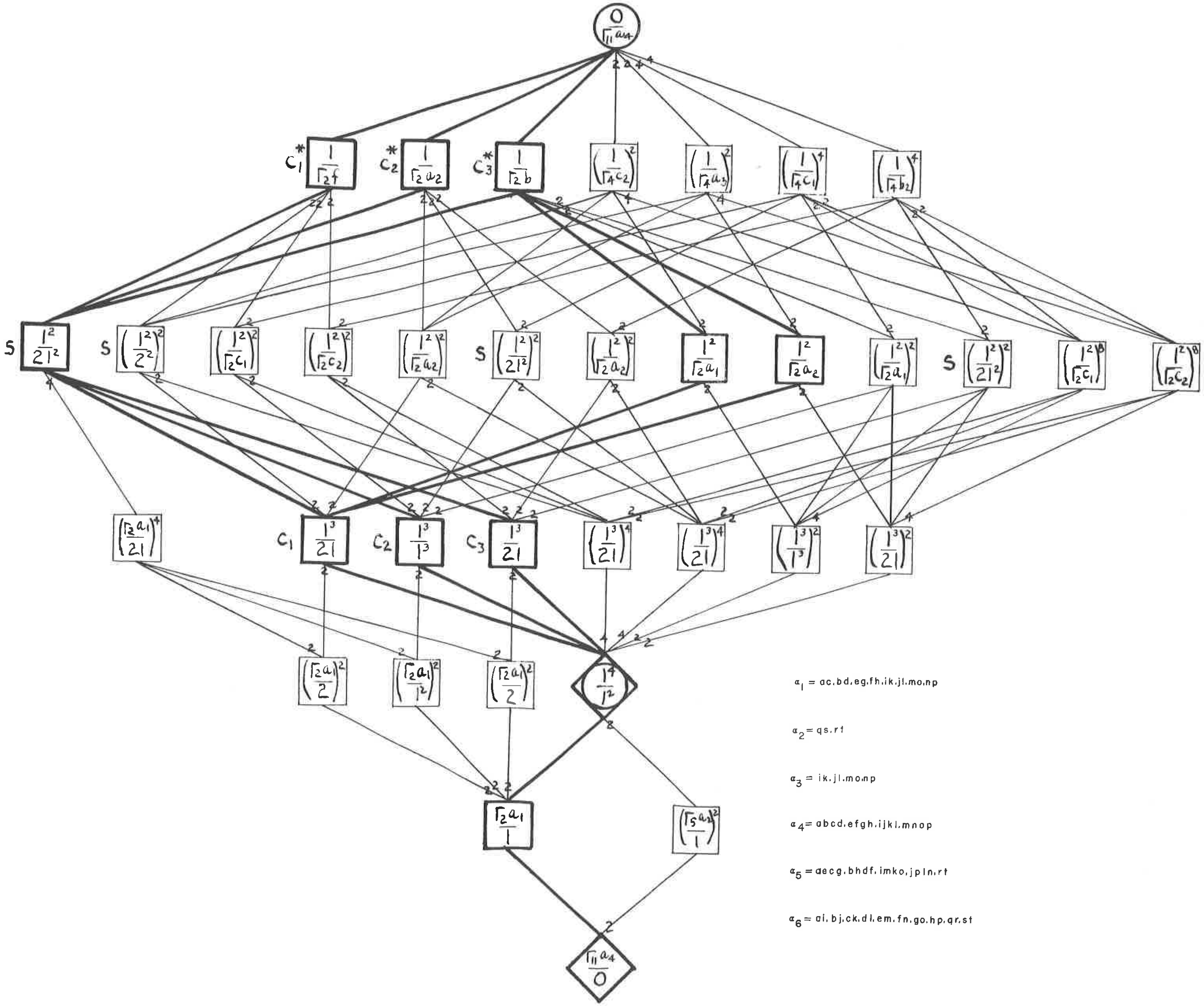
α_2 = ik.jl

α_3 = eg.fh

α_4 = abcd.efgh

α_5 = bd.fh.ijkl

α_6 = ae.bf.cg.dh.jl



α_1 = ac.bd.eg.fh.ik.jl.mo.np

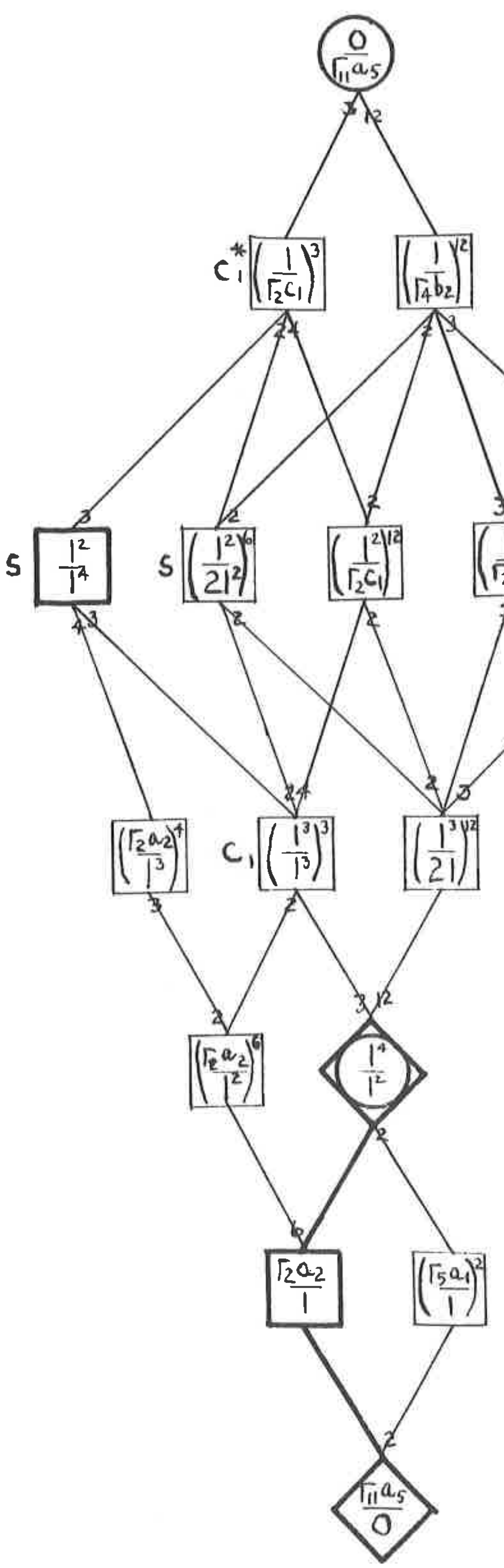
α_2 = qs.rt

α_3 = ik.jl.mo.np

α_4 = abcd.efgh.ijkl.mnop

α_5 = aecg.bhdf.imko.jpln.rt

α_6 = ai.bj.ck.dl.em.fn.go.hp.qr.st



α_1 = ac.bd.eg.fh

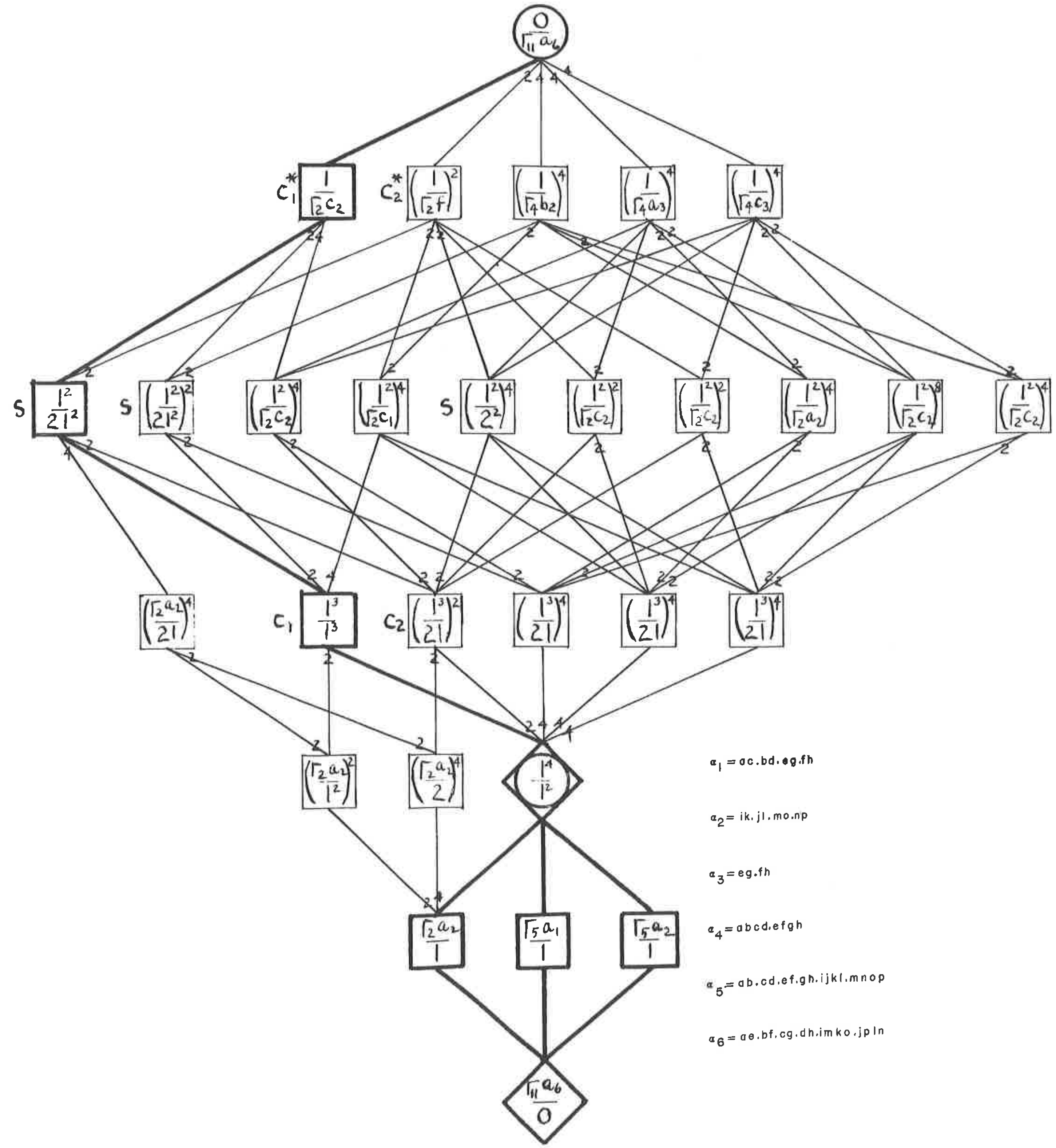
α_2 = ik.jl.mo.np

α_3 = eg.fh

α_4 = bd.fh

α_5 = ab. cd.ef. gh.ijkl.mnop

α_6 = ae.bf.cg.dh.imko.jpln



α_1 = ac.bd.eg.fh

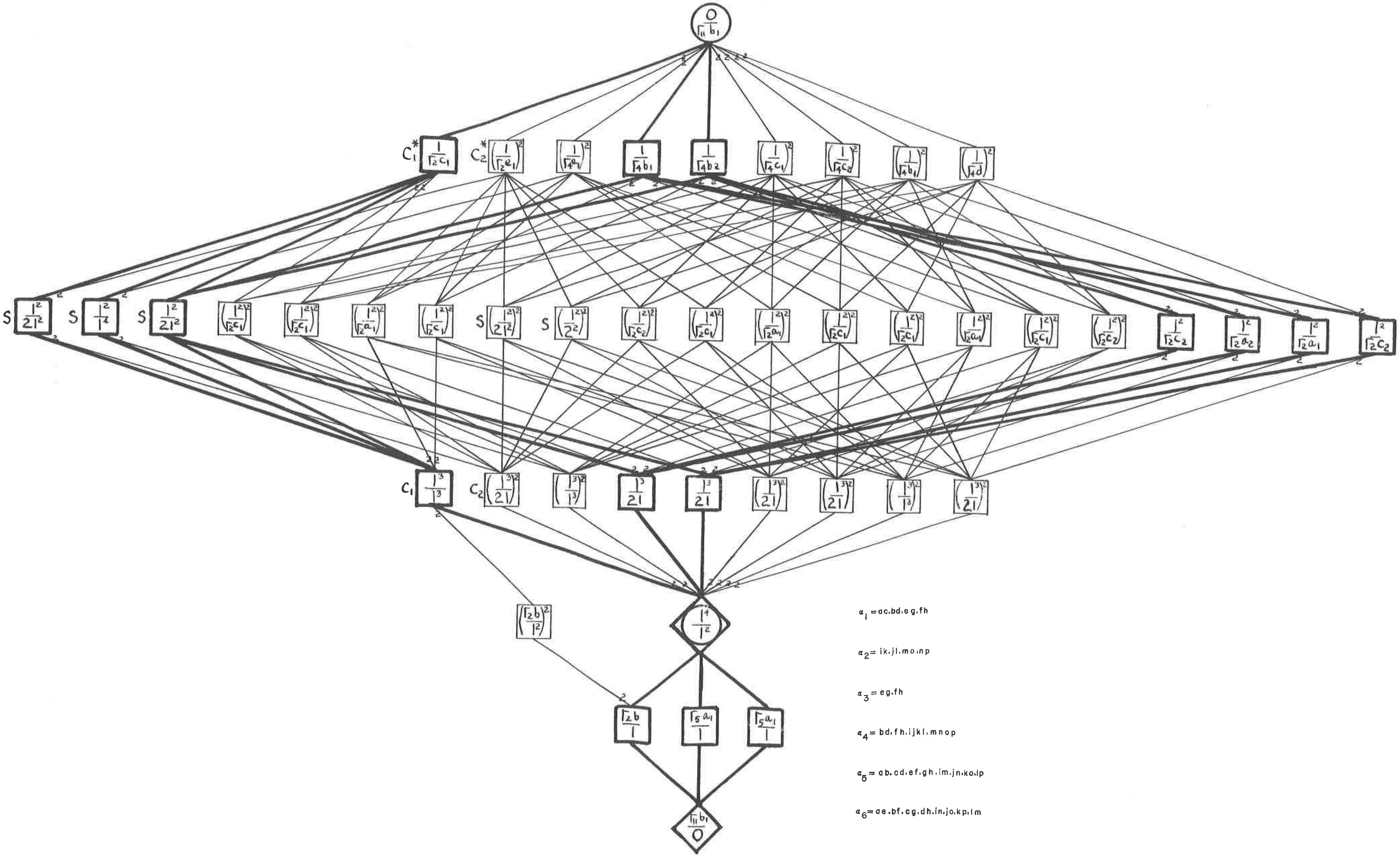
α_2 = ik.jl.mo.np

α_3 = eg.fh

α_4 = abcd.efgh

α_5 = ab.cd.ef.gh.ijkl.mnop

α_6 = ae.bf.cg.dh.imko.jpln



$\alpha_1 = ac.bd.eg.fh$

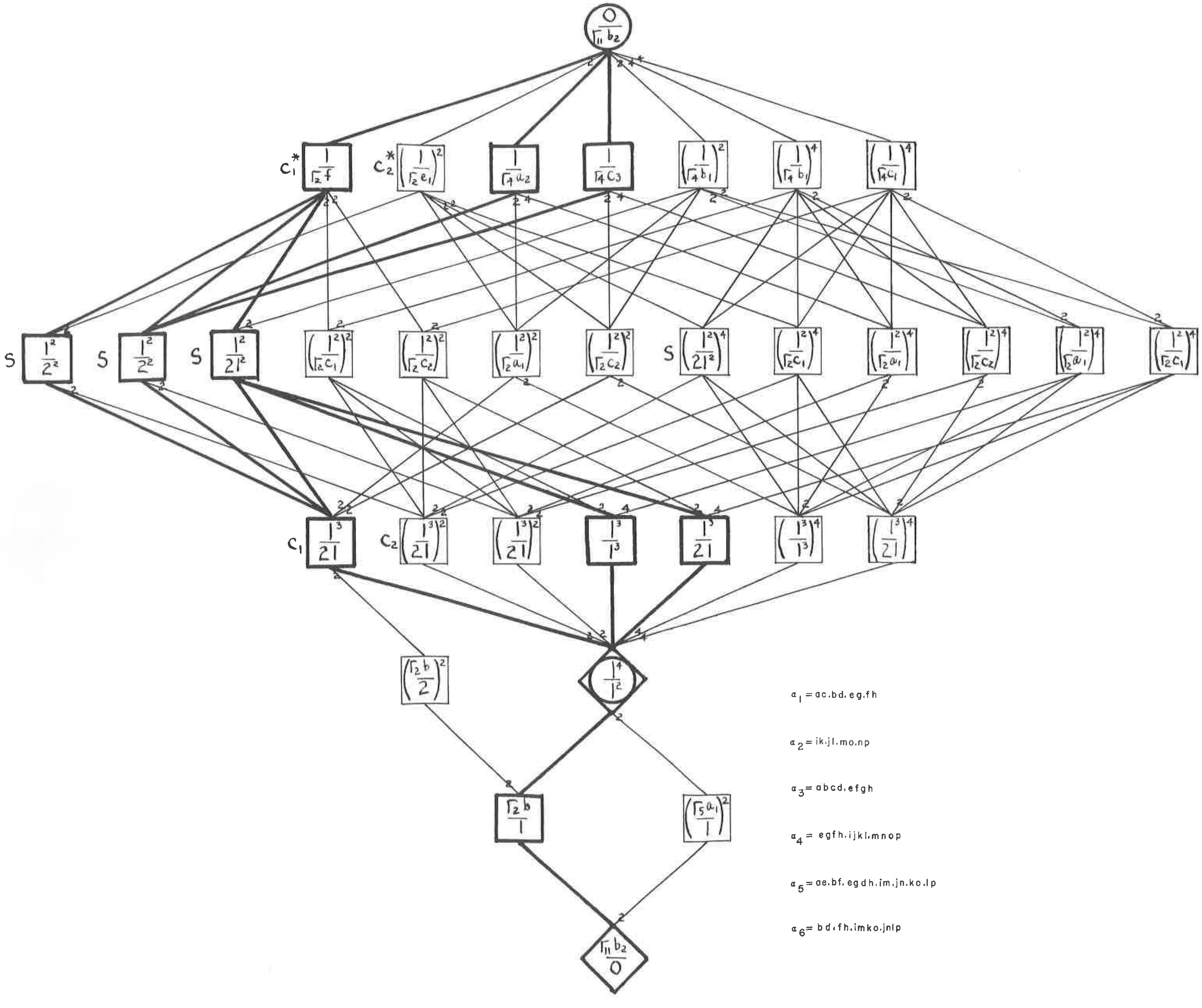
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = eg.fh$

$\alpha_4 = bd.fh.ijkl.mnop$

$\alpha_5 = ab.cd.ef.gh.im.jn.ko.lp$

$\alpha_6 = ae.bf.cg.dh.in.jo.kp.lm$



$\alpha_1 = ac.bd.eg.fh$

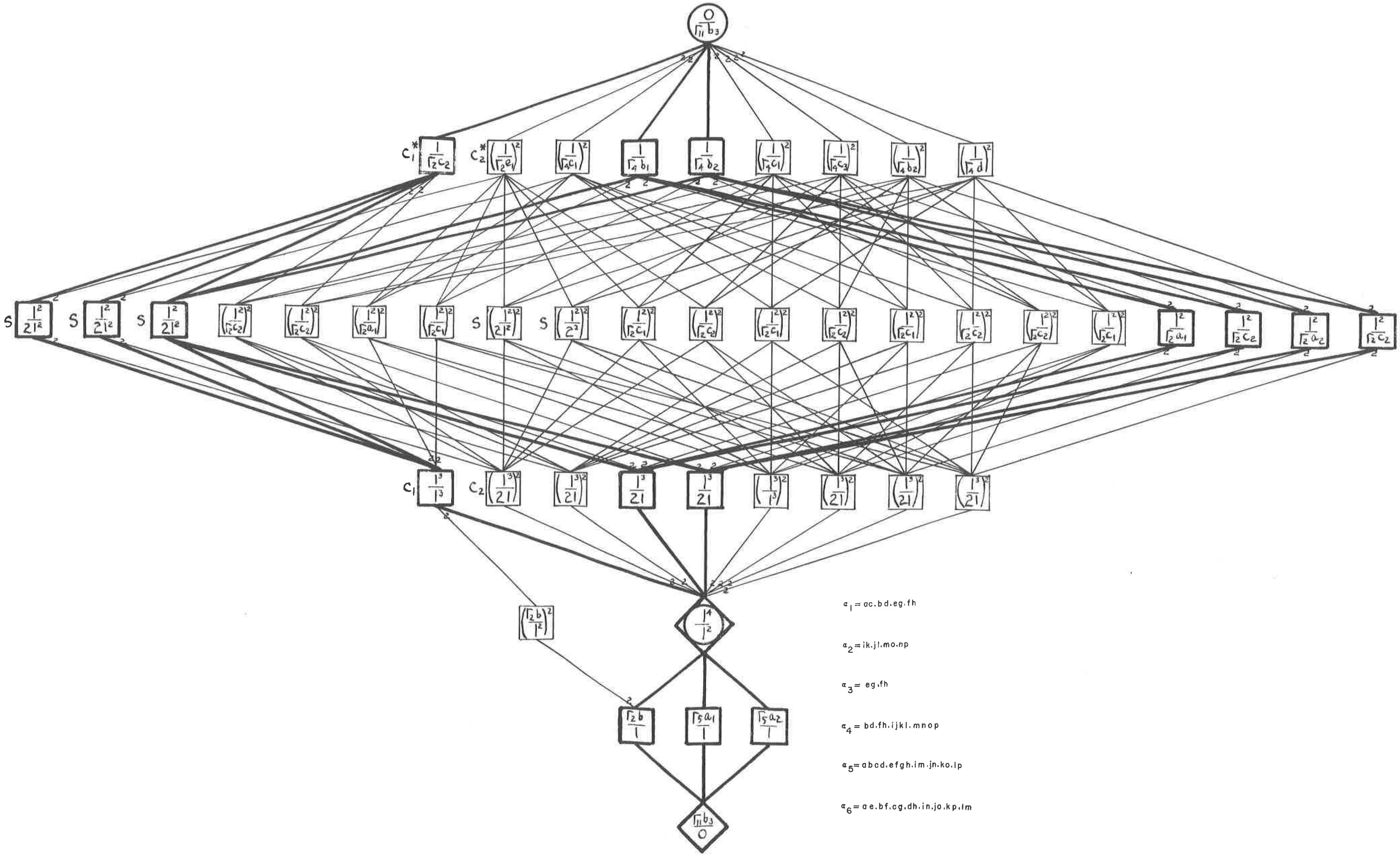
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = abcd.efgh$

$\alpha_4 = egfh.ijkl.mnop$

$\alpha_5 = ae.bf.egdh.im.jn.ko.lp$

$\alpha_6 = bd.fh.imko.jnlp$



$\alpha_1 = ac.bd.eg.fh$

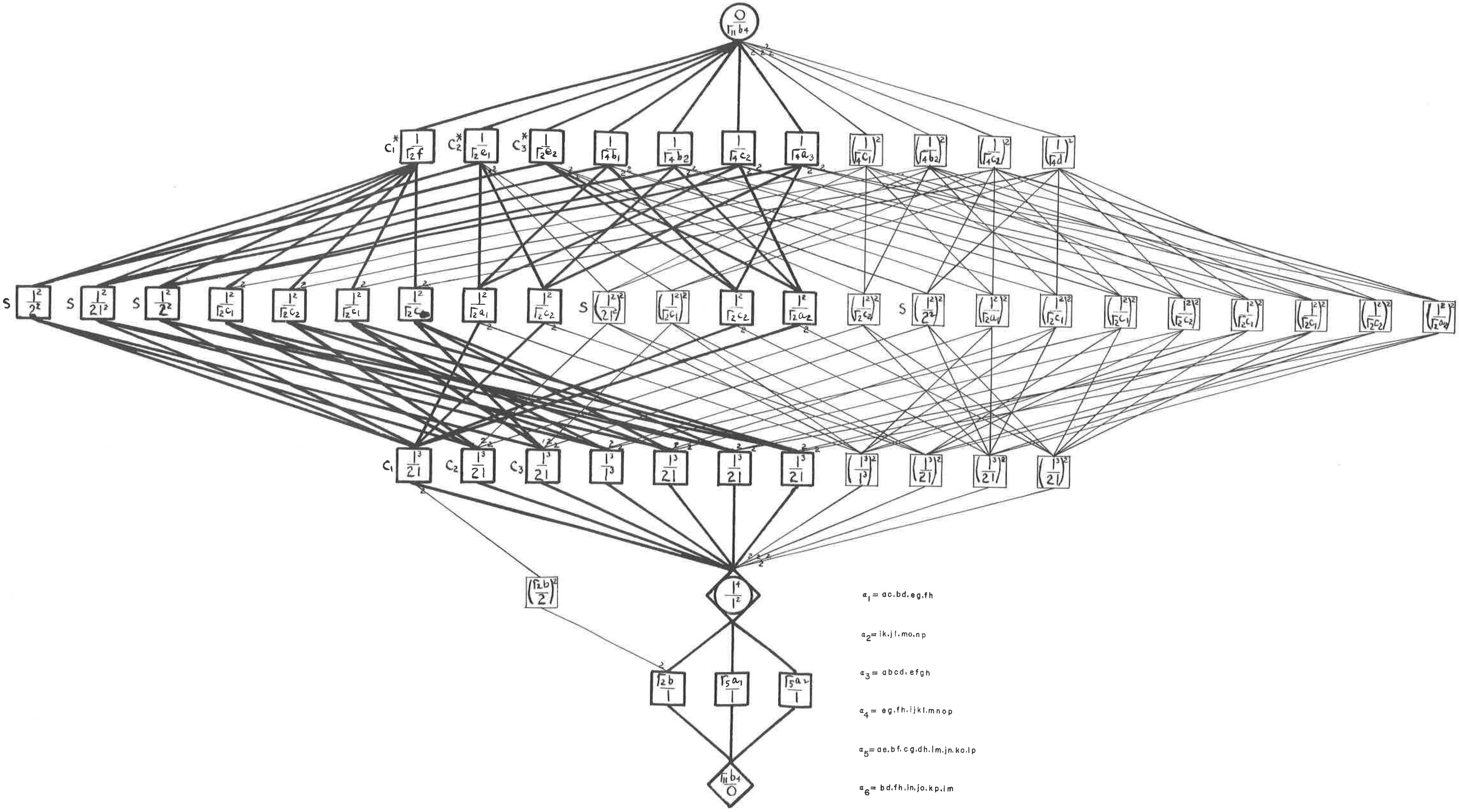
$\alpha_2 = ik.jl.mo.np$

$\alpha_3 = eg.fh$

$\alpha_4 = bd.fh.ijkl.mnop$

$\alpha_5 = abcd.efgh.im.jn.ko.lp$

$\alpha_6 = ae.bf.cg.dh.in.jo.kp.lm$



α_1 = ac.bd.eg.fh

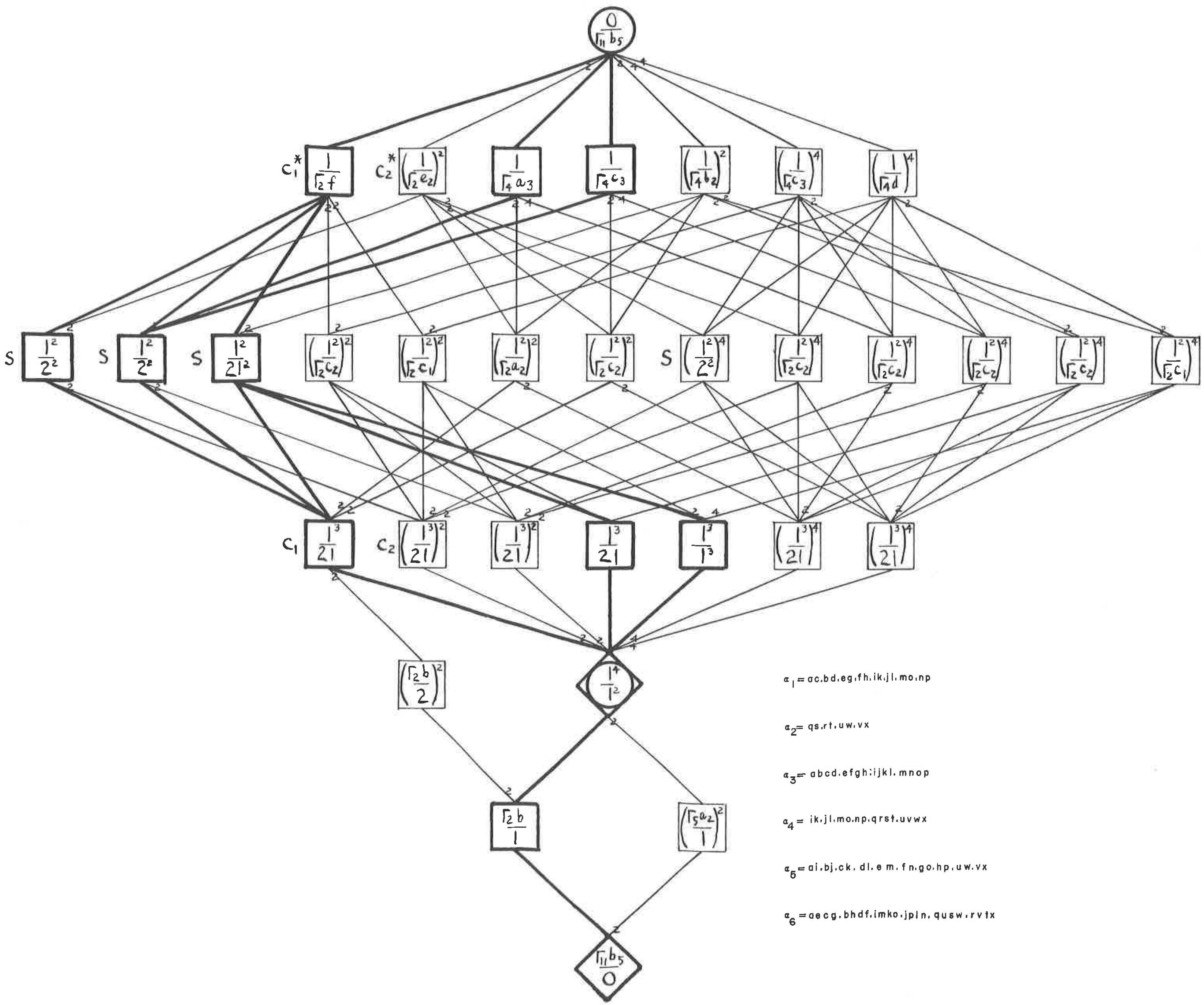
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = eg.fh.ijkl.mnop

α_5 = ae.bf.cg.dh.lm.jn.ko.lp

α_6 = bd.fh.in.jo.kp.lm



α_1 = ac.bd.eg.fh.ik.jl.mo.np

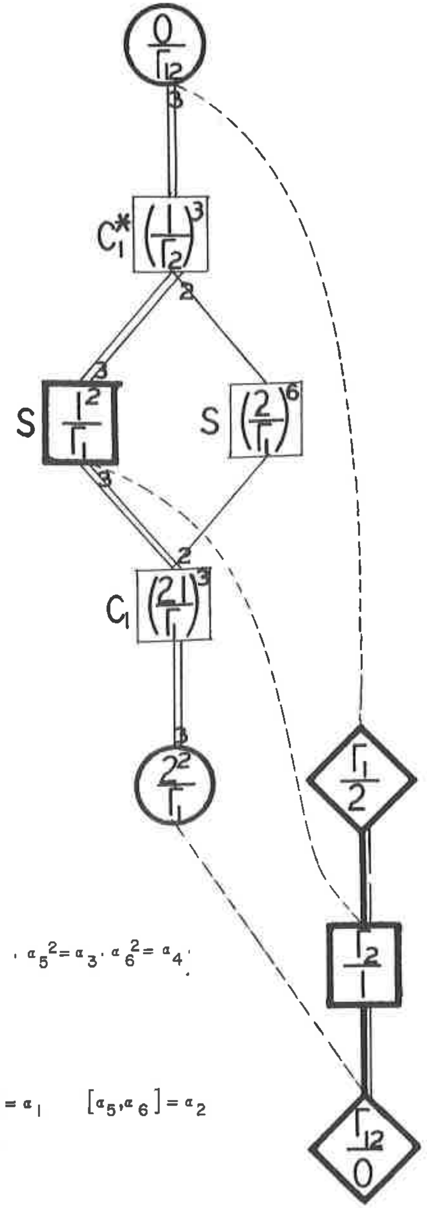
α_2 = qs.rt.uw.vx

α_3 = abcd.efgh:ijkl.mnop

α_4 = ik.jl.mo.np.qrst.uvwx

α_5 = ai.bj.ck. dl.e m. fn.go.hp.uw.vx

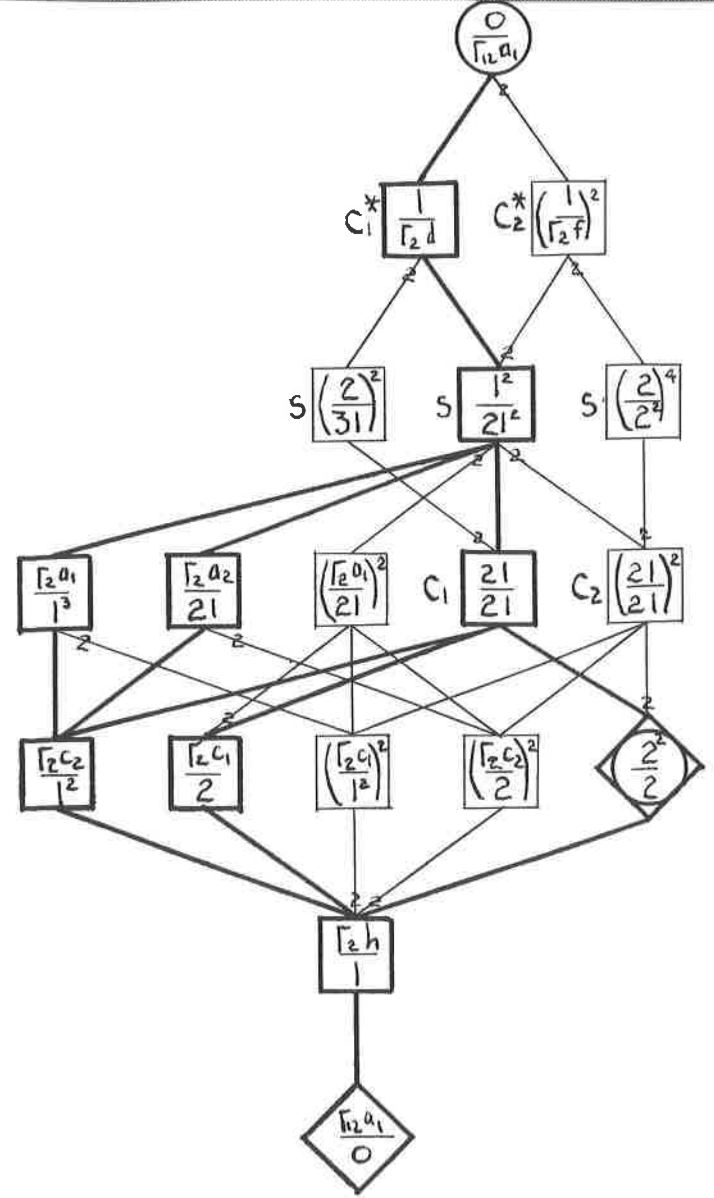
α_6 = aecg.bhdf.imko.jpln. qusw.rvtx



$\alpha_1^2 = 1 \qquad \alpha_2^2 = \alpha_1 \quad , \alpha_5^2 = \alpha_3 , \alpha_6^2 = \alpha_4 .$

$\alpha_3^2 \equiv \alpha_4^2 \equiv 1$

$[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1 \qquad [\alpha_5, \alpha_6] = \alpha_2$



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

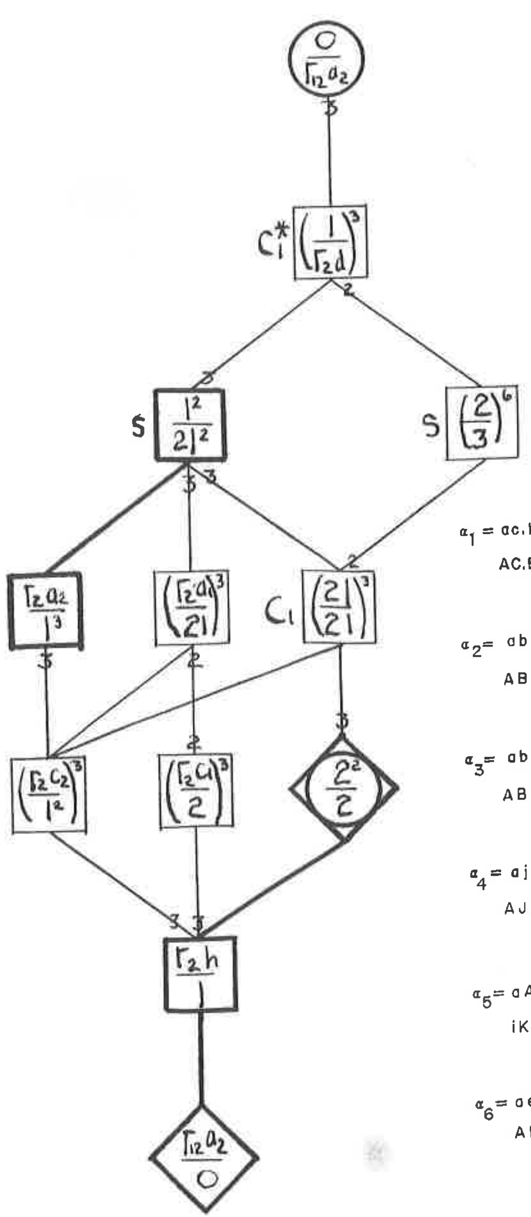
$\alpha_2 =$ abcd.efgh.ijkl.mnop

$\alpha_3 =$ ajcl.bkdi.engp.fohm

$\alpha_4 =$ eg.fh.mo.np

$\alpha_5 =$ aejncglp,bfkodhim

$\alpha_6 =$ ikjl.efgh.mpon



α_1 = ac.bd.eg.fh.ik.jl.mo.np
AC.BD.EG.FH.IK.JL.MO.NP

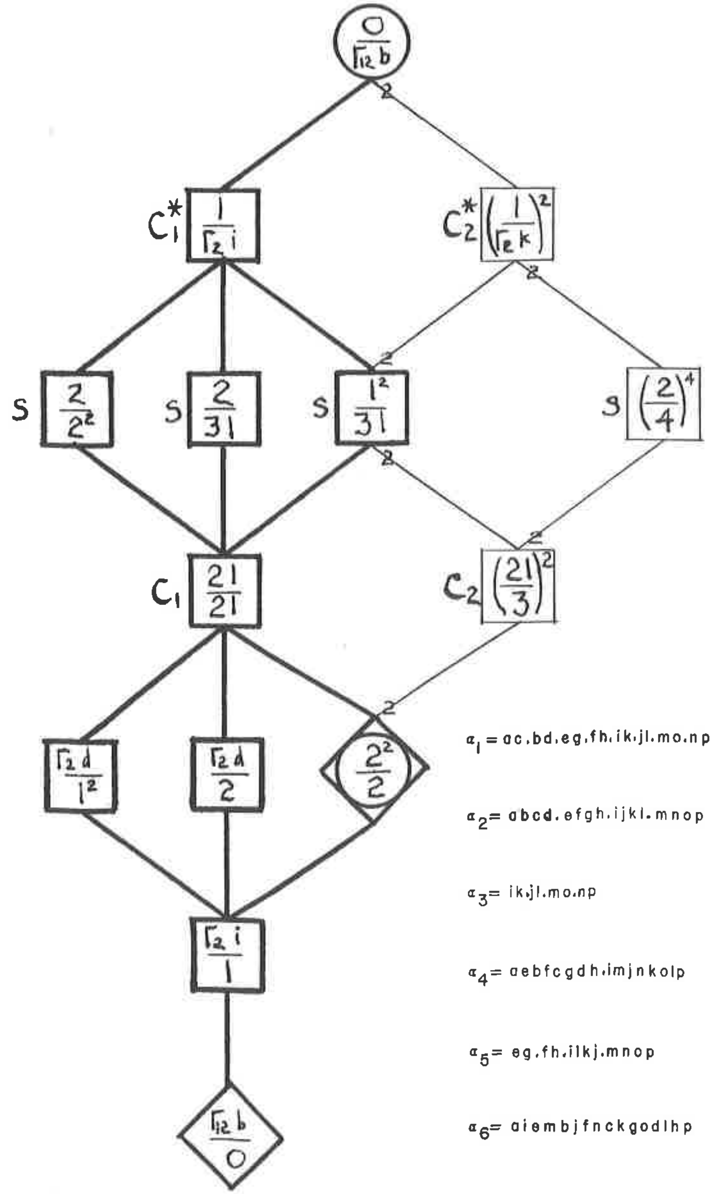
α_2 = abcd.efgh.ijkl.mnop
ABCD.EFGH.IJKL.MNOP

α_3 = abcd.ehgf.ijkl.mpon
ABCD.EHGF.IJKL.MPON

α_4 = ajcl.bkdi.engp.fohm
AJCL.BKDI.ENGP.FOHM

α_5 = aAbBcCdD.eHhGgFfE
iKjLkIlJ.mNpMoPnO

α_6 = aejncglp.bfkodhlm
AEJNCGLP.BFKODHIM



α_1 = ac.bd.eg.fh.ik.jl.mo.np

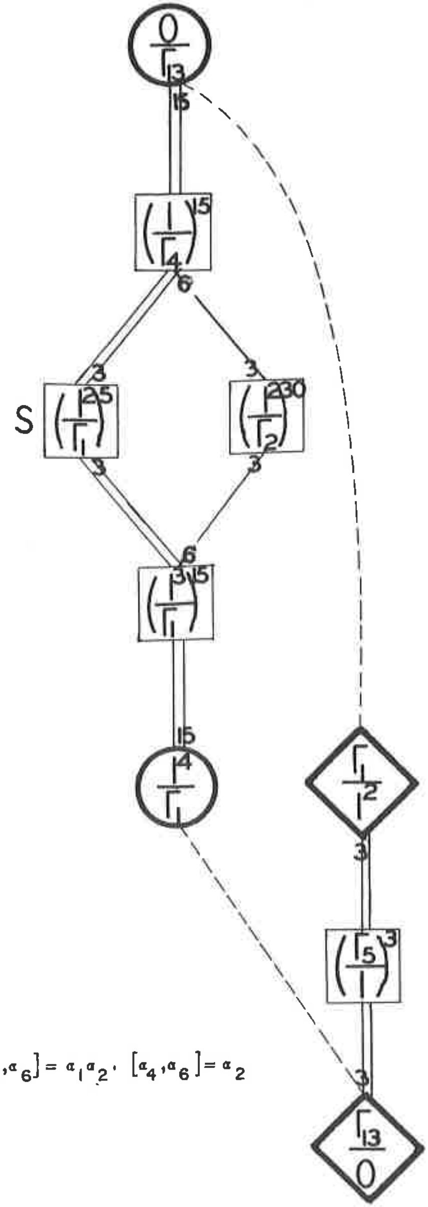
α_2 = abcd.efgh.ijkl.mnop

α_3 = ik.jl.mo.np

α_4 = aebfcgdh.imjnkolp

α_5 = eg.fh.ilkj.mnop

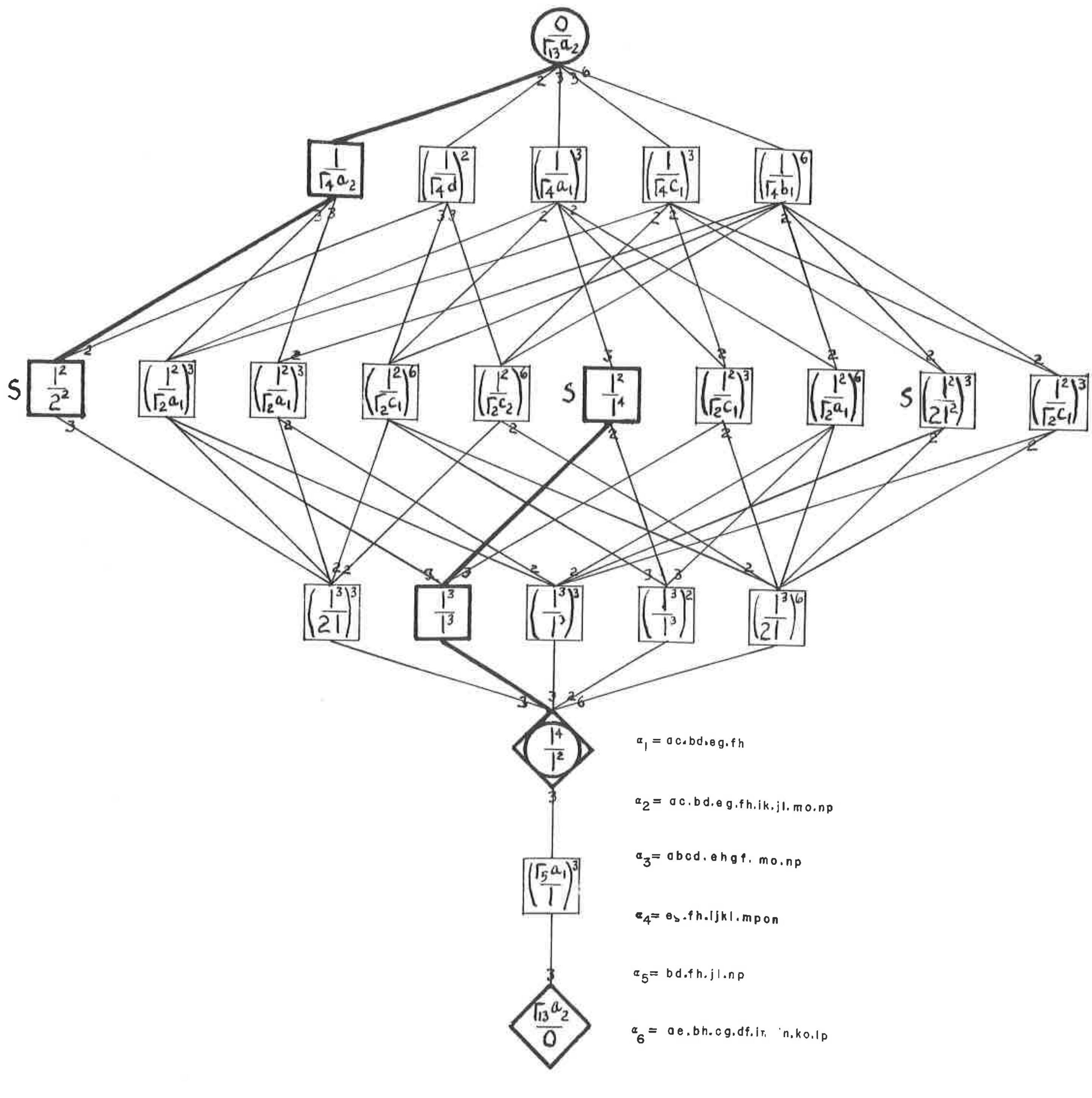
α_6 = aiembjfnckgodlhp



$$\alpha_1{}^2 = \alpha_2{}^2 = 1$$

$$\alpha_3{}^2 \equiv \alpha_4{}^2 \equiv \alpha_5{}^2 \equiv \alpha_6{}^2 \equiv 1$$

$$[\alpha_3, \alpha_5] = \alpha_1 \quad [\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1 \alpha_2, \quad [\alpha_4, \alpha_6] = \alpha_2$$



$\alpha_1 =$ ac.bd.eg.fh

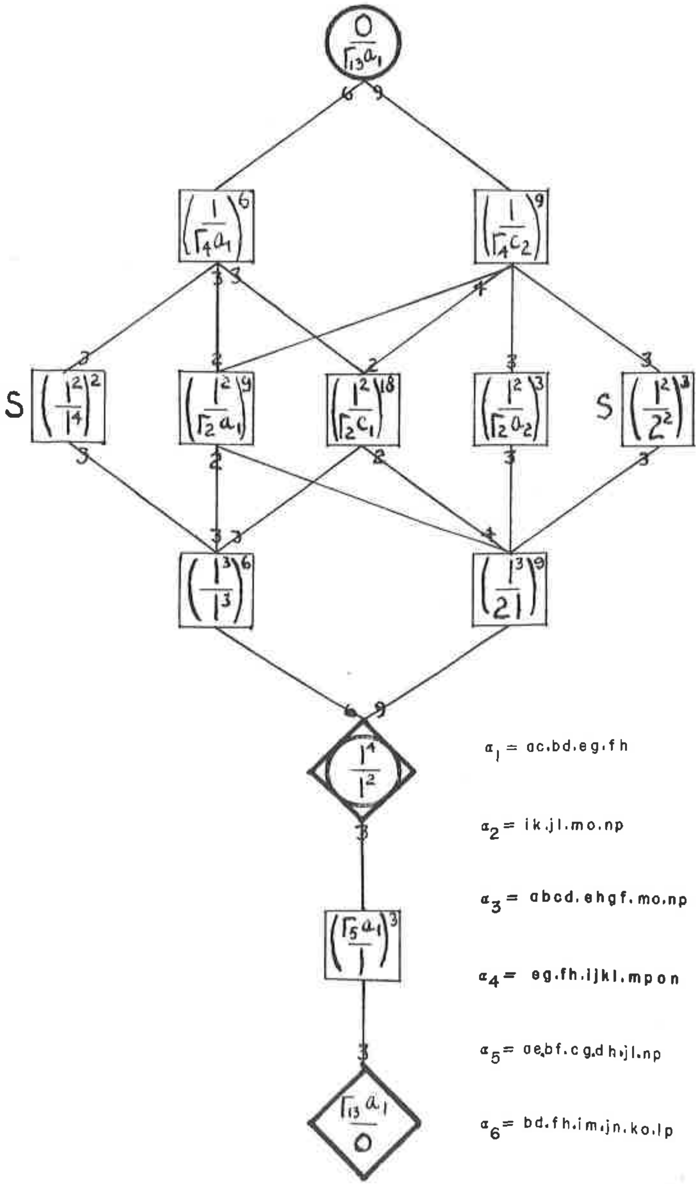
$\alpha_2 =$ ac.bd.eg.fh.ik.jl.mo.np

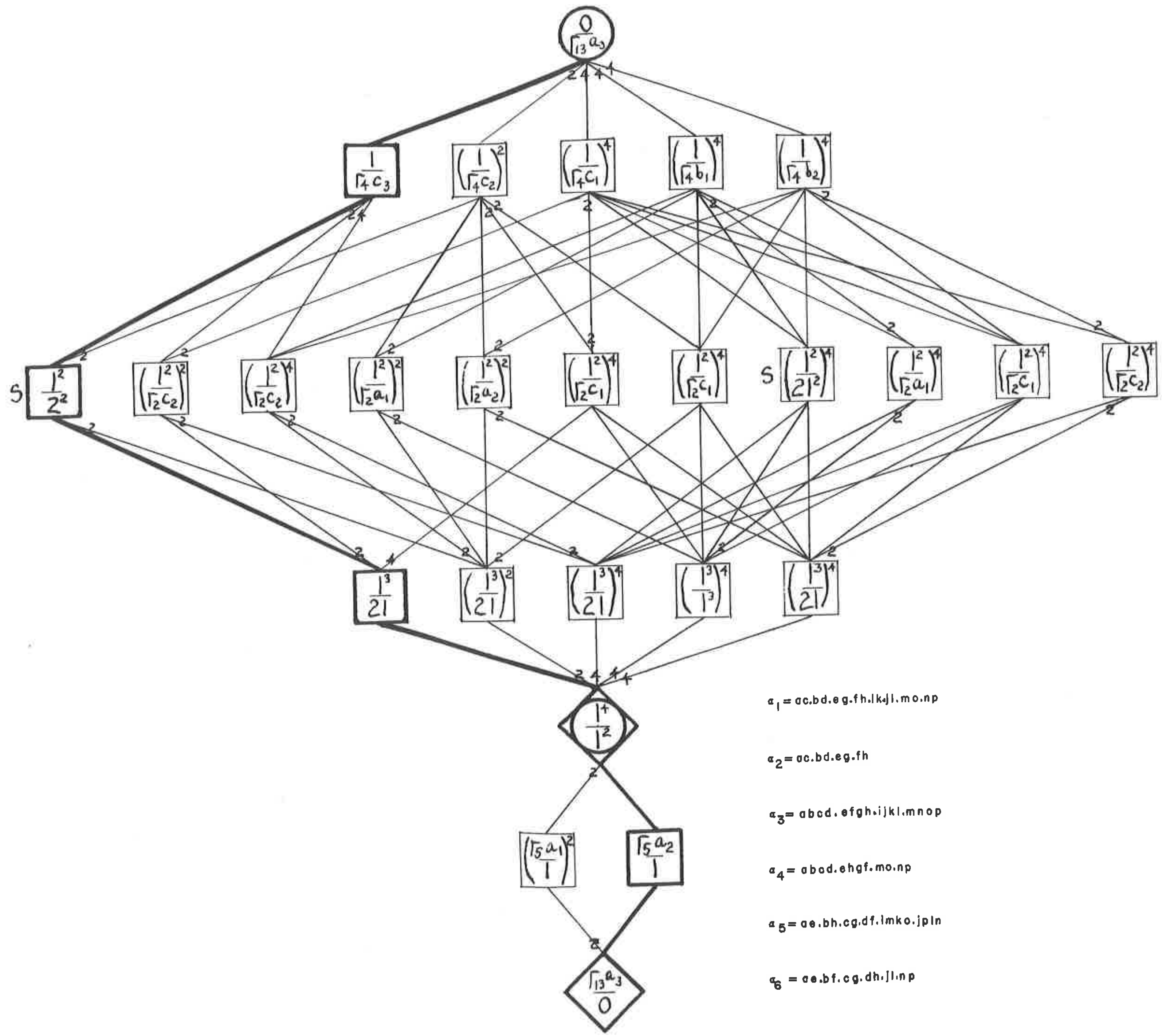
$\alpha_3 =$ abcd.ehgf. mo.np

$\alpha_4 =$ e[illegible].fh.ijkl.mpon

$\alpha_5 =$ bd.fh.jl.np

$\alpha_6 =$ ae.bh.cg.df.i[illegible]n.ko.lp


α1 = ac.bd.eg.fh
α2 = ik.jl.mo.np
α3 = abcd.ehgf.mo.np
α4 = eg.fh.ijkl.mpon
α5 = ae.bf.cg.dh.jl.np
α6 = bd.fh.im.jn.ko.lp



α_1 = ac.bd.eg.fh.lk.jl.mo.np

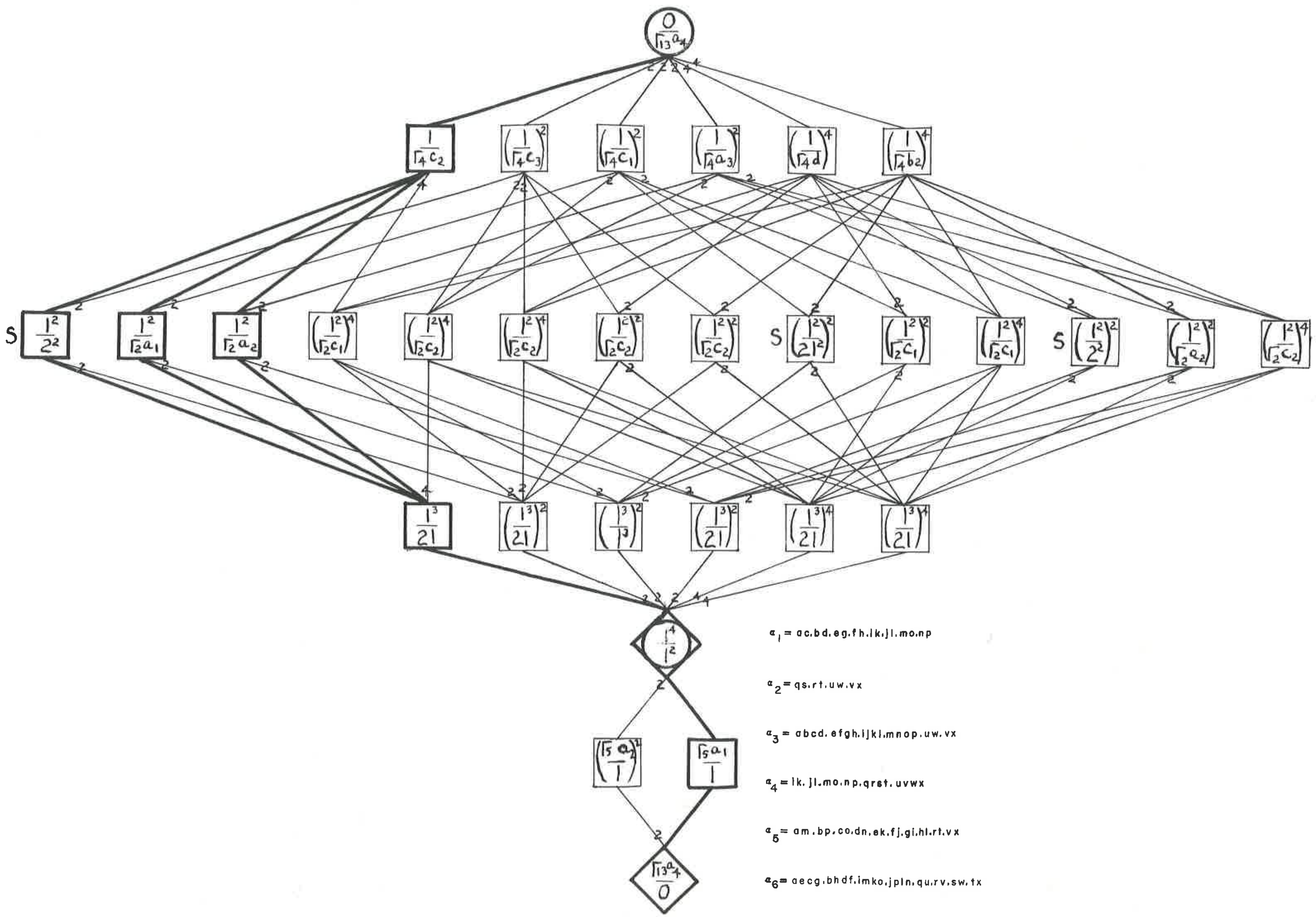
α_2 = ac.bd.eg.fh

α_3 = abcd.efgh.ijkl.mnop

α_4 = abcd.ehgf.mo.np

α_5 = ae.bh.cg.df.lmko.jpln

α_6 = ae.bf.cg.dh.jl.np



α_1 = ac.bd.eg.fh.ik.jl.mo.np

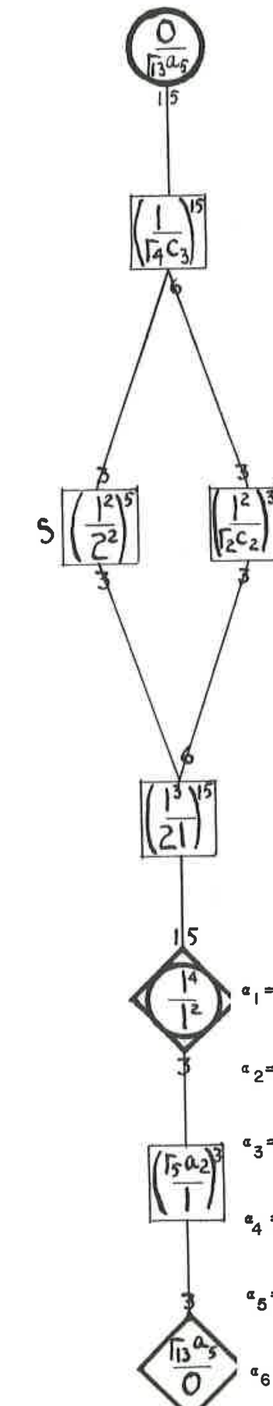
α_2 = qs.rt.uw.vx

α_3 = abcd.efgh.ijkl.mnop.uw.vx

α_4 = lk.jl.mo.np.qrst.uvwx

α_5 = am.bp.co.dn.ek.fj.gi.hl.rt.vx

α_6 = aecg.bhdf.imko.jpln.qu.rv.sw.tx



α_1 = ac.bd.eg.fh.ik.jl.mo.np

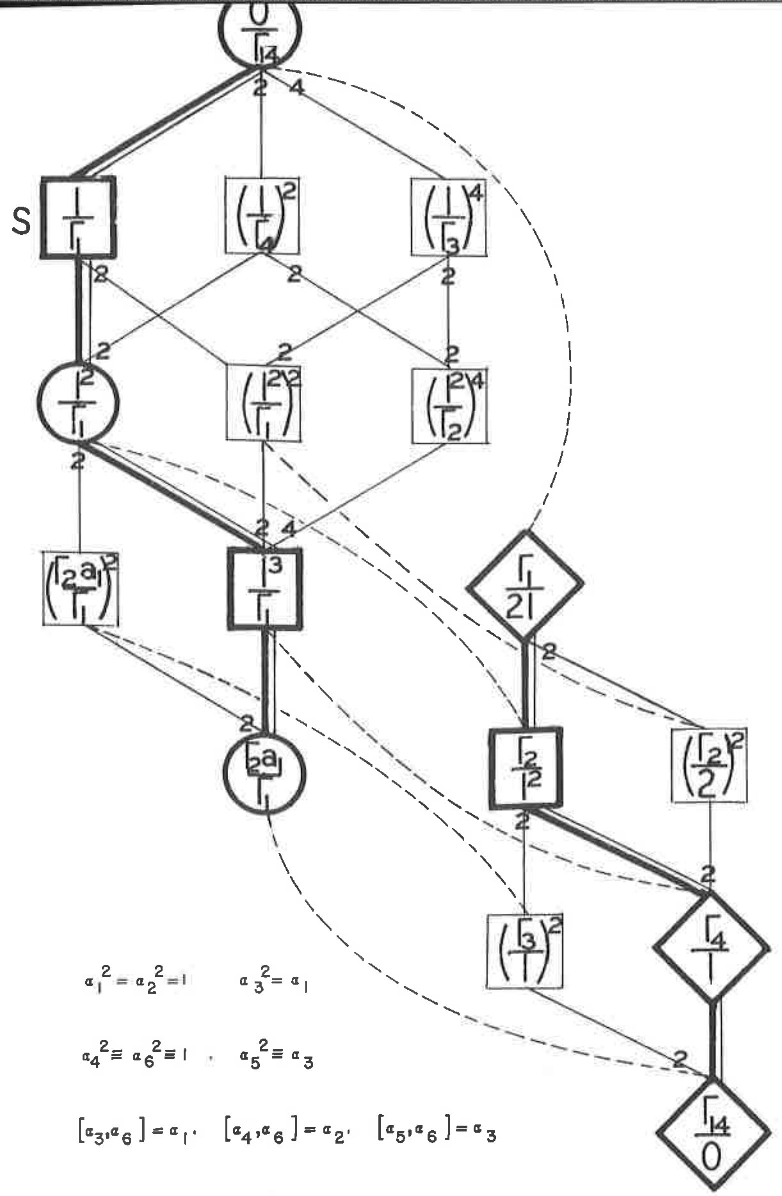
α_2 = AC.BD.EG.FH.IK.JL.MO.NP

α_3 = abcd.ehgf.ijkl.mpon.EG.FH.MO.NP

α_4 = eg.fh.mo.np.ABCD.EHGF.IJKL.MPON

α_5 = ae.bf.cg.dh.im.jn.ko.lp.AICK.BLDJ.EMGO.FPHN

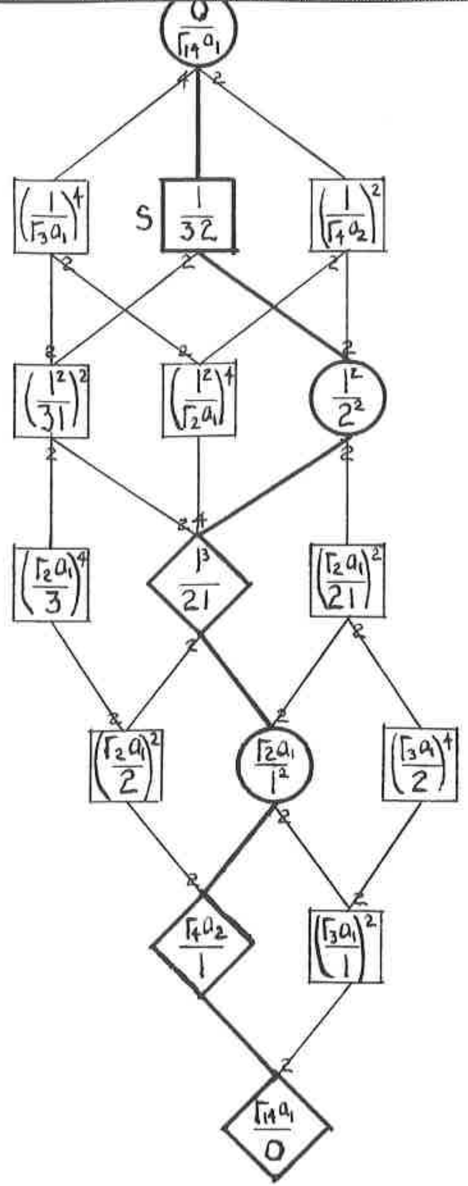
α_6 = aick.bldj.emgo.fphn.AGCE.BHDF.IOKM.JPLN



$\alpha_1^2 = \alpha_2^2 = 1 \qquad \alpha_3^2 = \alpha_1$

$\alpha_4^2 \equiv \alpha_6^2 \equiv 1 \;, \quad \alpha_5^2 \equiv \alpha_3$

$[\alpha_3, \alpha_6] = \alpha_1, \quad [\alpha_4, \alpha_6] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3$



α_1 = ac.bd.eg.fh

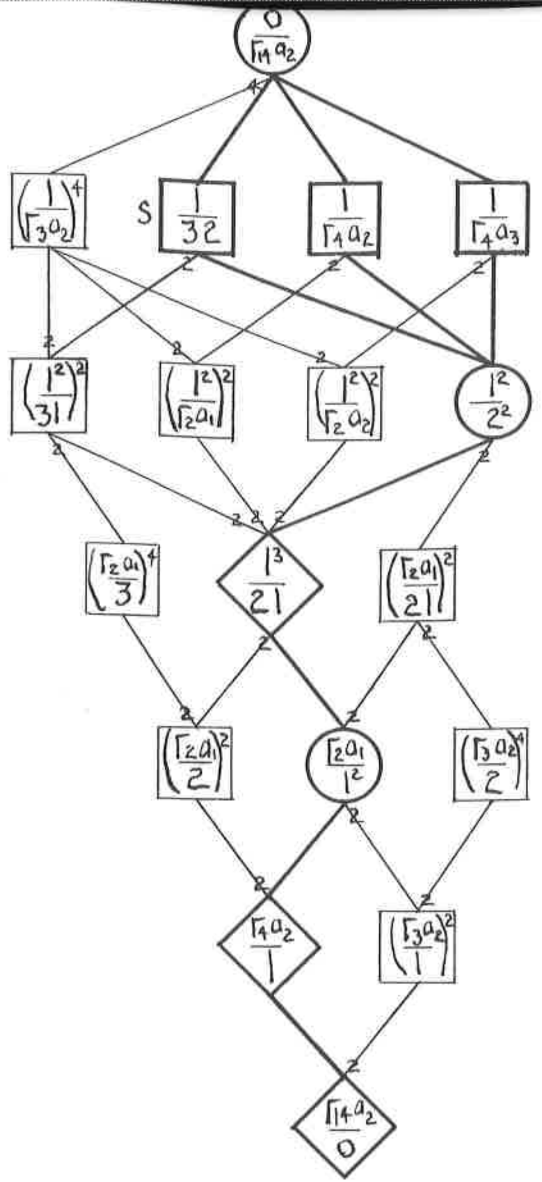
α_2 = ik.jl

α_3 = adcb.ehgf

α_4 = ijkl

α_5 = aebfcgdh

α_6 = bd.eh.fg.jl



α_1 = ac.bd.eg.fh

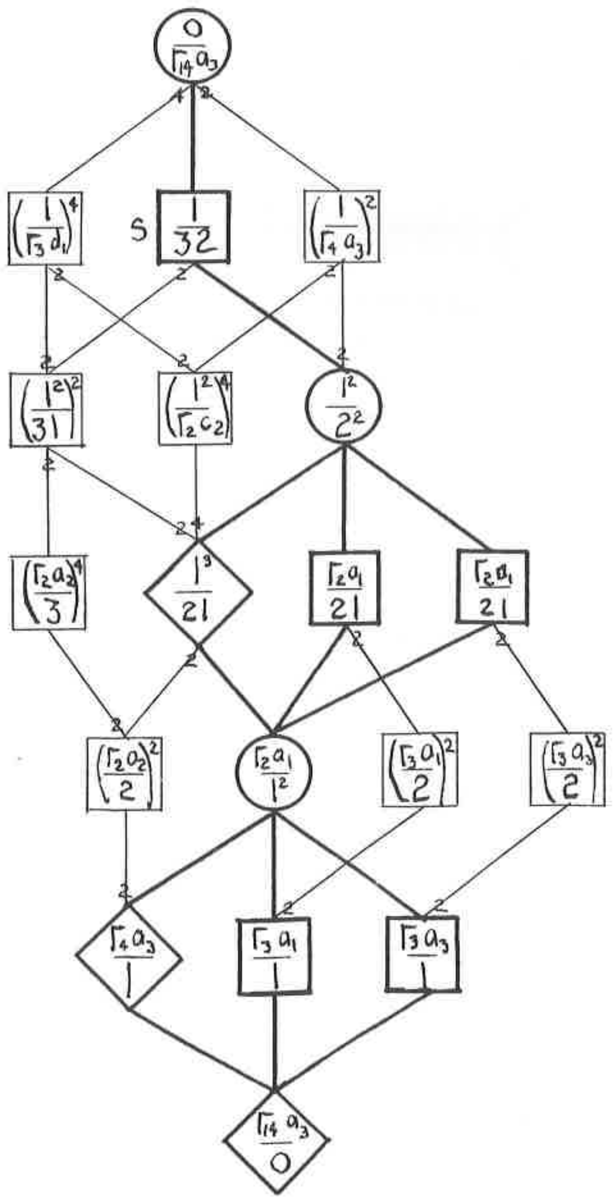
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ijkl

α_5 = aebfcgdh

α_6 = bd.ef.gh.jl



α_1 = ac.bd.eg.fh

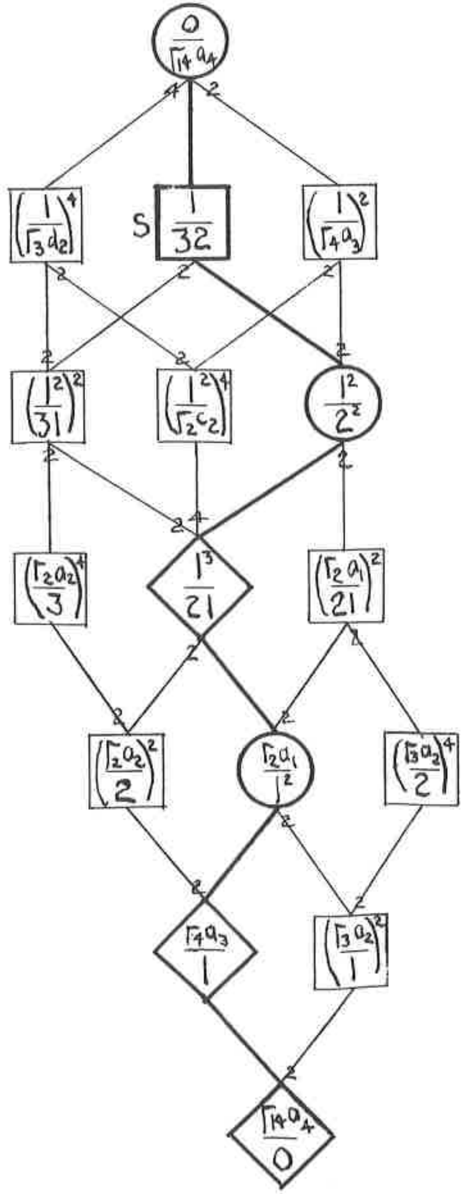
α_2 = ik .jl.mo.np

α_3 = adcb.ehgf

α_4 = ijkl. mnop

α_5 = aebfcgdh

α_6 = bd.eh.fg.imko.jpln



α_1 = ac.bd.eg.fh

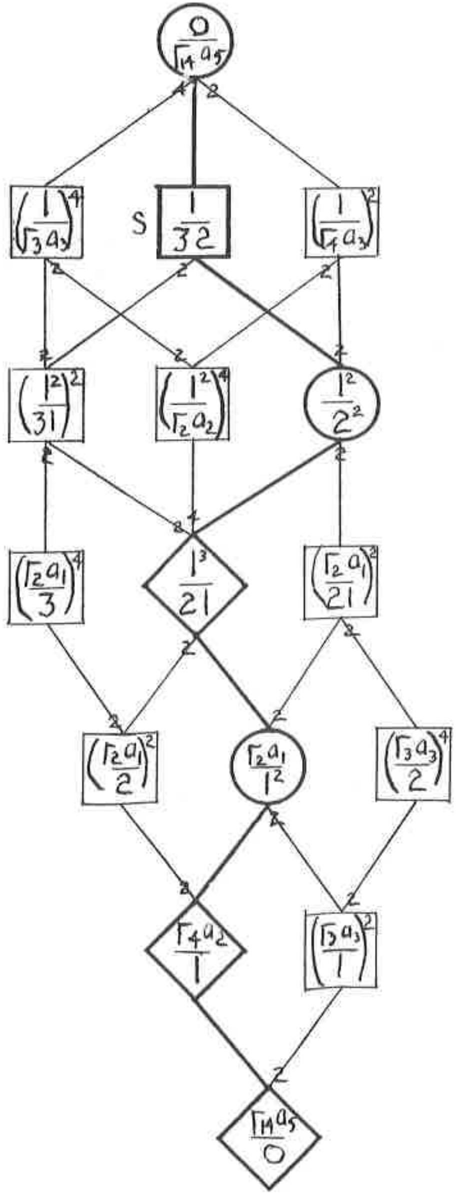
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = ijkl. mnop

α_5 = aebfcgdh

α_6 = bd.ef.gh.imko.jpln



α_1 = ac.bd.eg.fh.ik.jl.mo.np

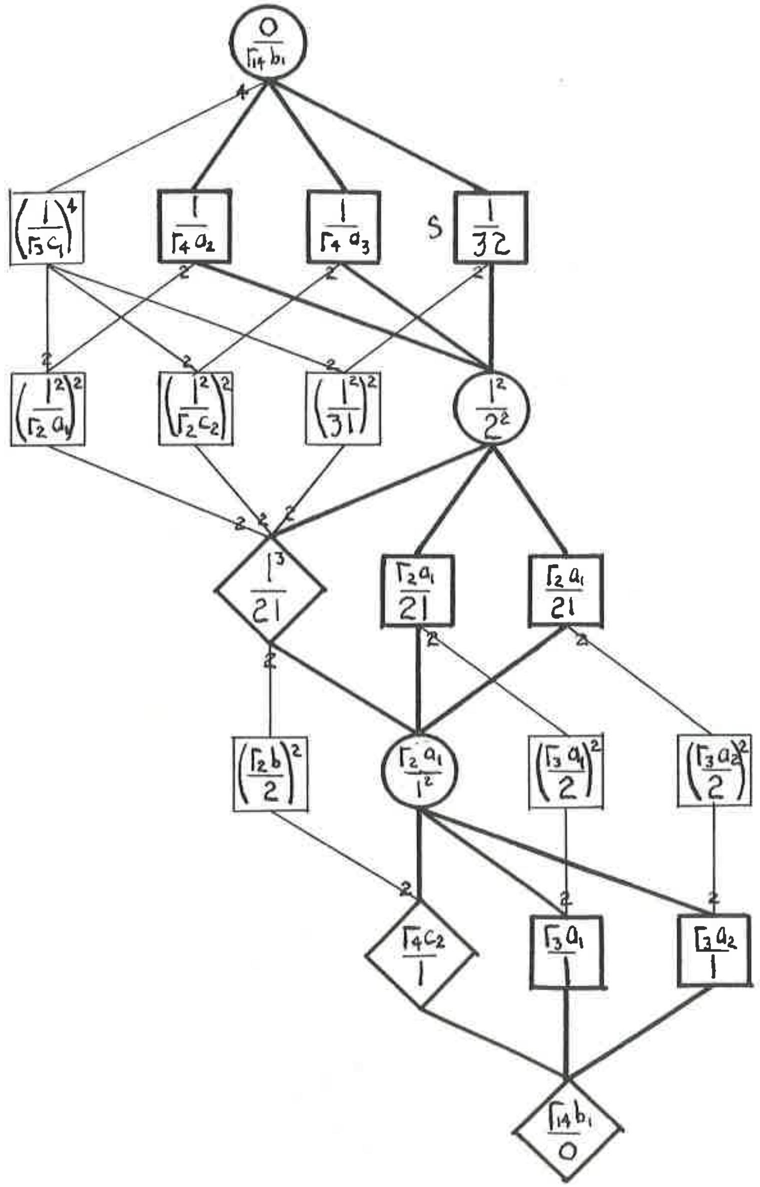
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = qrst

α_5 = aebfcgdh.imjnkolp

α_6 = aick.bldj.epgn.fohm.rt



α_1 = ac.bd.eg.fh

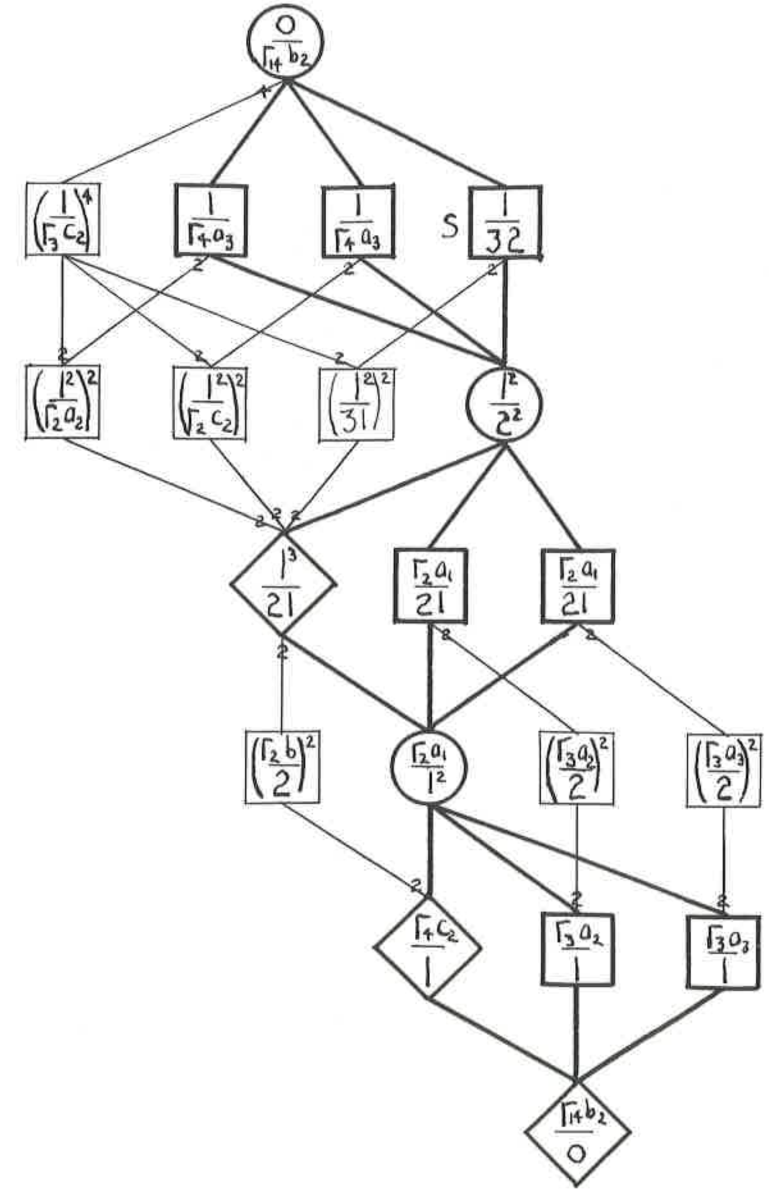
α_2 = ik.jl.mo.np

α_3 = adcb.ehgf.ik.jl.mo.np

α_4 = ijkl.mpon

α_5 = aebfcgdh.mo.np

α_6 = bd.eh.fg.im.jn.ko.lp



α_1 = ac.bd.eg.fh

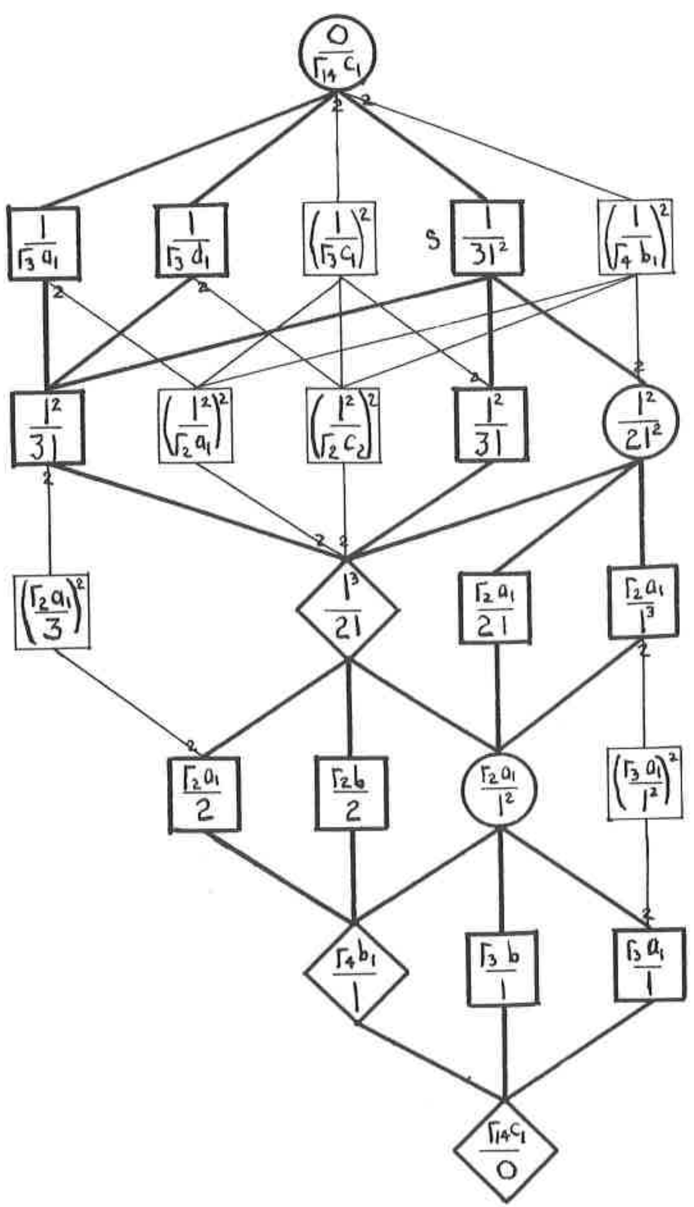
α_2 = ac.bd.eg.fh.ik.jl.mo.np

α_3 = abcd.efgh.ik.jl.mo.np

α_4 = abcd.efgh.ijkl.mpon

α_5 = aebfcgdh.mo.np

α_6 = aecg.bhdf.im.jn.ko.lp



α_1 = ac.bd.eg.fh

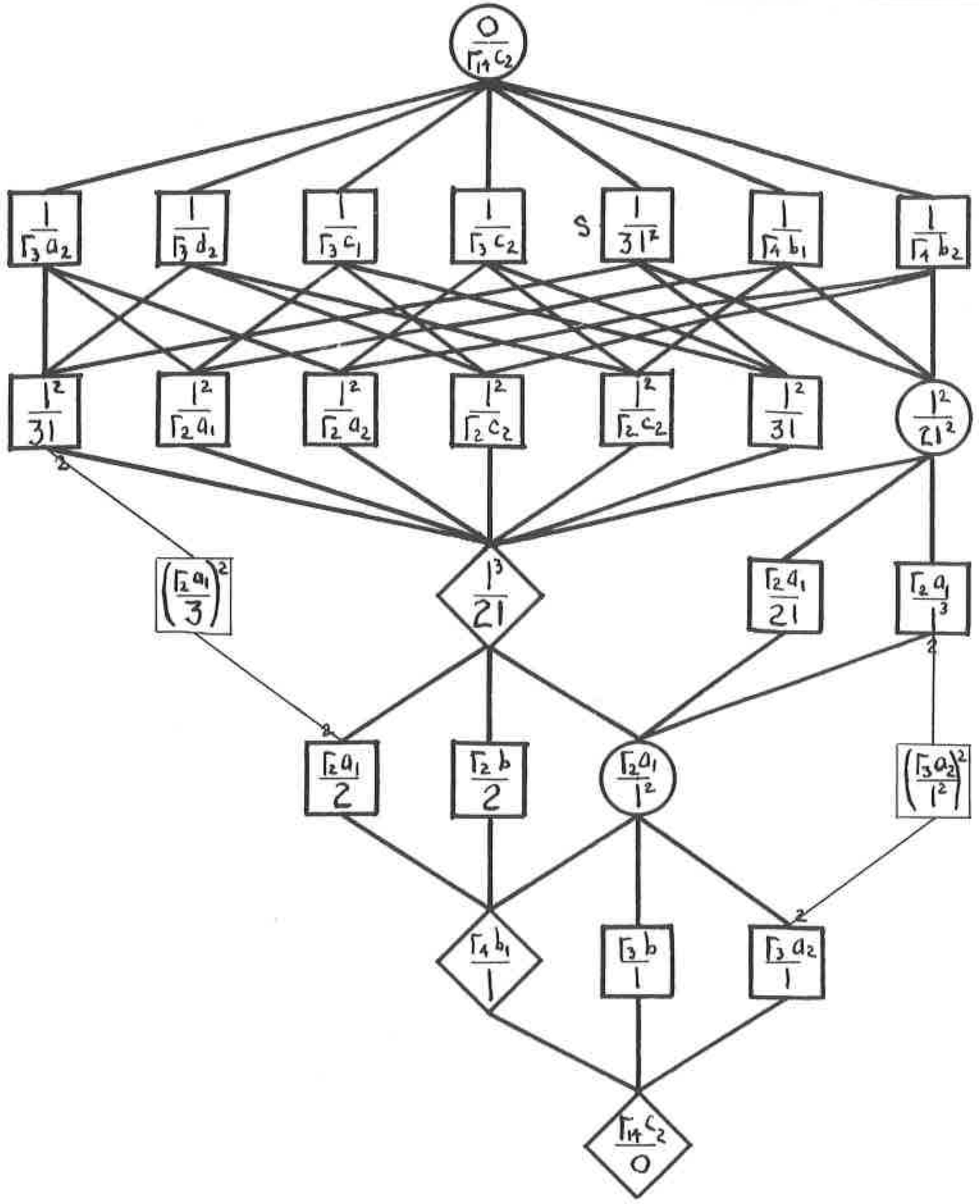
α_2 = ik.jl

α_3 = adcb.ehgf

α_4 = ij.kl

α_5 = aebfcgdh

α_6 = bd.eh.fg.jl



α_1 = ac.bd.eg.fh

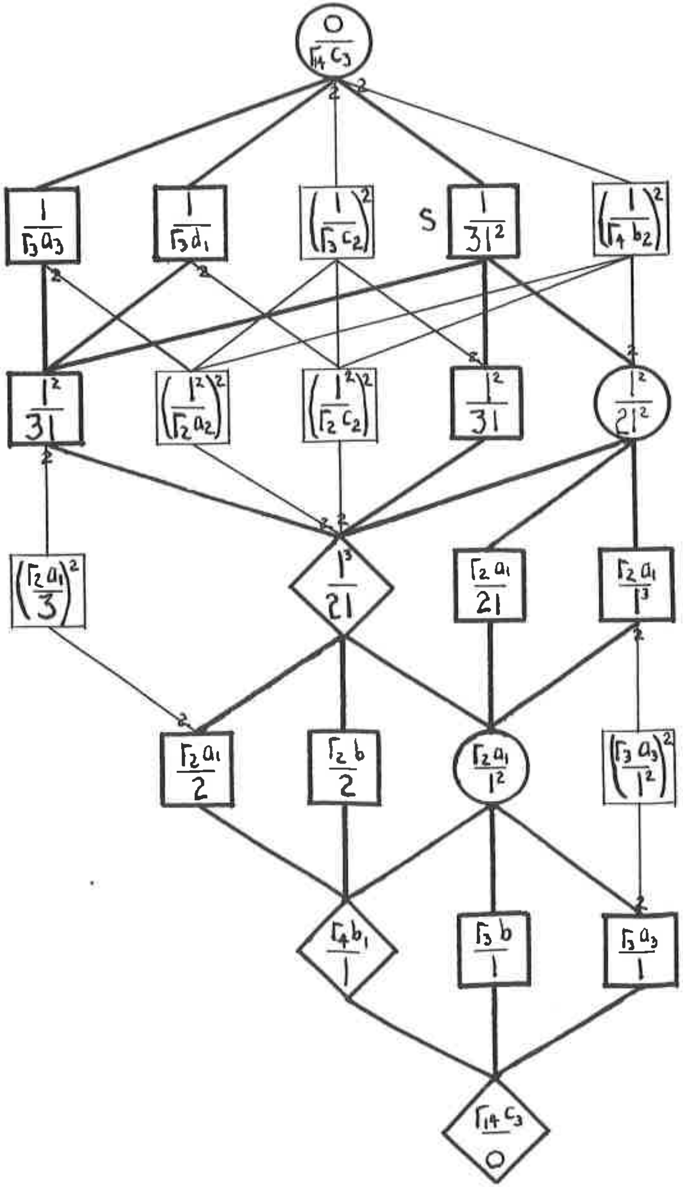
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ij.kl

α_5 = aebfcgdh

α_6 = bd.ef.gh.jl



α_1 = ac.bd.eg.fh.ik.jl.mo.np

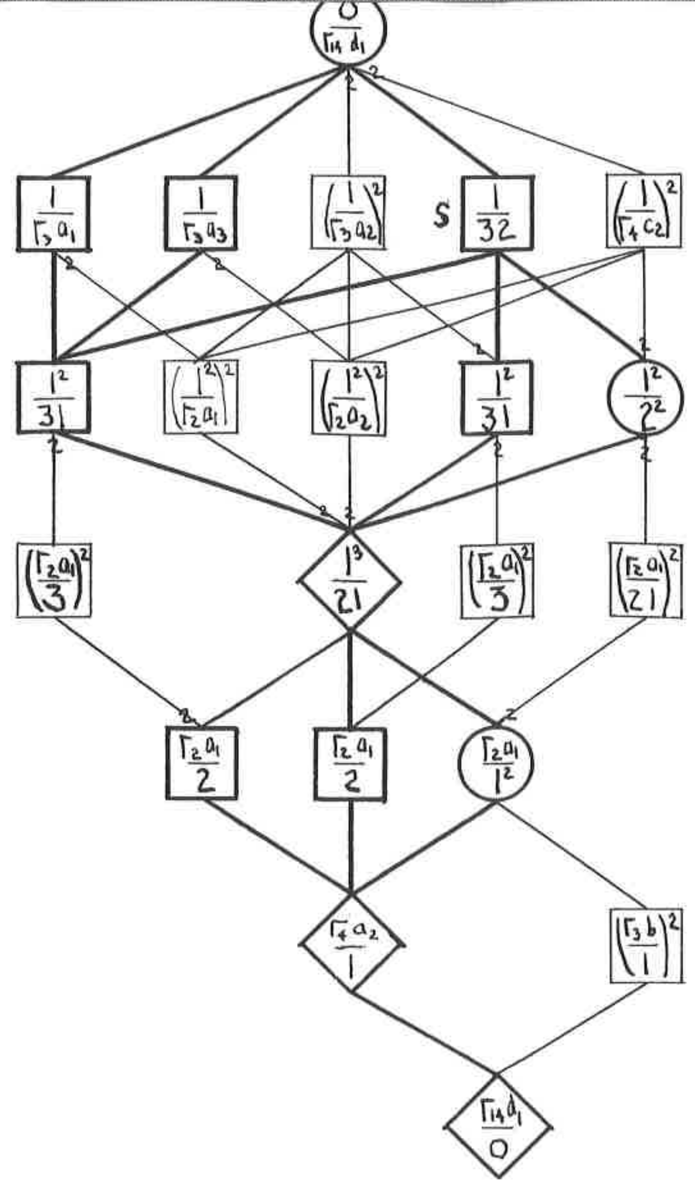
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = qr.st

α_5 = aebfcgdh.imjnkolp

α_6 = aick.bldj.epgn.fohm.rt



α_1 = ac.bd.eg.fh.ik.jl.mo.np

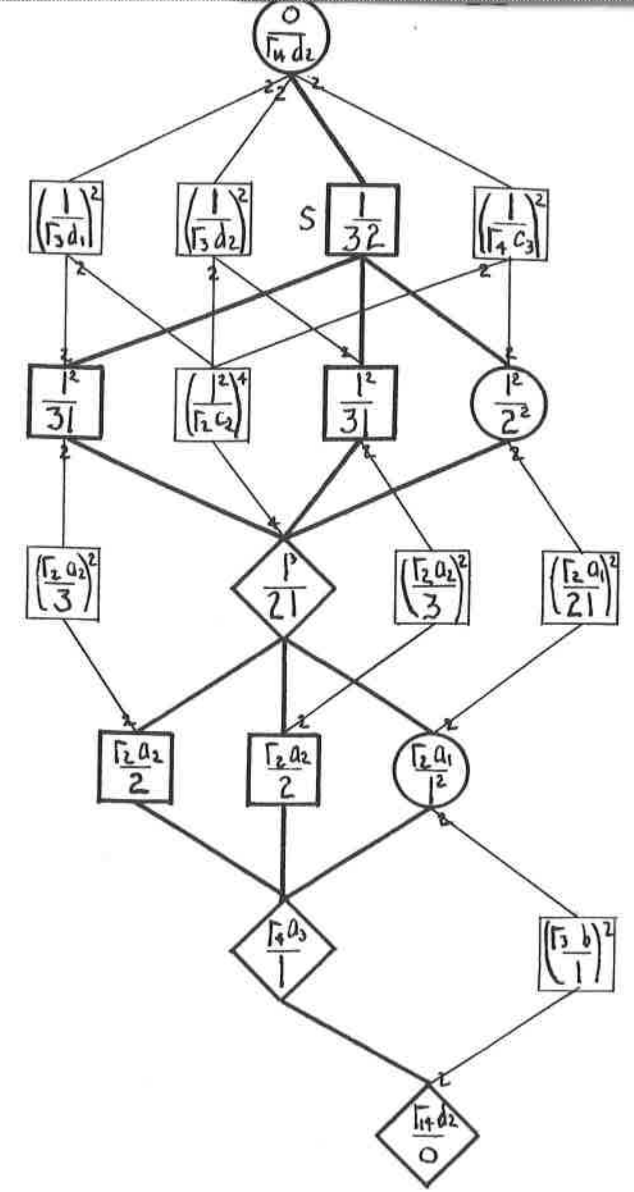
α_2 = qs.rt

α_3 = adcb.ehgf.llkj.mpon

α_4 = aick.bjdl.emgo.fnhp.qrst

α_5 = aebfcgdh.imjnkolp

α_6 = bd.eh.fg.jl.mp.no.rt



α_1 = ac.bd.eg.fh.ik.jl.mo.np

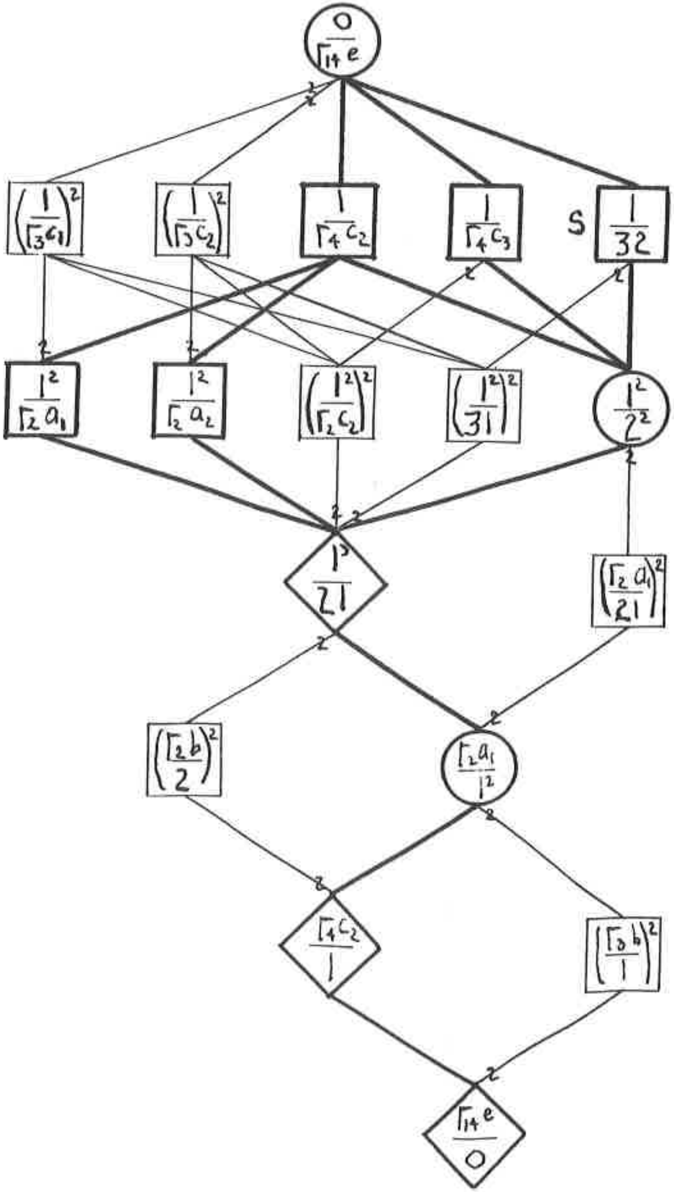
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.llkj.mpon

α_4 = aick.bjdl.emgo.fnhp.qrst.uvwx

α_5 = aebfcgdh.imjnkolp

α_6 = bd.eh.fg.jl.mp.no.qusw.rxtv



α_1 = ac.bd.eg.fh.ik.jl.mo.np

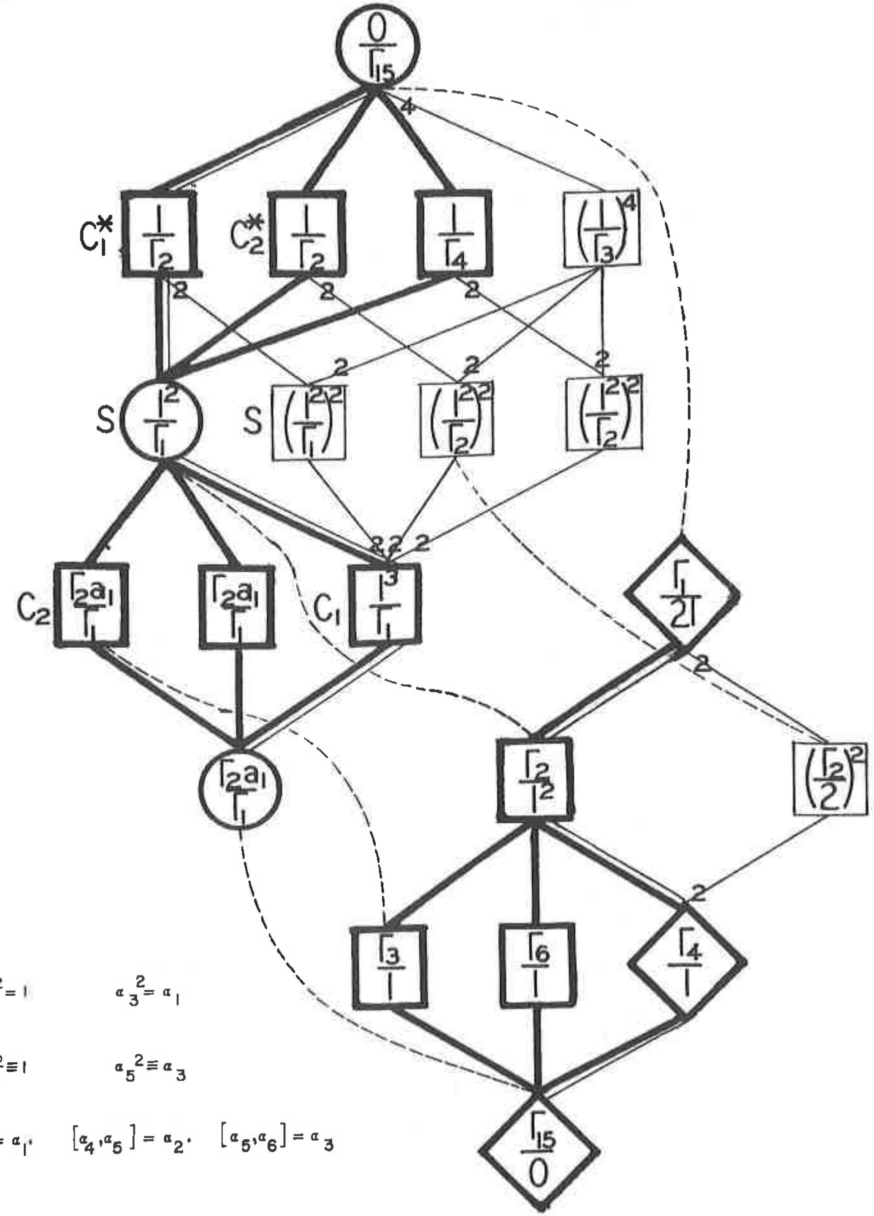
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon.qs.rt.uw.vx

α_4 = aick.bjdl.emgo.fnhp.qrst.uxwv

α_5 = aebfcgdh.imjnkolp.uw.vx

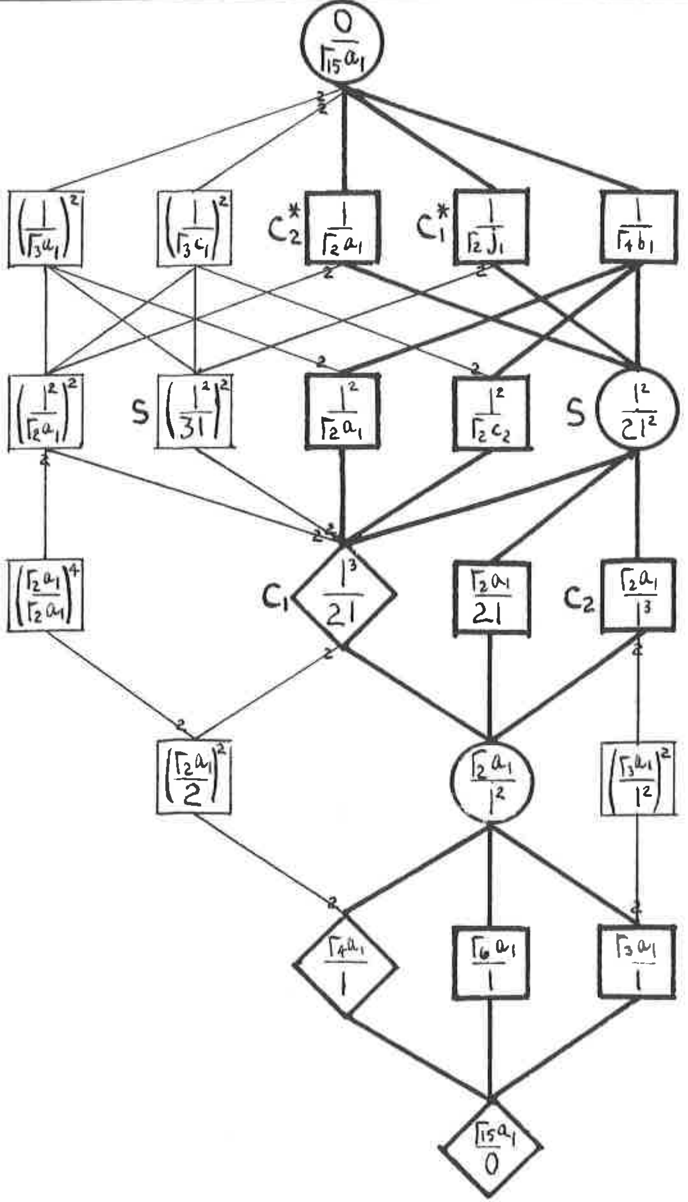
α_6 = bd.eh.fg.jl.mp.on.qu.rv.sw.tx



$\alpha_1^2 = \alpha_2^2 = 1$ $\alpha_3^2 = \alpha_1$

$\alpha_4^2 \equiv \alpha_6^2 \equiv 1$ $\alpha_5^2 \equiv \alpha_3$

$[\alpha_3, \alpha_6] = \alpha_1$, $[\alpha_4, \alpha_5] = \alpha_2$, $[\alpha_5, \alpha_6] = \alpha_3$



$\alpha_1 = ac.bd.eg.fh$

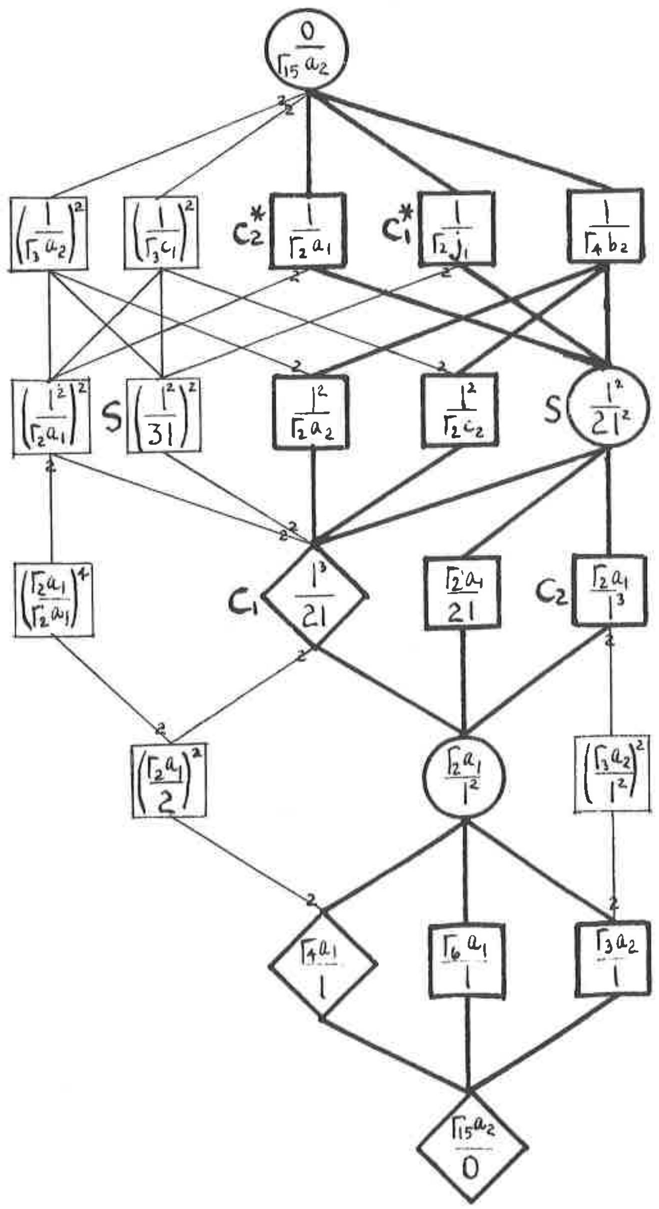
$\alpha_2 = ik.jl$

$\alpha_3 = adcb.ehgf$

$\alpha_4 = lj.kl$

$\alpha_5 = aebf\ cgdh.jl$

$\alpha_6 = bd.eh.fg$



α_1 = ac.bd.eg.fh

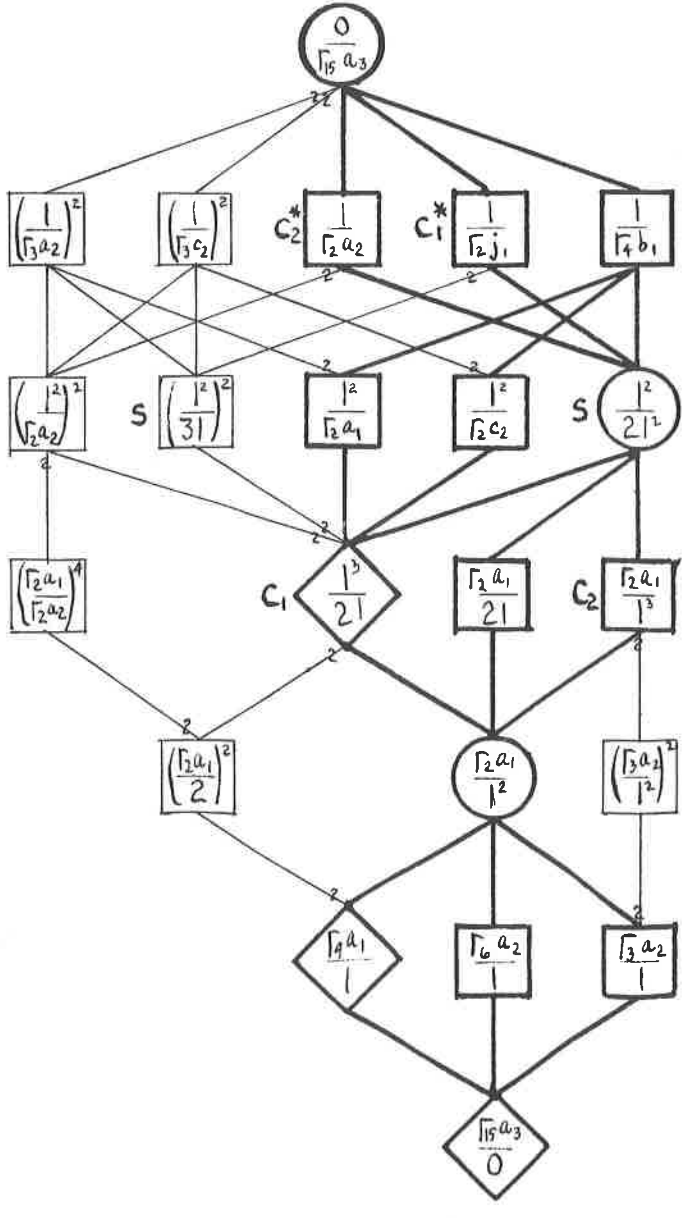
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ij.kl

α_5 = aebfcgdh.jl

α_6 = bd.ef.gh



α_1 = ac.bd.eg.fh

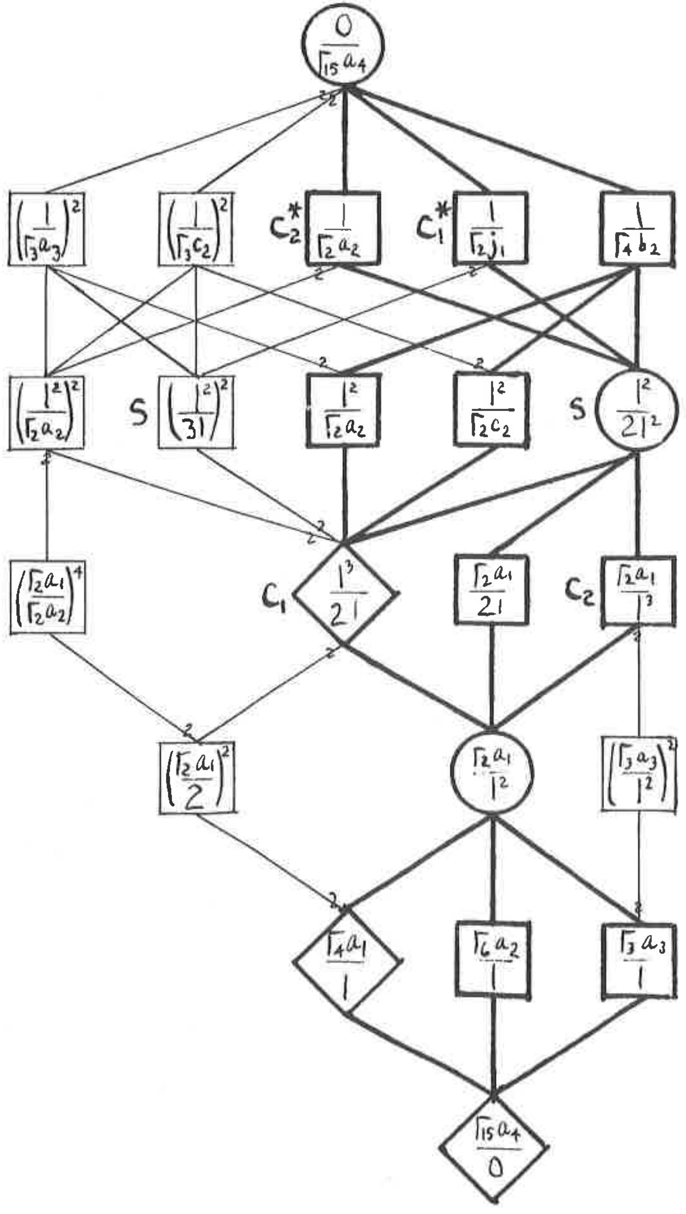
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ij.kl

α_5 = aebfcgdh.jl

α_6 = aecg.bhdf



α_1 = ac.bd.eg.fh.ik.jl.mo.np

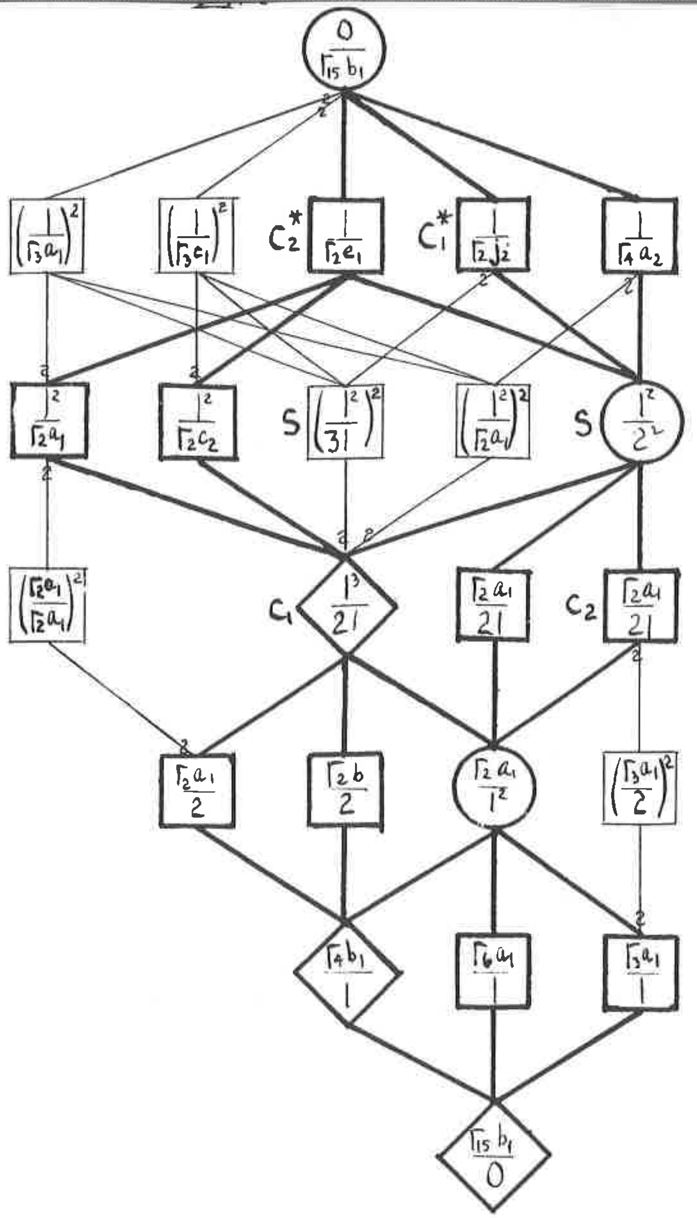
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = qr.st

α_5 = aebfcgdh.imjnkolp.rt

α_6 = alck.bldj.engp.fohm



α_1 = ac.bd.eg.fh

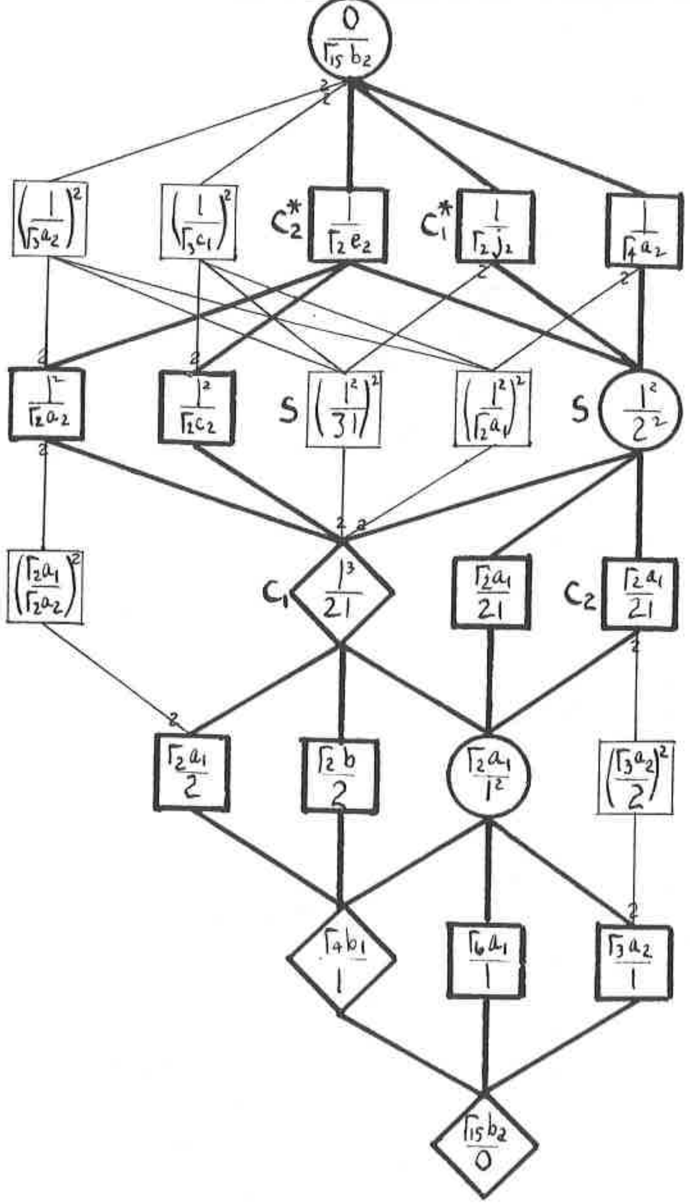
α_2 = lk.jl

α_3 = adcb.ehgf

α_4 = ijkl

α_5 = aebfcgdh.jl

α_6 = bd.eh.fg



α_1 = ac.bd.eg.fh

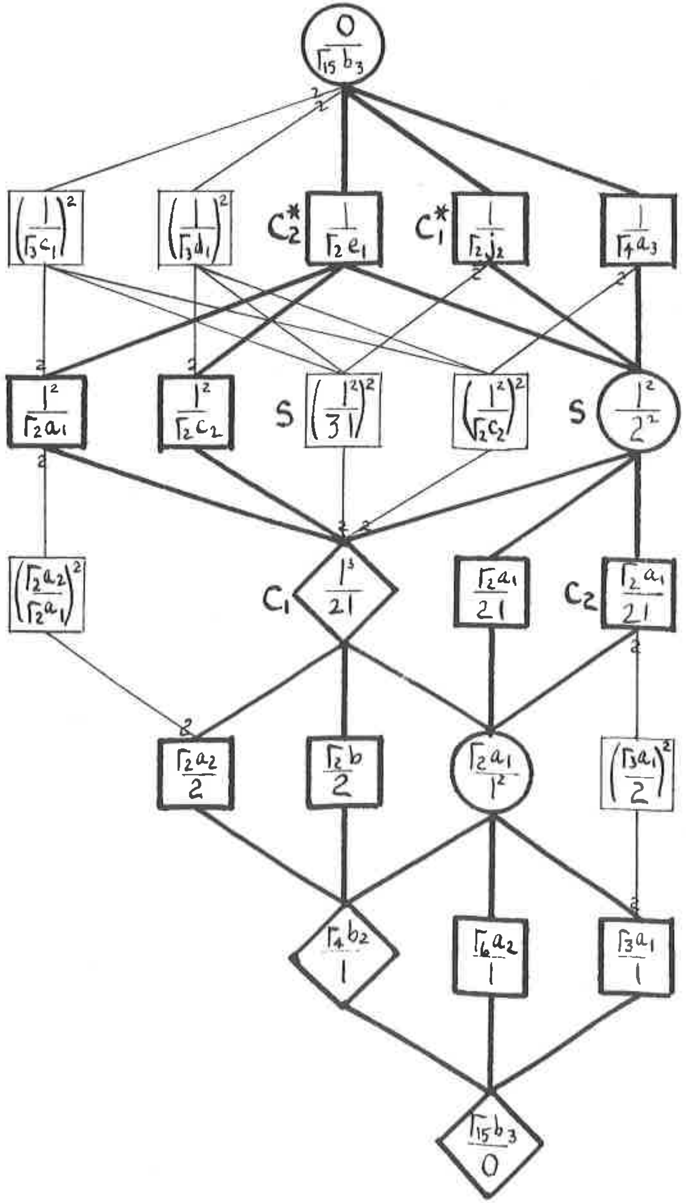
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ijkl

α_5 = aebfcgdh.jl

α_6 = aecg.bhdf



α_1 = ac.bd.eg.fh

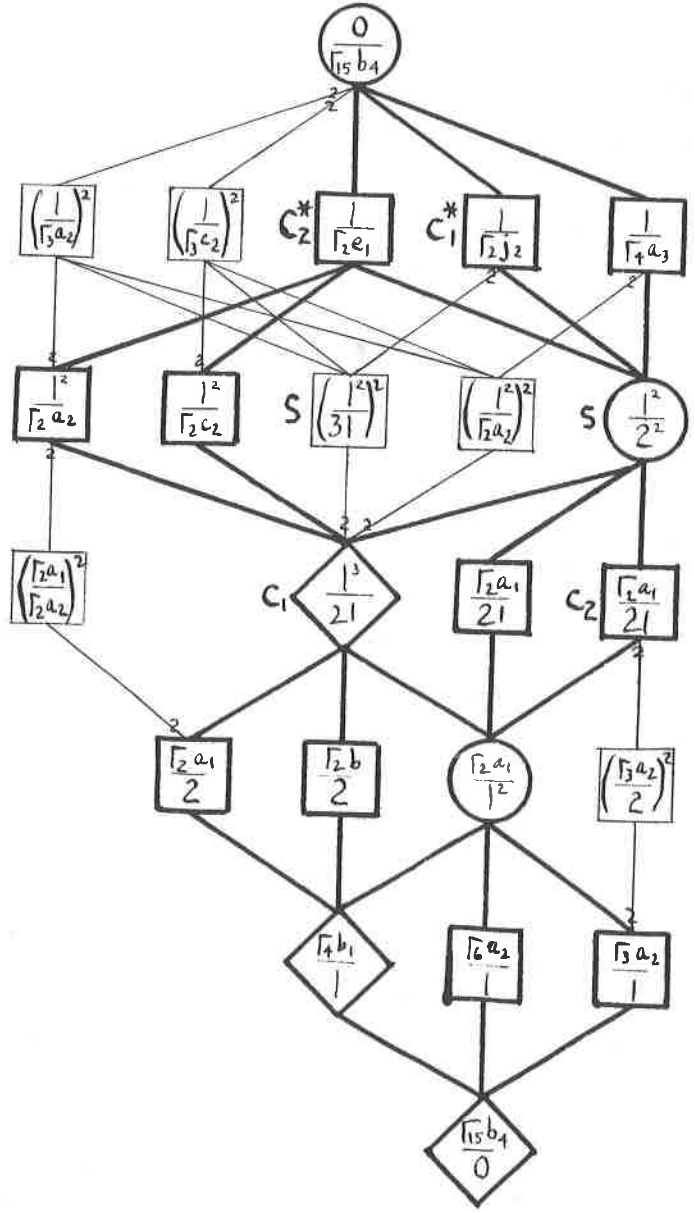
α_2 = ik.jl.mo.np

α_3 = adcb.ehgf

α_4 = ijkl.mnop

α_5 = aebfcgdh.imko.jpln

α_6 = bd.eh.fg



α_1 = ac. bd. eg. fh

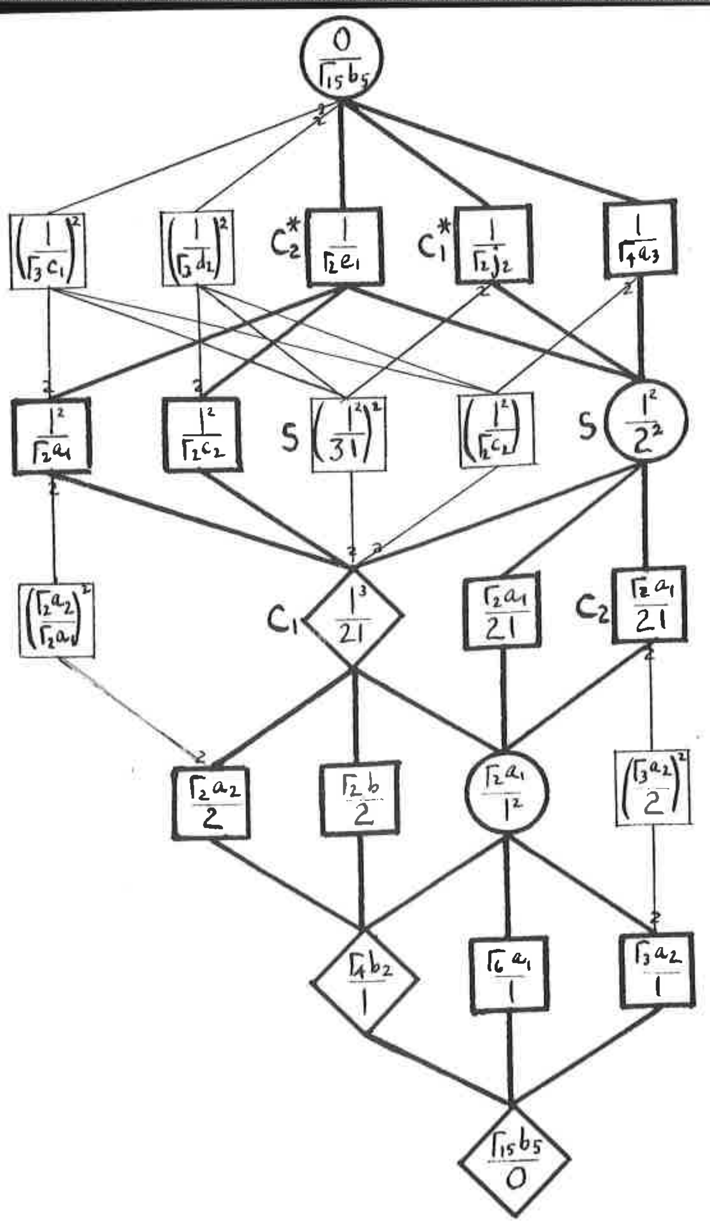
α_2 = ik.jl

α_3 = abcd.efgh

α_4 = ijkl

α_5 = aebfcgdh.jl

α_6 = bd.ef.gh



α_1 = ac.bd.eg.fh

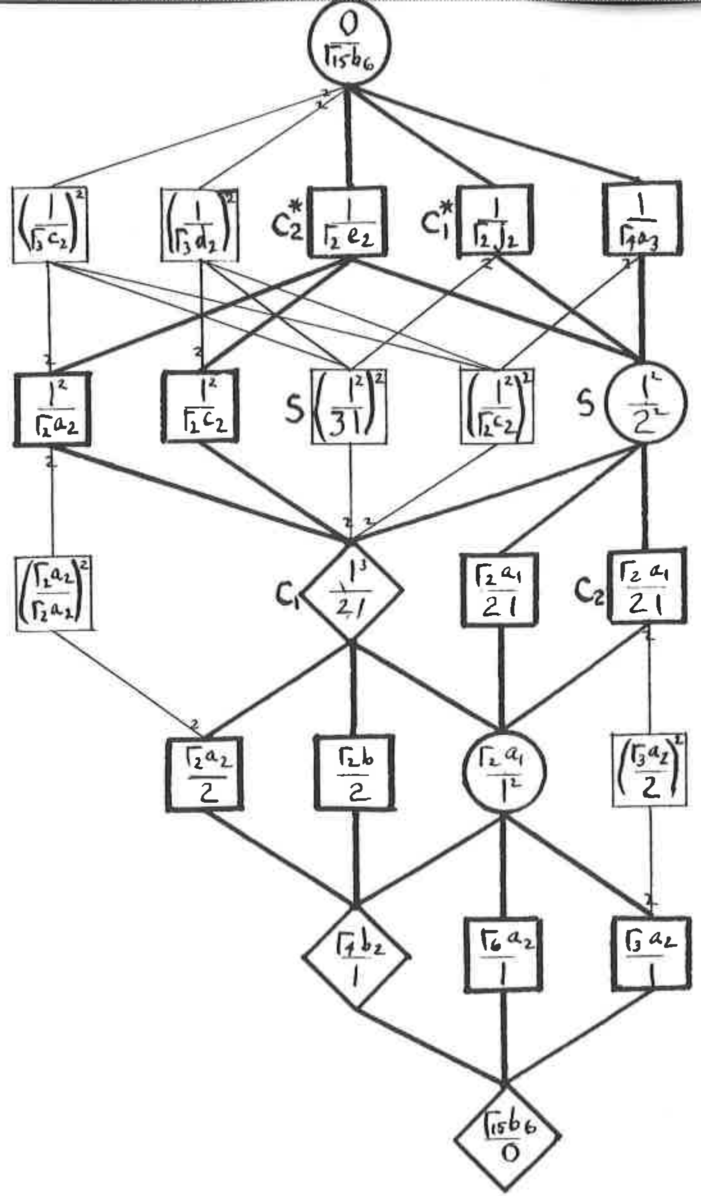
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = ijkl.mnop

α_5 = aebfcgdh.imko.jpln

α_6 = bd.ef.gh



α_1 = ac.bd.eg.fh

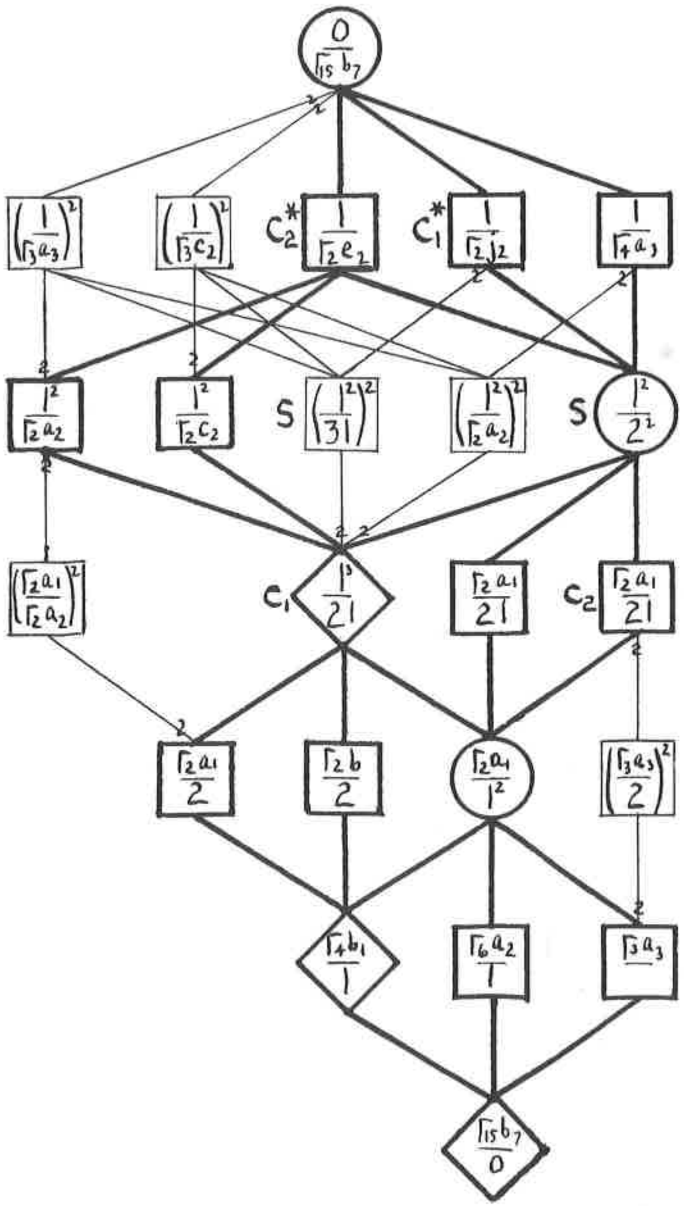
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = ijkl.mnop

α_5 = aebfcgdh.imko.jpln

α_6 = aecg.bhdf



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

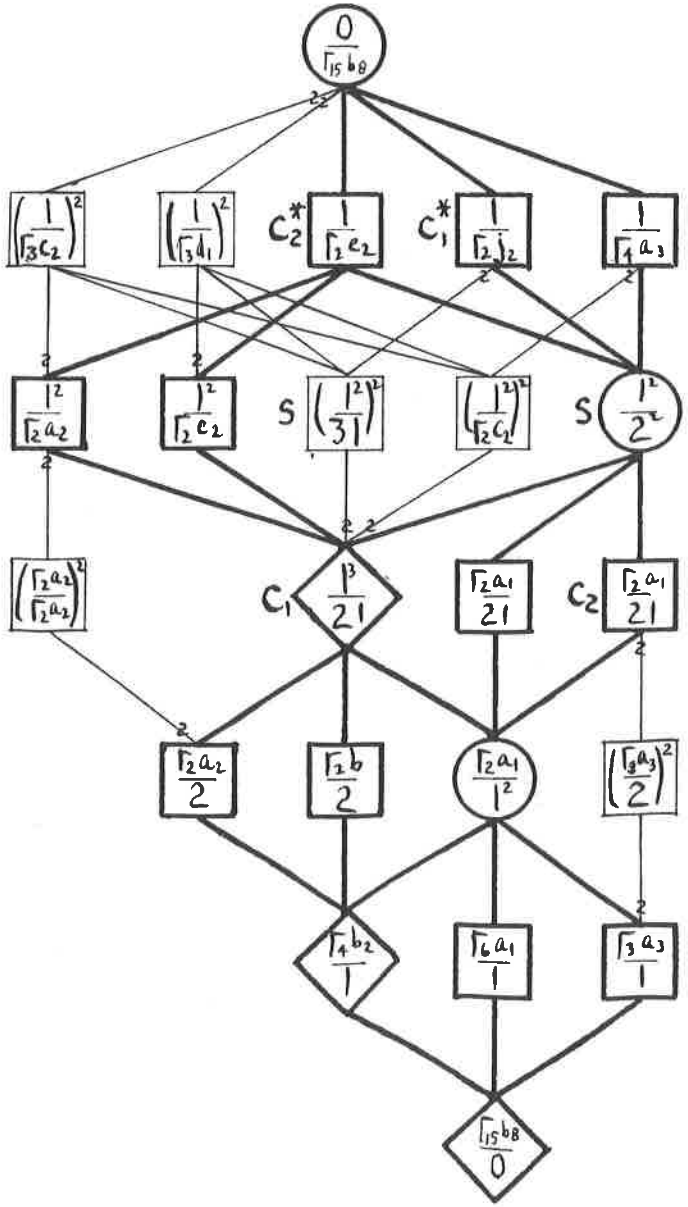
$\alpha_2 =$ qs.rt

$\alpha_3 =$ adcb.ehgf.ilkj.mpon

$\alpha_4 =$ qrst

$\alpha_5 =$ aebfcgdh.imjnkolp.rt

$\alpha_6 =$ alck.bldj.epgn.fohm



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

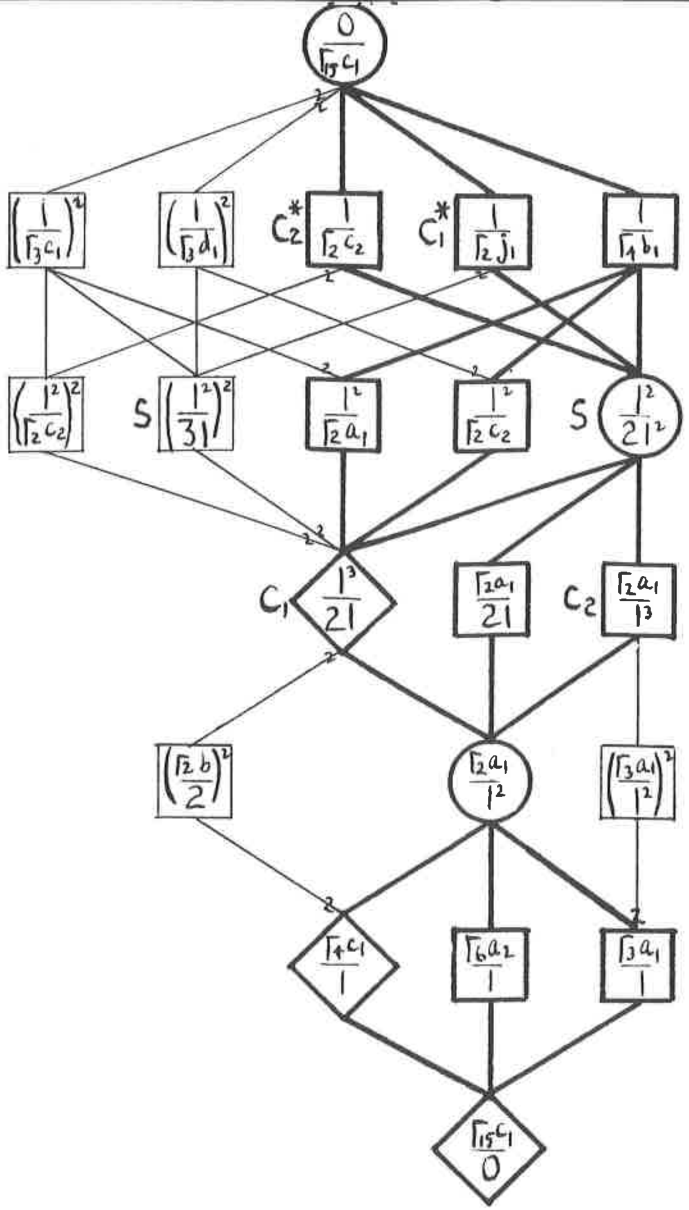
$\alpha_2 =$ qs.rt.uw.vx

$\alpha_3 =$ adcb.ehgf.ilkj.mpon

$\alpha_4 =$ qrst.uvwx

$\alpha_5 =$ aebfcgdh.imjnkolp.qusw.rxtv

$\alpha_6 =$ alck.bldj.epgn.fohm



α_1 = ac.bd.eg.fh

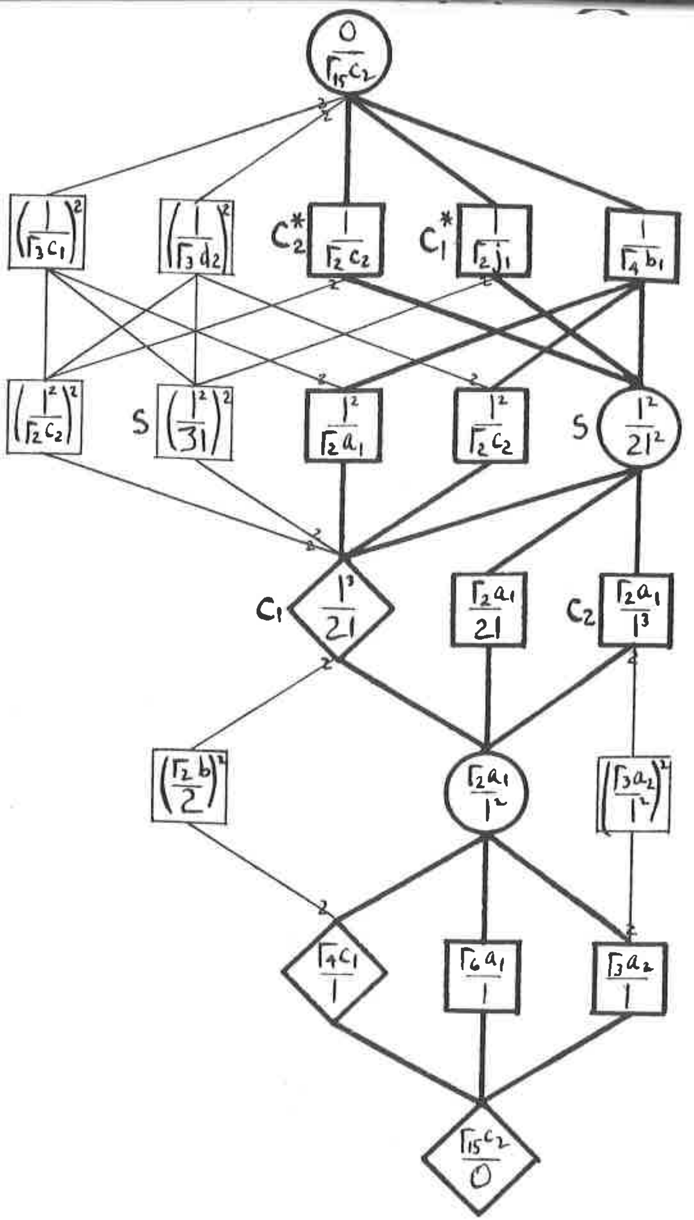
α_2 = lk.jl.mo.np

α_3 = adcb.ehgf

α_4 = lm.jn.ko.lp

α_5 = aebfcgdh.in.jo.kp.lm

α_6 = bd.eh.fg.ljkl.mnop



α_1 = ac.bd.eg.fh

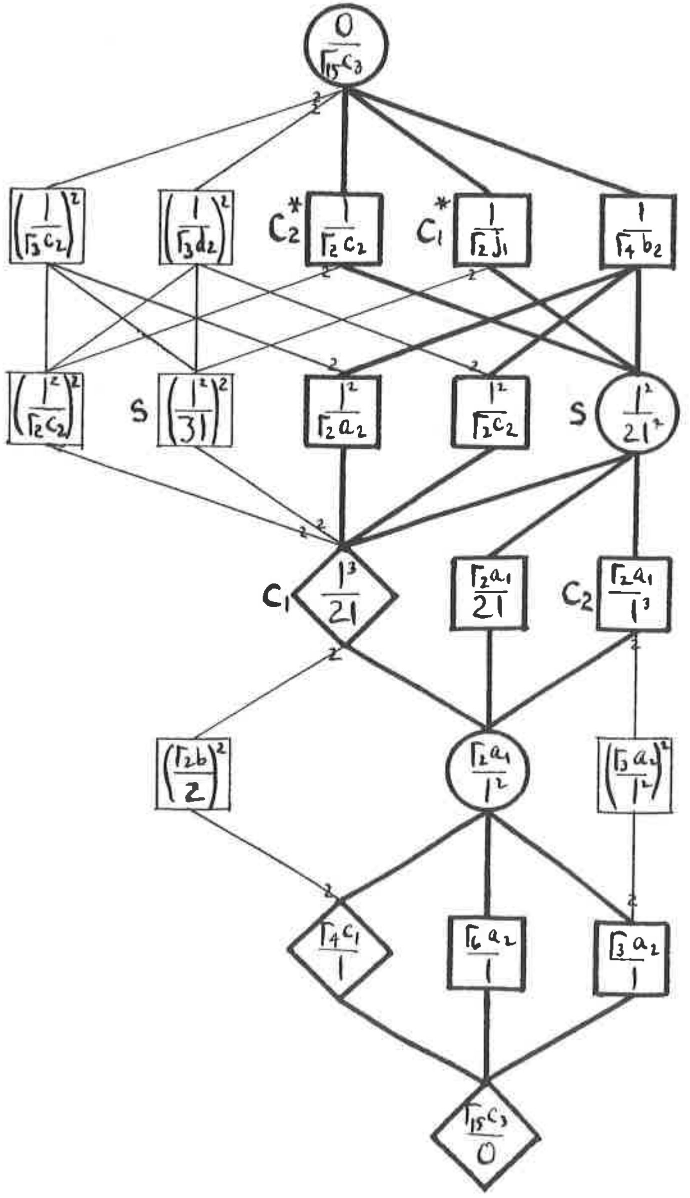
α_2 = lk.jl.mo.np

α_3 = abcd.efgh

α_4 = lm.jn.ko.lp

α_5 = aebfcgdh.in.jo.kp.lm

α_6 = aecg.bhdf.ijkl.mnop



α_1 = ac.bd.eg.fh

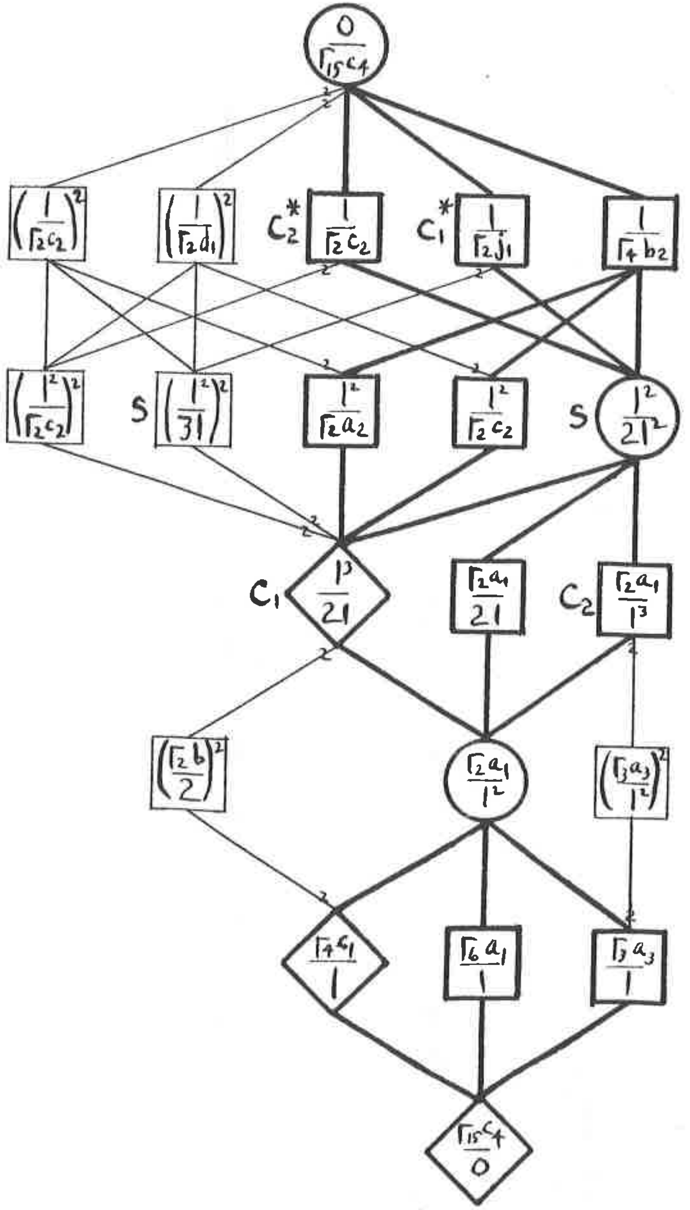
α_2 = ik.jl.mo.np

α_3 = abcd.efgh

α_4 = lm.jn.ko.lp

α_5 = aebfcgdh.in.jo.kp.lm

α_6 = bd.ef.gh.ijkl.mnop



α_1 = ac.bd.eg.fh.ik.jl.mo.np

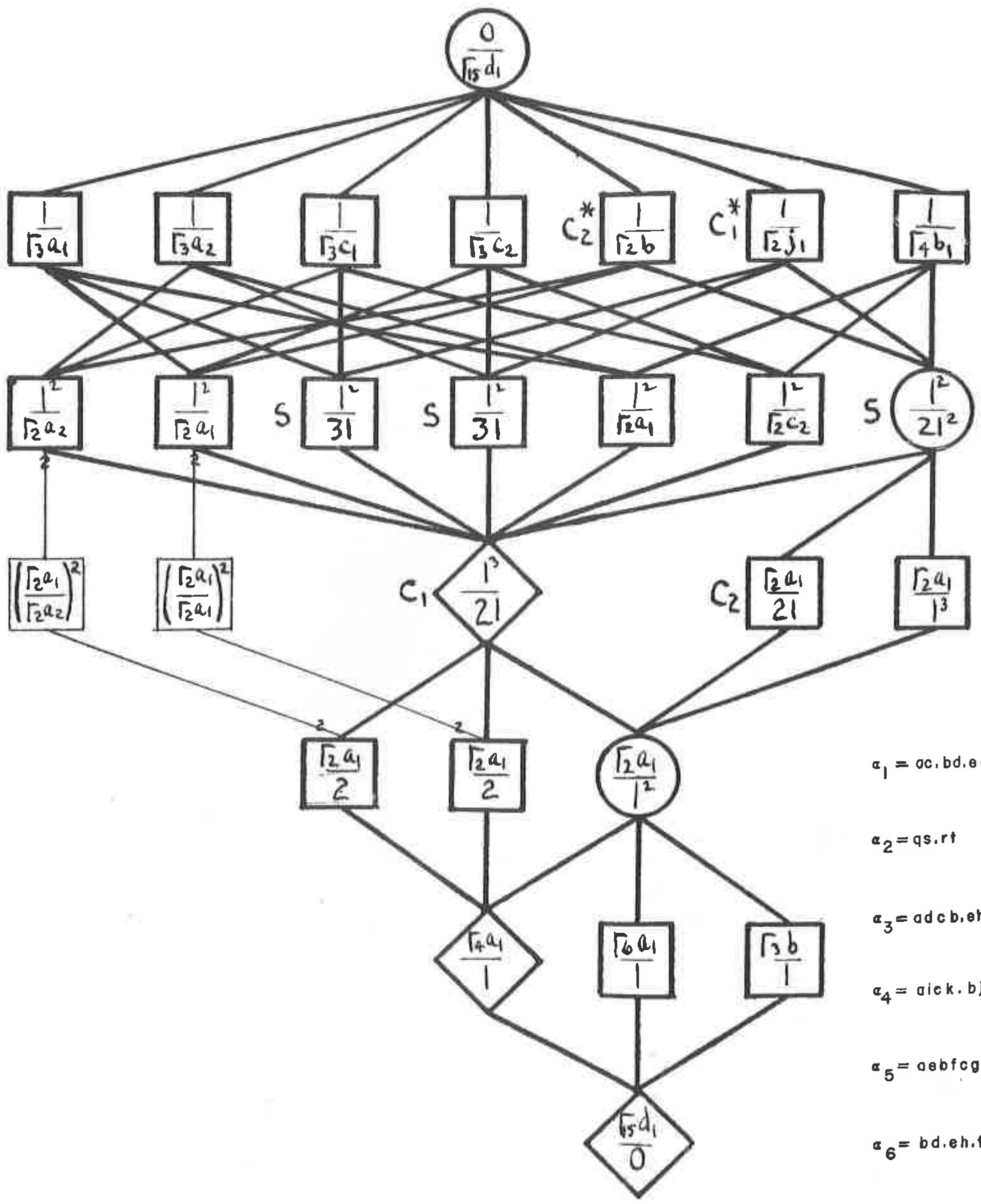
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon

α_4 = qu.rv.sw.tx

α_5 = aebfcgdh.lmjnkolp.qv.rw.sx.tu

α_6 = aick.bldj.epgn.fohm.qrst.uvwx



α_1 = ac, bd, eg, fh, ik, jl, mo, np

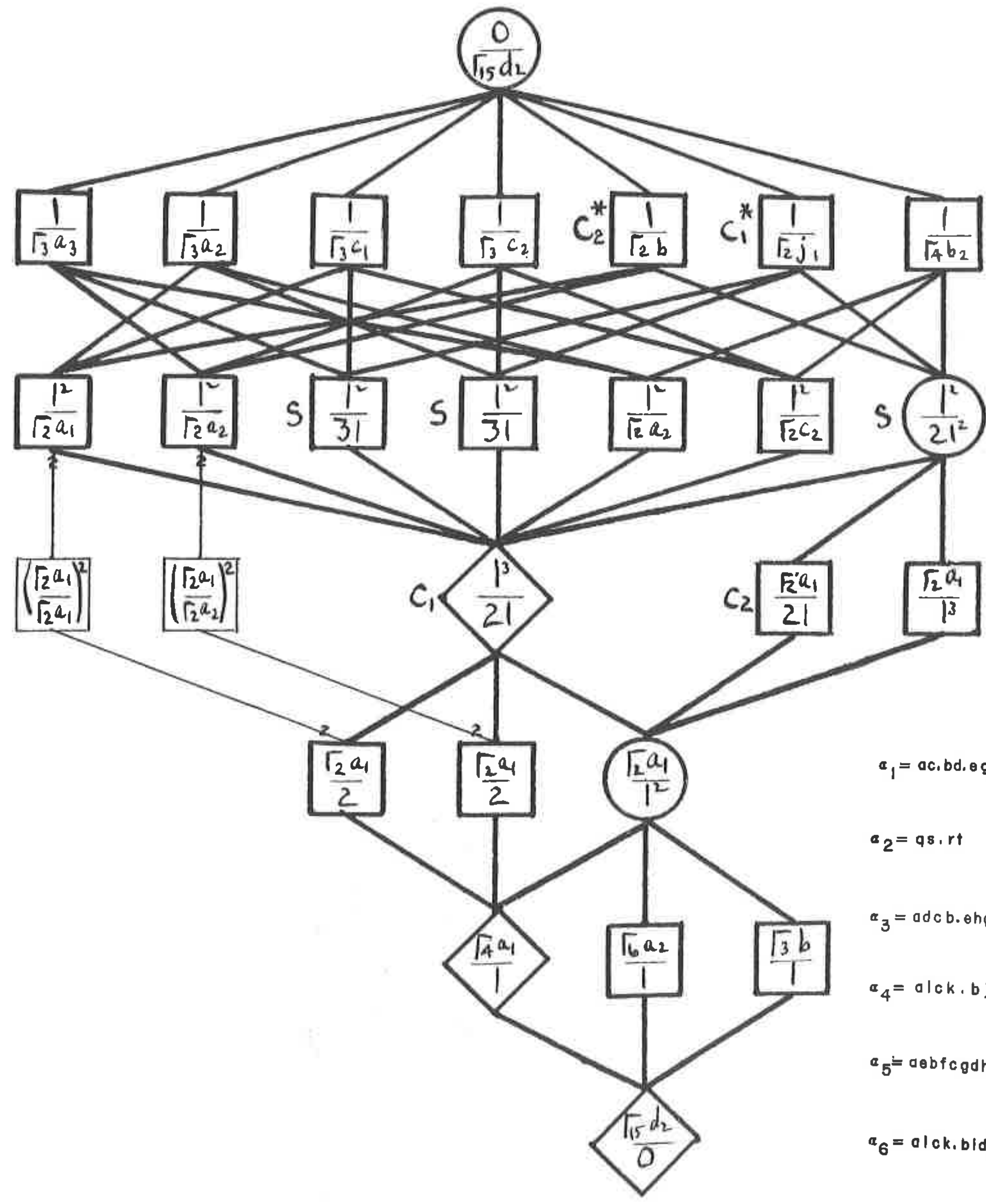
α_2 = qs, rt

α_3 = adcb, ehgf, ilkj, mpon

α_4 = aick, bjdl, emgo, fnhp, qr, st

α_5 = aebfcgdh, imjnkolp, rt

α_6 = bd, eh, fg, jl, mp, on



α_1 = ac, bd, eg, fh, ik, jl, mo, np

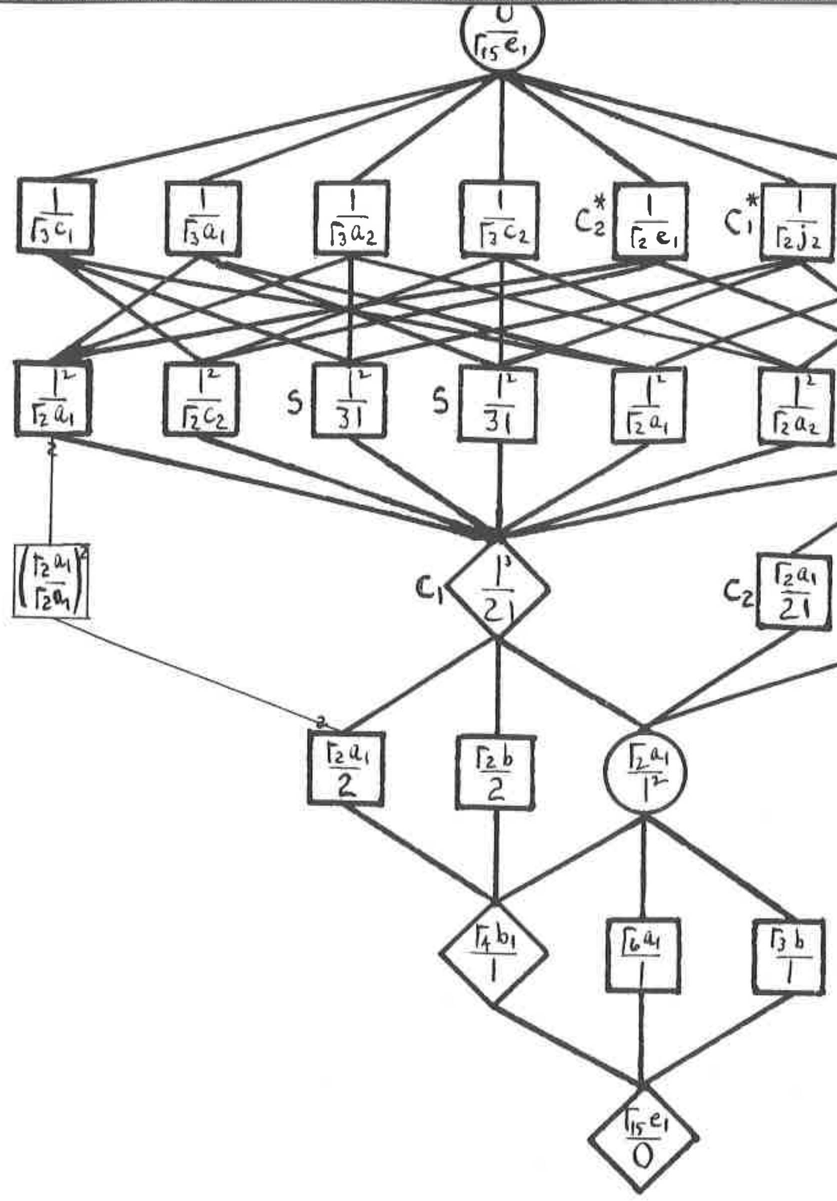
α_2 = qs, rt

α_3 = adcb, ehgf, ilkj, mpon

α_4 = alck, bjdl, emgo, fnhp, qr, st

α_5 = aebfcgdh, lmjnkolp, rt

α_6 = alck, bldj, epgn, fohm



α_1 = ac.bd.eg.fh.ik.jl.mo.np

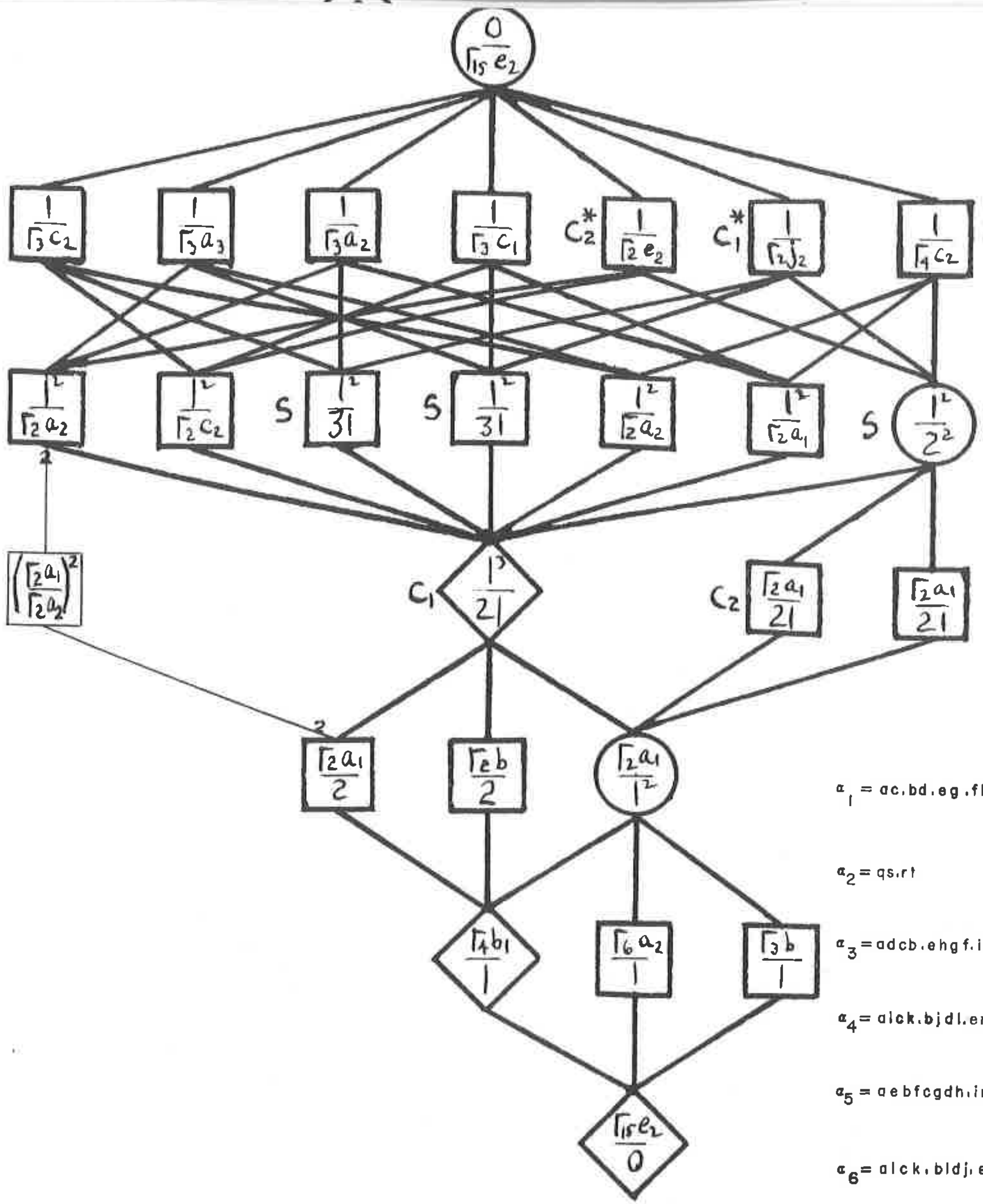
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = aick.bjdl.emgo.fnhp.qrst

α_5 = aebfcgdh.imjnkolp.rt

α_6 = bd.eh.fg.jl.mp.on



α_1 = ac.bd.eg.fh.ik.jl.mo.np

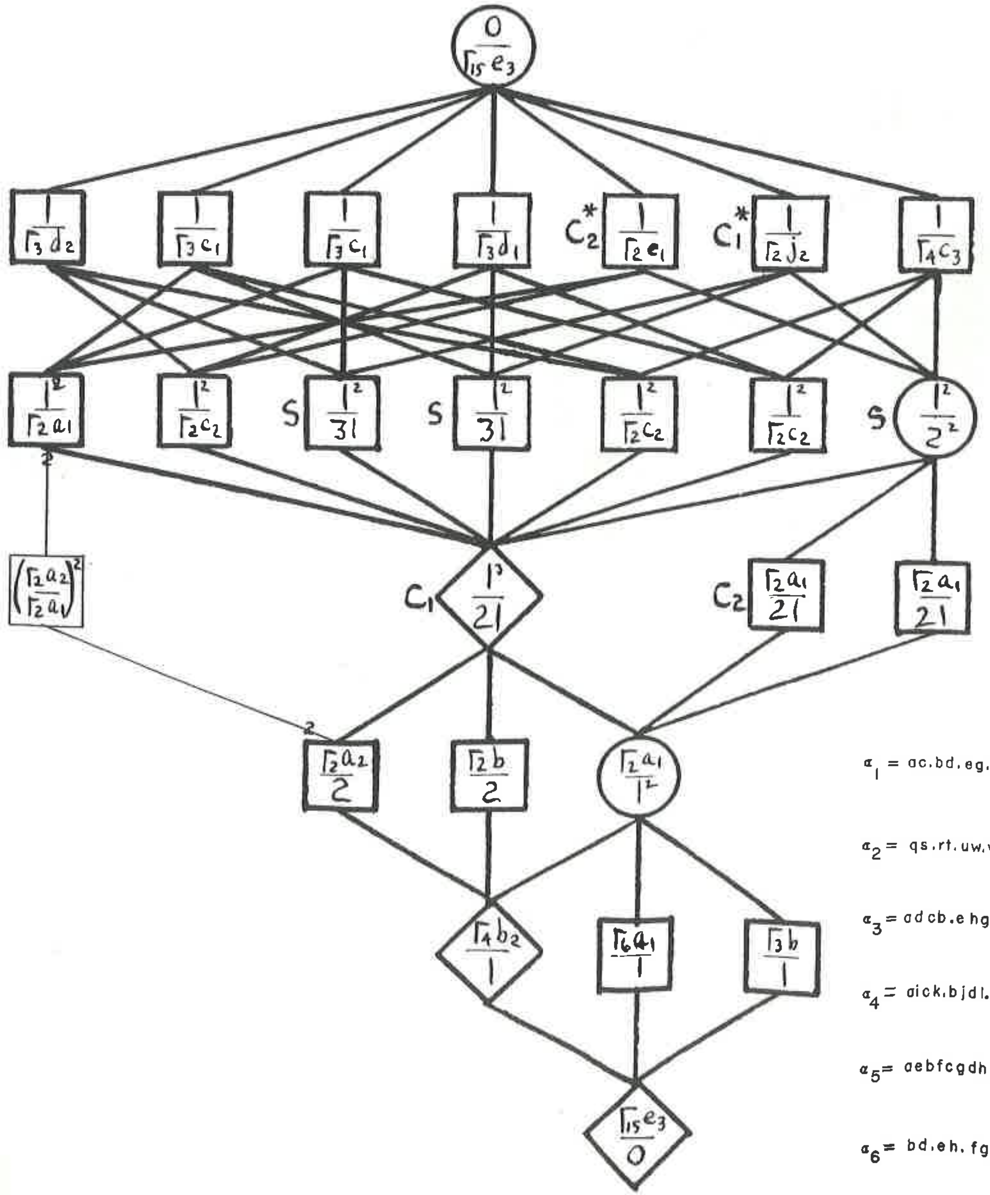
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = alck.bjdl.emgo.fnhp.qrst

α_5 = aebfcgdh.imjnkolp.rt

α_6 = alck.bldj.epgn.fohm



α_1 = ac.bd.eg.fh.ik.jl.mo.np

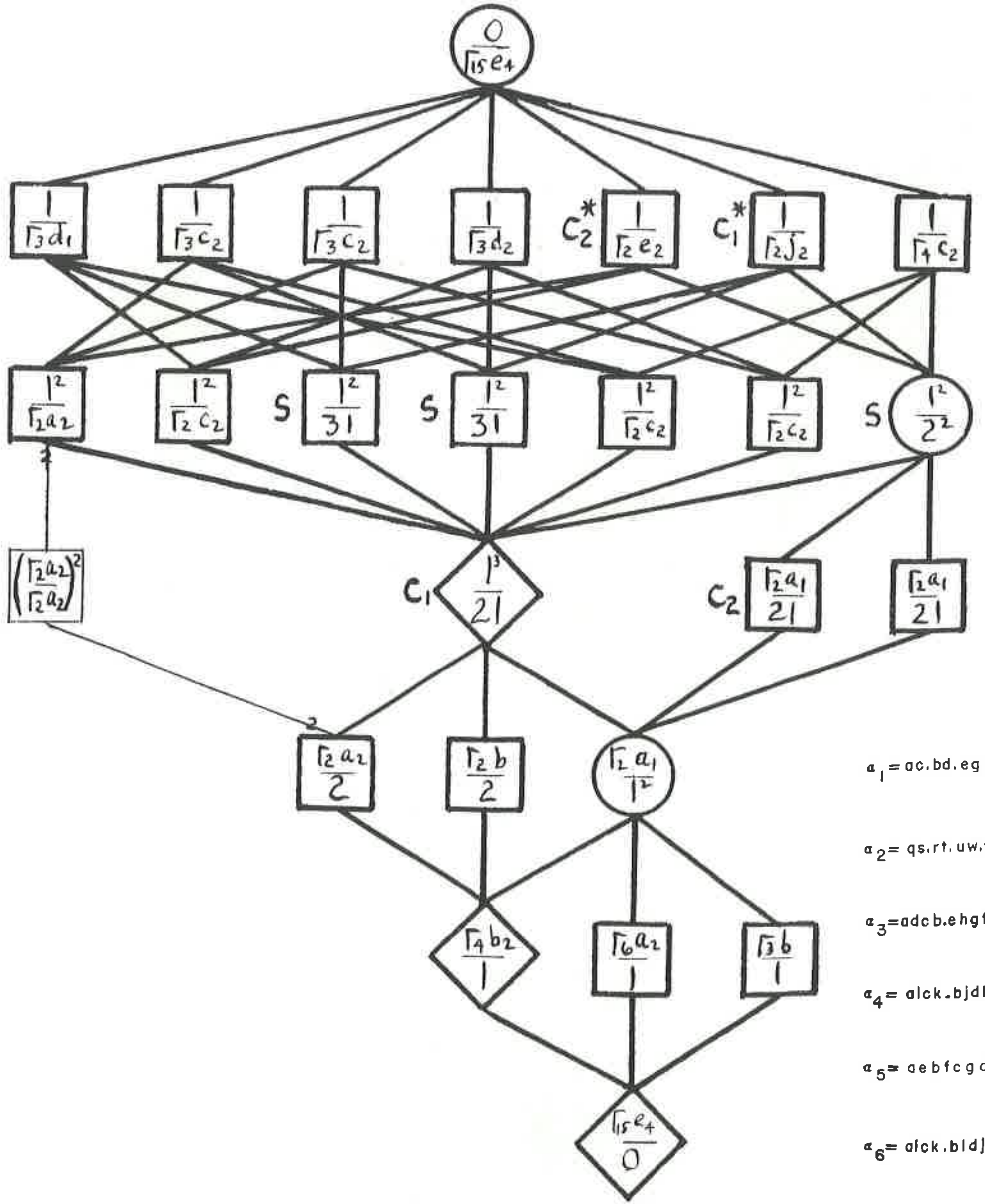
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon

α_4 = aick.bjdl.emgo.fnhp.qrst.uvwx

α_5 = aebfcgdh.imjnkolp.qu.sw.rx.tv

α_6 = bd.eh.fg.jl.mp.on



α_1 = ac.bd.eg.fh.ik.jl.mo.np

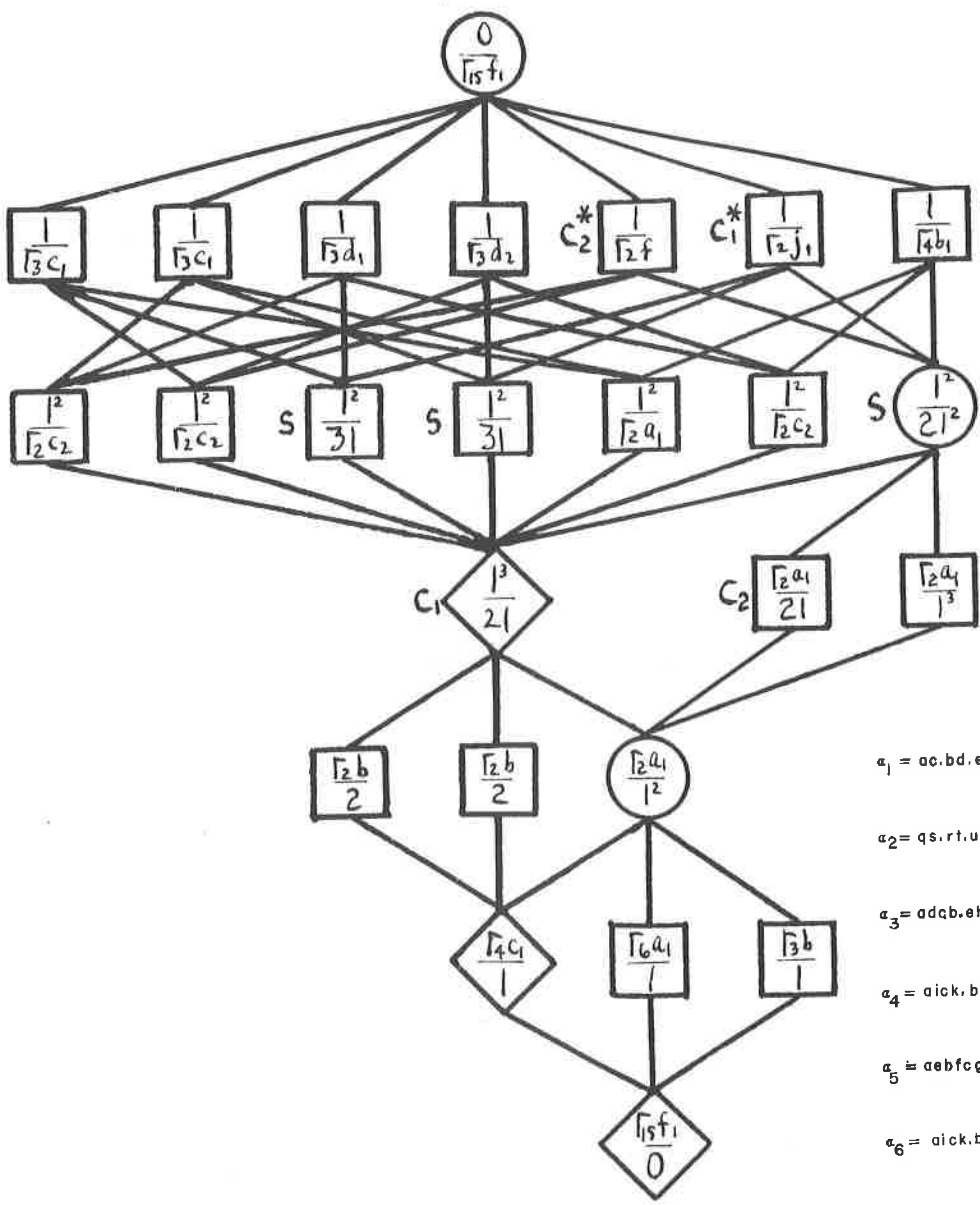
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon

α_4 = alck.bjdl.emgo.fnhp.qrst.uvwx

α_5 = aebfcgdh.imjnkolp.qusw.rxtv

α_6 = alck.bldj.epgn.fohm



α_1 = ac.bd.eg.fh.ik.jl.mo.np

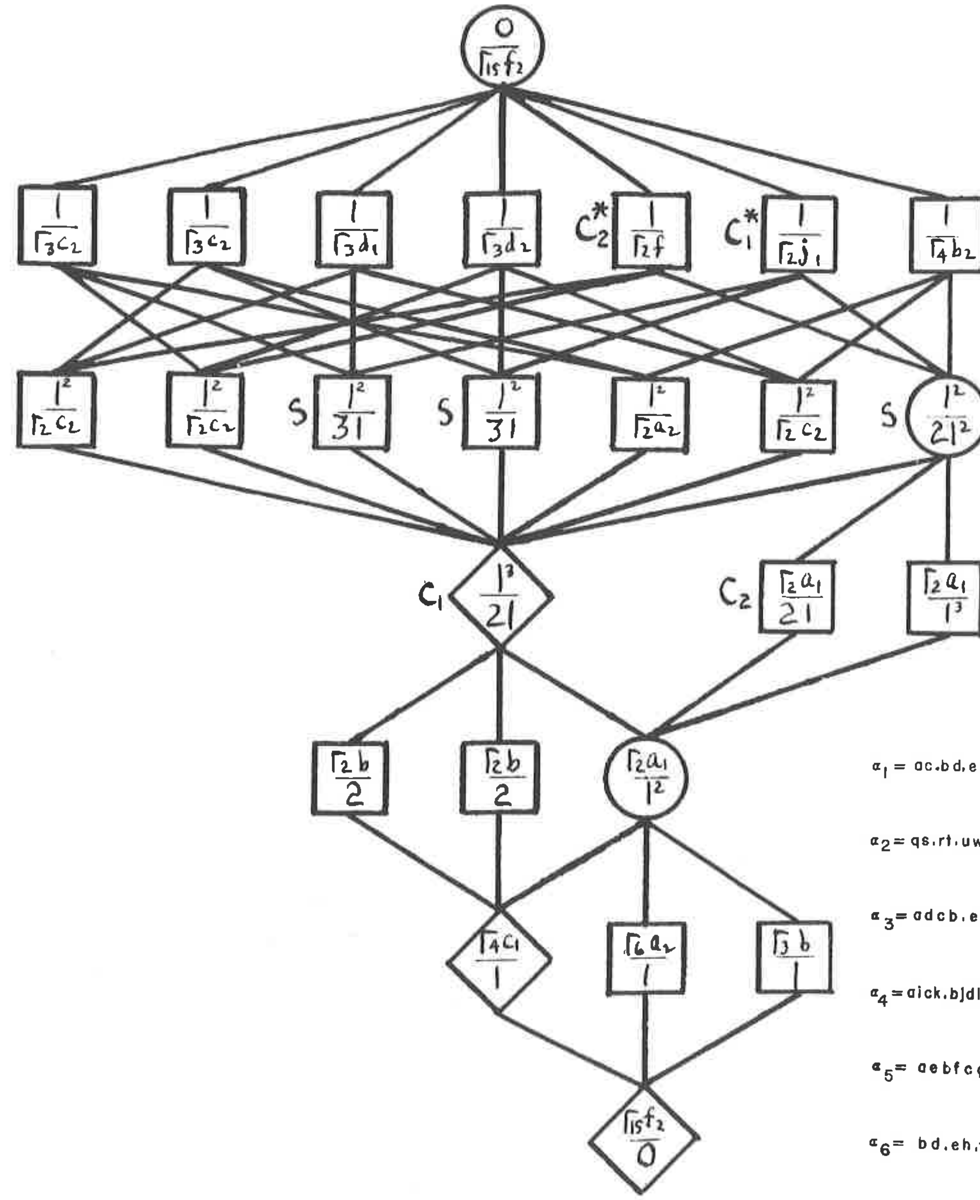
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon

α_4 = aick.bjdl.emgo.fnhp. qu.rv.sw.tx

α_5 = aebfcgdh.imjnkolp.qv.rw.sx.tu

α_6 = aick.bldj.epgn.fohm.qrst.uvwx



α_1 = ac.bd.eg.fh.ik.jl.mo.np

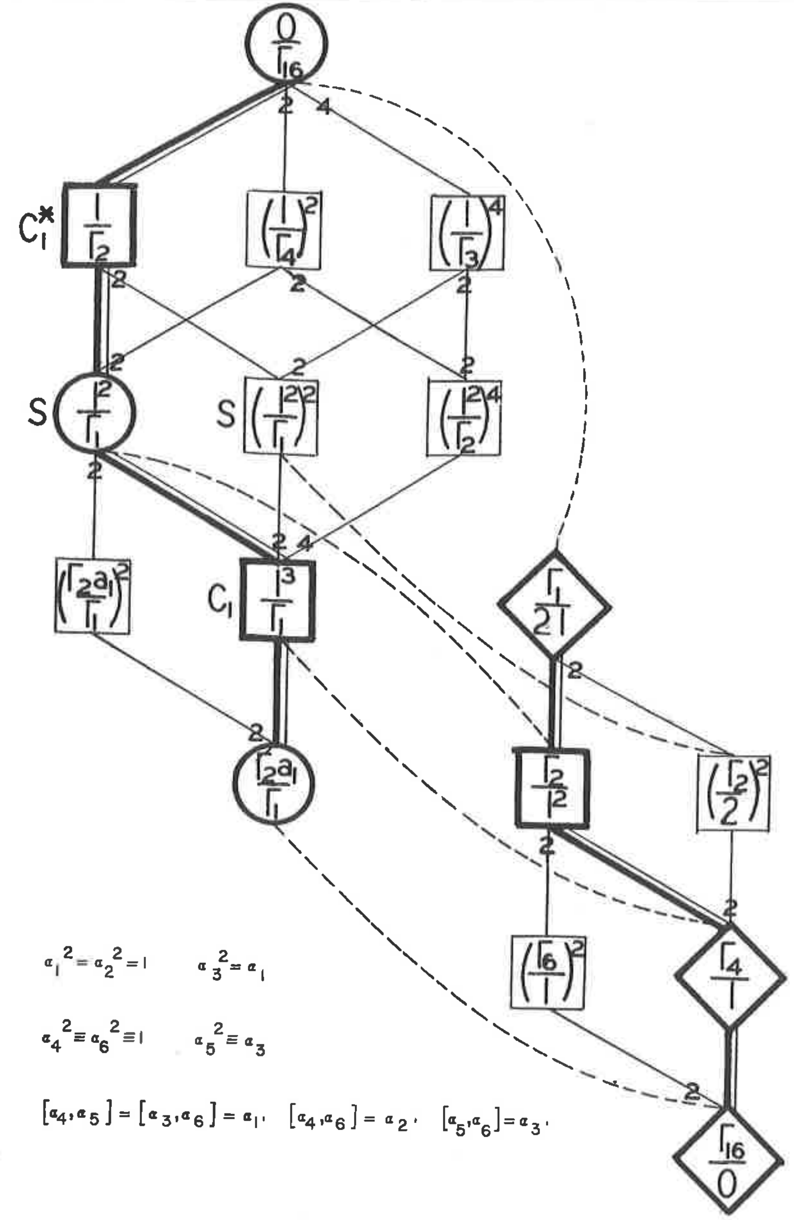
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf. ilkj.mpon

α_4 = aick.bjdl.emgo.fnhp. qu.rv.sw.tx

α_5 = aebfcgdh.imjnkolp.qv.rw. sx.tu

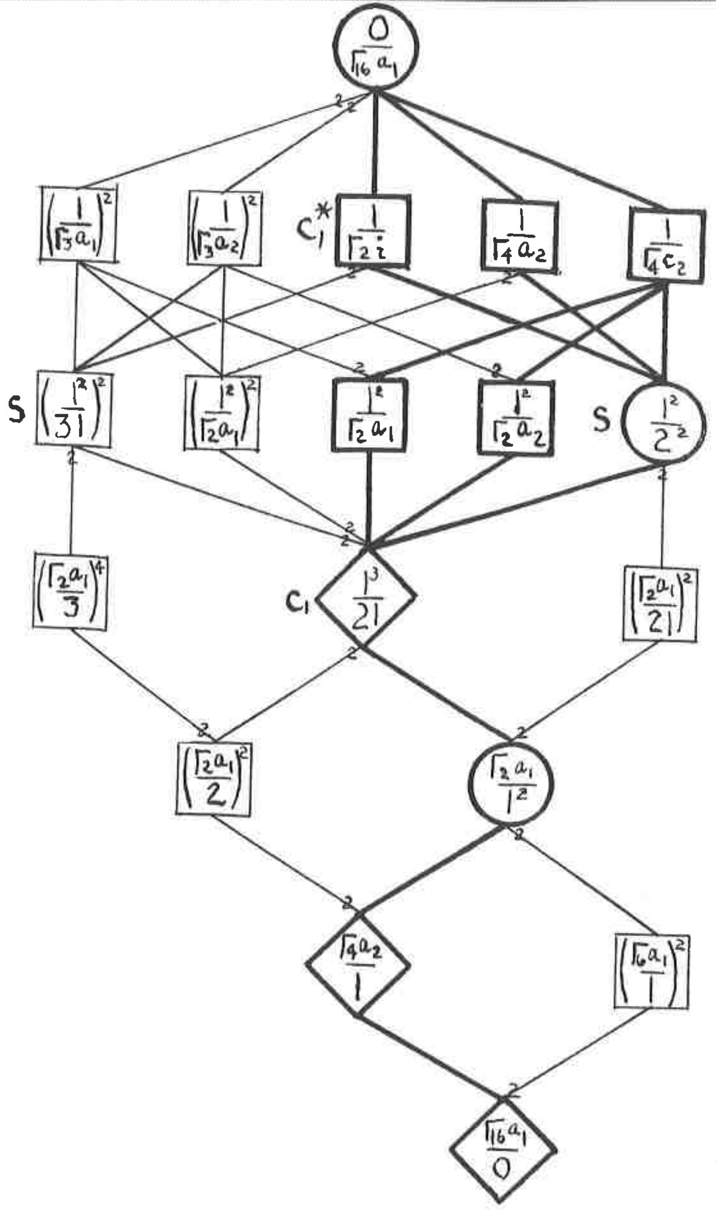
α_6 = bd.eh.fg. jl.mp.on.qrst.uvwx



$\alpha_1^2 = \alpha_2^2 = 1 \qquad \alpha_3^2 = \alpha_1$

$\alpha_4^2 \equiv \alpha_6^2 \equiv 1 \qquad \alpha_5^2 \equiv \alpha_3$

$[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_1, \quad [\alpha_4, \alpha_6] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3.$



$\alpha_1 =$ ac.bd.eg.fh

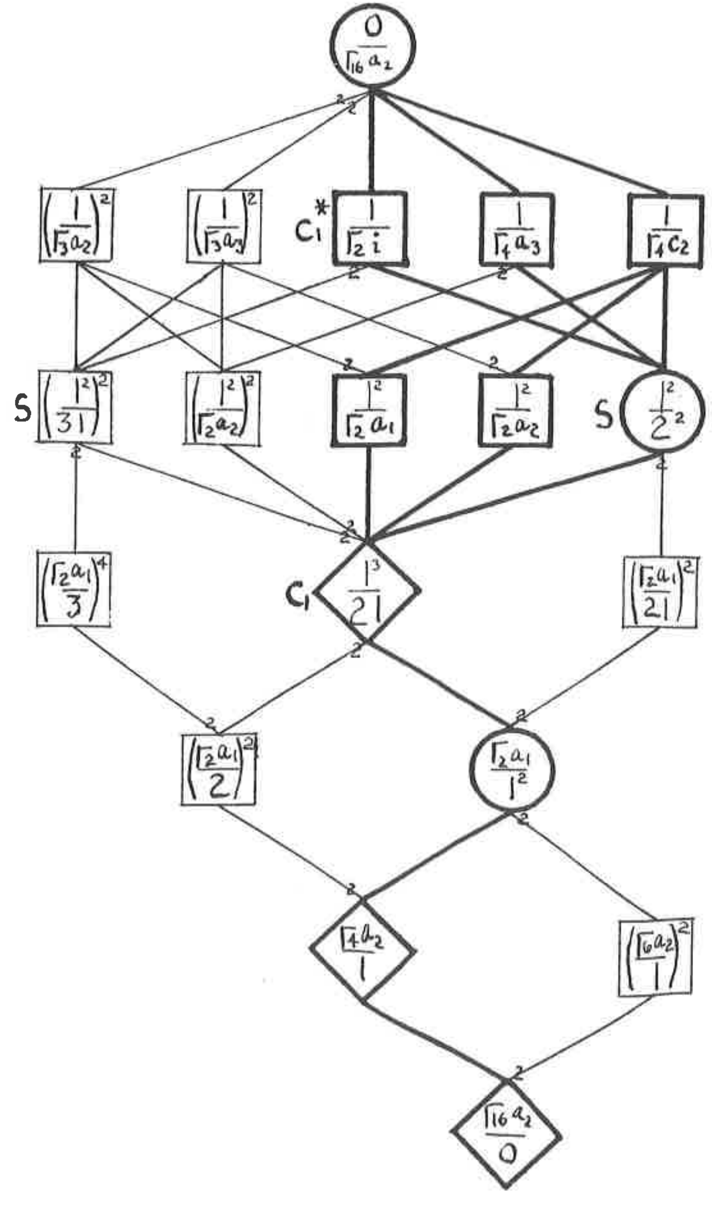
$\alpha_2 =$ lk,jl

$\alpha_3 =$ adob.ehgf

$\alpha_4 =$ eg.fh.ijkl

$\alpha_5 =$ aebfcgdh

$\alpha_6 =$ bd,eh.fg,jl



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

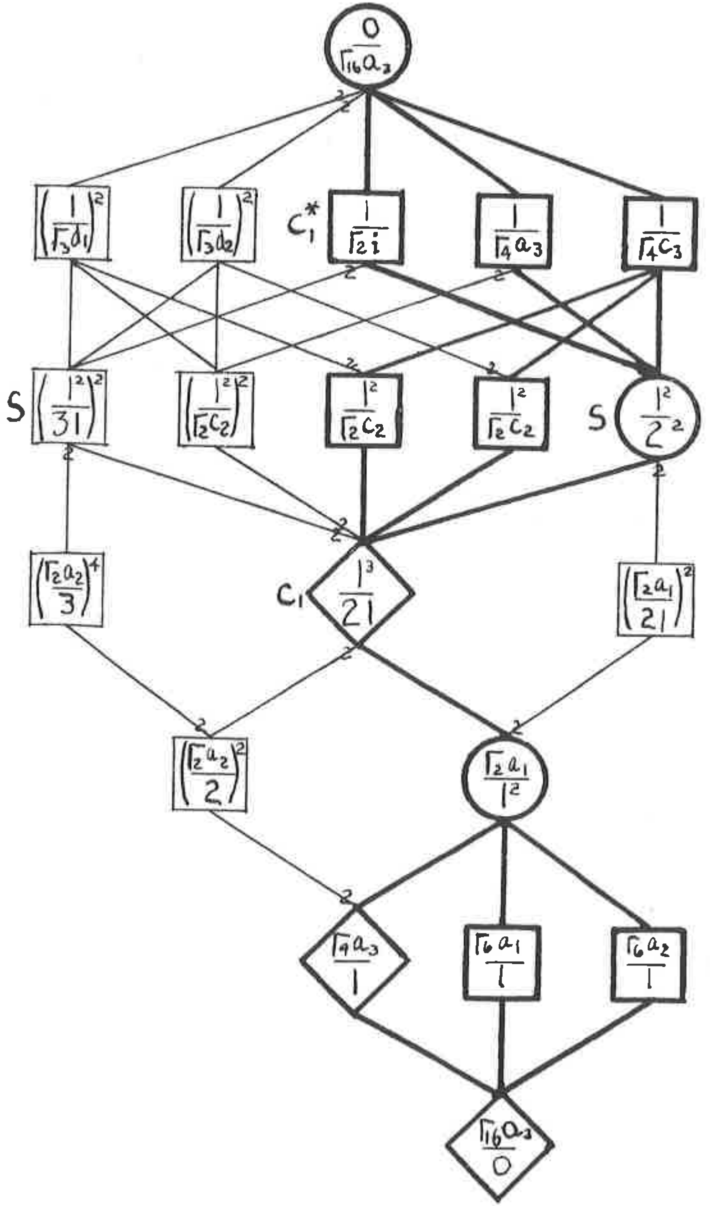
$\alpha_2 =$ qs.rt

$\alpha_3 =$ adcb.ehgf.ilkj.mpon

$\alpha_4 =$ eg.fh.mo.np.qrst

$\alpha_5 =$ aebfcgdh.imjnkolp

$\alpha_6 =$ alck.bldj.epgn.fohm.rt



$\alpha_1 =$ ac.bd.eg.fh

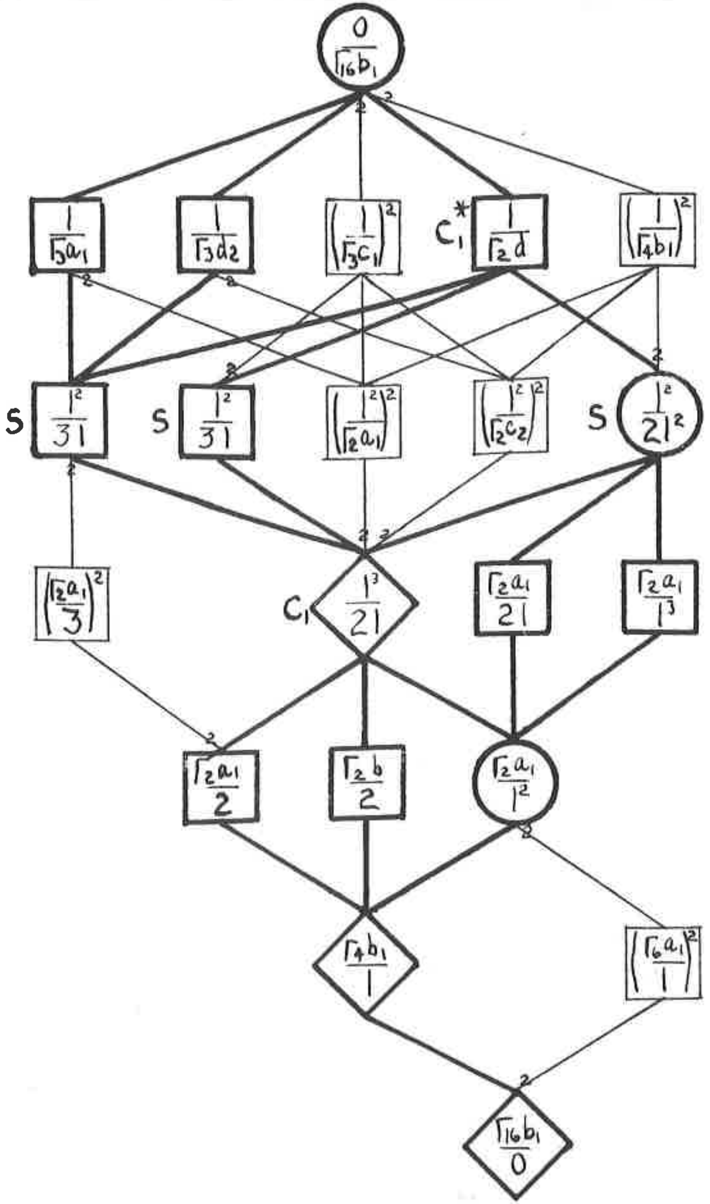
$\alpha_2 =$ ik.jl.mo.np

$\alpha_3 =$ adcb.ehgf

$\alpha_4 =$ eg.fh.ljkl.mnop

$\alpha_5 =$ aebfcgdh

$\alpha_6 =$ bd.eh.fg.lmko.jpln



α_1 = ac.bd.eg.fh

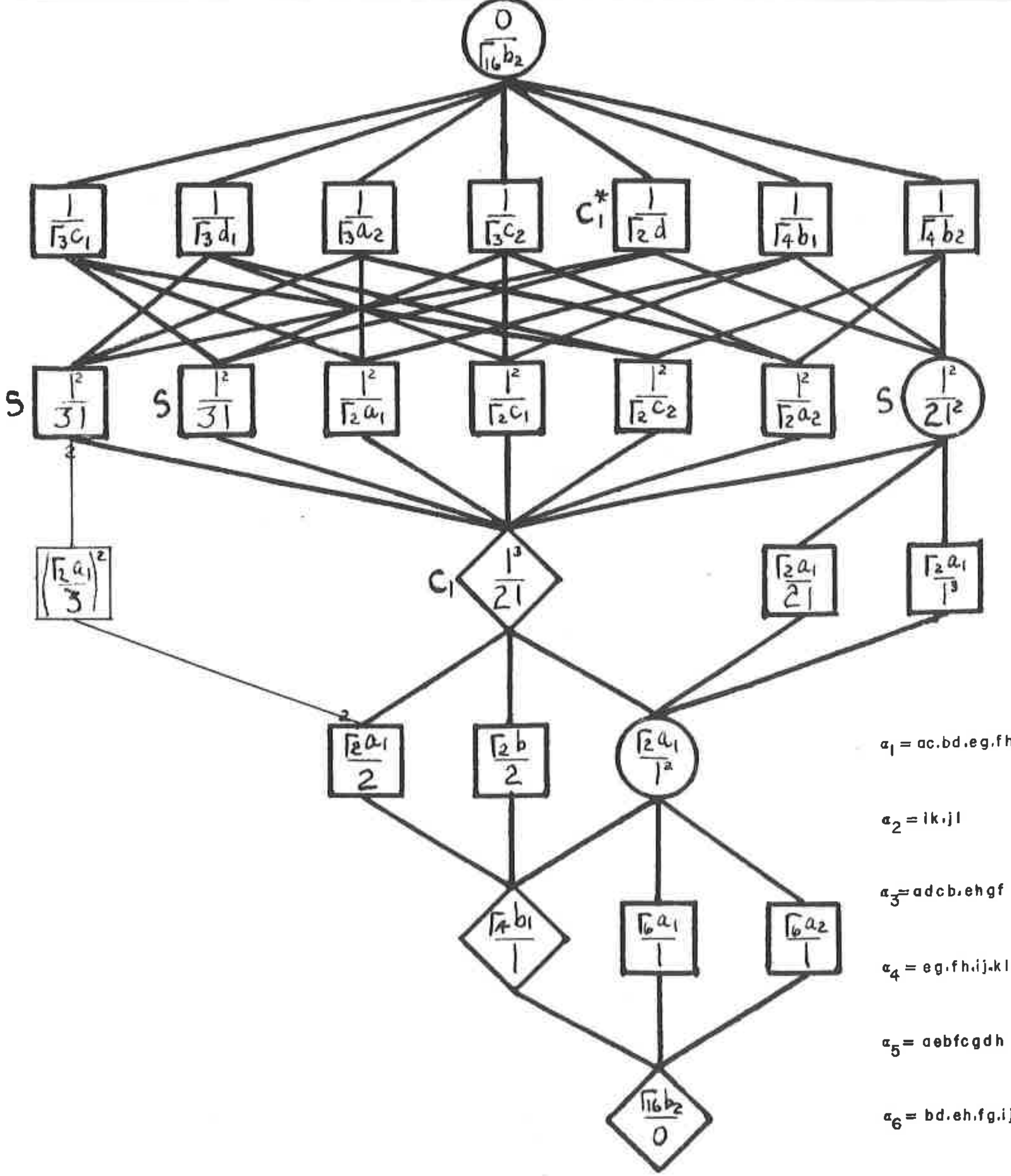
α_2 = ik.jl

α_3 = adcb.ehgf

α_4 = eg.fh.ij.kl

α_5 = aebfcgdh

α_6 = bd.eh.fg.jl



α_1 = ac.bd.eg.fh

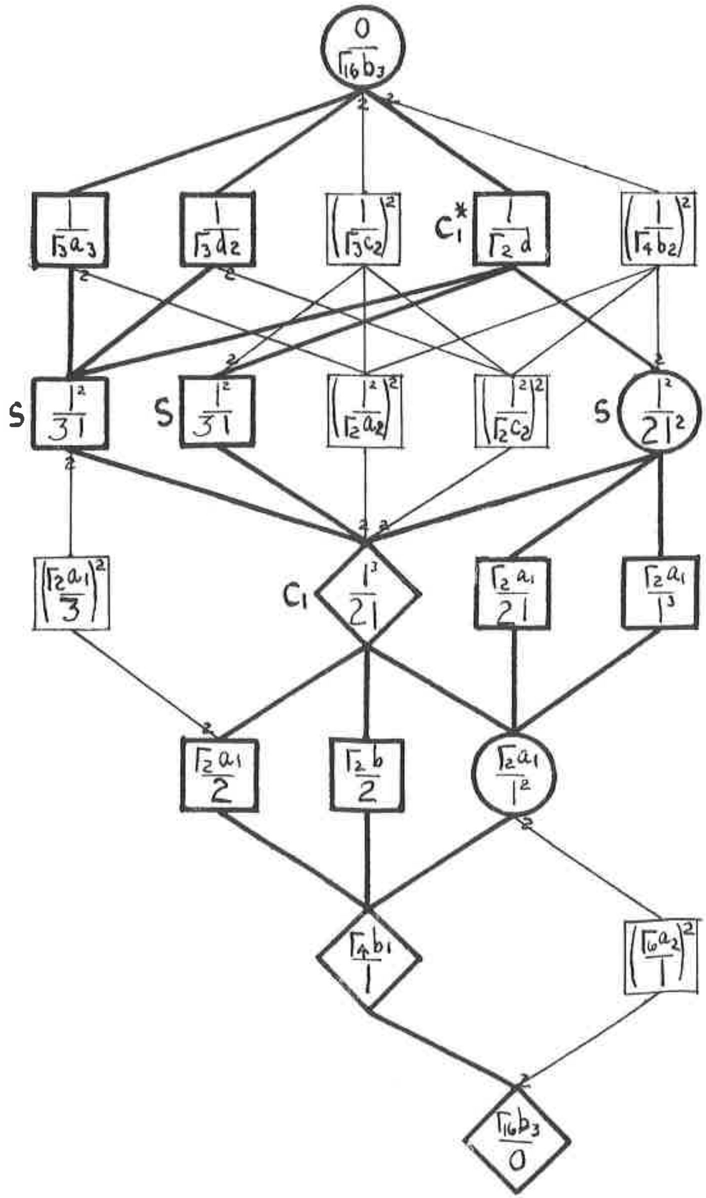
α_2 = ik.jl

α_3 = adcb.ehgf

α_4 = eg.fh.ij.kl

α_5 = aebfcgdh

α_6 = bd.eh.fg.ijkl



α_1 = ac.bd.eg.fh.ik.jl.mo.np

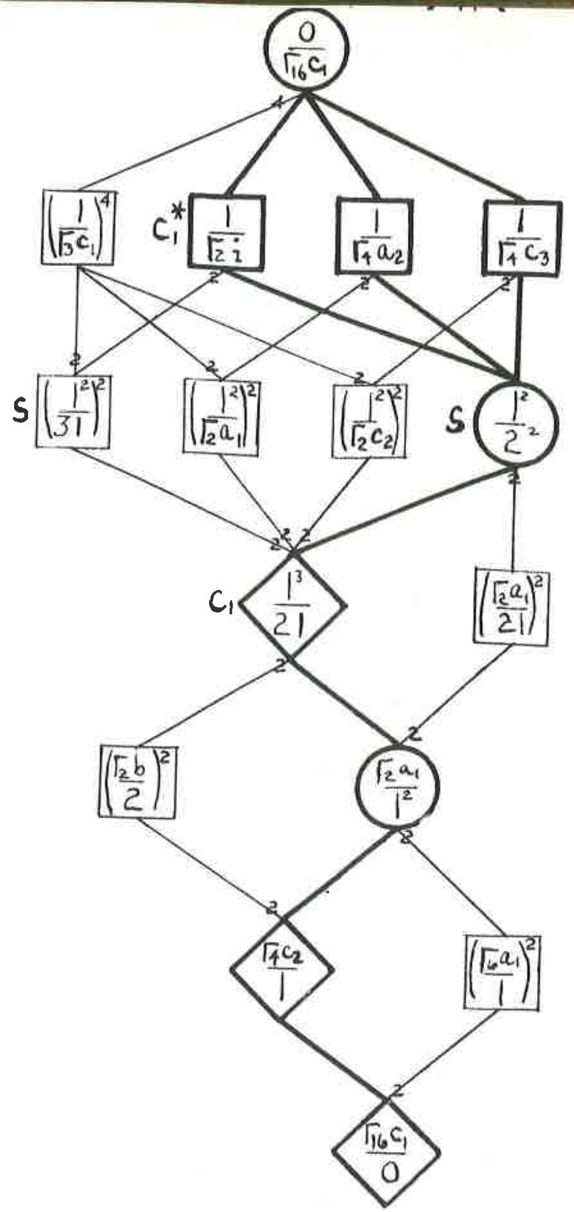
α_2 = qs.rt

α_3 = adcb.ehgf.ilkj.mpon

α_4 = eg.fh.mo.np.qr.st

α_5 = aebfcgdh.imjnkolp

α_6 = aick.bldj.epgn.fohm.rt



α_1 = ac.bd.eg.fh

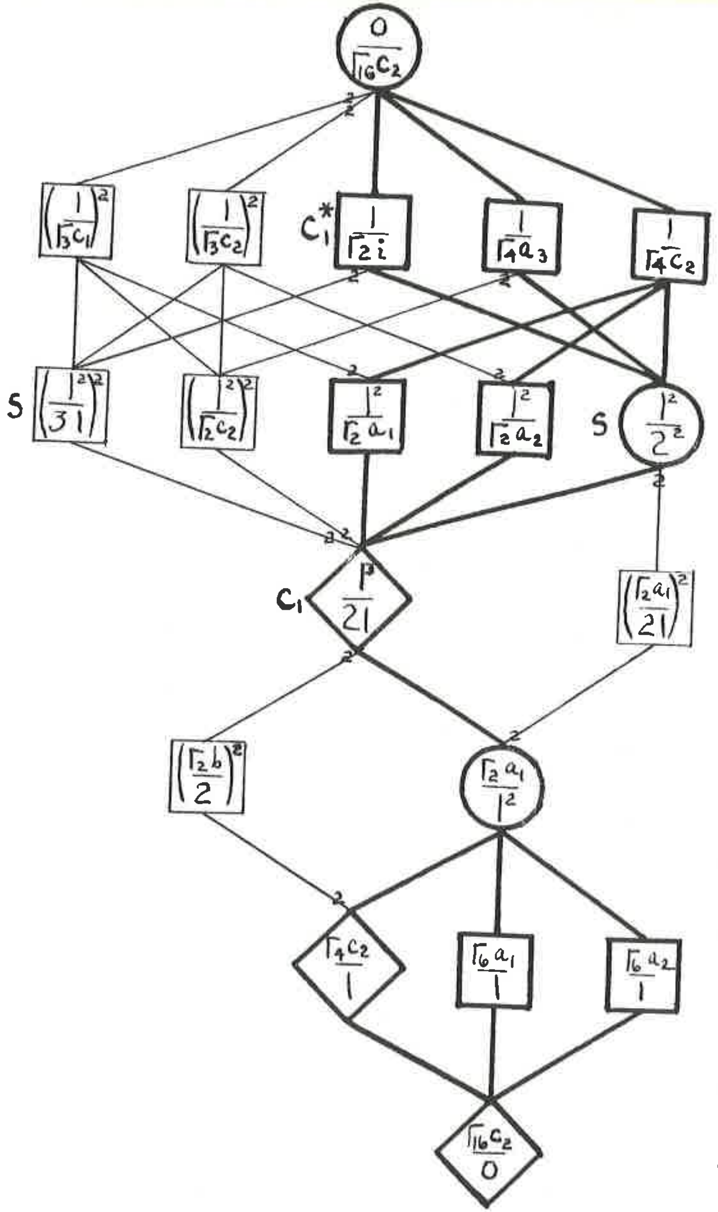
α_2 = ik.jl.mo.np

α_3 = adcb.ehgf

α_4 = eg.fh.ijkl.mpon

α_5 = aebfcgdh.ijkl.mnop

α_6 = bd.eh.fg.im.jn.ko.lp



α_1 = ac.bd.eg.fh

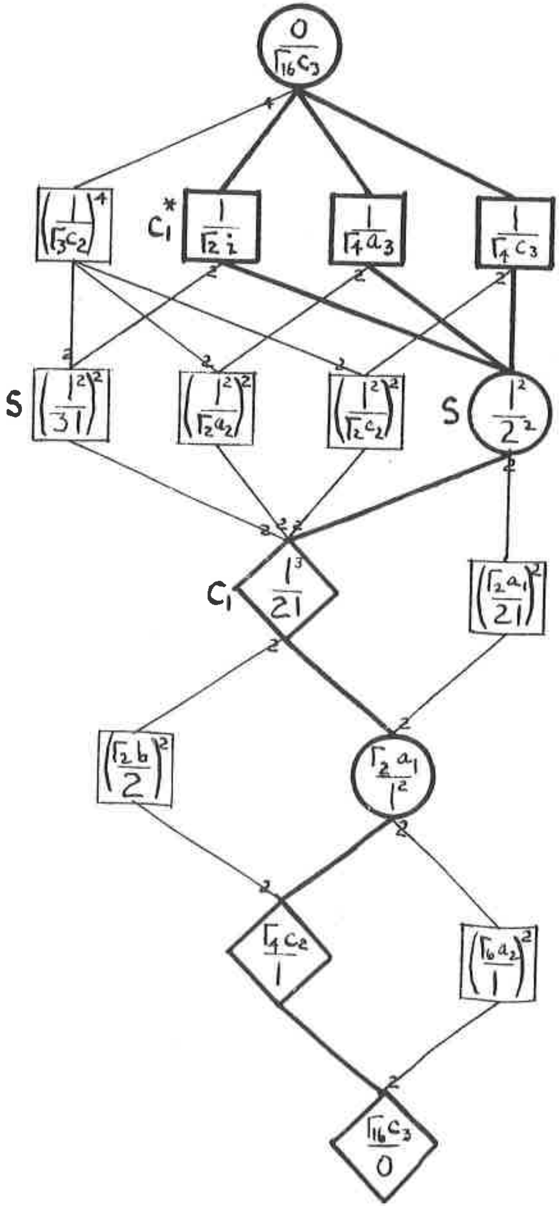
α_2 = ik.jl.mo.np

α_3 = adcb.ehgf

α_4 = eg.fh.ijkl.mpon

α_5 = aebfcgdh.ijkl.mnop

α_6 = bd.eh.fg.imko.jnlp



α_1 = ac.bd.eg.fh.ik.jl.mo.np

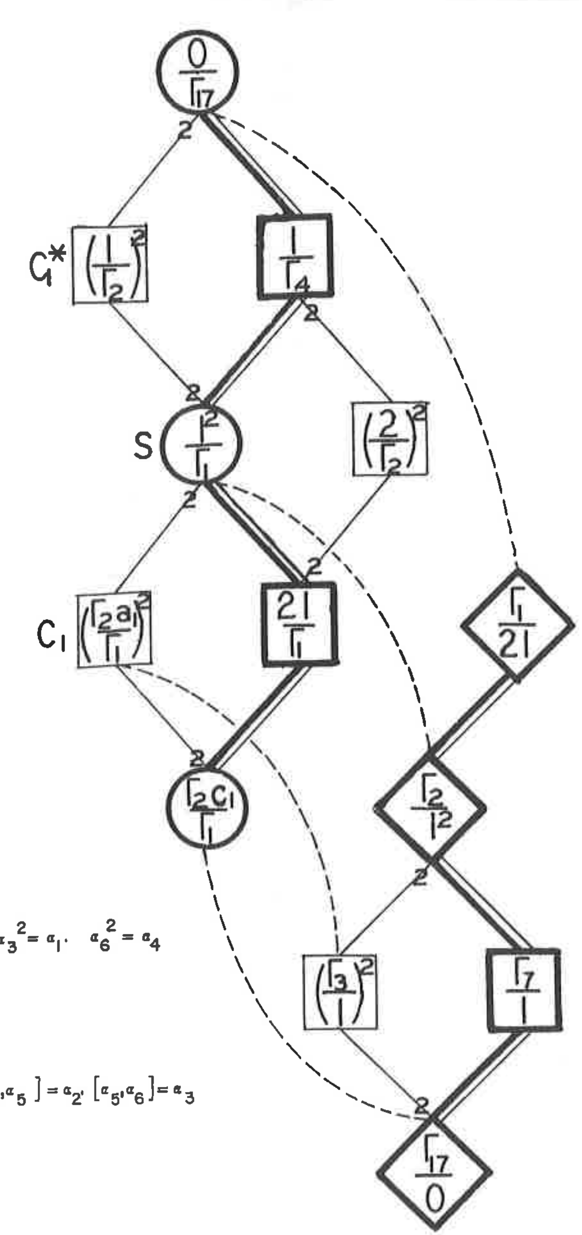
α_2 = qs.rt.uw.vx

α_3 = adcb.ehgf.ilkj.mpon

α_4 = eg.fh.mo.np.qrst.uxwv

α_5 = aebfcgdh.imjnkolp.qrst.uvwx

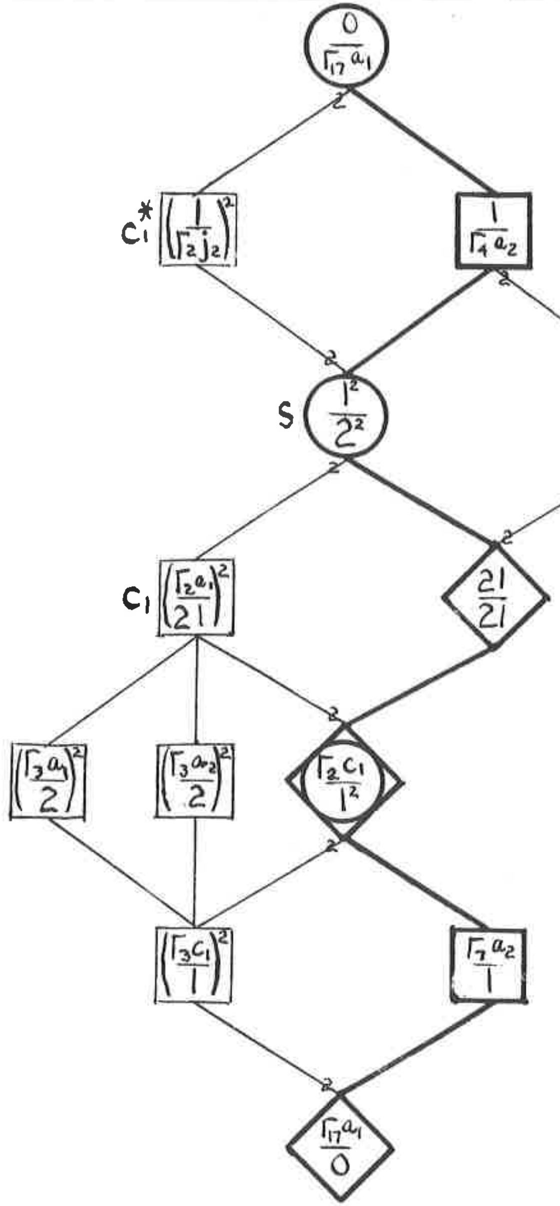
α_6 = aick.bldj.epgn.fohm.qu.rv.sw.tx



$\alpha_1^2 = \alpha_2^2 = 1, \quad \alpha_3^2 = \alpha_1, \quad \alpha_6^2 = \alpha_4$

$\alpha_4^2 \equiv \alpha_5^2 \equiv 1$

$[\alpha_3, \alpha_5] = \alpha_1, \quad [\alpha_3, \alpha_6] = \alpha_1 \alpha_2, \quad [\alpha_4, \alpha_5] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3$



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

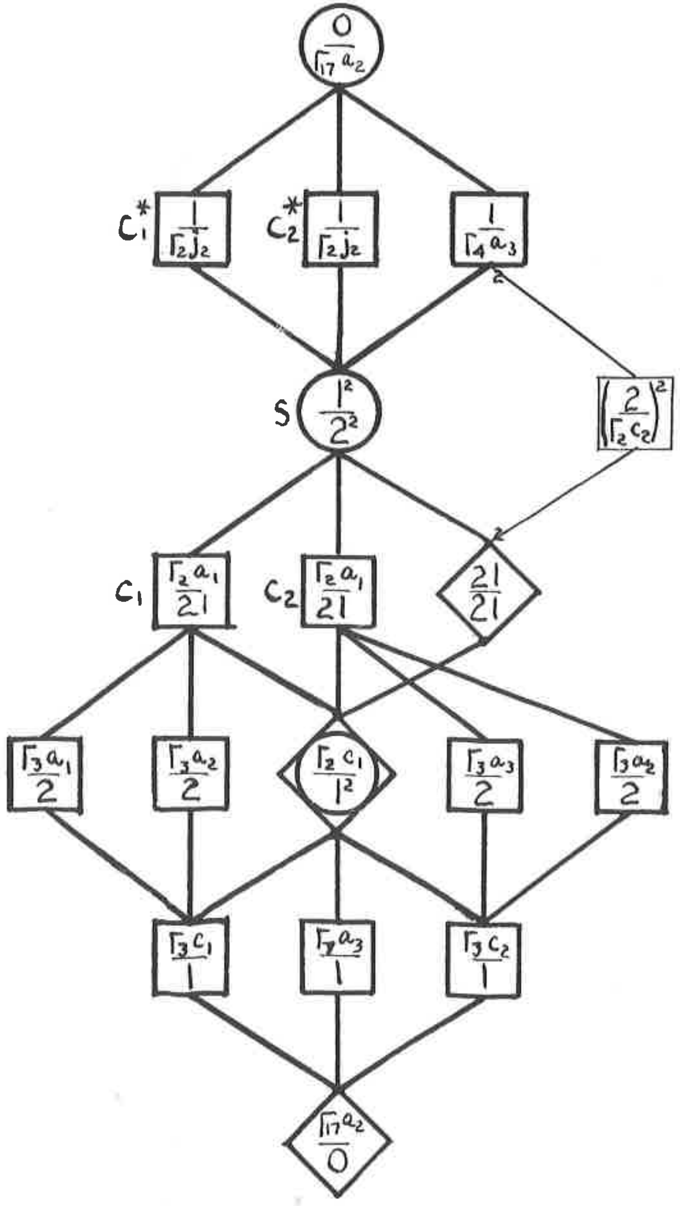
$\alpha_2 =$ ik.jl.mo.np

$\alpha_3 =$ abcd.efgh.ijkl.mnop

$\alpha_4 =$ ijkl.mnop

$\alpha_5 =$ bd.eh.fg.jl.mp.on

$\alpha_6 =$ ae.bh.cg.df.imjn kolp



α_1 = ac.bd.eg.fh.lk.jl.mo.np

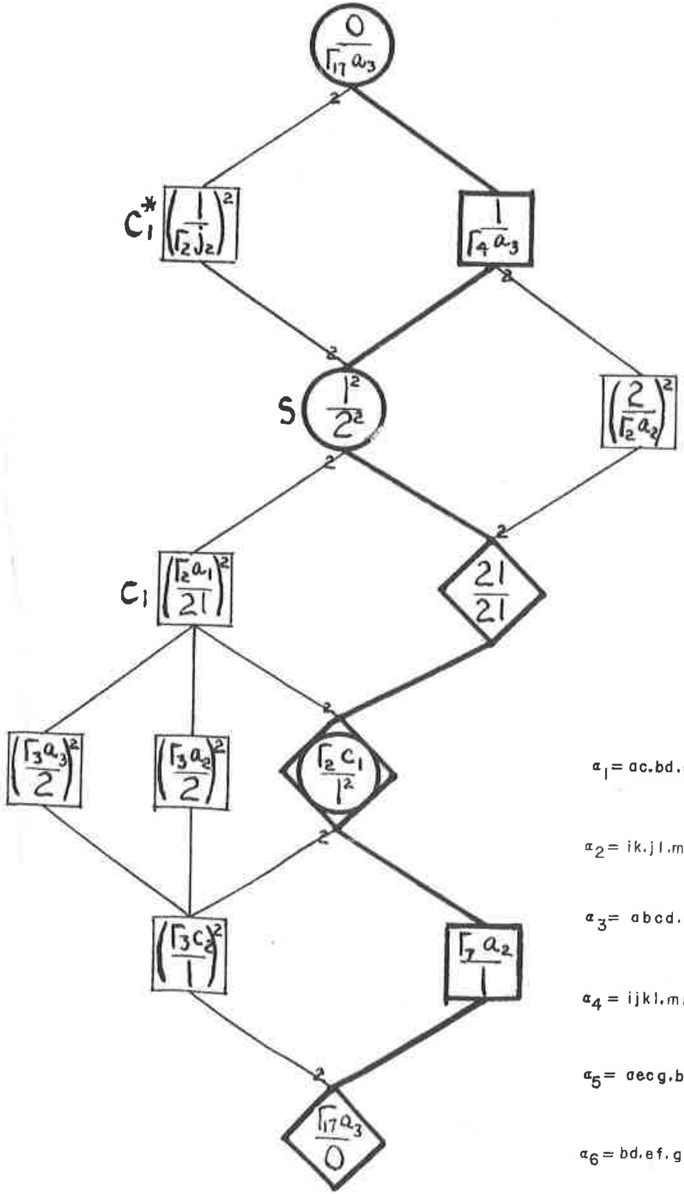
α_2 = ik.jl.mo.np

α_3 = abcd.efgh.ilkj.mpon

α_4 = ijkl.mnop

α_5 = bd.eh.fg.imko.jpln

α_6 = ae.bh.cg.df.imjn kolp



α_1 = ac.bd.eg.fh.ik.jl.mo.np

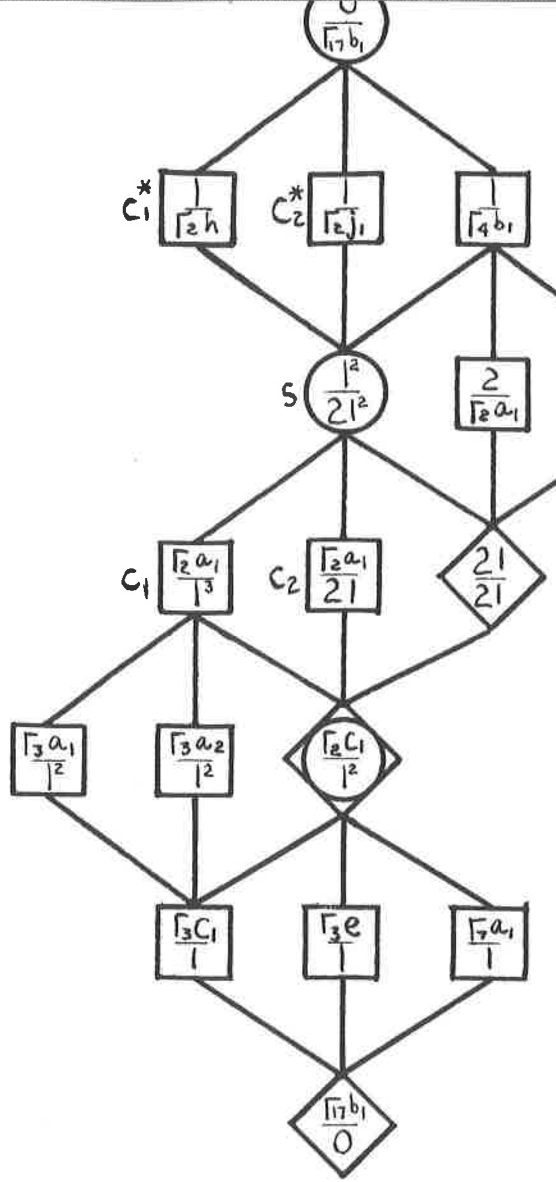
α_2 = ik.jl.mo.np

α_3 = abcd.efgh.ilkj.mpon

α_4 = ijkl.mnop

α_5 = aecg.bhdf.imko.jpln

α_6 = bd.ef.gh.im.jn.ko.lp



α_1 = ac.bd.eg.fh.ik.jl.mo.np

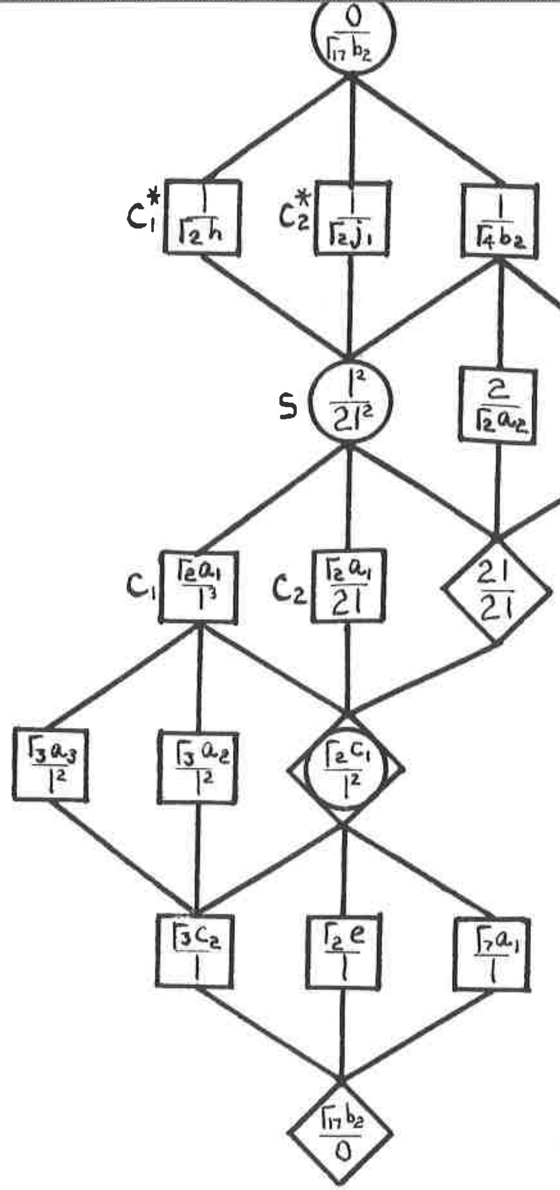
α_2 = ac.bd.eg.fh

α_3 = abcd.ehgf.ilkj.mpon

α_4 = ac.bd.ik.jl.mo.np

α_5 = aebf.cgdh.jl.mn.op

α_6 = abcd.imko.jpln



α_1 = ac.bd.eg.fh.ik.jl.mo.np

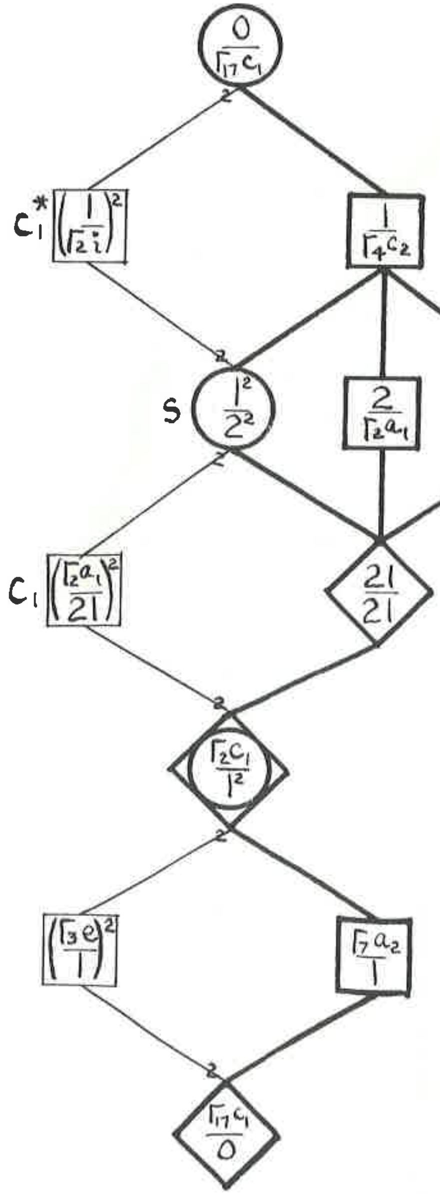
α_2 = ac.bd.eg.fh

α_3 = abcd.ehgf.ijkl.mnop

α_4 = ac.bd

α_5 = aecg.bfdh.imko.jpln

α_6 = abcd.jl.mn.op



α_1 = ac.bd.eg.fh.ik.jl.mo.np

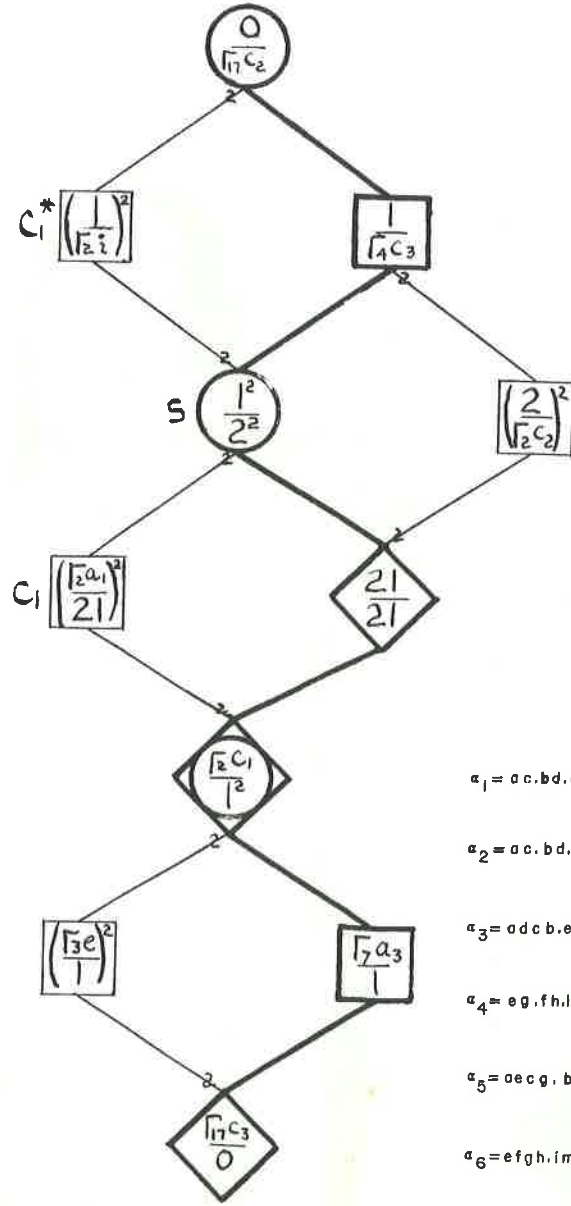
α_2 = ac.bd.eg.fh

α_3 = adcb.efgh.ijkl.mpon

α_4 = eg.fh.ijkl.mnop

α_5 = ae.bf.cg.dh.imjnkolp

α_6 = efgh.imjnkolp



α_1 = ac.bd.eg.fh.ik.jl.mo.np

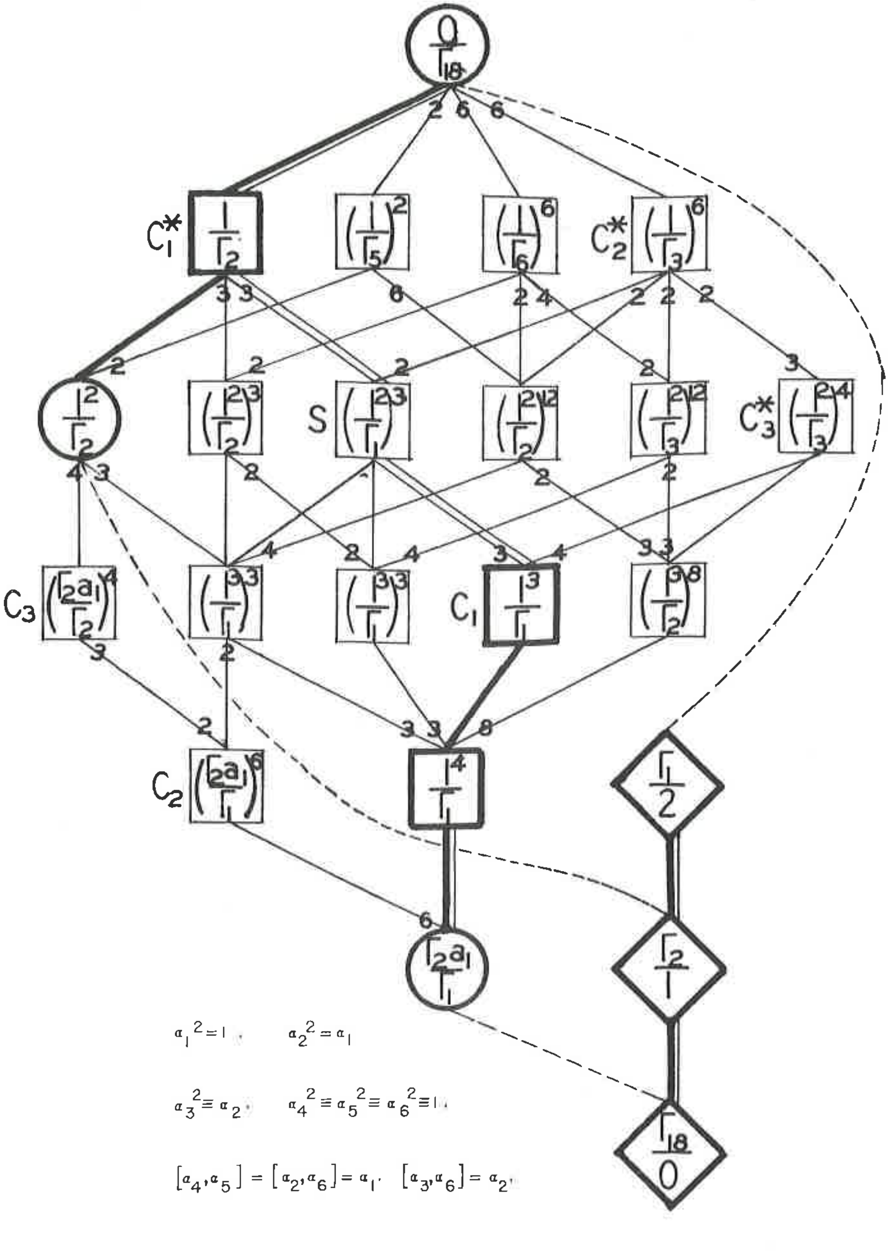
α_2 = ac.bd.eg.fh

α_3 = adcb.efgh.ijkl.mpon

α_4 = eg.fh.ijkl.mnop

α_5 = aecg.bfdh.im.jn.ko.lp

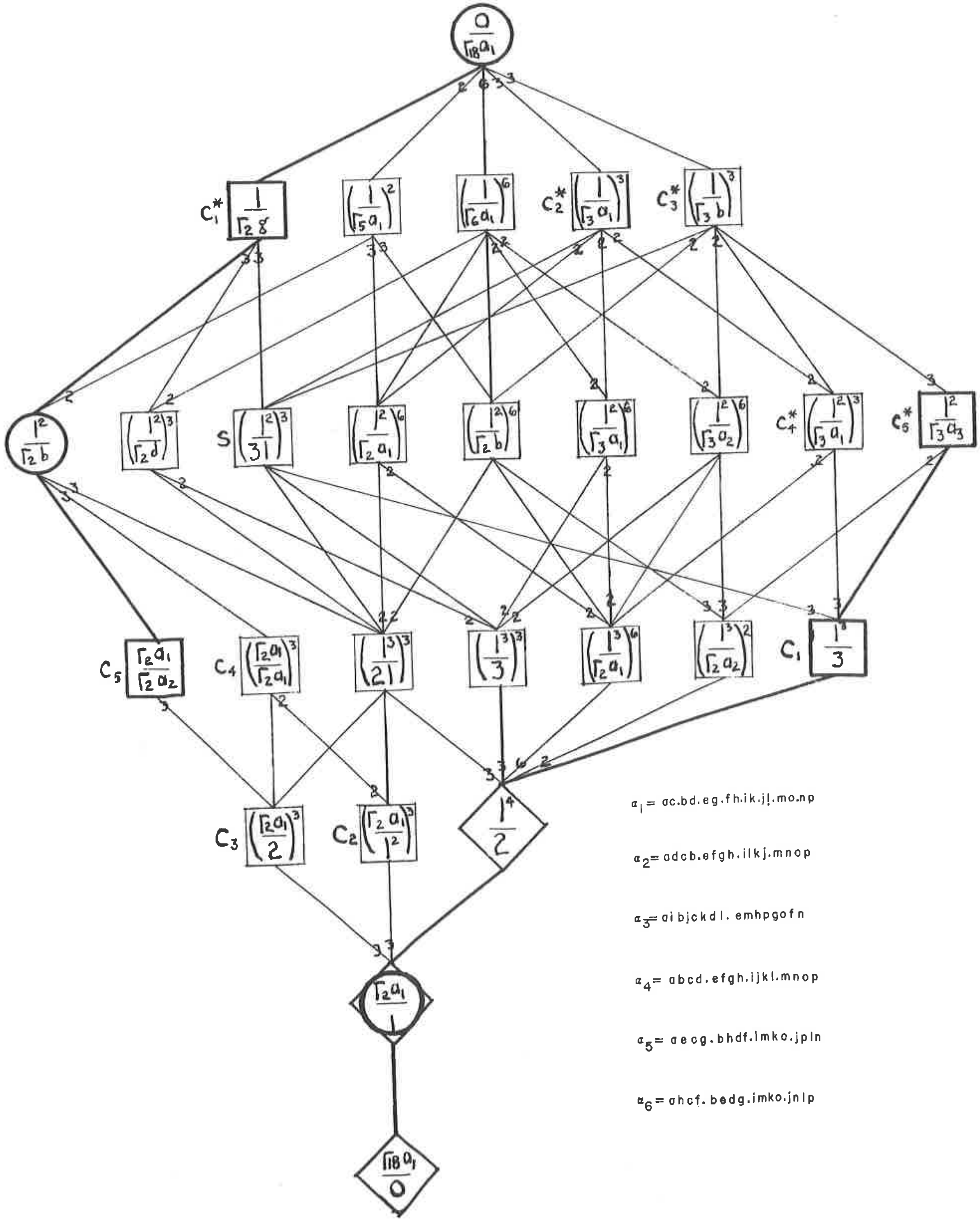
α_6 = efgh.imjnkolp



$\alpha_1^{\,2} = 1, \qquad \alpha_2^{\,2} = \alpha_1$

$\alpha_3^{\,2} \equiv \alpha_2, \qquad \alpha_4^{\,2} \equiv \alpha_5^{\,2} \equiv \alpha_6^{\,2} \equiv 1,$

$[\alpha_4, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1, \quad [\alpha_3, \alpha_6] = \alpha_2,$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

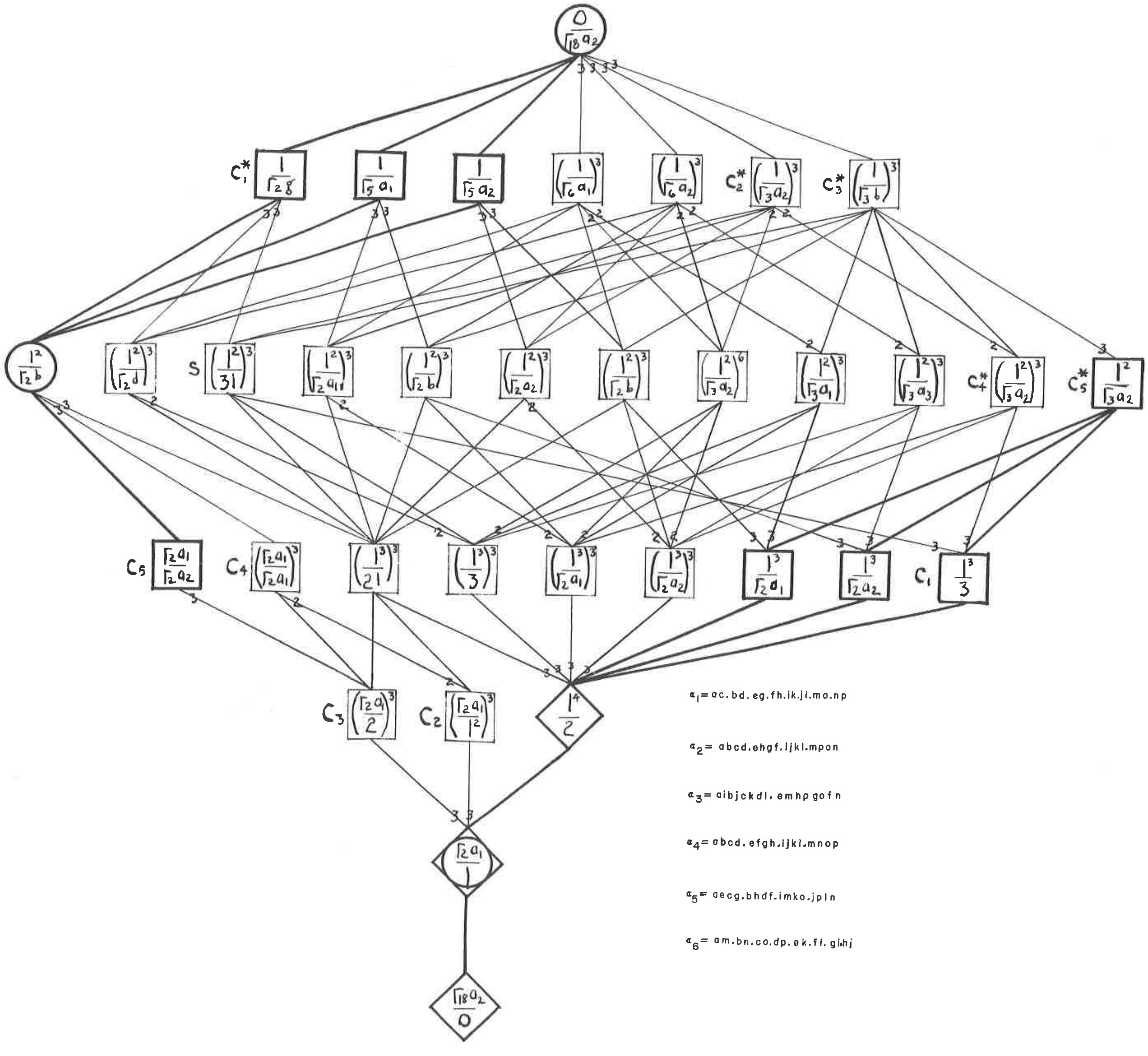
α_2 = adcb.efgh.ilkj.mnop

α_3 = aibjckdl. emhpgofn

α_4 = abcd.efgh.ijkl.mnop

α_5 = aecg.bhdf.lmko.jpln

α_6 = ahcf. bedg.imko.jnlp



α_1 = ac. bd. eg. fh. ik. jl. mo. np

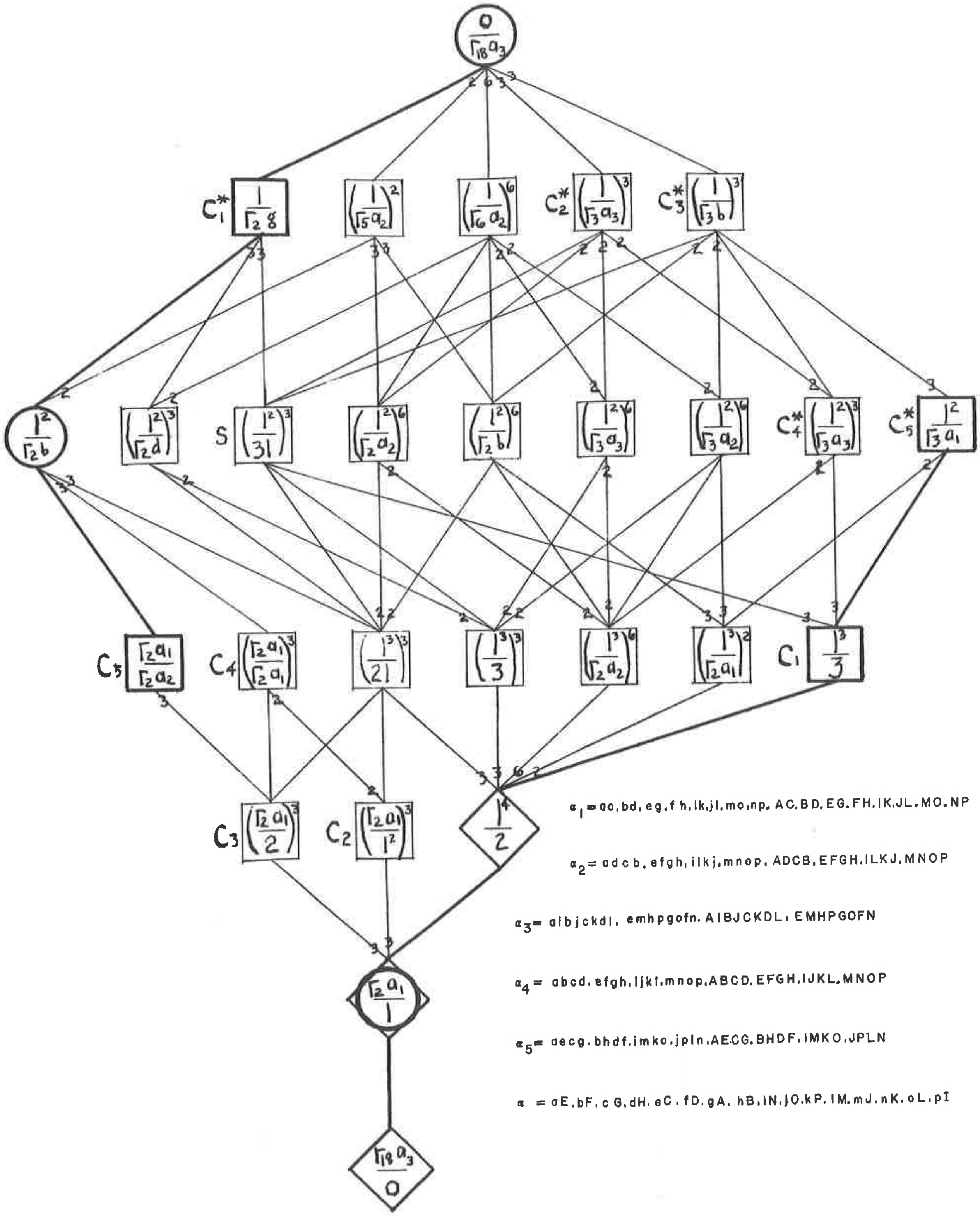
α_2 = abcd. ehgf. ijkl. mpon

α_3 = aibjckdl. emhpgofn

α_4 = abcd. efgh. ijkl. mnop

α_5 = aecg. bhdf. imko. jpln

α_6 = am. bn. co. dp. ek. fl. gi. hj



α_1 = ac, bd, eg, fh, ik, jl, mo, np. AC, BD, EG, FH, IK, JL, MO, NP

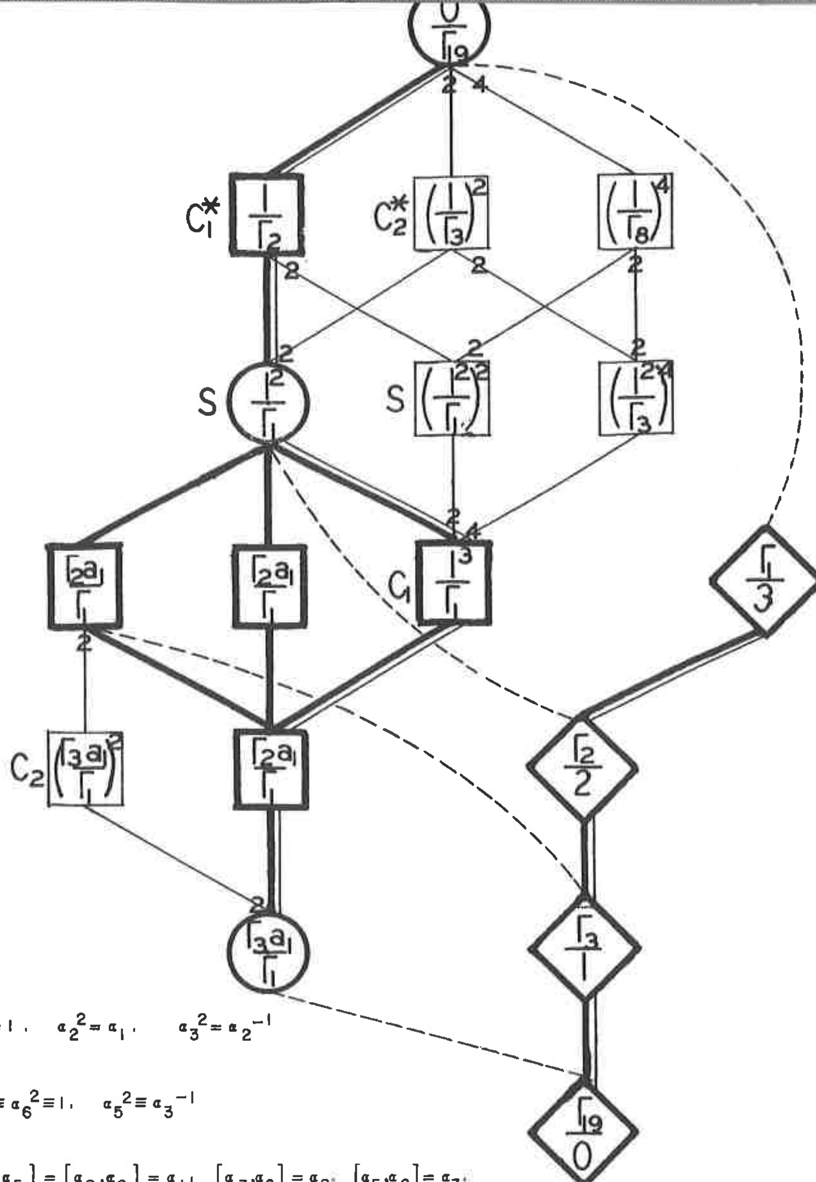
α_2 = adcb, efgh, ilkj, mnop. ADCB, EFGH, ILKJ, MNOP

α_3 = aibjckdl, emhpgofn. AIBJCKDL, EMHPGOFN

α_4 = abcd, efgh, ijkl, mnop, ABCD, EFGH, IJKL, MNOP

α_5 = aecg, bhdf, imko, jpln, AECG, BHDF, IMKO, JPLN

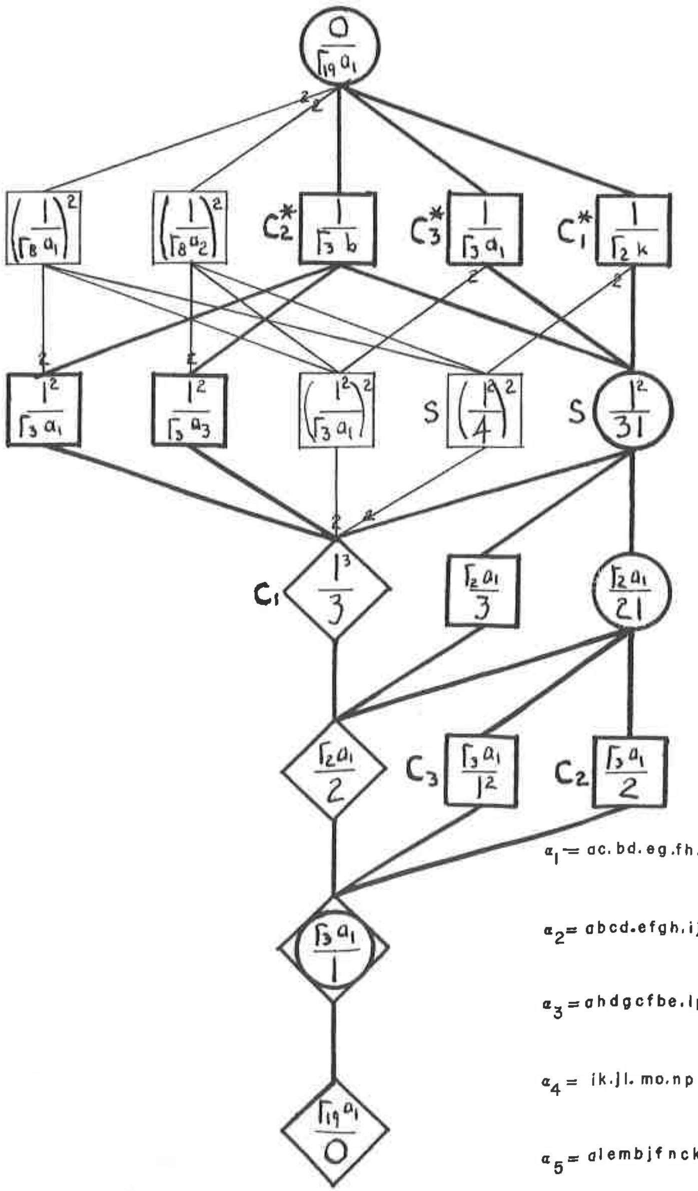
α = aE, bF, cG, dH, eC, fD, gA, hB, iN, jO, kP, lM, mJ, nK, oL, pI



$\alpha_1^2 = 1, \quad \alpha_2^2 = \alpha_1, \quad \alpha_3^2 = \alpha_2^{-1}$

$\alpha_4^2 \equiv \alpha_6^2 \equiv 1, \quad \alpha_5^2 \equiv \alpha_3^{-1}$

$[\alpha_4, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1, \quad [\alpha_3, \alpha_6] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3.$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

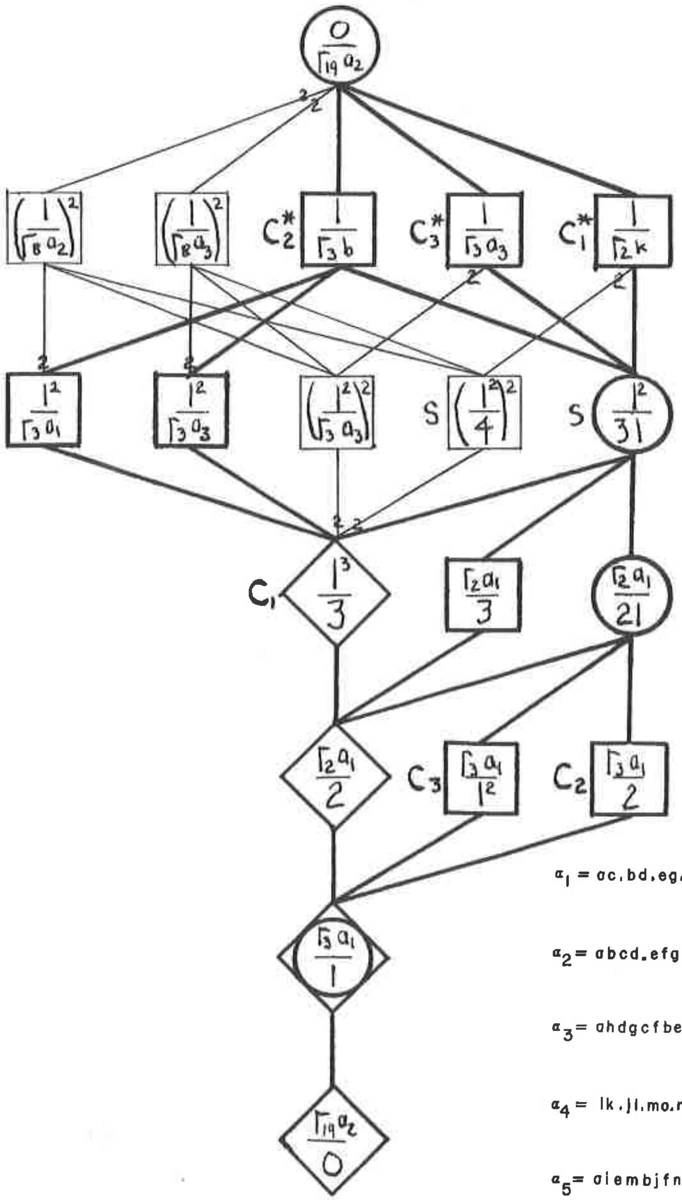
α_2 = abcd.efgh.ijkl.mnop

α_3 = ahdgcfbe.lploknjm

α_4 = ik.jl.mo.np

α_5 = alembjfnckgodlhp

α_6 = bd.eh.fg.lp.jo.kn.lm



α_1 = ac.bd.eg.fh.ik.jl.mo.np.AC.BD.EG.FH.IK.JL.MO.NP

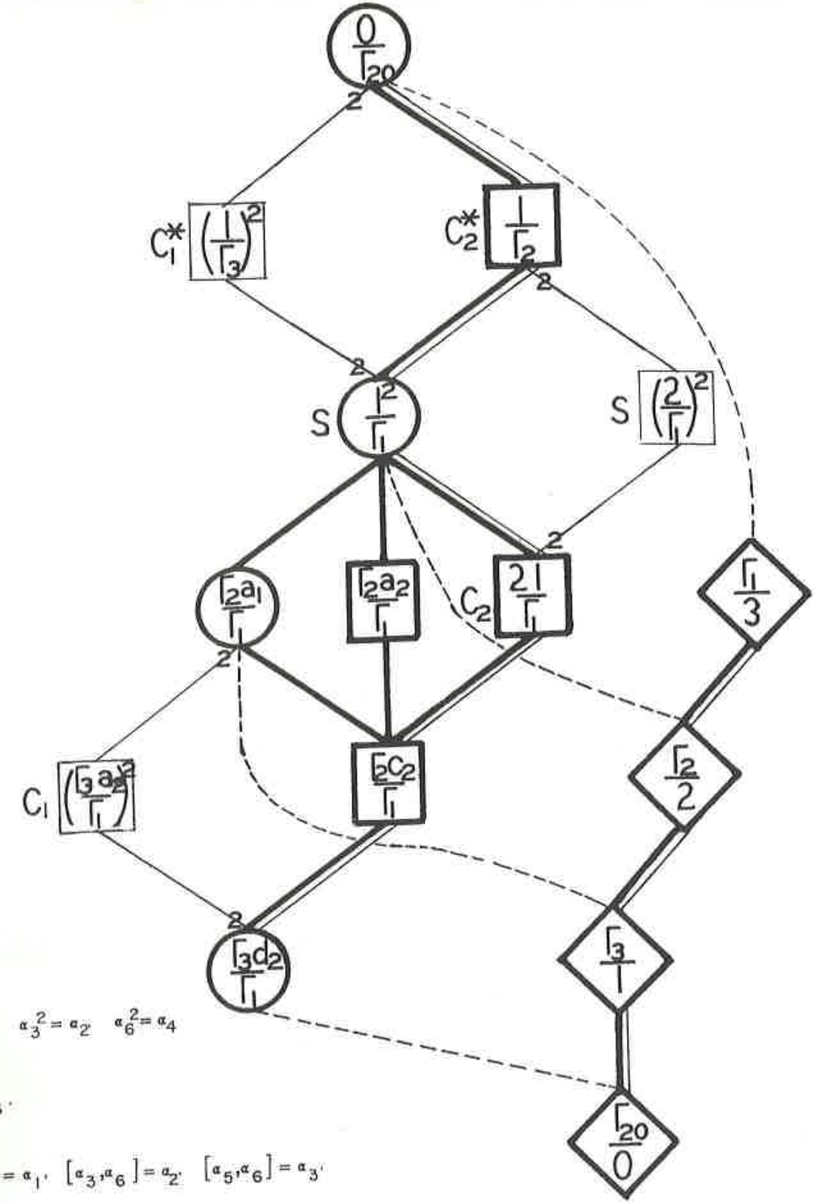
α_2 = abcd.efgh.ijkl.mnop.ABCD.EFGH.IJKL.MNOP

α_3 = ahdgcfbe.lploknjm.AHDGCFBE.IPLOKNJM

α_4 = lk.jl.mo.np.IK.JL.MO.NP

α_5 = alembjfnckgodlhp.AIEMBJFNCKGODLHP

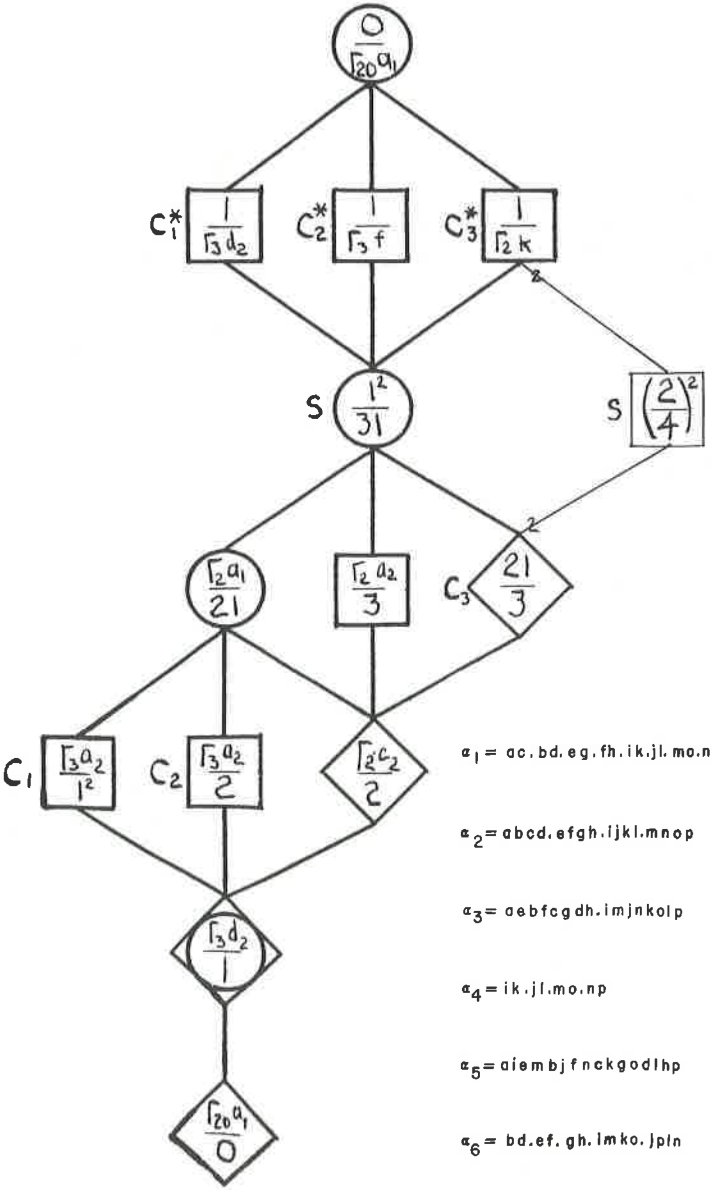
α_6 = aAcC.bDdB.eHgF.fGhE.lPkN.JOIM.mLoJ.nKpI



$\alpha_1{}^2 = 1, \quad \alpha_2{}^2 = \alpha_1, \quad \alpha_3{}^2 = \alpha_2, \quad \alpha_6{}^2 = \alpha_4$

$\alpha_4{}^2 \equiv 1, \quad \alpha_5{}^2 \equiv \alpha_3.$

$[\alpha_4, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1, \quad [\alpha_3, \alpha_6] = \alpha_2, \quad [\alpha_5, \alpha_6] = \alpha_3.$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

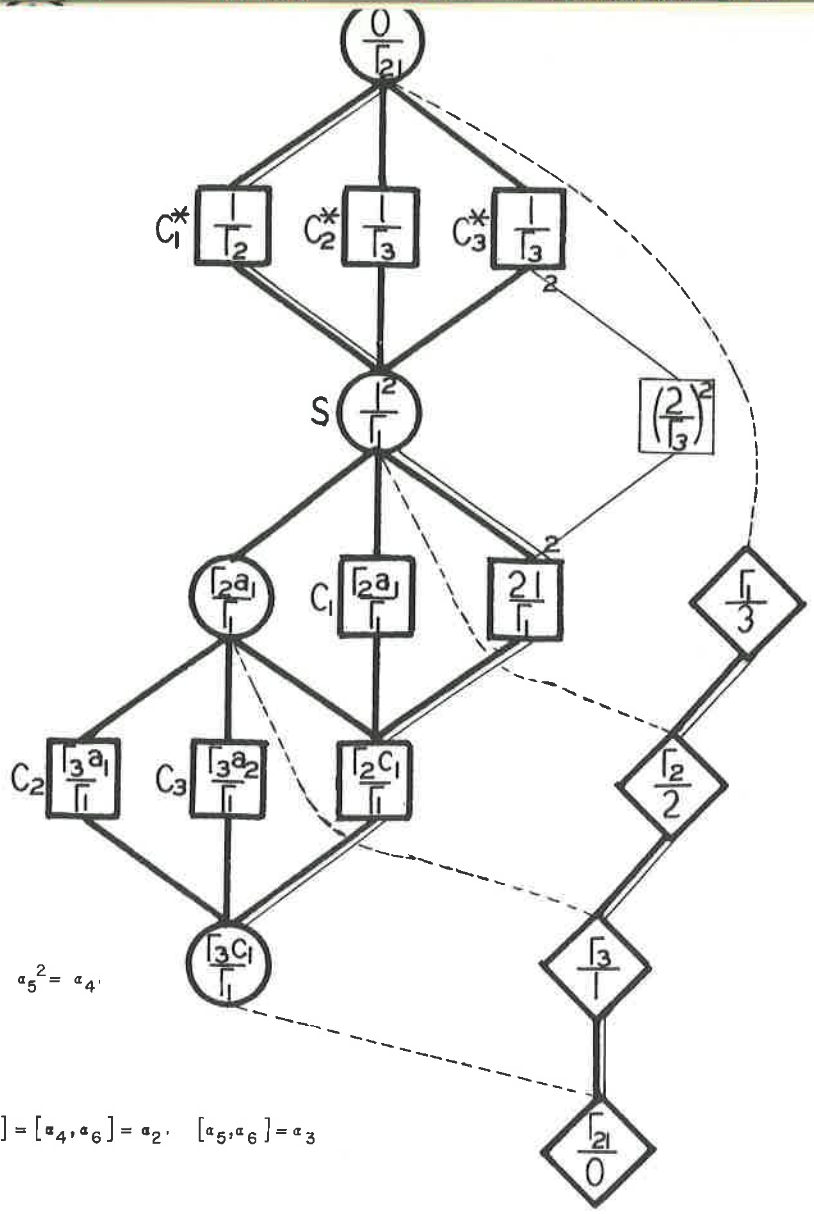
α_2 = abcd.efgh.ijkl.mnop

α_3 = aebfcgdh.imjnkolp

α_4 = ik.jl.mo.np

α_5 = aiembjfnckgodlhp

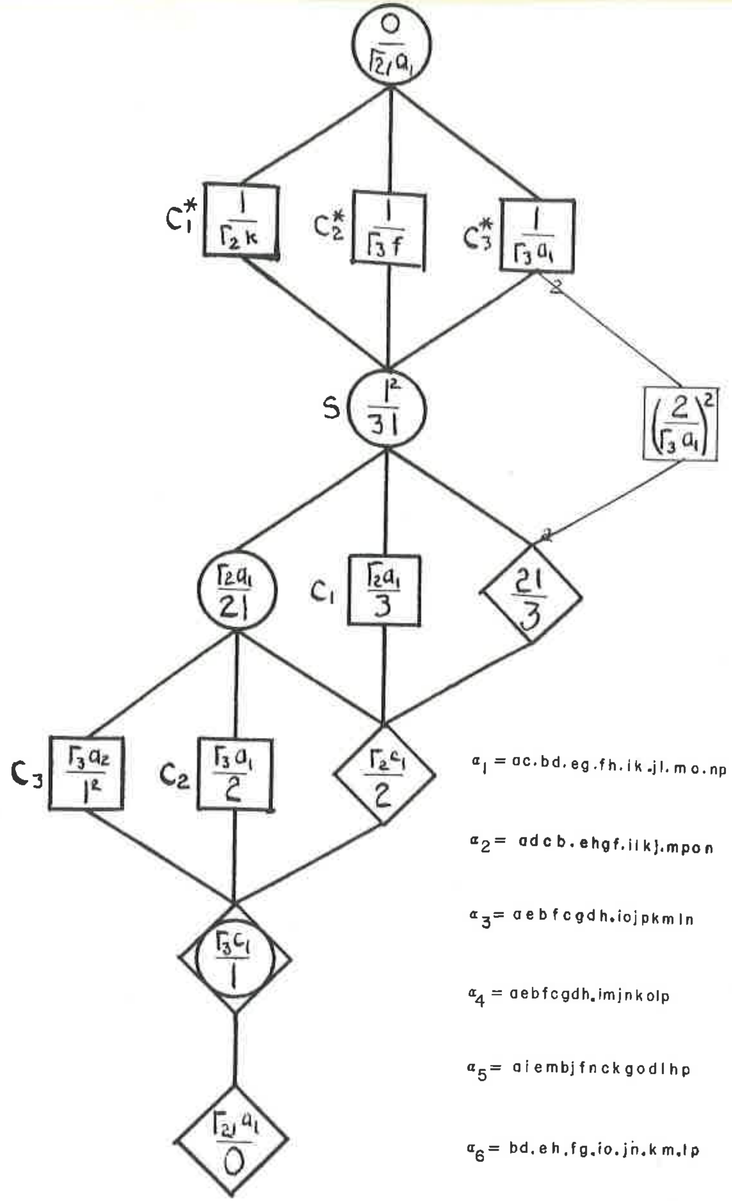
α_6 = bd.ef. gh. lmko. jpln



$\alpha_1{}^2 = 1, \quad \alpha_2{}^2 = \alpha_1, \quad \alpha_3{}^2 = \alpha_2^{-1}, \quad \alpha_5{}^2 = \alpha_4,$

$\alpha_4{}^2 \equiv \alpha_3, \quad \alpha_6{}^2 \equiv 1$

$[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1. \quad [\alpha_3, \alpha_6] = [\alpha_4, \alpha_6] = \alpha_2. \quad [\alpha_5, \alpha_6] = \alpha_3$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

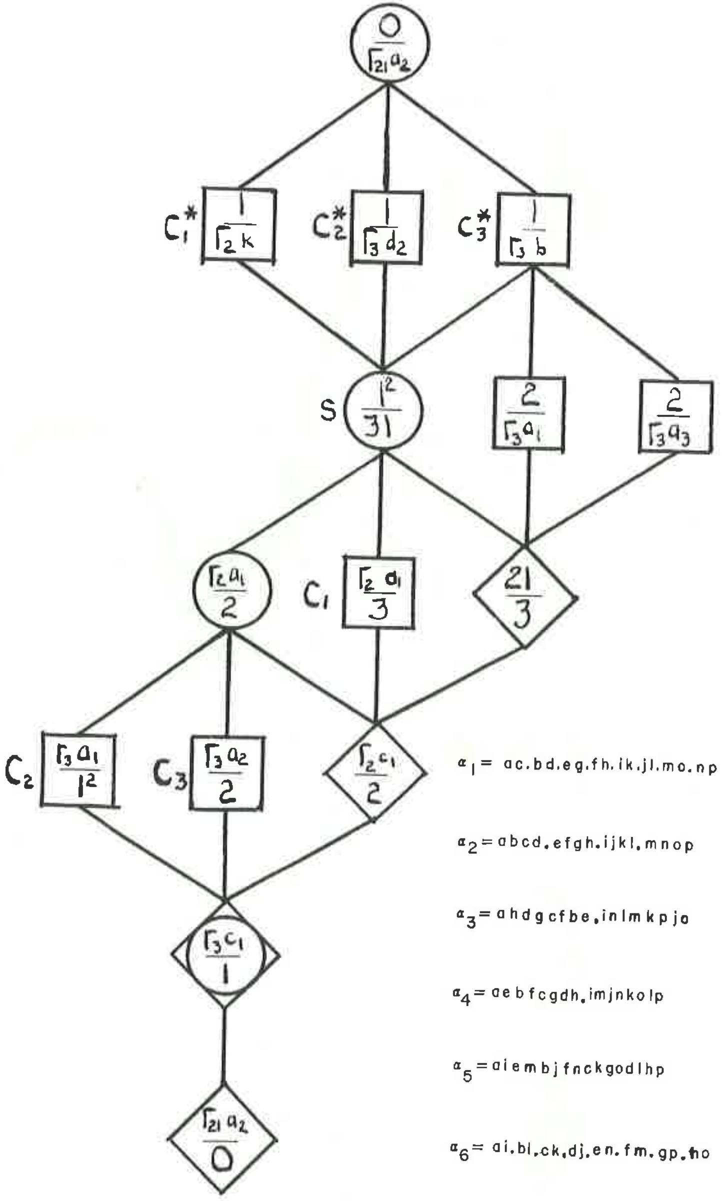
α_2 = adcb.ehgf.ilkj.mpon

α_3 = aebfcgdh.iojpkmln

α_4 = aebfcgdh.imjnkolp

α_5 = aiembjfnckgodlhp

α_6 = bd.eh.fg.io.jn.km.lp



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

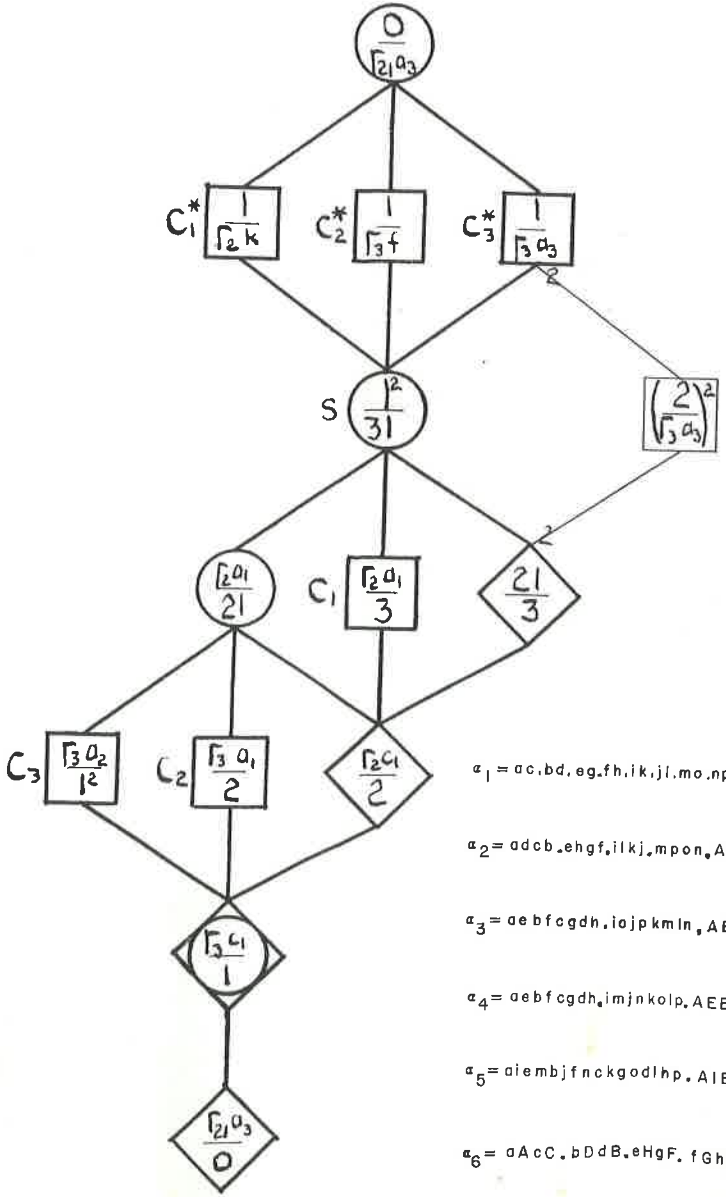
$\alpha_2 =$ abcd.efgh.ijkl.mnop

$\alpha_3 =$ ahdgcfbe.inlmkpjo

$\alpha_4 =$ aebfcgdh.imjnkolp

$\alpha_5 =$ aiembjfnckgodlhp

$\alpha_6 =$ ai.bl.ck.dj.en.fm.gp.ho



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np.AC.BD.EG.FH.IK.JL.MO.NP

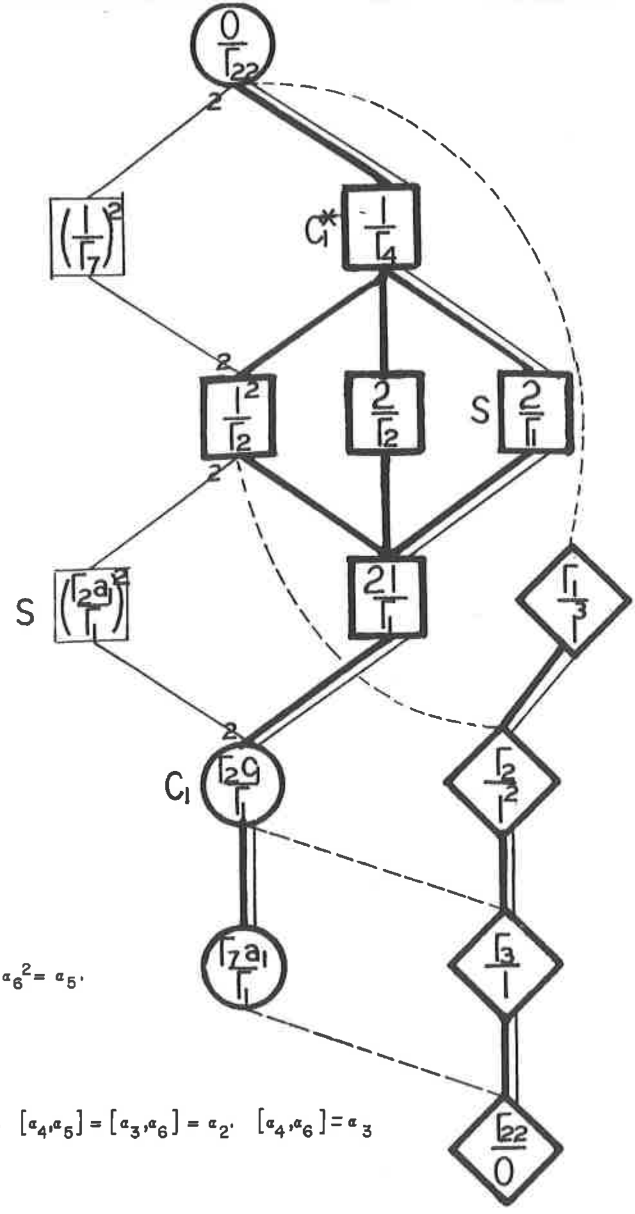
$\alpha_2 =$ adcb.ehgf.ilkj.mpon.ADCB.EHGF.ILKJ.MPON

$\alpha_3 =$ aebfcgdh.iojpkmln.AEBFCGDH.IOJPKMLN

$\alpha_4 =$ aebfcgdh.imjnkolp.AEBFCGDH.IMJNKOLP

$\alpha_5 =$ aiembjfnckgodlhp.AIEMBJFNCKGODLHP

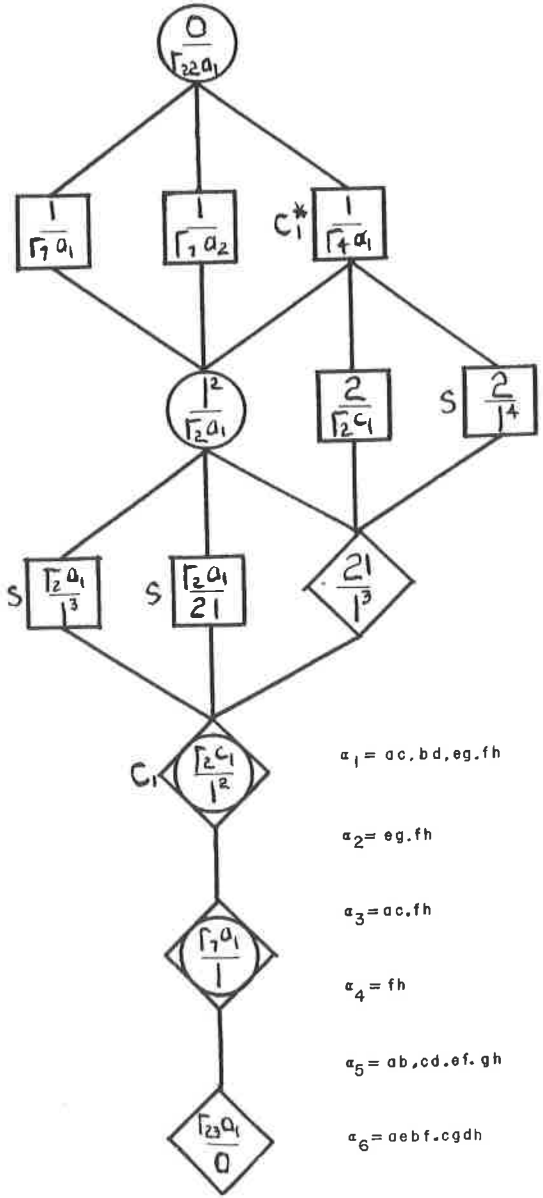
$\alpha_6 =$ aAcC.bDdB.eHgF.fGhE.iOkM.jNlP.mKoI.nJpL



$\alpha_1{}^2 = \alpha_2{}^2 = \alpha_3{}^2 = 1.$ $\alpha_6{}^2 = \alpha_5.$

$\alpha_4{}^2 \equiv \alpha_5{}^2 \equiv 1$

$[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1.$ $[\alpha_4, \alpha_5] = [\alpha_3, \alpha_6] = \alpha_2.$ $[\alpha_4, \alpha_6] = \alpha_3$



α_1 = ac, bd, eg. fh

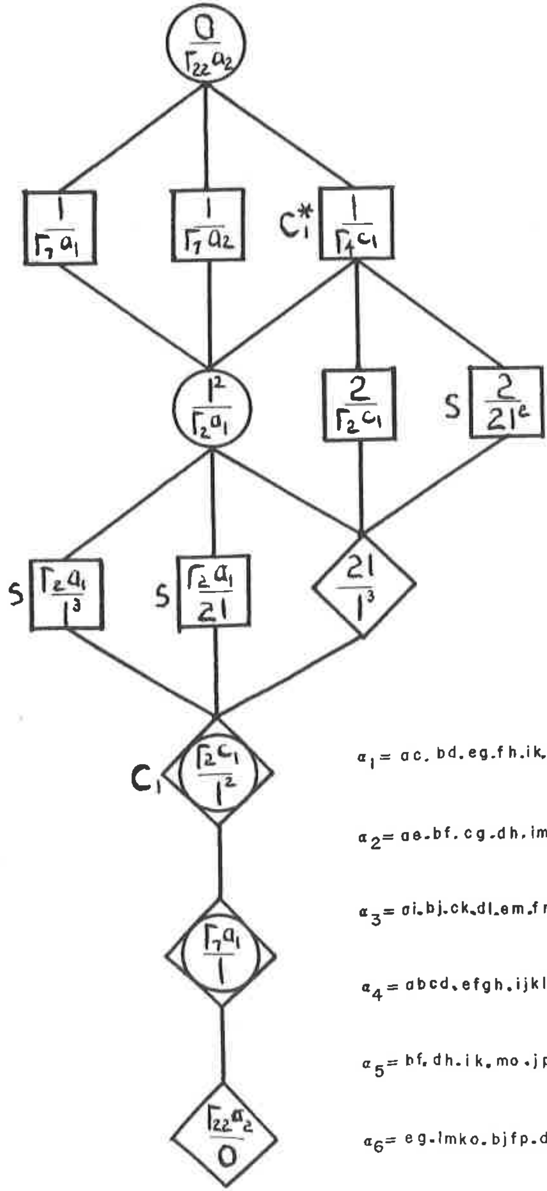
α_2 = eg. fh

α_3 = ac. fh

α_4 = fh

α_5 = ab, cd. ef. gh

α_6 = aebf. cgdh



α_1 = ac. bd. eg. fh. ik. jl. mo. np

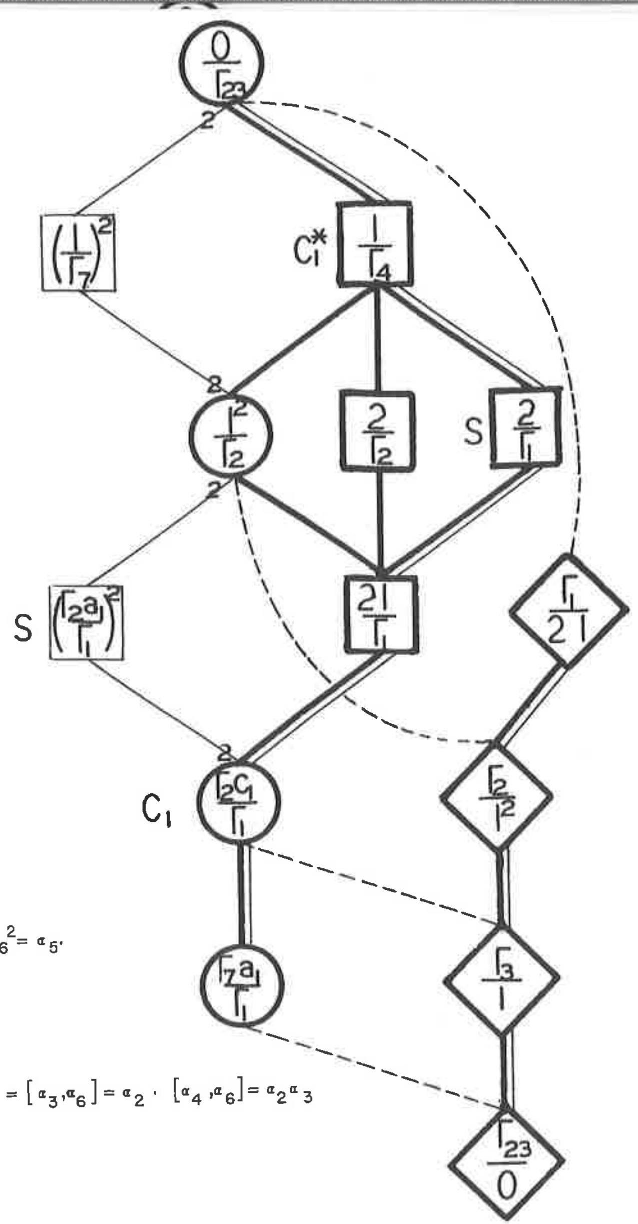
α_2 = ae. bf. cg. dh. im. jn. ko. lp

α_3 = ai. bj. ck. dl. em. fn. go. hp

α_4 = abcd. efgh. ijkl. mnop

α_5 = bf. dh. ik. mo. jp. ln

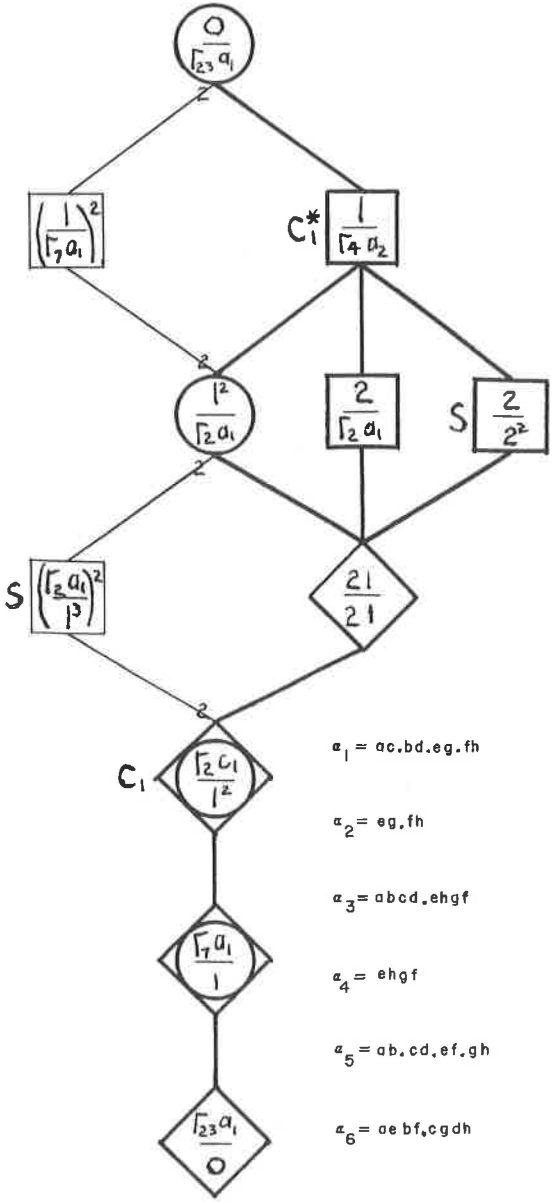
α_6 = eg. lmko. bjfp. dlhn



$\alpha_1^2=\alpha_2^2=1$. $\alpha_3^2=\alpha_1$. $\alpha_6^2=\alpha_5$.

$\alpha_4^2\equiv\alpha_2$. $\alpha_5^2\equiv 1$.

$[\alpha_3,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1$. $[\alpha_4,\alpha_5]=[\alpha_3,\alpha_6]=\alpha_2$. $[\alpha_4,\alpha_6]=\alpha_2\alpha_3$



α_1 = ac.bd.eg.fh

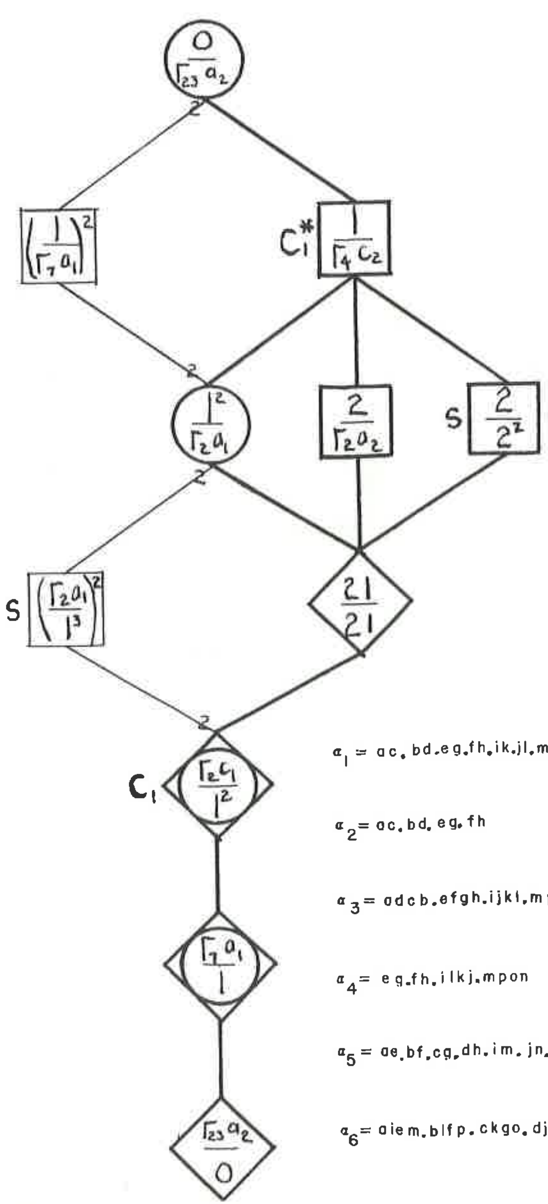
α_2 = eg.fh

α_3 = abcd.ehgf

α_4 = ehgf

α_5 = ab.cd.ef.gh

α_6 = ae bf.cgdh



$\alpha_1 =$ ac. bd.eg.fh.ik.jl.mo.np

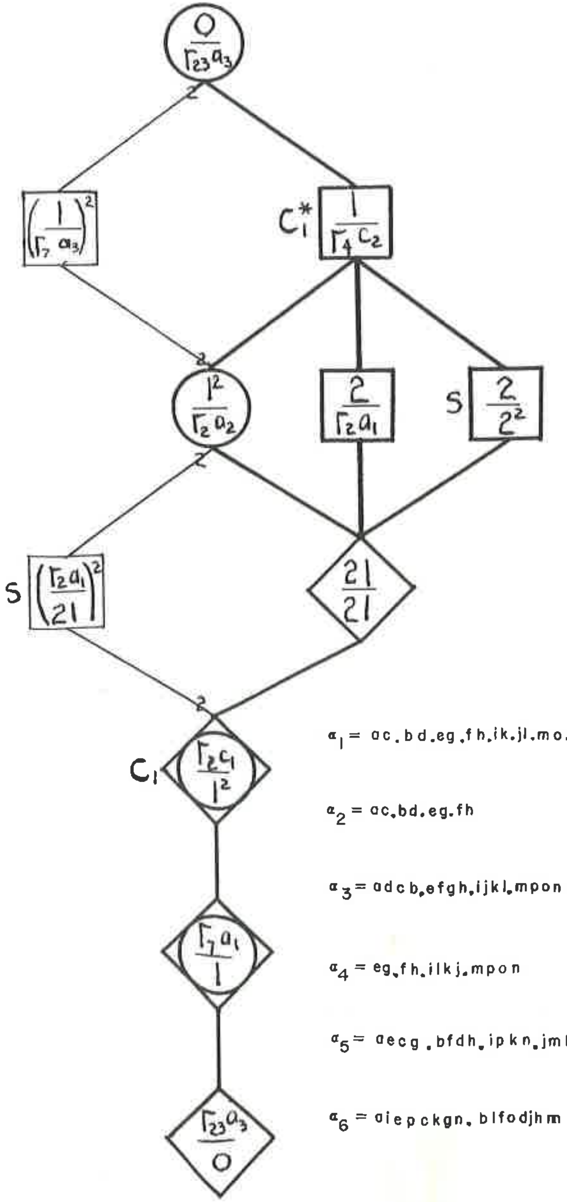
$\alpha_2 =$ ac. bd. eg. fh

$\alpha_3 =$ adcb.efgh.ijkl.mpon

$\alpha_4 =$ eg.fh.ilkj.mpon

$\alpha_5 =$ ae.bf.cg.dh.im. jn.ko.lp

$\alpha_6 =$ aiem.blfp.ckgo.djhn



$\alpha_1 =$ ac. bd.eg.fh.ik.jl.mo.np

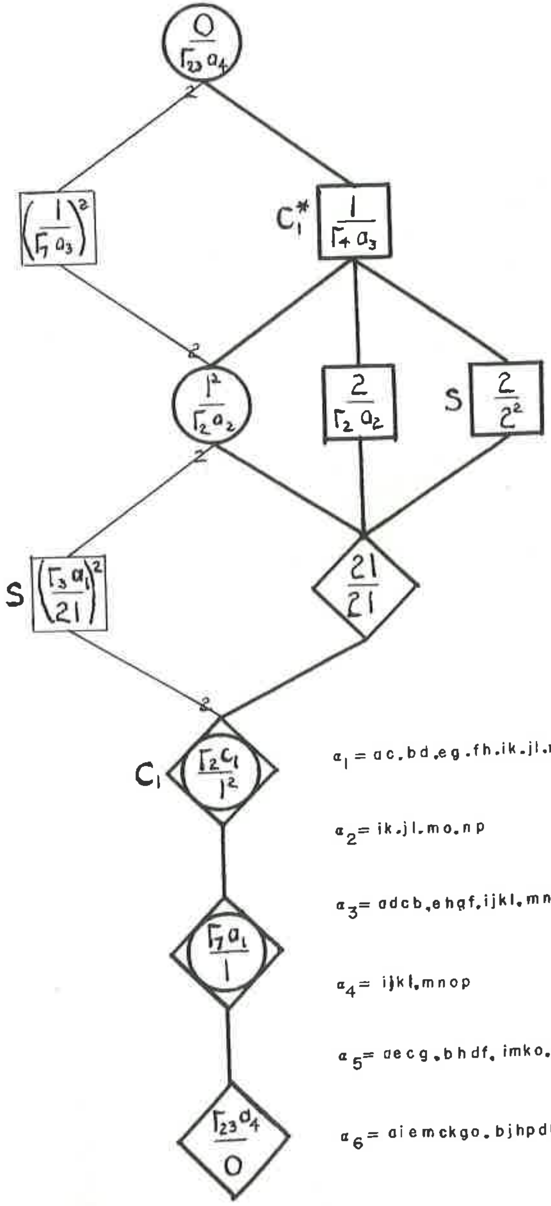
$\alpha_2 =$ ac.bd.eg.fh

$\alpha_3 =$ adcb.efgh.ijkl.mpon

$\alpha_4 =$ eg.fh.ilkj.mpon

$\alpha_5 =$ aecg .bfdh.ipkn.jmlo

$\alpha_6 =$ aiepckgn. blfodjhm



$\alpha_1 =$ ac.bd.eg.fh.ik.jl.mo.np

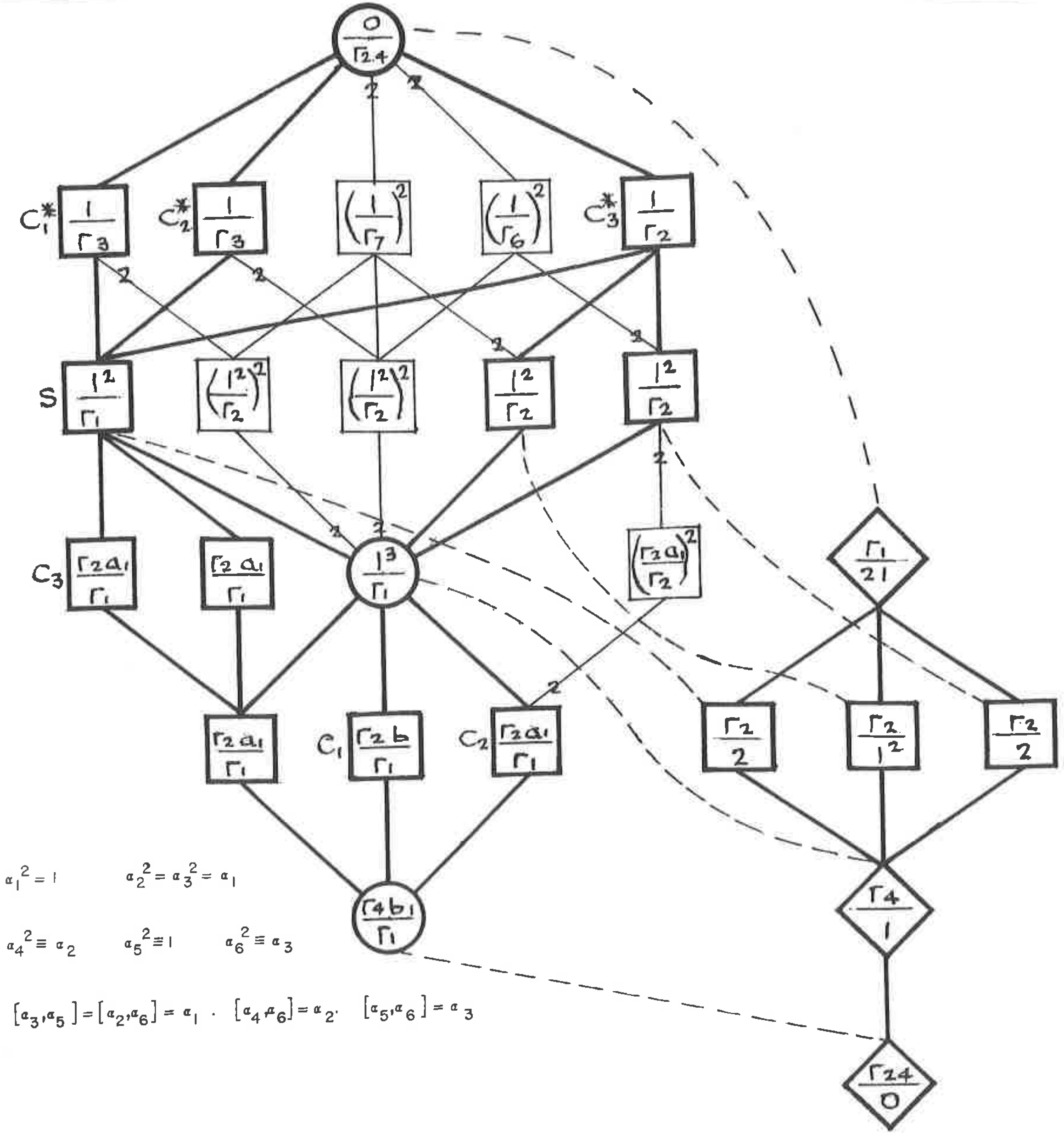
$\alpha_2 =$ ik.jl.mo.np

$\alpha_3 =$ adcb.ehgf.ijkl.mnop

$\alpha_4 =$ ijkl.mnop

$\alpha_5 =$ aecg.bhdf. imko.jpln

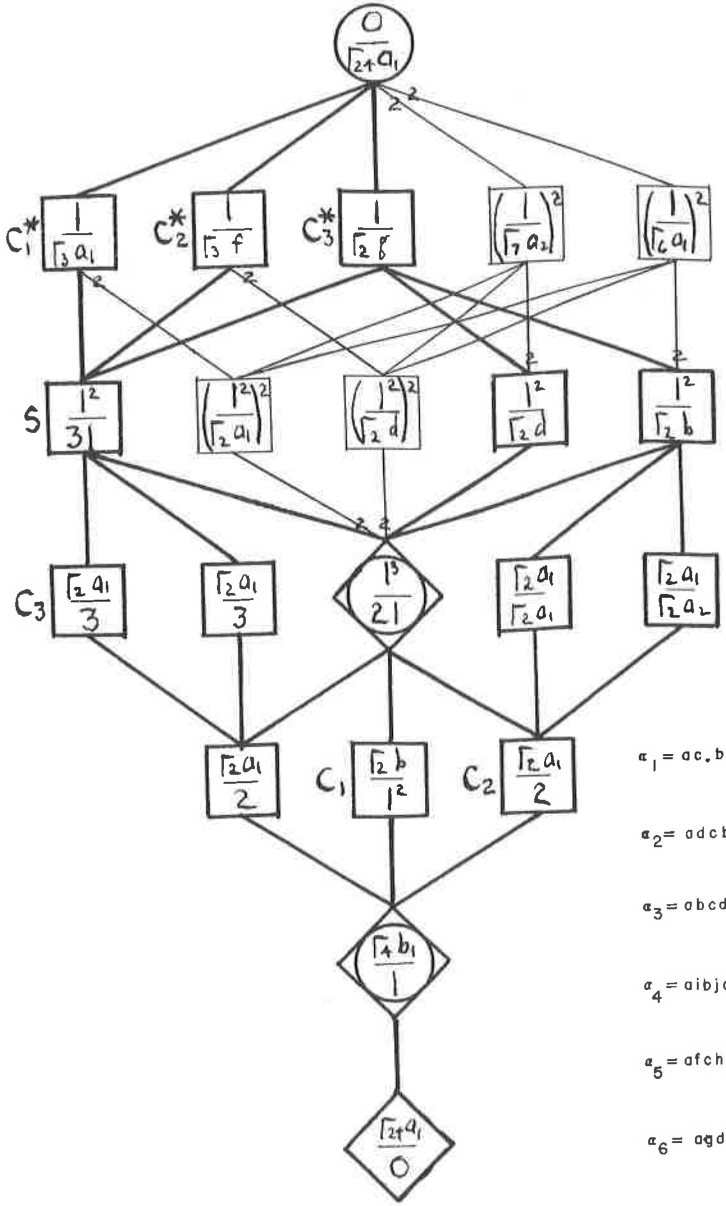
$\alpha_6 =$ aiemckgo. bjhpdlfn



$\alpha_1{}^2 = 1$ $\alpha_2{}^2 = \alpha_3{}^2 = \alpha_1$

$\alpha_4{}^2 \equiv \alpha_2$ $\alpha_5{}^2 \equiv 1$ $\alpha_6{}^2 \equiv \alpha_3$

$[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1$. $[\alpha_4, \alpha_6] = \alpha_2$. $[\alpha_5, \alpha_6] = \alpha_3$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

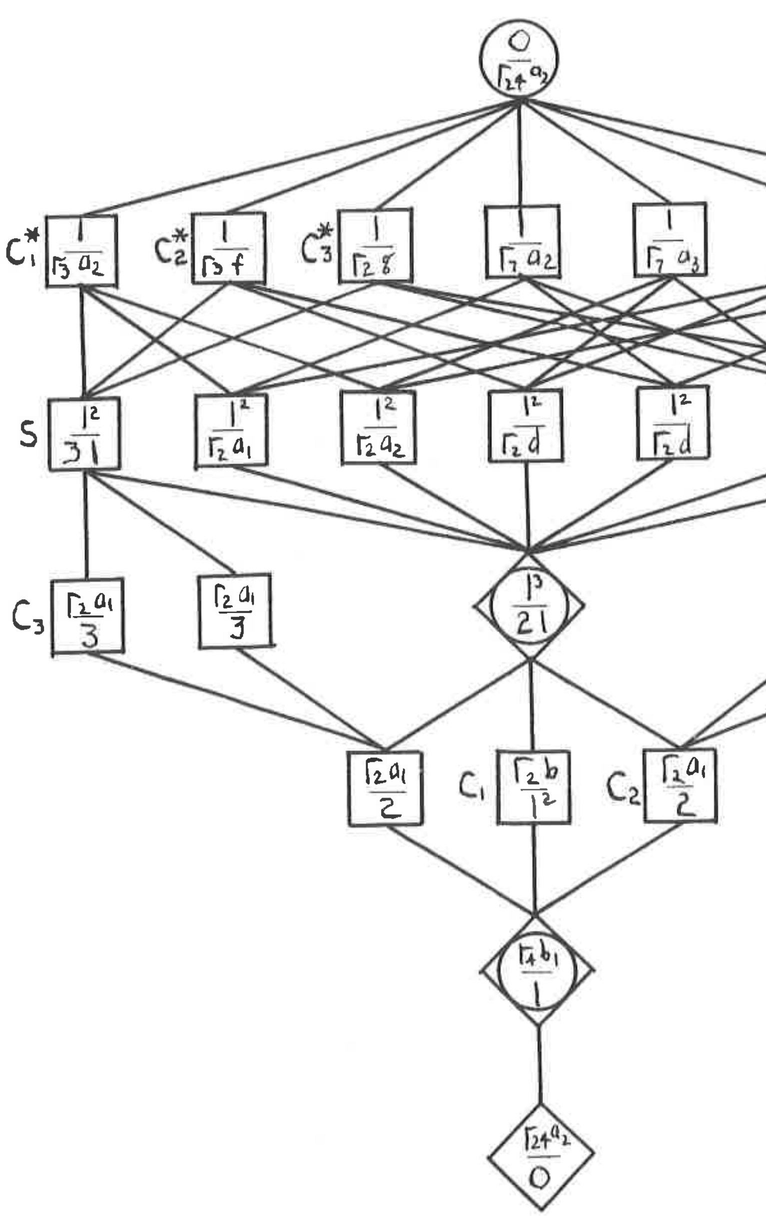
α_2 = adcb.efgh.ilkj.mnop

α_3 = abcd.efgh.ijkl.mnop

α_4 = aibjckdl.emhpgofn

α_5 = afch.bedg.inkp.jmlo

α_6 = agdfce bh.iploknjm



α_1 = ac. bd .eg.fh.ik.jl.mo.np

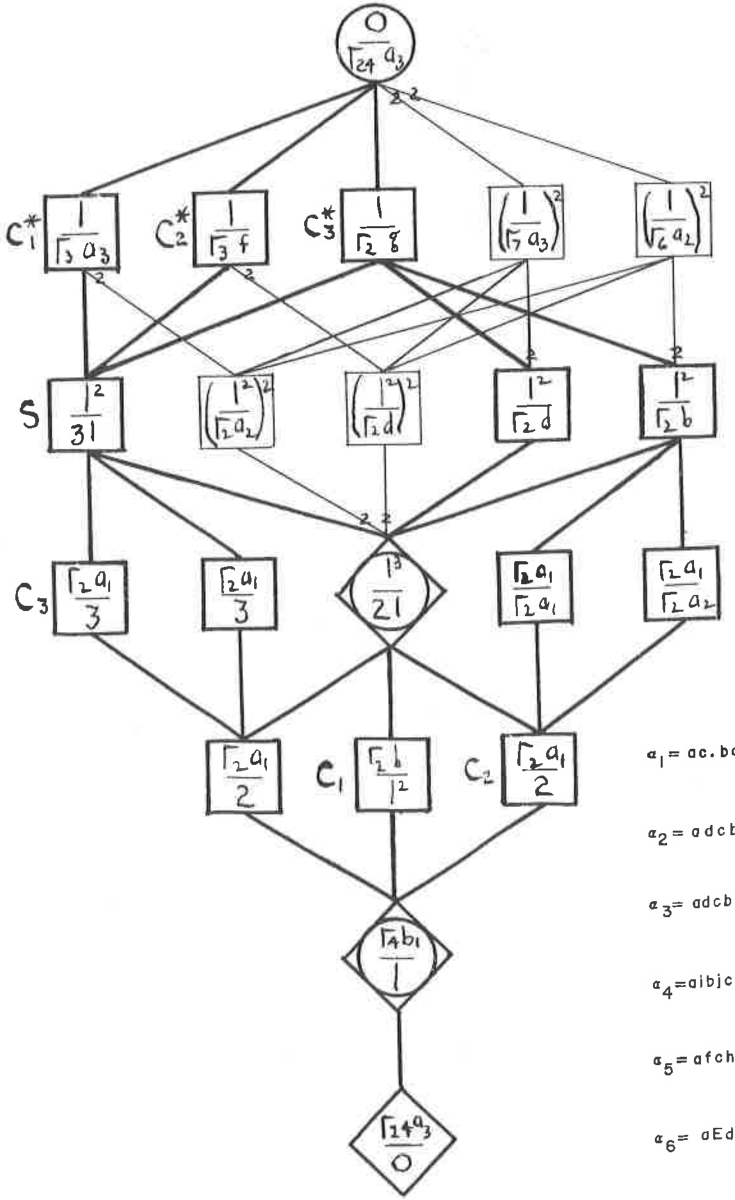
α_2 = abcd. ehgf.ijkl. mpon

α_3 = abcd.efgh.ijkl.mnop

α_4 = aibjckdl. emhpgofn

α_5 = afch.bedg. ink p. jmlo

α_6 = ambncodp.elfigjhk



α_1 = ac.bd.eg.fh.ik.jl.mo.np.AC.BD.EG.FH.IK.JL.MO.NP

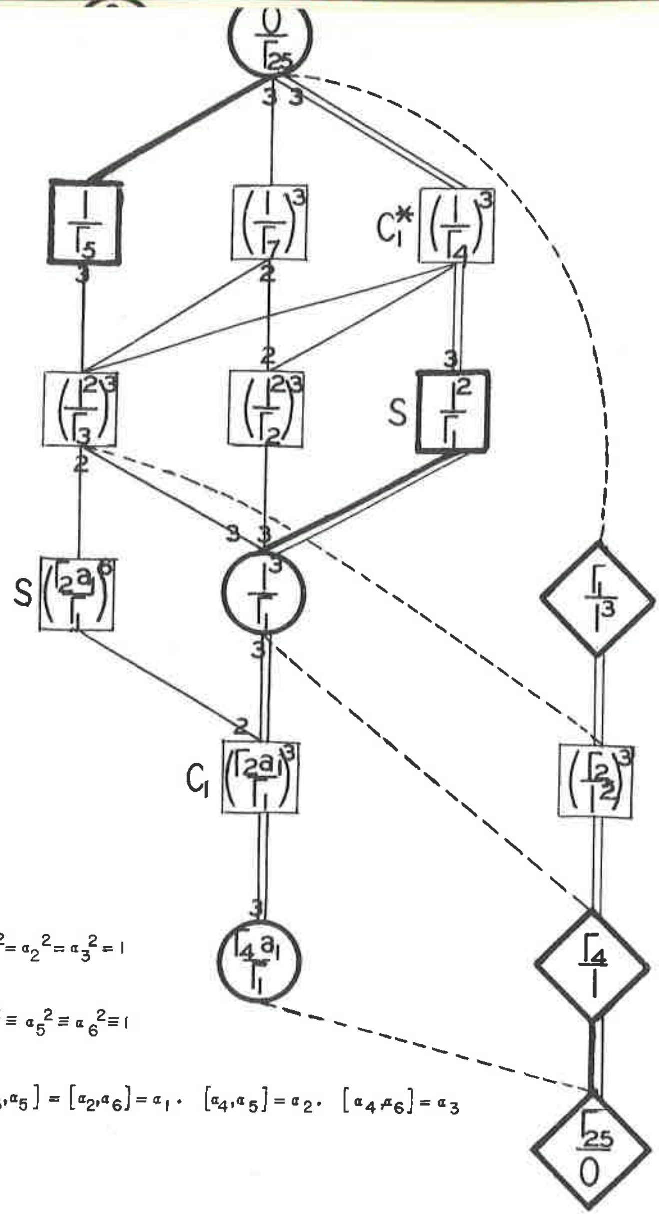
α_2 = adcb.efgh.ilkj.mnop. ADCB. EFGH.ILKJ.MNOP

α_3 = adcb.ehgf. ilkj.mpon.ADCB.EHGF.ILKJ.MPON

α_4 = aibjckdl.emhpgofn.AIBJCKDL.EMHPGOFN

α_5 = afch.bedg.inkp.jmlo.AFCH.BEDG.INKP.JMLO

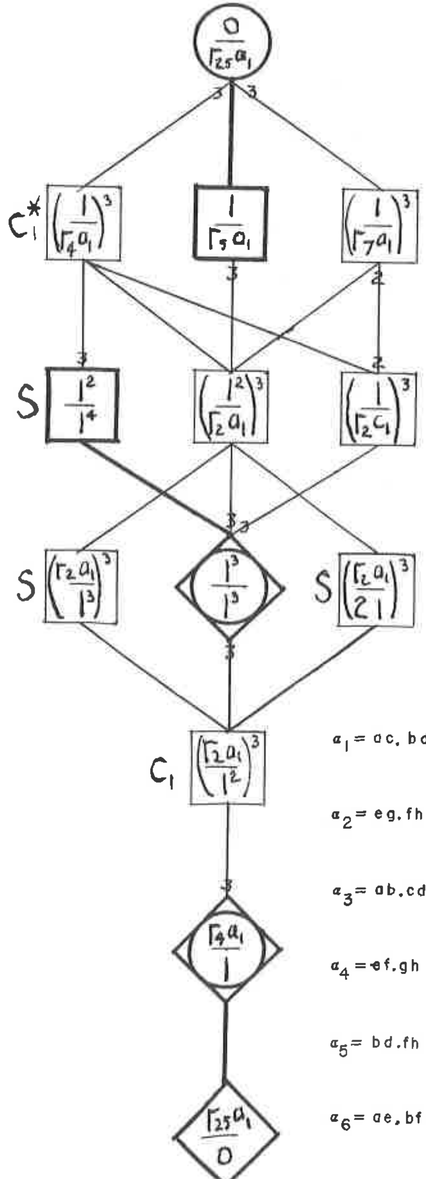
α_6 = aEdHcGbF eBhAgDfC.iNIMkPjO.mIpLoKnJ



$\alpha_1^2=\alpha_2^2=\alpha_3^2=1$

$\alpha_4^2\equiv\alpha_5^2\equiv\alpha_6^2\equiv 1$

$[\alpha_3,\alpha_5]=[\alpha_2,\alpha_6]=\alpha_1. \quad [\alpha_4,\alpha_5]=\alpha_2. \quad [\alpha_4,\alpha_6]=\alpha_3$



α_1 = ac, bd.eg.fh

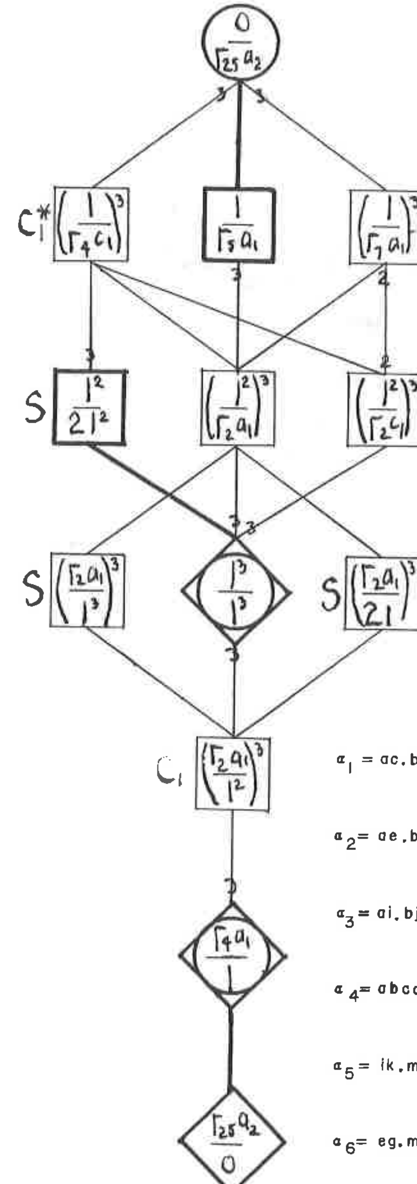
α_2 = eg.fh

α_3 = ab.cd.ef.gh

α_4 = ef.gh

α_5 = bd.fh

α_6 = ae, bf.cg.dh



α_1 = ac.bd.eg.fh.ik.jl.mo.np

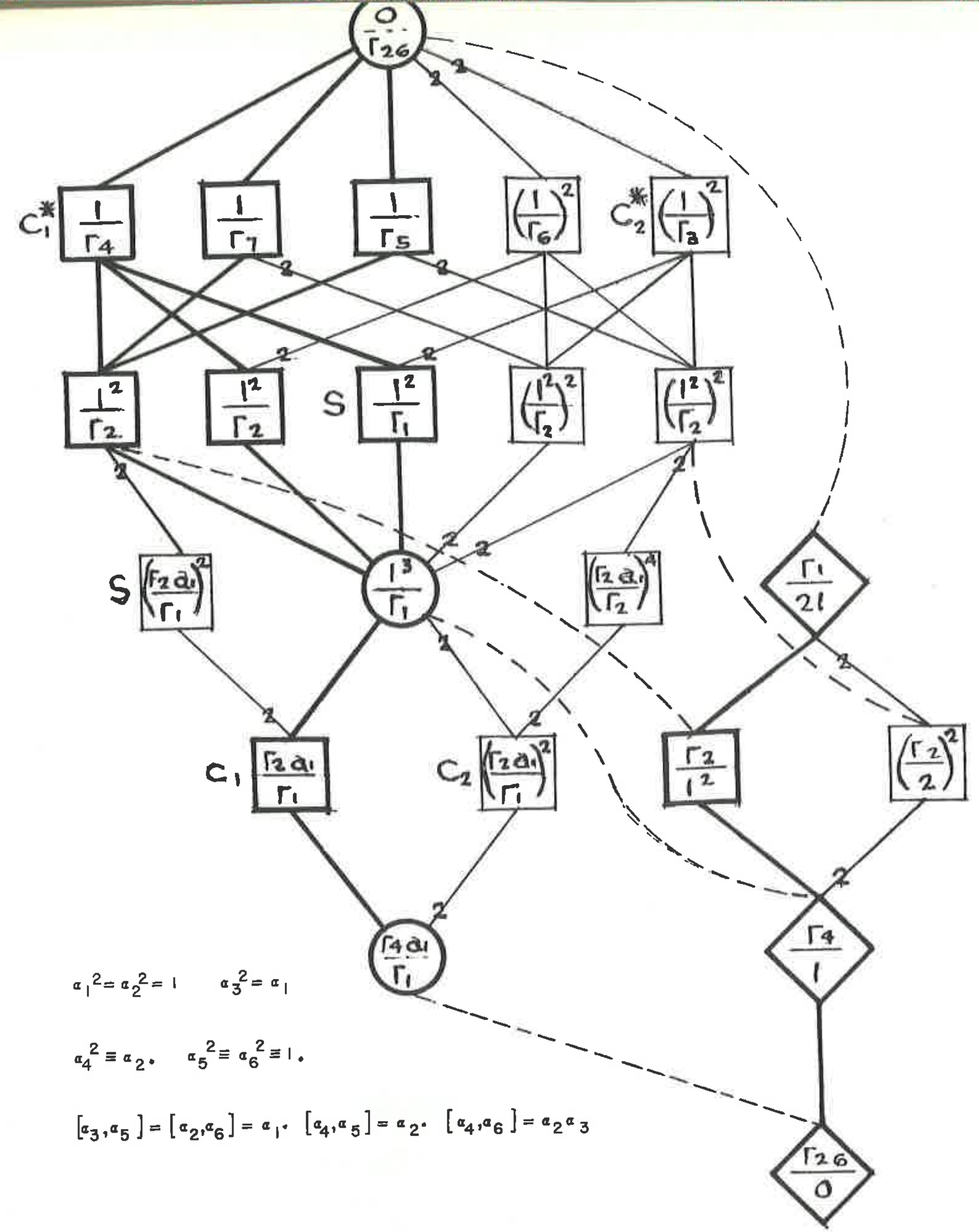
α_2 = ae.bf.cg.dh.im.jn.ko.lp

α_3 = ai.bj.ck.dl.em.fn.go.hp

α_4 = abcd.efgh.ijkl.mnop

α_5 = ik.mo.bf.jp.dh.ln

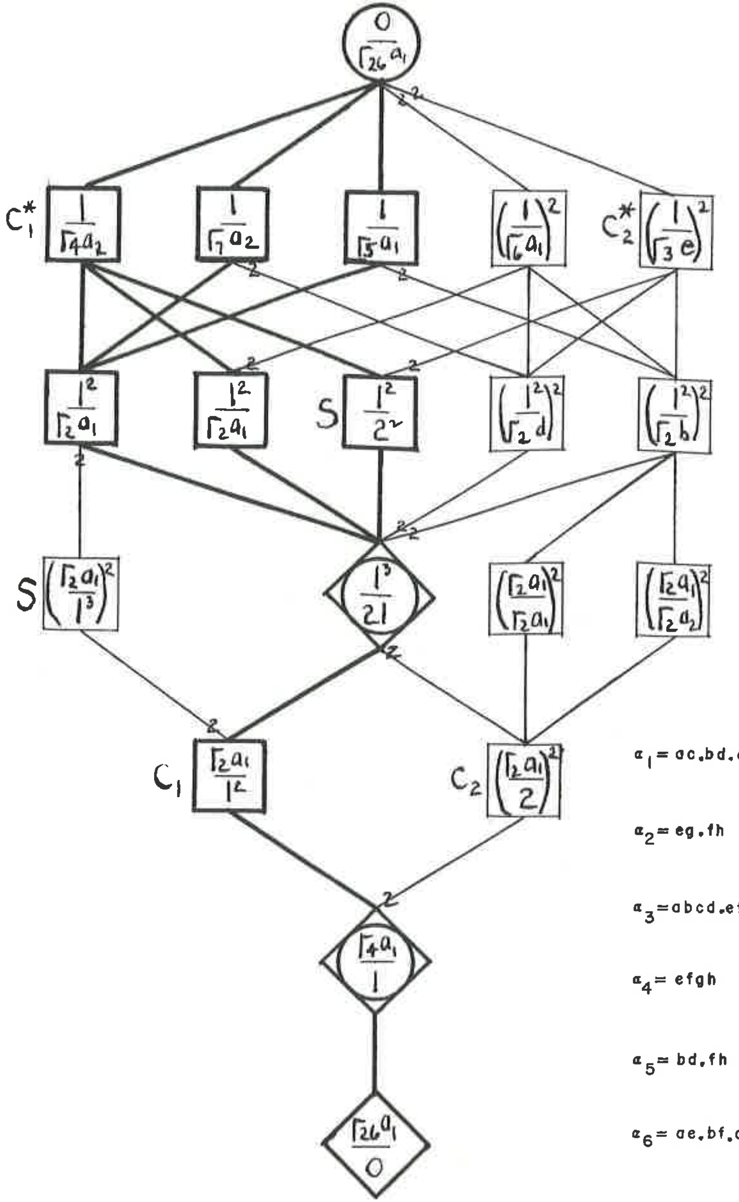
α_6 = eg.mo.bj.fp.dl.hn



$\alpha_1^2 = \alpha_2^2 = 1 \qquad \alpha_3^2 = \alpha_1$

$\alpha_4^2 \equiv \alpha_2. \qquad \alpha_5^2 \equiv \alpha_6^2 \equiv 1.$

$[\alpha_3, \alpha_5] = [\alpha_2, \alpha_6] = \alpha_1. \quad [\alpha_4, \alpha_5] = \alpha_2. \quad [\alpha_4, \alpha_6] = \alpha_2 \alpha_3$



$\alpha_1 = ac.bd.eg.fh$

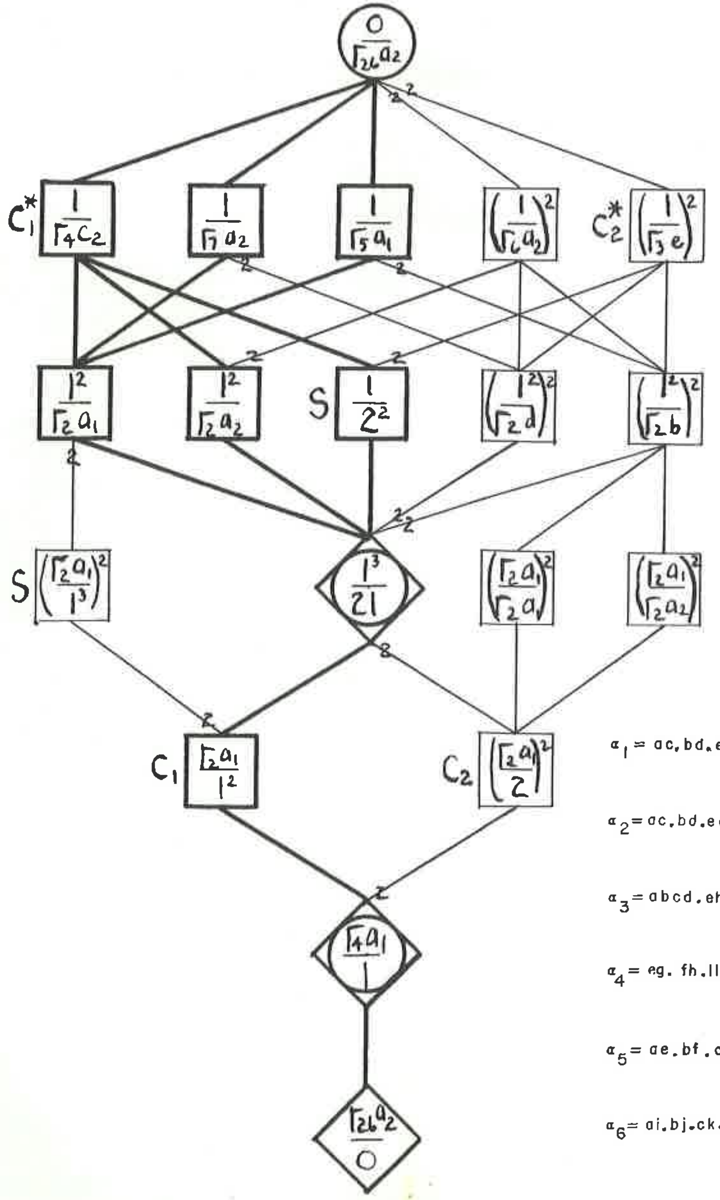
$\alpha_2 = eg.fh$

$\alpha_3 = abcd.efgh$

$\alpha_4 = efgh$

$\alpha_5 = bd.fh$

$\alpha_6 = ae.bf.cg.dh$



α_1 = ac.bd.eg.fh.ik.jl.mo.np

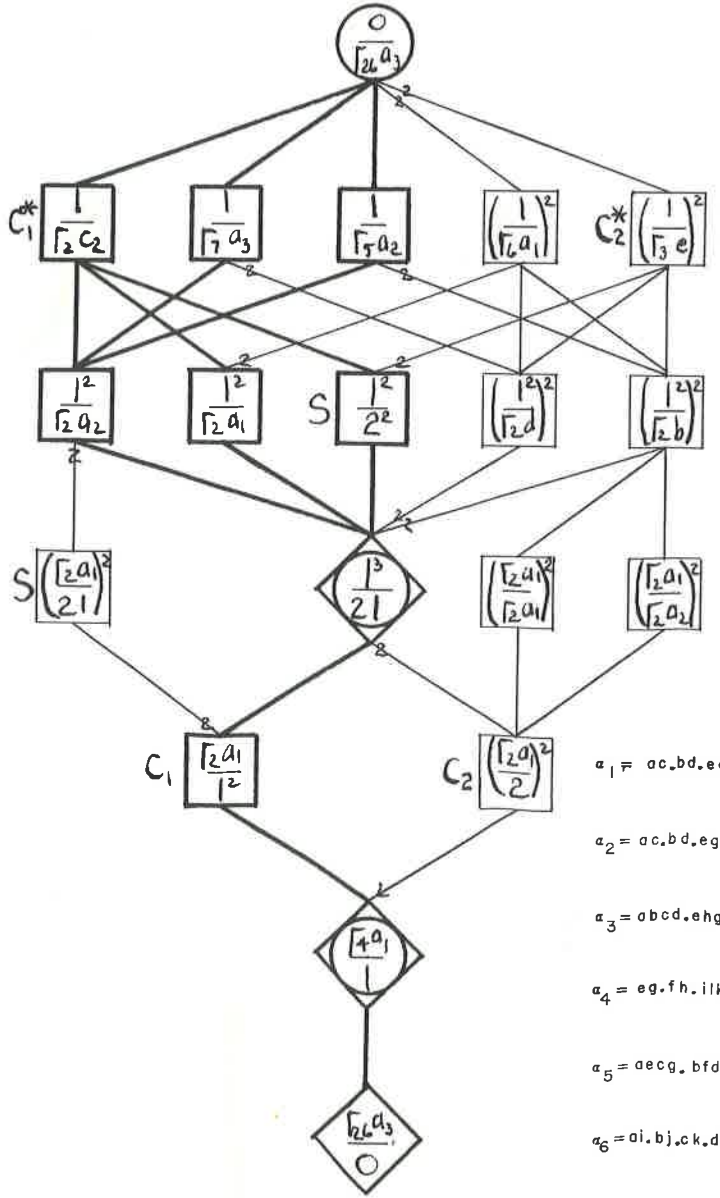
α_2 = ac.bd.eg.fh

α_3 = abcd.ehgf.ijkl.mpon

α_4 = eg. fh.llkj.mpon

α_5 = ae.bf.cg.dh.im.jn.ko.lp

α_6 = ai.bj.ck.dl.em.fn.go.hp



α_1 = ac.bd.eg.fh.ik.jl.mo.np

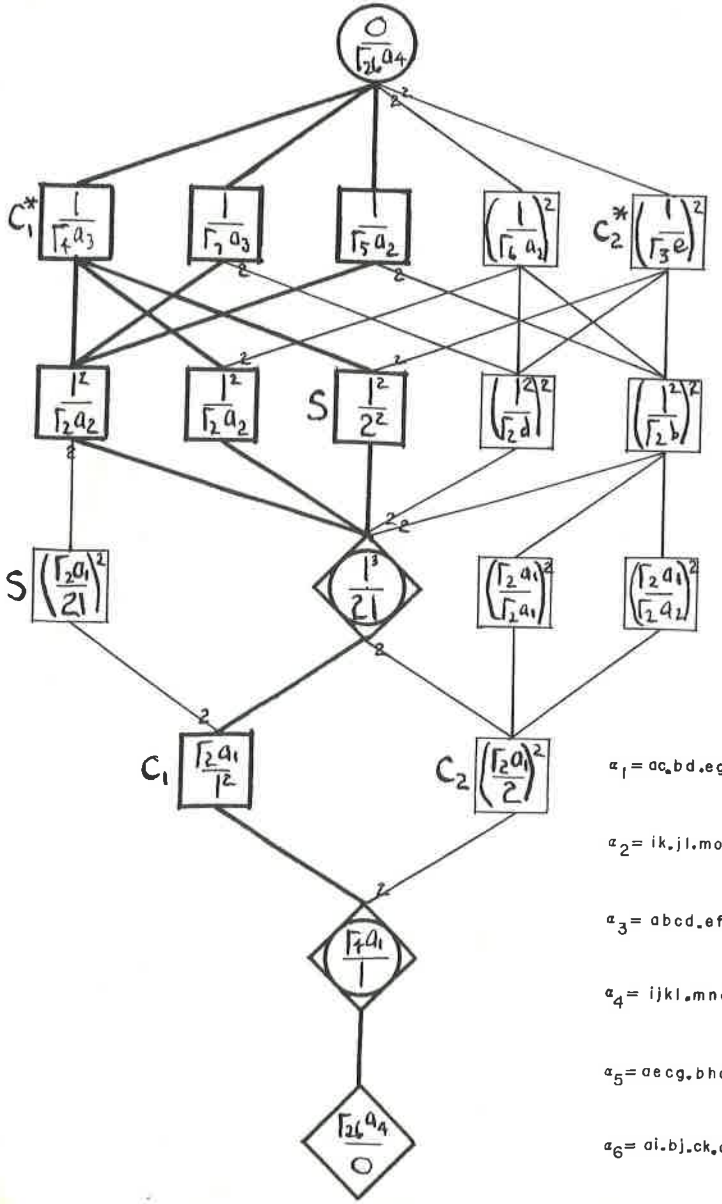
α_2 = ac.bd.eg.fh

α_3 = abcd.ehgf.ijkl.mpon

α_4 = eg.fh.ilkj.mpon

α_5 = aecg.bfdh.ipkn.jmlo

α_6 = ai.bj.ck.dl.ep.fm.gn.ho



α_1 = ac.bd.eg.fh.ik.jl.mo.np

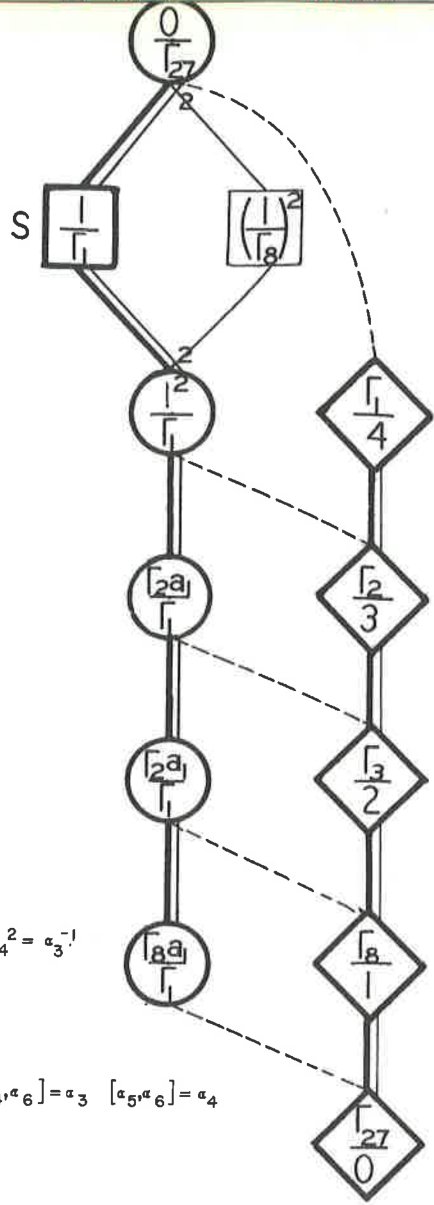
α_2 = ik.jl.mo.np

α_3 = abcd.efgh.ijkl.mnop

α_4 = ijkl.mnop

α_5 = aecg.bhdf.imko.jpln

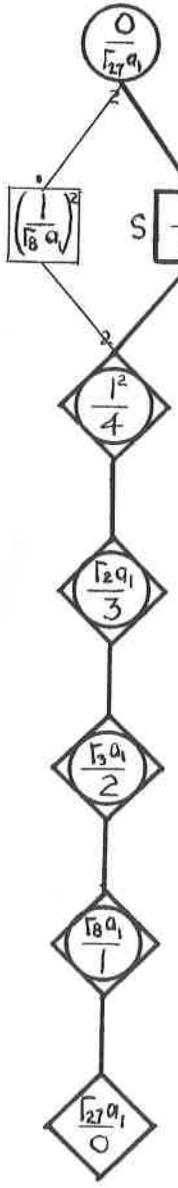
α_6 = ai.bj.ck.dl.em.fn.go.hp



$\alpha_1^2 = 1, \quad \alpha_2^2 = \alpha_1, \quad \alpha_3^2 = \alpha_2^{-1} \quad \alpha_4^2 = \alpha_3^{-1}$

$\alpha_5^2 \equiv \alpha_4^{-1} \qquad \alpha_6^2 \equiv 1$

$[\alpha_2, \alpha_6] = \alpha_1 \quad [\alpha_3, \alpha_6] = \alpha_2 \quad [\alpha_4, \alpha_6] = \alpha_3 \quad [\alpha_5, \alpha_6] = \alpha_4$



α1 = ac. bd. eg. fh. ik. jl. mo. np
AC. BD. EG. FH. IK. JL. MO. NP

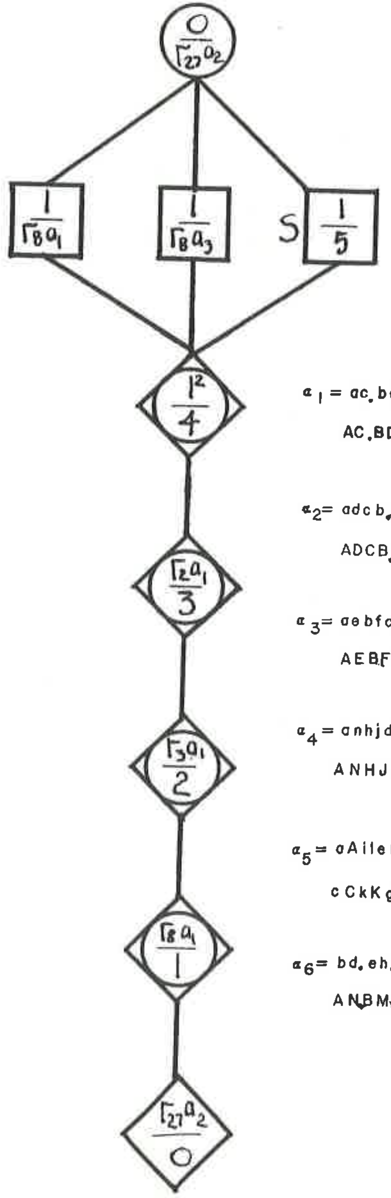
α2 = adcb. ehgf. ilkj. mpon
ADCB. EHGF. ILKJ. MPON

α3 = aebfcgdh. imjnkolp
AEBFCGDH. IMJNKOLP

α4 = aphldogkcnfjbmei
APHLDOGKCNFJBMEI

α5 = a AIIeE mMbBjJfFnN—
cCkKgGoOdDILhHpP

α6 = bd. eh. fg. ip. jo. kn. lm.
AP. BO. CN. DM. EL. FK. GJ. HI



α1 = ac. bd. eg. fh. ik. jl. mo. np
AC. BD. EG. FH. IK. JL. MO. NP

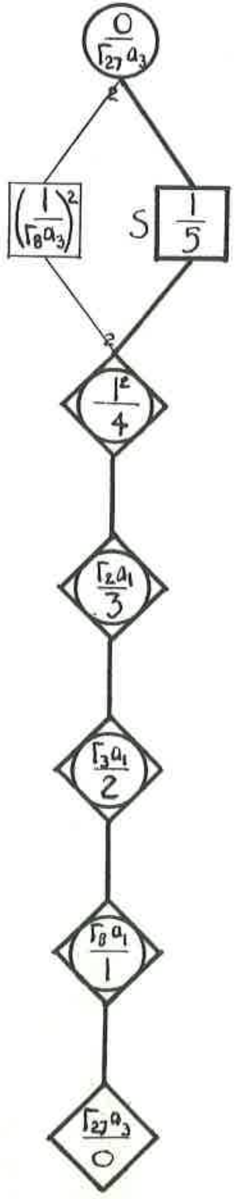
α2 = adcb. ehgf. ilkj. mpon
ADCB. EHGF. ILKJ. MPON

α3 = aebfcgdh. Imjnkolp
AEBFCGDH. IMJNKOLP

α4 = anhjdmgicpflboek.
ANHJDMGICPFLBOEK

α5 = aAileEmMbBjJfFnN —
cCkKgGoOdDIL hHpP

α6 = bd. eh. fg. ip. jo. kn. lm.
AN. BM. CP. DO. EJ. FI. GL. HK



α1 = ac. bd. eg. fh. ik. jl. mo. np
AC. BD. EG. FH. IK. JL. MO. NP
a'c'. b'd'. e'g'. f'h'. i'k'. j'l'. m'o'. n'p'
A'C'. B'D'. E'G'. F'H'. I'K'. J'L'. M'O'. N'P'

α2 = adcb. ehgf. ilkj. mpon
ADCB. EHGF. ILKJ. MPON
a'd'c'b'. e'h'g'f'. i'l'k'j'. m'p'o'n'
A'D'C'B'. E'H'G'F'. I'L'K'J'. M'P'O'N'

α3 = aebfcgdh. imjnkolp
AEBFCGDH. IMJNKOLP
a'e'b'f'c'g'd'h'. i'm'j'n'k'o'l'p'
A'E'B'F'C'G'D'H'. I'M'J'N'K'O'L'P'

α4 = aphldogkcnfjbmei.
APHLDOGKCNFJBMEI.
a'p'h'l'd'o'g'k'c'n'f'j'b'm'e'i'
A'P'H'L'D'O'G'K'C'N'F'J'B'M'E'I'

α5 = aAileEmMbBjJfFnN—
cCkKgGoOdDILhHpP.
a'A'i'I'e'E'm'M'b'B'j'J'f'F'n'N'—
c'C'k'K'g'G'o'O' d'D'l'L'h'H'p'P'.

α6 = aa'cc'. bd'db'. eh'gf'. fg'he'.
ip'kn'. jo'lm'. ml'oj'. nk'pi'.
AP'CN'. BO'DM'. EL'GJ'. FK'HI'.
IH'KF'. JG'LE'. MD'OB'. NC'PA'.