Chapter 1

Introduction to Clones

Boolean logic (or propositional logic as we prefer to call it) is named after George Boole, a professor at University College Cork, who first thought about an algebraic system of logic in the chapter "Of Hypotheticals" of his 1847 book "The Mathematical Analysis of Logic".

The ideas of George Boole (that can be traced back to Leibniz, actually) have reached their final form in the formalisation of mathematical logic at the beginning of the 20th century, which, among other things, lead to the clear distinction between the *syntax* and the *semantics* of logical systems. In case of Boolean logic, the syntactic part consists of propositional formulas, while the semantics is provided by Boolean functions, e.g.

$$p \vee (q \wedge \neg r) \qquad \text{vs.} \qquad \begin{array}{ccc|c} p & q & r & f(p,q,r) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

It is an easy obsetvation that every Boolean function $f(x_1,\ldots,x_n)$ in n unknowns which is not identically equal to 0 can be represented by a formula of the propositional calculus as follows:

$$f(x_1,\ldots,x_n) \quad = \bigvee_{\substack{\varepsilon_1,\ldots,\varepsilon_n: \\ f(\varepsilon_1,\ldots,\varepsilon_n)=1}} x_1^{\varepsilon_1} \wedge \ldots \wedge x_n^{\varepsilon_n}$$

where

$$x^{\varepsilon} \text{ is a replacement for } \begin{cases} x & \text{if } \varepsilon = 1, \\ \neg x & \text{if } \varepsilon = 0. \end{cases}$$

(If f is identically equal to 0 then $f(x_1,\dots,x_n) = x_1 \wedge \neg x_1$.) Therefore, $\{\wedge,\vee,\neg\}$ is a *complete* set of Boolean operations in the sense that every Boolean function can be "obtained from $\{\wedge,\vee,\neg\}$" using superpositions and usual manipulation of variables.

Are there any other complete sets of Boolean functions? Well, clearly $\{\neg,\wedge\}$, $\{\neg,\vee\}$ and $\{\neg,\Rightarrow\}$ are complete.

It is usually tedious but easy to show that a certain set is complete. But how can one show that some set of Boolean functions is *not* complete?

Example 1.1 (a) $\{\wedge,\vee\}$ is not complete: since $0\wedge 0 = 0\vee 0 = 0$, every function f that can be obtained from these two functions also fulfills $f(0,\dots,0) = 0$. Therefore, $\neg$ cannot be obtained from $\wedge$ and $\vee$.

(b) $\{\Rightarrow\}$ is not complete: since $1 \Rightarrow 1 = 1$, every function f that can be obtained from this function has the property that $f(1,\dots,1) = 1$. Therefore, $\neg$ cannot be obtained from $\Rightarrow$.

(c) $\{\neg,\Leftrightarrow\}$ is not complete. To see this, note that $\neg x = 1+x$ and $x \Leftrightarrow y = 1+x+y$, where $+$ is the addition in $GF(2)$. Now it is easy to show that if a function f can be obtained from $\neg$ and $\Leftrightarrow$ then $f(x_1,\dots,x_n) = b + a_1x_1 + \ldots + a_nx_n$ for some $b,a_1,\dots,a_n \in \{0,1\}$. On the other hand, $x\wedge y$ is not of this form, so $\{\neg,\Leftrightarrow\}$ is not complete.

Problem 1.2 (a) *Make precise the meaning of the phrase "a function can be 'obtained' from a set of functions".*

(b) *Given a set of functions F, decide whether F is complete.*

Note. Our major reference for general clone theory is [13], and we rely on [22] for the applications of clone theory in universal algebra. The section on abstract clones follows the idea presented by B. Csákány in his addendum to the Hungarian translation of "A course in universal algebra" by S. Burris and H. P. Sankappanavar [4]. Most unreferenced statements in this text can be found (in this or a similar form) in one of the three books.

1.1 Clones

Throughout this text, A is a finite set with at least two elements. Let $\mathscr{O}_A^{(n)} = A^{A^n}$ be a set of all n-ary operations on A, $n \geqslant 1$, and $\mathscr{O}_A = \bigcup_{n\geqslant 1} \mathscr{O}_A^{(n)}$ be the set of all finitary operations on A. The arity of an operation f is denoted by $\mathrm{ar}(f)$. For $F \subseteq \mathscr{O}_A$ let $F^{(n)} = F \cap \mathscr{O}_A^{(n)}$.

Let π_i^n denote the *i-th n-ary projection*:

$$\pi_i^n(x_1,\dots,x_i,\dots,x_n) = x_i$$

and let Π_A denote the set of all projections of all finite arities on A. Let $f \in \mathscr{O}_A^{(n)}$ and $g_1,\dots,g_n \in \mathscr{O}_A^{(m)}$. The *superposition of f and $g_1,\dots,g_n$* is an operation $h = f(g_1,\dots,g_n)$ of arity m defined by

$$h(x_1,\dots,x_m) = f(g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m)).$$

Definition 1.3 A set $C \subseteq \mathscr{O}_A$ is called a *clone of operations on A* if

- $\Pi_A \subseteq C$, and
- for each $f, g_1,\dots,g_n \in C$ such that $\mathrm{ar}(f) = n$ and $\mathrm{ar}(g_1) = \dots = \mathrm{ar}(g_n)$, we have $f(g_1,\dots,g_n) \in C$.

The legend says that the name *clone* came around 1936 from Marshall Hall (1910–1990) as a convenient abbreviation for the "closed one". The requirement that the clone contain projections makes it easy to formalise "usual manipulations with variables" as the following examples show.

Example 1.4 Let C be a clone and $f(x,y,z,u,v) \in C$. Then

$$\begin{aligned}
&f(y,y,y,x,x) \in C \quad \text{since} \quad f(y,y,y,x,x) = f(\pi_2^5,\pi_2^5,\pi_2^5,\pi_1^5,\pi_1^5)(x,y,z,u,v),\\
&f(z,x,u,y,v) \in C \quad \text{since} \quad f(z,x,u,y,v) = f(\pi_3^5,\pi_1^5,\pi_4^5,\pi_2^5,\pi_5^5)(x,y,z,u,v).
\end{aligned}$$

It is obvious that the intersection of an arbitrary family of clones is a clone. So for an $F \subseteq \mathscr{O}_A$ let $\mathrm{Clo}(F)$ denote the least clone that contains F. This clone is said to be *generated* by F. A clone C is said to be *finitely generated* if $C = \mathrm{Clo}(\{f_1,\dots,f_n\})$ for some $f_i \in \mathscr{O}_A$. Now it is easy to show:

Theorem 1.5 *Let $F \subseteq \mathscr{O}_A$ and let $\mathscr{F}$ be an algebraic type such that $\mathbf{A} = (A,F)$ is of type $\mathscr{F}$. Then $g \in \mathrm{Clo}(F)$ if and only if there is an $\mathscr{F}$-term t such that $g = t^{\mathbf{A}}$.*

Theorem 1.6 *All clones of operations on a finite set A form an algebraic lattice $\mathscr{L}_A$ under set inclusion. The least element of the lattice is Π_A, the greatest element is $\mathscr{O}_A$, and the lattice operations are given by:*

$$\bigwedge_\alpha C_\alpha = \bigcap_\alpha C_\alpha, \quad \textit{and} \quad \bigvee_\alpha C_\alpha = \mathrm{Clo}(\bigcup_\alpha C_\alpha).$$

The last theorem tells us that it is possible to introduce and algebraic structure on $\mathscr{O}_A$ in such a way that $C \subseteq \mathscr{O}_A$ is a clone if and only if it is a subuniverse of the algebra. Presenting such an algebra explicitely is easy. Consider the following four operations on $\mathscr{O}_A$:

- for $f \in \mathscr{O}_A^{(m)}$ and $g \in \mathscr{O}_A^{(n)}$ let

$$(f * g)(x_1, \dots, x_{m+n-1}) = f(g(x_1, \dots, x_n), x_{n+1}, \dots, x_{m+n-1});$$

- for $f \in \mathscr{O}_A^{(1)}$ let $\zeta f = \tau f = \triangle f = f$;
- for $f \in \mathscr{O}_A^{(n)}$, $n \geqslant 2$, let

$$\begin{aligned}(\zeta f)(x_1, x_2, \dots, x_n) &= f(x_2, \dots, x_n, x_1)\\ (\tau f)(x_1, x_2, x_3, \dots, x_n) &= f(x_2, x_1, x_3, \dots, x_n)\\ (\triangle f)(x_1, x_2, \dots, x_{n-1}) &= f(x_1, x_1, x_2, \dots, x_{n-1}).\end{aligned}$$

The algebra $(\mathscr{O}_A, *, \zeta, \tau, \triangle, \pi_1^2)$ is referred to as the *Mal'cev algebra*.

Theorem 1.7 *A set $C \subseteq \mathscr{O}_A$ is a clone if and only if it is a subuniverse of the Mal'cev algebra.*

Definition 1.8 A set $F \subseteq \mathscr{O}_A$ is *complete* if $\mathrm{Clo}(F) = \mathscr{O}_A$.

Now that we have firmly established the terminology, we can finally start looking for a completeness criterion. First, one can easily show that

Proposition 1.9 *A set $F \subseteq \mathscr{O}_A$ is complete if and only if $F \nsubseteq C$ for every clone $C \neq \mathscr{O}_A$.*

Needless to say that this criterion is pretty useless. We are going to turn it into a much more usefull criterion by focusing of some very *special* clones. The structure of the lattice of clones given in Theorem 1.6 suggests that the lattice of clones could be dual-atomic, which actually *is the case*. The key argument is the following statement due to Yablonskiĭ:

Theorem 1.10 (Yablonskiĭ 1958, [23]) *Let C be a clone on a finite set. Then C is finitely generated if and only if there exist maximal subclones of C, every proper subclone of C is contained in a maximal subclone of C and maximal subclones of C are finite in number.*

The proof of this theorem requires some preparation. Let $F \subseteq \mathscr{O}_A$ and $g \in \mathscr{O}_A^{(n)}$ for some integer $n \geqslant 1$. We say that *an operation g preserves operations from F* if $g(f_1, \dots, f_n) \in F$ whenever $f_1, \dots, f_n \in F$ are of the same arity.

Lemma 1.11 *Let $n \geqslant 1$ be an integer, let C be a clone and let $F \subset C^{(n)}$. There exists at most one maximal subclone D of C such that $D^{(n)} = F$.*

Proof. Let D be a maximal subclone of C such that $D^{(n)} = F$ and let us show that D has to be unique. Let $P_C(F)$ be the set of all operations in C that preserve operations from F and let us show that $D = P_C(F)$.

It is easy to see that $P_C(F)$ is a clone and that $D \subseteq P_C(F) \subseteq C$. Since D is a maximal subclone of C it follows that either $D = P_C(F)$ or $P_C(F) = C$. Suppose that $P_C(F) = C$ and let us show that this leads to a contradiction. Take any $f \in C^{(n)}$. Then $f \in C = P_C(F)$, so f preserves operations from F. According to the choice of D we have that $F = D^{(n)}$, so F contains $\pi_1^n, \dots, \pi_n^n$. Now, f preserves operations from F, so $f(\pi_1^n, \dots, \pi_n^n) \in F$, i.e. $f \in F$. This shows that $C^{(n)} \subseteq F$ – contradiction.

Therefore, $P_C(F) \neq C$ and the maximality of D now yields $P_C(F) = D$, which shows that D is uniquely determined by F. □

Let us now go back to the proof of Theorem 1.10.

Proof. (of Theorem 1.10) ($\Leftarrow$) Let $D_1, \dots, D_k$ be maximal proper subclones of C. Take any $f_1 \in C \setminus D_1, \dots, f_k \in C \setminus D_k$ and let $F = \{f_1, \dots, f_k\}$. From $F \subseteq C$ it follows that $\mathrm{Clo}(F) \subseteq C$. Now, if $\mathrm{Clo}(F) \subset C$, then it is a proper subclone of C and, by assumption, it is contained in a maximal subclone of C, say D_i. But, this is not possible, since $f_i \in \mathrm{Clo}(F) \setminus D_i$. Therefore, $\mathrm{Clo}(F) = C$.

($\Rightarrow$) Assume that C is finitely generated, say, $C = \mathrm{Clo}(\{f_1, \dots, f_k\})$. Since C is finitely generated, it follows immediately that the union of every chain of proper subclones of C is again a proper subclone of C, so by Zorn's Lemma, we get that maximal subclones of C exist and that every proper subclone of C is contained in a maximal subclone of C.

Let us now show that there are only finitely many maximal subclones of C. Let $n_i = \mathrm{ar}(f_i)$, $N = \max\{n_1, \dots, n_k\}$, let $\mathscr{M}$ be the set of all maximal subclones of C and consider the mapping

$$\varphi : \mathscr{M} \to \mathscr{P}(\mathscr{O}_A^{(N)}) : D \mapsto D^{(N)}.$$

Since $\mathrm{Clo}(C^{(N)}) = C$ it follows that $D^{(N)} \subset C^{(N)}$ for every $D \in \mathscr{M}$. Lemma 1.11 now imples that φ is injective, so $|\mathscr{M}| \leqslant |\mathscr{P}(\mathscr{O}_A^{(N)})|$, and $\mathscr{P}(\mathscr{O}_A^{(N)})$ is a finite set due to the fact that A is finite. $\square$

Theorem 1.12 *$\mathscr{O}_A$ is finitely generated.*

Proof. If $|A| = 2$ we have seen at the very beginning that $\mathscr{O}_A$ is finitely generated. For $|A| \geqslant 3$ we follow the straightforward idea of Werner and Wille. Fix two distinct elements from A and call them 0 and 1. For $a \in A$ let

$$c_a(x) = a, \quad \text{and} \quad \chi_a(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a. \end{cases}$$

Choose binary operations $+, \cdot \in \mathscr{O}_A^{(2)}$ so that $0 + x = x + 0 = 0$, $x \cdot 0 = 0$ and $x \cdot 1 = x$ for all $x \in A$. Then it is easy to see that

$$f(x_1, \dots, x_n) = \sum_{(a_1,\dots,a_n) \in A^n} c_{f(a_1,\dots,a_n)}(x_1) \cdot \chi_{a_1}(x_1) \cdot \chi_{a_2}(x_2) \cdot \ldots \cdot \chi_{a_n}(x_n).$$

Note that $+$ and $\cdot$ need not be associative, and that the representation is valid regardless of the actual order of taking sums and taking products.

Therefore, $\mathscr{O}_A$ can be generated by the following finite set of functions: $\{+, \cdot\} \cup \{c_a : a \in A\} \cup \{\chi_a : a \in A\}$. $\square$

Corollary 1.13 *$\mathscr{O}_A$ has maximal subclones, they are finite in number, and every proper subclone of $\mathscr{O}_A$ is contained in one of the maximal subclones.*

Definition 1.14 Maximal subclones of $\mathscr{O}_A$ are called *maximal clones on A*.

Finally, we come to a much better completeness criterion, which tells us that in order to prove that a set F is complete, it suffices to check only finitely many clones. Nevertheless, the result needs some more refinements.

Theorem 1.15 *A set $F \subseteq \mathscr{O}_A$ is complete if and only if $F \nsubseteq C$ for every maximal clone C on A.*

We shall conclude this section by two important completeness criteria. The Słupecki completeness criterion says that all unary operations together with an essential operation (to be defined shortly) constitute a complete set. The theorem of Webb, on the other hand, shows that there exist one-element complete sets.

We say that an operation $f \in \mathscr{O}_A^{(n)}$ *depends on its i-th argument* if there exist $a, b, c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in A$ such that $a \neq b$ and

$$f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) \neq f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n).$$

An operation f is *essential* if it is surjective and depends on at least two of its arguments. Let $\mathrm{im}(f) = f(A^n)$ denote the set of images of f.

Theorem 1.16 (Słupecki 1939, [19]) *Let $|A| \geqslant 3$ and let $F \subseteq \mathscr{O}_A$. If F contains an essential operation and if $\mathscr{O}_A^{(1)} \subseteq F$, then F is complete.*

Theorem 1.17 (Webb 1935, [25]) *Let $A = \{0, 1, \ldots, k-1\}$ and $x \uparrow y = \max(x, y) + 1$, where $+$ denotes addition mod k. Then $\{\uparrow\}$ is a complete set of operations.*

Proof. Let $s(x) = x \uparrow x = x + 1$. Then $\max(x, y) = s^{k-1}(x \uparrow y)$. Next, we can obtain constant maps as

$$c_a(x) = s^{a+1}(\max\{s^j(x) : j \in A\}),$$

and characteristic functions as

$$\chi_a(x) = s(\max\{s^j(x) : j \in A \text{ and } a + j \neq k-1\}) = \begin{cases} 0, & x \neq a \\ k-1, & x = a \end{cases}$$

Next,

$$\overline{x} = (k-1) - x = \max\{s^{k-j}(\max(\chi_j(x), j)) : j \in A\}$$

and

$$\min(x, y) = \overline{\max(\overline{x}, \overline{y})}.$$

The statement now follows by the same argument as in the proof of Theorem 1.12, where we let max play the role of $+$, min the role of $\cdot$ and $k-1$ the role of 1. $\square$

Definition 1.18 A *Sheffer operation* is any operation $f \in \mathscr{O}_A$ such that $\{f\}$ is a complete set.

Sheffer operations were named after H. M. Sheffer who in 1913 discovered a Sheffer operation $x \uparrow y = \neg(x \wedge y)$ on $\{0, 1\}$ (see [18]). We see now that $\mathscr{O}_A$ has a Sheffer operation whenever A is a finite set with at least two elements.

1.2 Galois connections

We have seen that in order to show that a certain set of operations is complete it suffices to show how to produce operations from some other complete set of operations. But how does one show that a certain set F of functions is *not* complete? The idea is to find a property P which is preserved under superpositions, to show that every operation $f \in F$ has the property P, and to find an operation g which does not have the property. Then $g \notin \mathrm{Clo}(F)$ (since, by the choice of P, every operation in F has the property P) and F is not complete.

Example 1.19 (a) Let $A = \{1, \dots, n\}$ and let $\min(x,y)$, $\max(x,y)$ denote the usual binary minimum and maximum operations on integers. To show that $\{\min, \max\}$ is not complete, if suffices to note that $\min(1,1) = \max(1,1) = 1$ and hence for every $f \in \mathrm{Clo}(\min, \max)$ we have $f(1, \dots, 1) = 1$. On the other hand, the constant map $c_n(x) = n$ does not have this property.

(b) Let $A = \{0, 1, \dots, p-1\}$ where p is an odd prime and let $+$ be the addition mod p. Let $\sim x = 1 + x$ and let us show that $\{\sim, +\}$ is not complete. For every $f \in \mathrm{Clo}(\sim, +)$ there are $a_1, \dots, a_n, b \in A$ such that $f(x_1, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n + b$ whence follows that $\min \notin \mathrm{Clo}(\sim, +)$.

The standard approach is to encode properties by finitary relations and then interpret "f has property P" by "f preserves (an appropriately chosen relation) ρ". Let $\mathscr{R}_A^{(h)} = \mathscr{P}(A^h)$ denote the set of all h-ary relations on A and let $\mathscr{R}_A = \bigcup_{h \geqslant 1} \mathscr{R}_A^{(h)}$ be the set of all finitary relations on A. If $\rho \in \mathscr{R}_A^{(h)}$ and $\rho \neq \varnothing$, we write $\mathrm{ar}(\rho) = h$. We take $\mathrm{ar}(\varnothing) = 0$.

Definition 1.20 An operation $f \in \mathscr{O}_A^{(n)}$ *preserves* a relation $\rho \in \mathscr{R}_A^{(h)}$ if

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{h1} \end{bmatrix} \in \rho, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{h2} \end{bmatrix} \in \rho, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{hn} \end{bmatrix} \in \rho \text{ implies } \begin{bmatrix} f(a_{11}, a_{12}, \dots, a_{1n}) \\ f(a_{21}, a_{22}, \dots, a_{2n}) \\ \vdots \\ f(a_{h1}, a_{h2}, \dots, a_{hn}) \end{bmatrix} \in \rho,$$

or, equivalently, if ρ is a subuniverse of $(A, f)^h$. We also say that ρ is an *invariant relation* of f.

Example 1.21 (a) Let $\leqslant$ be a partial order on A. Then "f preserves $\leqslant$" is equivalent to the fact that f is monotonous with respect to $\leqslant$.

(b) "f preserves $\{a\}$" is equivalent to $f(a, \dots, a) = a$.

(c) If ε is an equivalence relation on A then "f preserves ε" is equivalent to ε being a congruence of (A, f).

Let O and R be nonempty sets and let $\rho \subseteq O \times R$ be a binary relation. Define $\overrightarrow{\rho} : \mathscr{P}(O) \to \mathscr{P}(R)$ and $\overleftarrow{\rho} : \mathscr{P}(R) \to \mathscr{P}(O)$ by

$$\overrightarrow{\rho}(F) = \{r \in R : \forall f \in F\ (f \rho r)\} \quad \text{and} \quad \overleftarrow{\rho}(Q) = \{f \in O : \forall r \in Q\ (f \rho r)\},$$

where $F \subseteq O$ and $Q \subseteq R$. Then the pair $(\overrightarrow{\rho}, \overleftarrow{\rho})$ is called the *Galois connection* between $\mathscr{P}(O)$ and $\mathscr{P}(R)$ with respect to ρ. The proof of the following theorem can be found e.g. in [10]:

Theorem 1.22 *Let $(\overrightarrow{\rho}, \overleftarrow{\rho})$ be a Galois connection between $\mathscr{P}(O)$ and $\mathscr{P}(R)$ with respect to ρ, and let $F, F_1, F_2 \in \mathscr{P}(O)$ and $Q, Q_1, Q_2 \in \mathscr{P}(R)$.*

(1) If $F_1 \subseteq F_2$ then $\overrightarrow{\rho}(F_1) \supseteq \overrightarrow{\rho}(F_2)$.
If $Q_1 \subseteq Q_2$ then $\overleftarrow{\rho}(Q_1) \supseteq \overleftarrow{\rho}(Q_2)$.

(2) $F \subseteq \overleftarrow{\rho}(\overrightarrow{\rho}(F))$ and $Q \subseteq \overrightarrow{\rho}(\overleftarrow{\rho}(Q))$.

(3) $\overrightarrow{\rho}(F) = \overrightarrow{\rho}(\overleftarrow{\rho}(\overrightarrow{\rho}(F)))$ and $\overleftarrow{\rho}(Q) = \overleftarrow{\rho}(\overrightarrow{\rho}(\overleftarrow{\rho}(Q)))$.

(4) $L_O = \{\overleftarrow{\rho}(Q) : Q \subseteq R\}$ and $L_R = \{\overrightarrow{\rho}(F) : F \subseteq O\}$ are dually isomorphic complete lattices with respect to $\subseteq$. The dual isomorphisms are

$$\overrightarrow{\rho} : L_O \to L_R \quad \text{and} \quad \overleftarrow{\rho} : L_R \to L_O.$$

The relation "... preserves..." generates a Galois connection between operations and relations and the corresponding operators $\overrightarrow{\rho}$ and $\overleftarrow{\rho}$ are commonly denoted by $\operatorname{Pol} Q$ and $\operatorname{Inv} F$, $Q \subseteq \mathscr{R}_A$, $F \subseteq \mathscr{O}_A$:

$$\operatorname{Pol} Q = \{f \in \mathscr{O}_A : f \text{ preserves every } \rho \in Q\}$$
$$\operatorname{Inv} F = \{\rho \in \mathscr{R}_A : \text{ every } f \in F \text{ preserves } \rho\}.$$

It is easy to show that $\operatorname{Pol} Q$ is a clone of operations for every Q. But the converse is also true. Before we move on to the proof, let us introduce some more notation. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be tuples understood as column-vectors. Then $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a column-vector obtained by applying f to the rows of the matrix $[\mathbf{x}_1 \ \ldots \mathbf{x}_n]$, i.e.

$$\text{if} \quad \mathbf{x}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{h1} \end{bmatrix}, \ldots, \mathbf{x}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{hn} \end{bmatrix} \quad \text{then} \quad f(\mathbf{x}_1, \ldots \mathbf{x}_n) = \begin{bmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{h1}, a_{h2}, \ldots, a_{hn}) \end{bmatrix}.$$

$$
\begin{array}{ccccc}
a & a & a & \dots & a \\
a & a & a & \dots & b \\
\vdots & \vdots & \vdots & & \vdots \\
z & z & z & \dots & y \\
z & z & z & \dots & z
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccc}
\begin{bmatrix} a \\ a \\ \vdots \\ z \\ z \end{bmatrix} & \begin{bmatrix} a \\ a \\ \vdots \\ z \\ z \end{bmatrix} & \begin{bmatrix} a \\ a \\ \vdots \\ z \\ z \end{bmatrix} & \dots & \begin{bmatrix} a \\ b \\ \vdots \\ y \\ z \end{bmatrix} \\
\uparrow & \uparrow & \uparrow & \dots & \uparrow \\
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_n
\end{array}
$$

Figure 1.1: The special relation from the proof of Theorem 1.23

Theorem 1.23 (Bondarčuk, Kalužnin, Kotov, Romov 1969, [2]) *Let $C \subseteq \mathscr{O}_A$. The following statements are equivalent:*

(1) C is a clone.

(2) $C = \mathrm{Pol}\, Q$ for some $Q \subseteq \mathscr{O}_A$.

(3) $C = \mathrm{Pol}\,\mathrm{Inv}\, C$.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is easy. In order to show (1) $\Rightarrow$ (3) it suffices to show $\mathrm{Pol}\,\mathrm{Inv}\, C \subseteq C$ since the other inclusion is true for any Galois connection. Take any $g \in \mathrm{Pol}\,\mathrm{Inv}\, C$ and let $n = \mathrm{ar}(g)$. We shall now construct a special relation of arity $|A|^n$. List all n-tuples from A^n in lexicographic order, denote the column-vectors by $\mathbf{x}_1, \dots, \mathbf{x}_n$, Fig. 1.1, and let

$$\theta_n(C) = \{f(\mathbf{x}_1, \dots, \mathbf{x}_n) : f \in C\}.$$

Since C is a clone, it is easy to show that $\theta_n(C) \in \mathrm{Inv}\, C$, so g preserves $\theta_n(C)$. Now, $\mathbf{x}_1, \dots, \mathbf{x}_n \in \theta_n(C)$ since $\pi_1^n, \dots, \pi_n^n \in C$, so from the fact that g preserves $\theta_n(C)$ it follows that $g(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \theta_n(C)$. Therefore, there is an $f \in C$ such that $g(\mathbf{x}_1, \dots, \mathbf{x}_n) = f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and from the construction of $\mathbf{x}_1, \dots, \mathbf{x}_n$ it follows $g = f \in C$. $\square$

Corollary 1.24 *Let $F \subseteq \mathscr{O}_A$.*

(a) $\mathrm{Clo}(F) = \mathrm{Pol}\,\mathrm{Inv}\, F$.

(b) *If every $f \in F$ preserves a relation ρ then every $f \in \mathrm{Clo}(F)$ preserves ρ.*

Galois closed sets on the relational side are somewhat more complicated. It is possible to define closed sets of relations using “trivial” relations and a requirement that the set be closed with respect to certain “superpositions”, but this approach is less usual. Later on, we shall treat relational clones in a more standard way, as subuniverses of a certain algebra on $\mathscr{R}_A$.

We start with the trivial relations. Let $n \geqslant 1$ and let ε be an equivalence relation on $\{1,\ldots,n\}$. Then *the n-ary ε-diagonal* is the following relation:

$$\delta_n^\varepsilon = \{(x_1,\ldots,x_n) \in A^n : \forall i,j\ ((i,j) \in \varepsilon \Rightarrow x_i = x_j)\}.$$

Let $\delta_0 = \varnothing$ be the *zero-ary diagonal* and let Δ_A denote the set of all diagonals on all arities $\geqslant 0$.

Example 1.25 Let $12|3$ denote the equivalence relation on $\{1,2,3\}$ with two blocks, $\{1,2\}$ and $\{3\}$. Then $\delta_3^{12|3} = \{(x,x,y) : x,y \in A\}$.

Let $\tau, \rho, \sigma \in \mathscr{R}_A$ be relations such that $\mathrm{ar}(\tau) = \mathrm{ar}(\rho) + \mathrm{ar}(\sigma)$, let t be an integer and $f : \{1,\ldots,t\} \to \{1,\ldots,m+n\}$ be a mapping where $m = \mathrm{ar}(\rho)$ and $n = \mathrm{ar}(\sigma)$. We define the *f-superposition of τ with ρ and σ* denoted by $[\tau,\rho,\sigma]_f$ as follows: $[\tau,\rho,\varnothing]_f = [\tau,\varnothing,\sigma]_f = [\varnothing,\varnothing,\varnothing]_\varnothing = \varnothing$ and in case $m \geqslant 1$, $n \geqslant 1$ we let

$$\begin{aligned}[\tau,\rho,\sigma]_f = \{(x_{f(1)}, x_{f(2)},\ldots,x_{f(t)}) \in A^t : (x_1,x_2,\ldots,x_{m+n}) \in \tau,\\ (x_1,\ldots,x_m) \in \rho \text{ and } (x_{m+1},\ldots,x_{m+n}) \in \sigma\}.\end{aligned}$$

Definition 1.26 A *relational clone on A* is any set $Q \subseteq \mathscr{R}_A$ such that

- $\Delta_A \subseteq Q$, and
- If $\tau, \rho, \sigma \in \mathscr{R}_A$ such that $\mathrm{ar}(\tau) = \mathrm{ar}(\rho) + \mathrm{ar}(\sigma)$, then for every $f : \{1,\ldots,t\} \to \{1,\ldots,m+n\}$, where $m = \mathrm{ar}(\rho)$ and $n = \mathrm{ar}(\sigma)$, we have $[\tau,\rho,\sigma]_f \in Q$.

It is easy to see that the intersection of an arbitrary nonempty family of relational clones is again a relational clone. Therefore, for every $Q \subseteq \mathscr{R}_A$ there exists the least relational clone that contains Q. We say that this *relational clone is generated by Q* and denote it by $\mathrm{Clr}(Q)$.

In order to present more conventional descriptions of relational clones, we have to introduce several operations on relations:

- for $\rho \in \mathscr{R}_A^{(m)}$ and $\sigma \in \mathscr{R}_A^{(n)}$ we define the *relational product* $\times$ and the *relational composition* $\circ$ by

$$\begin{aligned}\rho \times \sigma &= \{(x_1,\ldots,x_m,y_1,\ldots,y_n) : (x_1,\ldots,x_m) \in \rho \text{ and } (y_1,\ldots,y_n) \in \sigma\}\\ \rho \circ \sigma &= \{(x_1,\ldots,x_{m-1},y_2,\ldots,y_n) : \exists z \in A\ ((x_1,\ldots,x_{m-1},z) \in \rho \text{ and}\\ &\qquad (z,y_2,\ldots,y_n) \in \sigma)\};\end{aligned}$$

- for $\rho \in \mathscr{R}_A^{(1)}$ or $\rho = \varnothing$ let $\zeta\rho = \tau\rho = \triangle\rho = \rho$; for $\rho \in \mathscr{R}_A^{(n)}$, $n \geqslant 2$, let

$$\begin{aligned}\zeta\rho &= \{(x_1,x_2,\dots,x_n) : (x_2,\dots,x_n,x_1) \in \rho\},\\ \tau\rho &= \{(x_1,x_2,x_3,\dots,x_n) : (x_2,x_1,x_3,\dots,x_n) \in \rho\},\\ \triangle\rho &= \{(x_1,x_2,\dots,x_{n-1}) : (x_1,x_1,x_2,\dots,x_{n-1}) \in \rho\},\end{aligned}$$

  be the *cyclic permutation of variables*, *transposition of the first two variables* and the *identification of the first two variables*; and

- for $\rho \in \mathscr{R}_A^{(n)}$ with $n \geqslant 2$ and $i, j \in \{1,\dots,n\}$ such that $i < j$ let

$$\triangle_{i,j}(\rho) = \{(x_1,\dots,x_i,\dots,x_{j-1},x_{j+1},\dots,x_n) : (x_1,\dots,x_n) \in \rho \text{ and } x_i = x_j\}$$

  denote the variation of $\triangle$ that operates on arbitrary two coordinates;

- for $\rho \in \mathscr{R}_A^{(n)}$ and any mapping $f : \{1,\dots,m\} \to \{1,\dots,n\}$, let

$$\mathrm{pr}_f(\rho) = \{(x_{f(1)},x_{f(2)},\dots,x_{f(m)}) : (x_1,\dots,x_n) \in \rho\}.$$

  denote the *f-projection* of ρ. We shal often write simply $\mathrm{pr}_{i_1\dots i_k}$ e.g.

$$\mathrm{pr}_{522}(\rho) = \{(x_5,x_2,x_2) : (x_1,x_2,x_3,x_4,x_5) \in \rho\} \text{ where } f = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 2 & 2 \end{pmatrix}.$$

Theorem 1.27 *Let $Q \subseteq \mathscr{R}_A$. The following are equivalent:*

(1) Q is a relational clone;

(2) $\Delta_A \subseteq Q$ and Q is closed with respect to intersection of relations of the same arity, relational products and f-projections; and

(3) Q is a subuniverse of the algebra $(\mathscr{R}_A, \circ, \zeta, \tau, \triangle, \delta_3^{12|3})$.

Proof. (1) $\Leftrightarrow$ (2): Note that $[\tau,\rho,\sigma]_f = \mathrm{pr}_f(\tau \cap (\rho \times \sigma))$. On the other hand, $\mathrm{pr}_f(\rho) = [A^{n+1},\rho,A]_{f'}$, where $n = \mathrm{ar}(\rho)$ and $f' : \{1,\dots,t\} \to \{1,\dots,n+1\}$ is given by $f'(i) = f(i)$; $\rho \times \sigma = [A^{n+m},\rho,\sigma]_{\mathrm{id}}$, where $n = \mathrm{ar}(\rho)$ and $m = \mathrm{ar}(\sigma)$; and $\rho \cap \sigma = [\rho \times A,\sigma,A]_{f'}$, where $n = \mathrm{ar}(\rho)$ and $f' : \{1,\dots,n\} \to \{1,\dots,n+1\}$ is given by $f'(i) = i$.

(2) $\Rightarrow$ (3): Unary operations are eazy: $\zeta(\rho) = \mathrm{pr}_{n12\dots n-1}(\rho)$, $\tau(\rho) = \mathrm{pr}_{213\dots n}(\rho)$ and $\triangle(\rho) = \mathrm{pr}_{23\dots n}(\rho \cap \delta_n^{12|3|\dots|n})$. As for $\circ$ let $n = \mathrm{ar}(\rho)$, $m = \mathrm{ar}(\sigma)$ and let ε be the equivalence relation on $\{1,\dots,m+n\}$ whose only nontrivial block is $\{n,n+1\}$. Then $\rho \circ \sigma = \mathrm{pr}_{1\dots n-1,n+2\dots n+m}(\delta_{n+m}^{\varepsilon} \cap (\rho \times \sigma))$.

(3) $\Rightarrow$ (2): Note first that compositions of ζ and τ can achieve any permutation of variables. Now $\delta_3^{12|3} \circ \zeta\zeta(\delta_3^{12|3}) = \delta_4^{12|34}$, $\triangle\zeta^3(\delta_4^{12|34}) = \delta_3^{123}$ and $\triangle\triangle(\delta_3^{12|3}) =$

$\delta_1^1 = A$. Then $\rho \times \sigma = \rho \circ \delta_4^{12|34} \circ \sigma$, $A \circ \rho$ omits the first variable of ρ, while $\delta_3^{123} \circ \rho$ doubles the first variable of ρ. It is easy but rather technical to show that using the last three operations we can get all the diagonals and all the f-projections. Next we note that each $\triangle_{i,j}$ can be obtained from ζ, τ and $\triangle$. Finally, if ρ and σ are relations of arity n then $\rho \cap \sigma = \triangle_{n,n+1} \triangle_{n-1,n+1} \dots \triangle_{2,n+1} \triangle_{1,n+1} (\rho \times \sigma)$ which completes the proof. □

The proof of the following theorem is due to Bondarčuk, Kalužnin, Kotov and Romov (1969) and independently Geiger (1968).

Theorem 1.28 (Geiger 1968, Bondarčuk, Kalužnin, Kotov, Romov 1969, [7, 2]) *Let $Q \subseteq \mathscr{R}_A$. The following statements are equivalent:*

(1) Q is a relational clone.

(2) $Q = \operatorname{Inv} F$ for some $F \subseteq \mathscr{R}_A$.

(3) $Q = \operatorname{Inv} \operatorname{Pol} Q$.

Proof. (3) ⇒ (2) ⇒ (1) is easy. Let us show (1) ⇒ (3). Clearly $Q \subseteq \operatorname{Inv} \operatorname{Pol} Q$ since this is true for every Galois connection. Let us show that $\operatorname{Inv} \operatorname{Pol} Q \subseteq Q$. Let $C = \operatorname{Pol} Q$ and let us first show that $\theta_n(C) \in Q$ for all $n \geqslant 1$, where $\theta_n(C)$ is the relation defined in the proof of Theorem 1.23. Recall also the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ from the definition of $\theta_n(C)$, let $q = |A|^n$ and let

$$\gamma_n = \bigcap \{ \eta \in Q : \operatorname{ar}(\eta) = q \text{ and } \eta \supseteq \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \}.$$

It is obvious that $\gamma_n \in Q$ since Q is closed with respect to finite intersections. We are going to show that $\theta_n(C) = \gamma_n$ whence $\theta_n(C) \in Q$ follows immediately.

First, $\theta_n(C) \subseteq \gamma_n$ since $\mathbf{x}_1, \dots, x_n \in \gamma_n$ and γ_n is preserved by every $f \in C$ ($C = \operatorname{Pol} Q$, so $\operatorname{Inv} C = \operatorname{Inv} \operatorname{Pol} Q \supseteq Q \ni \gamma_n$). Assume that $\theta_n(C) \subset \gamma_n$, take any $\mathbf{r} = (u_1, \dots, u_q) \in \gamma_n \setminus \theta_n(C)$ and consider $g_\mathbf{r}$ defined by $g_\mathbf{r}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \mathbf{r}$. From $\mathbf{r} \notin \theta_n(C)$ it follows that $g_\mathbf{r} \notin C = \operatorname{Pol} Q$, so there is a $\rho \in Q$ such that $g_\mathbf{r}$ does not preserve ρ. Therefore, there exist $\mathbf{y}_1, \dots, \mathbf{y}_n \in \rho$ such that $g_\mathbf{r}(\mathbf{y}_1, \dots, \mathbf{y}_n) \notin \rho$. Let $m = \operatorname{ar}(\rho)$ and consider the tuples $\mathbf{x}_i \times \mathbf{y}_i$ of length $q+m$ (obtained by concatenating $\mathbf{x}_i$ and $\mathbf{y}_i$) as columns of an $(q+m) \times n$-matrix, Fig. 1.2. Let X be the top $q \times n$-submatrix formed by $\mathbf{x}_1, \dots, \mathbf{x}_n$ and let Y be the bottom $m \times n$-submatrix formed by $\mathbf{y}_1, \dots, \mathbf{y}_n$. Since the rows of X are *all posssible* n-tuples of elements from A and rows of Y are *some* n-tuples of elements of Y, every row of Y appears as a row in X. Assume that the j-th row of Y appears as the h_j-th row of X. Let ε be the equivalence relation on $\{1, \dots, q+m\}$ generated by the pairs $(h_j, q+j)$, $j \in \{1, \dots, m\}$, and let

$$\rho' = \operatorname{pr}_{1\dots q}(\delta_{q+m}^{\varepsilon} \cap (\gamma_n \times \rho)).$$

$$\begin{array}{rccccll}
h_j \rightarrow & \begin{bmatrix} \\ \mathbf{x}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \mathbf{x}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \mathbf{x}_n \\ \\ \end{bmatrix} & \begin{array}{l} 1 \\ \\ \\ q \end{array} & \text{submatrix } X \\
j \rightarrow & \begin{bmatrix} \mathbf{y}_1 \end{bmatrix} & \begin{bmatrix} \mathbf{y}_2 \end{bmatrix} & \cdots & \begin{bmatrix} \mathbf{y}_n \end{bmatrix} & \begin{array}{l} q+1 \\ q+m \end{array} & \text{submatrix } Y
\end{array}$$

Figure 1.2: The proof of Theorem 1.28

Clearly, $\rho' = [\delta^{\varepsilon}_{q+m}, \gamma_n, \rho]_{1\ldots q} \in Q$ and $\rho' \subseteq \gamma_n$. On the other hand, from $\mathbf{x}_i \times \mathbf{y}_i \in \delta^{\varepsilon}_{q+m} \cap (\gamma_n \times \rho)$ it follows that $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \rho'$.

Let us show that $\mathbf{r} \notin \rho'$. Since $g_{\mathbf{r}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (u_1, \ldots, u_q)$ and since the j-th row of Y is equal to the h_j-th row of X it follows that $g_{\mathbf{r}}(\mathbf{y}_1, \ldots, \mathbf{y}_n) = (u_{h_1}, \ldots, u_{h_m})$ and by the choice of $\mathbf{y}_1, \ldots, \mathbf{y}_n$ we have $(u_{h_1}, \ldots, u_{h_m}) \notin \rho$. Now, if $\mathbf{r} \in \rho'$ then $(u_1, \ldots, u_q, u_{h_1}, \ldots, u_{h_m}) \in \gamma_n \times \rho$, whence follows $(u_{h_1}, \ldots, u_{h_m}) \in \rho$, which is impossible. So, $\mathbf{r} \notin \rho'$.

Therefore, $\mathbf{r} \in \gamma_n \setminus \rho'$, whence $\rho' \subset \gamma_n$. But this is impossible since γ_n is constructed as the least relation in Q that contains $\mathbf{x}_1, \ldots, \mathbf{x}_n$. This shows $\gamma_n = \theta_n(C)$ and thus $\theta_n(C) \in Q$.

Finally, let us show that $\operatorname{Inv}\operatorname{Pol} Q \subseteq Q$. Take any $\sigma \in \operatorname{Inv}\operatorname{Pol} Q$, let $n = |\sigma|$ and $t = \operatorname{ar}(\sigma)$. Write all tuples in σ as column vectors and denote this matrix by M. The rows of matrix X (Fig. 1.2) are all possible n-tuples of elements of A, so there are indices $i_1, \ldots, i_t \in \{1, \ldots, q\}$ such that the k-th row of M is equal to the i_k-th row of X, $k \in \{1, \ldots, t\}$, so $\sigma = \operatorname{pr}_{i_1\ldots i_t}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})$. Let us show that $\sigma = \operatorname{pr}_{i_1\ldots i_t}(\theta_n(C))$. Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \theta_n(C)$ it immeditaly follows that $\sigma = \operatorname{pr}_{i_1\ldots i_t}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) \subseteq \operatorname{pr}_{i_1\ldots i_t}(\theta_n(C))$. To show the other inclusion, take any $(v_1, \ldots, v_t) \in \operatorname{pr}_{i_1\ldots i_t}(\theta_n(C))$. Then there is a $\mathbf{z} = (z_1, \ldots, z_q) \in \theta_n(C)$ such that $z_{i_1} = v_1, \ldots, z_{i_t} = v_t$. But $\mathbf{z} = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ for some $f \in C$ so $z_{i_j} = f(x_{1i_j}, x_{2i_j}, \ldots, x_{ni_j})$, where $\mathbf{x}_j = (x_{j1}, \ldots, x_{jq})$. Since $\sigma = pr_{i_1\ldots i_t}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})$ it follows that $(x_{1i_1}, \ldots, x_{1i_t}), \ldots, (x_{ni_1}, \ldots, x_{ni_t}) \in \sigma$. Now $f \in C$ and $\sigma \in \operatorname{Inv} C$ imply that f preserves σ, so $(v_1, \ldots, v_t) = (z_{i_1}, \ldots, z_{i_t}) \in \sigma$.

Therefore, $\sigma = \operatorname{pr}_{i_1\ldots i_t}(\theta_n(C))$. Since $\theta_n(C) \in Q$ we have $\sigma = \operatorname{pr}_{i_1\ldots i_t}(\theta_n(C)) \in Q$. This completes the proof that $\operatorname{Inv}\operatorname{Pol} Q = Q$. $\square$

Corollary 1.29 $\operatorname{Clr}(Q) = \operatorname{Inv}\operatorname{Pol} Q$.

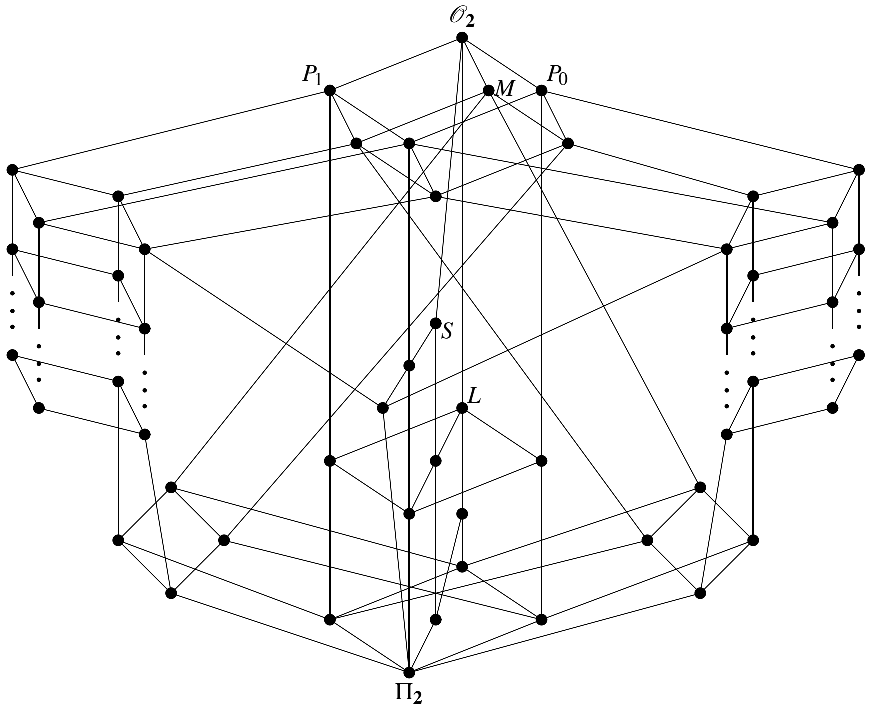

Figure 1.3: The lattice of clones on $\{0,1\}$ (E. Post)

1.3 On the number of clones

Emil Post classified all possible clones on $\{0,1\}$, and hence in a natural sense all possible 2-valued propositional logics. His work was first presented in 1920 as an addendum to his Ph. D. Thesis, and it was finally published in the book *Two-valued Iterative Systems of Mathematical Logic*, Princeton, 1941 (see [14]). The classification of E. Post was presented in a more modern notation by R. Lyndon in [9]. The lattice of clones on a two-element set is given in Fig. 1.3. Among other things, it explicitely shows that there are countably many clones of operations on A if $|A| = 2$.

Yu. I. Yanov and A. A. Muchnik showed in 1959 that in case of a finite set with $|A| \geqslant 3$ the number of clones is continuum.

Theorem 1.30 (Yanov, Muchnik 1959, [24]) *Let A be a finite set with at least three elements. Then the number of clones on A is continuum.*

Proof. Since clones are special subsets of the countably infinite set $\mathscr{O}_A$ it follows immediately that $|\mathscr{L}_A| \leqslant c$. To show that we have an equality, it suffices to construct a family of clones with continuum elements.

Let $0, 1, 2 \in A$ be three distinct elements of A and for each $n \geqslant 3$ define $f_n \in \mathscr{O}_A^{(n)}$ and $\rho_n, \sigma_n \in \mathscr{R}_A^{(n)}$ by

$$\sigma_n = \{(1,2,2,\dots,2,2),(2,1,2,\dots,2,2),\dots,(2,2,2,\dots,2,1)\},$$

$$f_n(x_1,\dots,x_n) = \begin{cases} 1, & (x_1,\dots,x_n) \in \sigma_n \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho_n = \sigma_n \cup \{(x_1,\dots,x_n) \in \{0,1,2\}^n : \exists i\ (x_i = 0)\}.$$

Let us first show that $f_n \notin \operatorname{Pol}\{\rho_n\}$ and that $f_k \in \operatorname{Pol}\{\rho_n\}$ whenever $k \neq n$. The first part of the claim is easy:

$$\begin{array}{cccccccc}
1 & 2 & 2 & \dots & 2 & 2 & \xrightarrow{f_n} & 1 \\
2 & 1 & 2 & \dots & 2 & 2 & \xrightarrow{f_n} & 1 \\
2 & 2 & 1 & \dots & 2 & 2 & \xrightarrow{f_n} & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & \dots & 1 & 2 & \xrightarrow{f_n} & 1 \\
2 & 2 & 2 & \dots & 2 & 1 & \xrightarrow{f_n} & 1 \\
\rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & \dots & \rotatebox{-90}{$\in$} & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\notin$} \\
\rho & \rho & \rho & & \rho & \rho & & \rho
\end{array}$$

As for the second part of the claim, assume that $k < n$ and take any $\mathbf{x}_1, \dots, \mathbf{x}_k \in \rho_n$. It at least one of the $\mathbf{x}_i$'s has a zero coordinate, then $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$ also has a zero coordinate and hence $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \rho_n$. If, however, $\mathbf{x}_1, \dots, \mathbf{x}_k \in \sigma_n$ then at least one of the rows will consist of 2's and $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$ will again have a zero coordinate, ensuring $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \rho_n$. E.g., if $k = 3$ and $n = 4$ this situation may be illustrated by

$$\begin{array}{ccccc}
1 & 2 & 2 & \xrightarrow{f_3} & 1 \\
2 & 1 & 2 & \xrightarrow{f_3} & 1 \\
2 & 2 & 1 & \xrightarrow{f_3} & 1 \\
2 & 2 & 2 & \xrightarrow{f_3} & 0 \\
\| & \| & \| & & \rotatebox{-90}{$\in$} \\
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & & \rho
\end{array}$$

Finally, let $k > n$ and take any $\mathbf{x}_1, \dots, \mathbf{x}_k \in \rho_n$. It at least one of the $\mathbf{x}_i$'s has a zero coordinate, then as in case $k < n$ we conclude that $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k)$ also has a zero coordinate and hence $f_k(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \rho_n$. If, however, $\mathbf{x}_1, \dots, \mathbf{x}_k \in \sigma_n$ then at least one of the rows will contain at least two 1's and $f_k(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ will again have a zero coordinate, ensuring $f_k(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in \rho_n$. E.g., if $k = 5$ and $n = 4$ this situation may be illustrated by

$$\begin{array}{ccccccc}
1 & 2 & 2 & 2 & 2 & \xrightarrow{f_5} & 1 \\
2 & 1 & 2 & 2 & 2 & \xrightarrow{f_5} & 1 \\
2 & 2 & 1 & 2 & 1 & \xrightarrow{f_5} & 0 \\
2 & 2 & 2 & 1 & 2 & \xrightarrow{f_5} & 1 \\
\shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & & \rotatebox{90}{$\in$} \\
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 & & \rho
\end{array}$$

We are now ready to complete the proof. Let $H = \{f_3, f_4, \ldots\}$ be the set of functions we have just constructed and define $\varphi : \mathscr{P}(H) \to \mathscr{L}_A$ by $\varphi(F) = \mathrm{Clo}(F)$. Let us show that φ is injective. Suppose $F_1 \neq F_2$ for some $F_1, F_2 \subseteq H$, but $\mathrm{Clo}(F_1) = \mathrm{Clo}(F_2)$. Since $F_1 \neq F_2$, there is an $n \geqslant 3$ such that $f_n \in F_1 \setminus F_2$ (or the other way around). So, $f_n \in F_1 \subseteq \mathrm{Clo}(F_1) = \mathrm{Clo}(F_2)$. Since $f_n \notin F_2$ we have that every $g \in F_2$ preserves ρ_n. Therefore, every $g \in \mathrm{Clo}(F_2)$ preserves ρ_n and consequently $f_n \in \mathrm{Clo}(F_2)$ preserves ρ_n – contradiction.

This shows that φ is injective and hence $|\mathscr{L}_A| \geqslant |\mathscr{P}(H)| = c$. □

1.4 Minimal clones

A clone C is called a *minimal clone* if $C \neq \Pi_A$ and $D \subseteq C$ implies $D = \Pi_A$ or $D = C$ for every clone D. Minimal clones are atoms in $\mathscr{L}_A$ and it is easy to see that every minimal clone is of the form $\mathrm{Clo}(f)$ for some $f \in \mathscr{O}_A \setminus \Pi_A$.

We are going to show that on a finte set every clone $\neq \Pi_A$ contains a minimal clone and that there are finitely many minimal clones. We also show that all minimal clones split into five types. We start by introducing some terminology. A *polymer* of an operation f is every operation that can be obtained from f by identifying certain variables. Note that f is *not* a polymer of itself: in order to obtain a polymer one *has to* identify at least two variables. For example, $g(x,y) = f(x,y,x,y)$ is a polymer of f. A ternary operation f on A is called a *majority operation* if

$$f(a,a,b) = f(a,b,a) = f(b,a,a) = a,$$

and a *minority operation* if

$$f(a,a,b) = f(a,b,a) = f(b,a,a) = b,$$

for all $a, b \in A$. For $n \geqslant 3$ and $k \in \{1, \ldots, n\}$, an n-ary operation f is called a *k-th n-ary semiprojection* if it is not a projection, but

$$f(a_1, \ldots, a_n) = a_k$$

whenever $a_1,\dots,a_n \in A$ have the property that $|\{a_1,\dots,a_n\}| < n$.

Lemma 1.31 (Świerczkowski 1960, [20]) *Let f be an n-ary operation on A and $n \geqslant 4$. Then f is a semiprojection if and only if every polymer of f is a projection.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) First let us note that

$$f(x_1,y,y,x_4,\dots,x_n) \notin \{\pi_2^{n-1},\pi_3^{n-1}\} \quad \text{or} \quad f(y,x_2,x_3,y,x_5,\dots,x_n) \notin \{\pi_1^{n-1},\pi_4^{n-1}\}.$$

(Suppose to the contrary that, e.g., $f(x_1,y,y,x_4,\dots,x_n) = \pi_2^{n-1}$ and $f(y,x_2,x_3,y,x_5,\dots,x_n) = \pi_1^{n-1}$. Then $f(z,y,y,z,x_5,\dots,x_n)$ would at the same time have to be the first and the second projection, which is impossible.) So, without loss of generality we may assume that $f(x_1,y,y,x_4,\dots,x_n) = \pi_1^{n-1}$, i.e. $f(x_1,y,y,x_3,\dots,x_n) = x_1$. Now take $i,j \in \{2,\dots,n\}$ such that $i<j$ and consider

$$f(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_{j-1},y,x_{j+1},\dots,x_n). \tag{1.1}$$

We know that it is a projection and since $f(x_1,y,y,\dots,y) = x_1$, the polymer (1.1) has to be the first projection. Using this fact one now easily shows that

$$f(y,x_2,\dots,x_{i-1},y,x_{i+1},\dots,x_n)$$

is again the first projection, for all $i \in \{2,\dots,n\}$. Therefore, f is a first semiprojection. $\square$

Theorem 1.32 (Rosenberg 1986, [17]) *Every clone $C \neq \Pi_A$ contains an operation which belongs to one of the following five classes of operations:*

(1) a nonidentical unary operation;

(2) a binary idempotent operation which is not a projection;

(3) a majority operation;

(4) a minority operation;

(5) a semiprojection.

Proof. Take an operation $f \in C \setminus \Pi_A$ of the least possible arity and let $n = \mathrm{ar}(f)$. If $n=1$ we have case (1). Otherwise, if $n \geqslant 2$ the choice of f implies that all polymers of f have to be projections. So, if $n=2$ we have case (2), while in case of $n \geqslant 4$ Lemma 1.31 yields that f is a semiprojection (case (5)).

Finally, let $n = 3$. Since all polymers of f are projections, $f(x,x,y)$, $f(x,y,x)$ and $f(y,x,x)$ are can be either π_1^2 or π_2^2 so we have eight cases to consider. Five cases are straightforward:

$$\begin{array}{llll}
f(x,x,y) = x, & f(x,y,x) = x, & f(y,x,x) = x: & \text{case (3)} \\
f(x,x,y) = y, & f(x,y,x) = y, & f(y,x,x) = y: & \text{case (4)} \\
f(x,x,y) = x, & f(x,y,x) = x, & f(y,x,x) = y: & \text{a first semiprojection} \\
f(x,x,y) = x, & f(x,y,x) = y, & f(y,x,x) = x: & \text{a second semiprojection} \\
f(x,x,y) = y, & f(x,y,x) = x, & f(y,x,x) = x: & \text{a third semiprojection}
\end{array}$$

In the remaining three cases (possibly after a permutation of variables) one obtains an operation that satisfies

$$f(x,x,y) = f(y,x,y) = f(y,x,x) = y$$

but then $g(x,y,z) = f(x, f(x,y,z), z)$ is a majority operation (case (3)). $\square$

Corollary 1.33 *For every minimal clone C we have that $C = \mathrm{Clo}(f)$, where f is an operation that belongs to one of the five classes of operations listed in Theorem 1.32. Moreover, every clone $C \neq \Pi_A$ contains a minimal clone, and minimal clones are finite in number.*

Proof. Let C be a minimal clone and let $f \in C$ be an operation whose existence is guaranteed by Theorem 1.32. Then $\mathrm{Clo}(f) \subseteq C$ and since $f \notin \Pi_A$, the minimality of C yields $\mathrm{Clo}(f) = C$.

For an arbitrary clone $C \neq \Pi_A$ take $f \in C$ as in Theorem 1.32. Then $\mathrm{Clo}(f)$ is a minimal clone contained in C.

To show that there are finitely many minimal clones on A it suffices to show that each of the classes (1)–(5) in Theorem 1.32 is finite. Classes (1)–(4) are obviously finite. As for class (5), note that if f is a semiprojection then $\mathrm{ar}(f) \leqslant |A|$ (a semiprojection of arity $> |A|$ would have to be a projection, which by definition is not a semiprojection). $\square$

The results presented in Theorem 1.32 and Corollary 1.33 have been a part of folklore for quite some time, and can be traced back to [6]. There are minimal clones that belong to each of the five types, as the following example shows.

Example 1.34 (1) Take an idempotent unary operation f which is not identity. Then $\mathrm{Clo}(f)$ is a minimal clone.

(2) Let $\wedge$ be a semilattice operation on A. Then $\mathrm{Clo}(\wedge)$ is a minimal clone.

(3) $\mathrm{Clo}(d)$ is a minimal clone if d is a dual discriminator on A, i.e.,

$$d(x,y,z) = \begin{cases} x, & x = y \\ z, & \text{otherwise.} \end{cases}$$

(4) If $(A,+)$ is an abelian group of exponent 2 and $f(x,y,z) = x+y+z$ then $\mathrm{Clo}(f)$ is a minimal clone.

(5) Let $n = |A| \geqslant 3$. Then $\mathrm{Clo}(l_n)$ is a minimal clone, where l_n is an n-ary operation defined by:

$$l_n(x_1,\dots,x_n) = \begin{cases} x_n, & \{x_1,\dots,x_n\} = A \\ x_1, & \text{otherwise.} \end{cases}$$

Theorem 1.35 (Rosenberg 1986, [17]) *Let C be a minimal clone generated by a unary operation f. Then $f \neq \mathrm{id}_A$ and either $f^2 = f$ or there is a prime number p such that $f^p = \mathrm{id}_A$.*

Let C be a minimal clone generated by a minority operation f. Then there is an abelian group $(A,+)$ of exponent 2 such that $f(x,y,z) = x+y+z$.

1.5 Maximal clones

We already know that there are finitely many maximal clones on a finite set and that every proper subclone of $\mathscr{O}_A$ is contained in a maximal clone. In this section we address one of the deepest and most influential results of clone theory, I. Rosenberg's classification of maximal clones. We start with a special case.

Theorem 1.36 (Yablonskiĭ 1958, [23]) *Let $|A| = n$ and let $\iota_A = \{(x_1,\dots,x_n) \in A^n : \exists i \neq j\ (x_i = x_j)\}$. Then $\mathrm{Pol}\{\iota_A\}$ is the set of all nonessential operations. It is a maximal clone, and it is the only maximal clone that contains $\mathscr{O}_A^{(1)}$.*

Proof. This is an easy consequence of the Słupecki completeness criterion (Theorem 1.16). It clearly contains all unary maps, and Lemma 1.11 ensures that no other maximal clone contains $\mathscr{O}_A^{(1)}$. □

For the general case, let us first show that each maximal clone is completely determined by a single relation.

Proposition 1.37 (Kuznecov 1961, [8]) *(a) If $\mathrm{Pol}\,Q = \mathscr{O}_A$ then $Q \subseteq \Delta_A$.*

(b) A clone $C \neq \mathscr{O}_A$ is a maximal clone if and only if $C = \mathrm{Pol}\{\rho\}$ for every $\rho \in \mathrm{Inv}\,C \setminus \Delta_A$.

Proof. (a) Take any $\rho \in Q$ and let $n = ar(\rho)$. Without loss of generality we may assume that there are no systematically repeated coordinates in ρ, i.e.,

$$\neg\, \exists i, j\ (i \neq j \text{ and } \forall (x_1, \dots, x_n) \in \rho\ (x_i = x_j))$$

for otherwise we can safely remove systematically repeated coordinates to obtain ρ' with the property $\mathrm{Pol}\{\rho\} = \mathrm{Pol}\{\rho'\}$. Therefore, for every $i, j \in \{1, \dots, n\}$ such that $i < j$ there exists an $\mathbf{x}^{ij} = (x_1^{ij}, \dots, x_n^{ij}) \in \rho$ such that $x_i^{ij} \neq x_j^{ij}$. Take arbitrary $a_1, \dots, a_n \in A$ and any map $f \in \mathscr{O}_A$ of arity $\binom{n}{2}$ which satisfies the following:

$$\begin{gathered} f(x_1^{12}, x_1^{13}, \dots, x_1^{n,n-1}) = a_1 \\ f(x_2^{12}, x_2^{13}, \dots, x_2^{n,n-1}) = a_2 \\ \vdots \\ f(x_n^{12}, x_n^{13}, \dots, x_n^{n,n-1}) = a_n \end{gathered}$$

Note that this requirement makes sense because each pair of $\binom{n}{2}$-tuples in the list above differs at at least one place. Since $\mathrm{Pol}\{\rho\} = \mathscr{O}_A$, this f preserves ρ so from $\mathbf{x}^{ij} \in \rho$ for all i, j it follows that $(a_1, \dots, a_n) \in \rho$. But $(a_1, \dots, a_n)$ was arbitrary, so $\rho = A^n \in \Delta_n$.

(b) First note that $\mathrm{Inv}\, C$ contains a nondiagonal relation whenever $C \neq \mathscr{O}_A$, since otherwise one has $\mathrm{Inv}\, C \subseteq \Delta_A$ whence $C = \mathrm{Pol}\,\mathrm{Inv}\, C \supseteq \mathrm{Pol}\,\Delta_A = \mathscr{O}_A$, which contradicts the fact that C is a proper subclone of $\mathscr{O}_A$.

$(\Rightarrow)$ Let C be a maximal clone and take any $\rho \in \mathrm{Inv}\, C \setminus \Delta_A$. Then $\mathrm{Pol}\{\rho\} \supseteq \mathrm{Pol}\,\mathrm{Inv}\, C = C$. Since (a) implies that $\mathrm{Pol}\{\rho\} \neq \mathscr{O}_A$, maximality of C yields $\mathrm{Pol}\{\rho\} = C$.

$(\Leftarrow)$ Suppose $C \neq \mathscr{O}_A$ is not a maximal clone. Then there is a maximal clone M such that $C \subset M$. Take any $\rho \in \mathrm{Inv}\, M \setminus \Delta_A$. Then $\rho \in \mathrm{Inv}\, C$ since $\mathrm{Inv}\, C \supseteq \mathrm{Inv}\, M \ni \rho$, but $C \neq \mathrm{Pol}\{\rho\}$ since $C \subset M = \mathrm{Pol}\{\rho\}$. $\square$

In particular, each maximal clone M takes the form $\mathrm{Pol}\{\rho\}$ for a nondiagonal relation ρ. One of the most influential results in clone theory is the explicite characterization of the maximal clones, obtained in 1970 by I. G. Rosenberg as the culmination of the work of many mathematicians. It is usually stated in terms of the following six classes of finitary relations on A (the so-called *Rosenberg relations*). For an $f \in \mathscr{O}_A^{(n)}$, let $f^\bullet$ denote the $(n+1)$-ary relation on A called the *graph of* f:

$$f^\bullet = \{(x_1, \dots, x_n, f(x_1, \dots, x_n)) : x_1, \dots, x_n \in A\}.$$

(R1) *Bounded partial orders.* These are partial orders on A with a least and a greatest element.

(R2) *Nontrivial equivalence relations.* These are equivalence relations on A distinct from $\delta_2^{12} = \{(x,x) : x \in A\}$ and A^2.

(R3) *Permutational relations.* These are relations of the form $\alpha^\bullet$ where α is a fixpoint-free permutation of A with all cycles of the same prime length p.

(R4) *Affine relations.* An *affine relation* is a relation of the form $f^\bullet$ where $f(x,y,z) = x-y+z$ for an elementary abelian p-group $(A,+)$ on A.

(R5) *Central relations.* All unary relations are central relations. For central relations of arity $h \geqslant 2$ the definition is as follows. The relation ρ is said to be *totally symmetric* if $(x_1,\dots,x_h) \in \rho$ implies $(x_{\pi(1)},\dots,x_{\pi(h)}) \in \rho$ for all permutations π, and it is said to be *totally reflexive* if $(x_1,\dots,x_h) \in \rho$ whenever there are $i \neq j$ such that $x_i = x_j$. A $c \in A$ is *central* if $(c,x_2,\dots,x_h) \in \rho$ for all $x_2,\dots,x_h \in A$. Finally, $\rho \neq A^h$ is called *central* if it is totally reflexive, totally symmetric and has a central element.

(R6) *Regular relations.* Let $\Theta = \{\theta_1,\dots,\theta_m\}$ be a family of equivalence relations. We say that Θ is an *h-regular family* if every θ_i has precisely h blocks, and additionally, if B_i is an arbitrary block of θ_i, $i \in \{1,\dots,m\}$, then $\bigcap_{i=1}^m B_i \neq \varnothing$.

An h-ary relation $\rho \neq A^h$ is said to be *h-regular* if $h \geqslant 3$ and there is an h-regular family Θ such that $(x_1,\dots,x_h) \in \rho$ if and only if for all $\theta \in \Theta$ there are distinct i, j with $(x_i,x_j) \in \theta$.

Theorem 1.38 (Rosenberg 1970, [16]) *A clone M is maximal if and only if there is a relation ρ from one of the classes (R1)–(R6) such that $M = \mathrm{Pol}\{\rho\}$.*

The proof of Rosenberg's theorem is very complicated. The shortest known proof comes from Quackenbush [15] and it is still rather complicated. The sketch of the proof that we are going to present follows the track of the Quackenbush's proof and relies on two nontrivial facts which we shall not prove. Recall that a finite algebra (A,F) is called *quasiprimal* if $\mathrm{Clo}(F)$ contains a discriminator.

Proposition 1.39 (a) *(Quackenbush [15]) If $F \nsubseteq \mathrm{Pol}\{\rho\}$ for every Rosenberg relation ρ, then $\mathrm{Clo}(F)$ contains a Mal'cev operation.*

(b) *(McKenzie) If (A,F) is a simple finite algebra which has no proper subalgebras and which has a Mal'cev term operation, then (A,F) is quasiprimal, or there is an elementary abelian p-group $(A,+)$ with the following property: for every $f \in F$ there are an $a \in A$ and endomorphisms ε_i of $(A,+)$ such that $f(x_1,\dots,x_n) = \varepsilon_1(x_1) + \dots + \varepsilon_n(x_n) + a$.*

(c) *(Pixley [11]) A finite algebra (A,F) is quasiprimal if and only if $\mathrm{Clo}(F) = \mathrm{Pol}\, Q$ where $Q = \{h^\bullet : h$ is an isomorphism between subalgebras of $(A,F)\}$.*

Proof. (*of Rosenberg's theorem*)

($\Leftarrow$) This direction is somewhat easier. We have to show that $\mathrm{Pol}\{\rho\}$ is a maximal clone for every Rosenberg relation ρ. We shall demonstrate main ideas in case of relations from (R1).

The strategy will be as follows. Let ρ be a Rosenberg relation and let $C = \mathrm{Pol}\{\rho\}$. According to Proposition 1.37 (a) we obtain $C \neq \mathscr{O}_A$, so by the statement (b) of the same proposition it suffices to show that for every $\sigma \in \mathrm{Inv}\,C \setminus \Delta_A$ we have $C = \mathrm{Pol}\{\sigma\}$. And this is equivalent to showing that for every $\sigma \in \mathrm{Inv}\,C \setminus \Delta_A$ we have $\rho \in \mathrm{Clr}(\sigma)$. (To see this, note that from $C = \mathrm{Pol}\{\sigma\}$ it follows that $\mathrm{Inv}\,C = \mathrm{Inv}\,\mathrm{Pol}\{\sigma\} = \mathrm{Clr}(\sigma)$ and $\rho \in \mathrm{Inv}\,C$ now yields $\rho \in \mathrm{Clr}(\sigma)$; conversely, if $\rho \in \mathrm{Clr}(\sigma)$ then $C = \mathrm{Pol}\{\rho\} \supseteq \mathrm{Pol}\{\sigma\}$, while from $\sigma \in \mathrm{Inv}\,C$ it follows that $\mathrm{Pol}\{\sigma\} \supseteq C$.)

Let $\leqslant$ be a bounded partial order on A with the least element 0 and the greatest element 1. Let $C = \mathrm{Pol}\{\leqslant\}$, take any $\sigma \in \mathrm{Inv}\,C \setminus \Delta_A$ and let us show that $\leqslant \in \mathrm{Clr}(\sigma)$. Without loss of generality we may assume that there are no systematically repeated coordinates in σ, i.e.

$$\neg\, \exists i, j\ (i \neq j \text{ and } \forall (x_1, \dots, x_n) \in \sigma\ (x_i = x_j)).$$

Suppose first that there exist $s \neq t$ such that $\mathrm{pr}_{st}(\sigma) \subseteq \leqslant$. Let us show that $\mathrm{pr}_{st}(\sigma) = \leqslant$. Let $a \leqslant b$, take any $(p_1, \dots, p_n) \in \sigma$ such that $p_s < p_t$ and define f by

$$f(x) = \begin{cases} a, & x \leqslant p_s \\ b, & \text{otherwise.} \end{cases}$$

Clearly $f \in \mathrm{Pol}\{\leqslant\}$, so f preserves σ as well. Therefore $(f(p_1), \dots, f(p_n)) \in \sigma$ and hence $(a,b) \in \mathrm{pr}_{st}(\sigma)$. This shows $\leqslant = \mathrm{pr}_{st}(\sigma) \in \mathrm{Clr}(\sigma)$.

Suppose now that for every $s \neq t$ we have $\mathrm{pr}_{st}(\sigma) \nsubseteq \leqslant$ and let us show that this leads to $\sigma = A^n$. Let $|\sigma| = m$ and denote the elements of σ by $(p_{i1}, \dots, p_{in})$, $i \in \{1, \dots, m\}$. For every $j \in \{1, \dots, n\}$ let $\mathbf{x}_j = (p_{1j}, \dots, p_{mj})$, Fig. 1.4. Take arbitrary $(q_1, \dots, q_n) \in A^n$ and define f by

$$f(y_1, \dots, y_m) = \begin{cases} q_i, & (y_1, \dots, y_m) = \mathbf{x}_i \\ 0, & \text{there exists a } j \text{ such that } (y_1, \dots, y_m) \leqslant \mathbf{x}_j \\ 1, & \text{otherwise.} \end{cases}$$

The assumption $\mathrm{pr}_{st}(\sigma) \nsubseteq \leqslant$ for all $s \neq t$ means that all $\mathbf{x}_j$'s are incomparable, so $f \in \mathrm{Pol}\{\leqslant\}$. Therefore, f preserves σ as well, whence $(q_1, \dots, q_n) \in \sigma$. But $(q_1, \dots, q_n)$ was arbitratily chosen, so $\sigma = A^n$.

($\Rightarrow$) We now know that $\mathrm{Pol}\{\rho\}$ is a maximal clone for every Rosenberg relatin ρ. Suppose there is a maximal clone C which is *not* of the form $\mathrm{Pol}\{\rho\}$ for a

$$
\begin{array}{ccccccc}
\mathbf{x}_1 & = & p_{11} & p_{21} & \dots & p_{m1} & \xrightarrow{f} & q_1 \\
\mathbf{x}_2 & = & p_{12} & p_{22} & \dots & p_{m2} & \xrightarrow{f} & q_2 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\mathbf{x}_n & = & p_{1n} & p_{2n} & \dots & p_{mn} & \xrightarrow{f} & q_n \\
 & & \in & \in & \dots & \in & & \\
 & & \rho & \rho & & \rho & &
\end{array}
$$

Figure 1.4: The proof that $\mathrm{Pol}\{\leqslant\}$ is a maximal clone

Rosenberg relation ρ. Then $C \not\subseteq \mathrm{Pol}\{\rho\}$ for every Rosenberg relation ρ, so by Proposition 1.39 (a), C contains a Mal'cev operation. In particular, since $C \not\subseteq \mathrm{Pol}\{\rho\}$ where ρ is an equivalence relation it follows that (A,C) is simple, while $C \not\subseteq \mathrm{Pol}\{\rho\}$ where ρ is a central relation implies (A,C) has no proper subalgebras. Then by Proposition 1.39 (b) it follows that (A,C) is quasiprimal, or that there is an elementary abelian p-group $(A,+)$ such that for every $f \in F$ there are an $a \in A$ and endomorphisms ε_i of $(A,+)$ and $f(x_1,\dots,x_n) = \varepsilon_1(x_1) + \dots + \varepsilon_n(x_n) + a$. The latter possibility would imply $C \subseteq \mathrm{Pol}\{(^\bullet x - y + z)\}$ which contradicts the fact that $C \not\subseteq \mathrm{Pol}\{\rho\}$ for every Rosenberg relation ρ. Therefore, (A,C) is quasiprimal.

We are going to show that (A,C) is primal, by showing that $\mathrm{Aut}(A,C) = \{\mathrm{id}\}$ (Proposition 1.39 (c)). Take any $\varphi \in \mathrm{Aut}(A,C)$. Let B be the set of all fixpoints of φ. Since B is a subalgebra of (A,C) and (A,C) has no proper subalgebras, we get $B = \varnothing$ or $B = A$. Therefore, $\varphi = \mathrm{id}$ or φ has no fixpoints. Assume $\varphi \neq \mathrm{id}$. The φ has no fixpoints. Let k be the length of the shortest cycle of φ and let

$$(a_{11} \dots a_{1k}), \dots, (a_{n1} \dots a_{nk})$$

be all the shortest cycles of φ. Then $\{a_{11},\dots,a_{1k},\dots,a_{n1},\dots,a_{nk}\}$ is the set of all the fixpoints of $\varphi^k \in \mathrm{Aut}(A,C)$ and hence $\{a_{11},\dots,a_{1k},\dots,a_{n1},\dots,a_{nk}\} = A$. From $k > 1$ it follows that there is a prime p such that $k = pm$ for some m. But then $\psi = \varphi^m$ is a fixpoint free automorphism of (A,C) of order p, and hence $C \subseteq \mathrm{Pol}\{\psi^\bullet\}$ – a contradiction.

Therefore, (A,C) is primal, i.e., $C = \mathscr{O}_A$, which contradicts the fact that C is a maximal clone. □

1.6 Describing clones by relations of bounded arity

Our next goal is to characterize clones uniquely determined by their invariant relations of arity at most k. Let $g \in \mathscr{O}_A^{(n)}$, $F \subseteq \mathscr{O}_A$ and let k be a positive integer.

Suppose that for every $S \subseteq A^n$ such that $|S| = k$ there is an $f \in F^{(n)}$ such that $g|_S = f|_S$. Then we say that g *can be k-approximated by F*. Let $\mathrm{Loc}_k(F)$ denote the set of all $g \in \mathscr{O}_A$ that can be k-approximated by F. We say that a clone C is *k-locally closed* if $C = \mathrm{Loc}_k(C)$.

Theorem 1.40 (Szábo 1978, Pöschel 1979 [21, 12]) *Let C be a clone and let k be a positive integer. Then the following are equivalent:*

(1) C is k-locally closed;

(2) $C = \mathrm{Pol}\ Q$ for some $Q \subseteq \mathscr{R}_A^{(k)}$;

(3) $C = \mathrm{Pol}\ \mathrm{Inv}^{(k)} C$.

Proof. (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Let $C = \mathrm{Pol}\ Q$ for some $Q \subseteq \mathscr{R}_A^{(k)}$. To show that C is k-locally closed it suffices to show that $\mathrm{Loc}_k(C) \subseteq C$, for the other inclusion is trivial. Take any $g \in \mathrm{Loc}_k(C)$ and let us show that $g \in \mathrm{Pol}\ Q = C$. Let $n = \mathrm{ar}(g)$. Take any $\rho \in Q$ and any $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \rho$. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{ik})$, $i \in \{1, \ldots, n\}$. Form new n-tuples $\mathbf{y}_j = (x_{1j}, \ldots, x_{nj})$, $j \in \{1, \ldots, k\}$, and let $S = \{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$.

$$
\begin{array}{ccccccc}
\mathbf{y}_1 & \rightarrow & x_{11} & x_{21} & \ldots & x_{n1} \\
\mathbf{y}_2 & \rightarrow & x_{12} & x_{22} & \ldots & x_{n2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\mathbf{y}_k & \rightarrow & x_{1k} & x_{2k} & \ldots & x_{nk} \\
 & & \uparrow & \uparrow & \ldots & \uparrow \\
 & & \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n
\end{array}
$$

Since $g \in \mathrm{Loc}_k(C)$ there is an $f \in C^{(n)}$ such that $g|_S = f|_S$, i.e., $g(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Now, $g \in C = \mathrm{Pol}\ Q$ and $\rho \in Q$ means that $g(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \rho$. Therefore, $f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \rho$. This shows that f preserves ρ, and since ρ was arbitrary, we get $f \in \mathrm{Pol}\ Q = C$.

(1) $\Rightarrow$ (3): Suppose $C \neq \mathrm{Pol}\ \mathrm{Inv}^{(k)} C$. Then $C = \mathrm{Pol}\ \mathrm{Inv} C \subset \mathrm{Pol}\ \mathrm{Inv}^{(k)} C$ because $\mathrm{Pol}\ \mathrm{Inv} C \subseteq \mathrm{Pol}\ \mathrm{Inv}^{(k)} C$ always holds. Take any $f \in (\mathrm{Pol}\ \mathrm{Inv}^{(k)} C) \setminus C$, $\mathrm{ar}(f) = n$, and let us show that $f \in \mathrm{Loc}_k(C)$. Take arbitrary $S = \{(x_{11}, \ldots, x_{n1}), \ldots, (x_{1k}, \ldots, x_{nk})\} \subseteq A^n$, $|S| = k$, let $\mathbf{x}_i = (x_{i1}, \ldots, x_{ik})$, $i \in \{1, \ldots, n\}$, and let $\gamma = \{g(\mathbf{x}_1, \ldots, \mathbf{x}_n) : g \in C\}$. Since C is a clone and $\mathrm{ar}(\gamma) = k$ we have $\gamma \in \mathrm{Inv}^{(k)} C$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in C$. Now, $f \in \mathrm{Pol}\ \mathrm{Inv}^{(k)} C$, so f preserves γ, whence follows that $f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \gamma$. Therefore, there is a $g \in C$ such that $f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = g(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, i.e., $f|_S = g|_S$. This shows that $f \in \mathrm{Loc}_k(C)$. But $f \notin C$ by the choice of f, so $C \neq \mathrm{Loc}_k(C)$. $\square$

We shall conclude this section by the famous Baker-Pixley theorem which states that a clone is uniquely deretmined by its k-ary invariant relations whenever it contains a special operation. An operation f of arity $\geqslant 3$ is called a *near-unanimity operation* if

$$f(y,x,x,\ldots,x,x) = f(x,y,x,\ldots,x,x) = \ldots = f(x,x,x,\ldots,x,y) = x$$

for all $x, y \in A$.

Theorem 1.41 (Baker, Pixley 1975, [1]) *Let $k \geqslant 2$ and suppose that a clone C contains an $(k+1)$-ary near-unanimity operation. Then $C = \operatorname{Pol}\operatorname{Inv}^{(k)} C$.*

Proof. Let $\nu \in C^{(k+1)}$ be a near-unanimity operation and let us show that $\operatorname{Pol}\operatorname{Inv}^{(k)} C \subseteq \operatorname{Pol}\operatorname{Inv} C$ since the other inclusion always holds.

Take any $f \in \operatorname{Pol}\operatorname{Inv}^{(k)} C$, any positive integer h and any $\rho \in \operatorname{Inv}^{(h)} C$. If $h = k$ we trivially have that f preserves ρ. If $h < k$ it suffices to note that $\rho \times A^{k-h} \in \operatorname{Inv}^{(k)} C$ and that f preserves ρ if and only if f preserves $\rho \times A^{k-h}$. Finally, assume $h > k$. For every sequence of indices $1 \leqslant i_1 < \ldots < i_s \leqslant h$ let $\rho_{i_1 \ldots i_s} = \operatorname{pr}_{i_1 \ldots i_s}(\rho)$. By the assumption, f preserves every $\rho_{i_1 \ldots i_k}$. Let us show that then f has to preserve ρ. We denostrate the main idea by considering a special case of $k = 2$. Let us start by showing that f preserves ρ_{ijl} for all $i < j < l$. Take any $(u_1, v_1, w_1), \ldots, (u_n, v_n, w_n) \in \rho_{ijl}$ and let $u = f(u_1, \ldots, u_n)$, $v = f(v_1, \ldots, v_n)$, $w = f(w_1, \ldots, w_n)$. Since f preserves ρ_{ij}, ρ_{il} and ρ_{jl} we have that $(u, v) \in \rho_{ij}$, $(u, w) \in \rho_{il}$ and $(v, w) \in \rho_{jl}$. But ρ_{ij}, ρ_{il} and ρ_{jl} are projections of ρ_{ijl}, so there exist x, y, $z \in A$ such that

$$(u,v,x) \in \rho_{ijl}, \quad (u,y,w) \in \rho_{ijl}, \quad (z,v,w) \in \rho_{ijl}.$$

Now, $\nu \in C$ preserves all relations in $\operatorname{Inv} C$ and in particular it preserves ρ_{ijl} so

$$(\nu(u,u,z), \nu(v,y,v), \nu(x,w,w)) = (u,v,w) \in \rho_{ijl}.$$

This shows that f preserves all ρ_{ijl}. Next, let us show that f preserves ρ_{ijlm} for all $i < j < l < m$. Take any $(u_1, v_1, w_1, p_1), \ldots, (u_n, v_n, w_n, p_n) \in \rho_{ijlm}$ and let $u = f(u_1, \ldots, u_n)$, $v = f(v_1, \ldots, v_n)$, $w = f(w_1, \ldots, w_n)$, $p = f(p_1, \ldots, p_n)$. Since f preserves ρ_{ijl}, ρ_{ijm} and ρ_{ilm} we have that $(u, v, w) \in \rho_{ijl}$, $(u, v, p) \in \rho_{ijl}$ and $(u, w, p) \in \rho_{ilm}$. But ρ_{ijl}, ρ_{ijm} and ρ_{ilm} are projections of ρ_{ijlm}, so there exist x, y, $z \in A$ such that

$$(u,v,w,x) \in \rho_{ijlm}, \quad (u,v,y,p) \in \rho_{ijlm}, \quad (u,z,w,p) \in \rho_{ijlm},$$

whence

$$(\nu(u,u,u), \nu(v,v,z), \nu(w,y,w), \nu(x,p,p)) = (u,v,w,p) \in \rho_{ijlm}.$$

Now, using induction on s we can show that f preserves $\rho_{i_1 \ldots i_s}$ for all $1 \leqslant i_1 < \ldots < i_s \leqslant h$, and hence f preserves $\rho = \rho_{12\ldots h}$. Therefore, $f \in \mathrm{Pol}\,\mathrm{Inv}\,C = C$ and this completes the proof. □

Corollary 1.42 *Let C be a clone that contains a near-unanimity operation. Then the order-filter $\uparrow C = \{D \subseteq \mathscr{O}_A : D$ is a clone and $D \supseteq C\}$ is finite.*

Proof. Let $\nu \in C$ be a near-unanimity operation and let $k = \mathrm{ar}(\nu)$. Then $\nu \in D$ for every $D \in \uparrow C$, so $D = \mathrm{Pol}\,\mathrm{Inv}^{(k)} D$ for every $D \in \uparrow C$ and hence the mapping $\varphi : \uparrow C \to \mathscr{P}(\mathscr{R}_A^{(k)})$ given by $\varphi(D) = \mathrm{Inv}^{(k)} D$ is injective. Since $\mathscr{P}(\mathscr{R}_A^{(k)})$ is finite, $\uparrow C$ is also finite. □

1.7 Primitive-positive clones

There is another important Galois connection this time between operations on A. We say that *operations $f \in \mathscr{O}_A^{(n)}$ and $g \in \mathscr{O}_A^{(m)}$ commute* if (see Fig. 1.5):

$$f(g(a_{11}, a_{12}, \ldots, a_{1m}), \ldots, g(a_{n1}, a_{n2}, \ldots, a_{nm})) = \\ = g(f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{1m}, a_{2m}, \ldots, a_{nm})).$$

$$\begin{array}{|cccc|cc|}
\hline
a_{11} & a_{12} & \ldots & a_{1m} & \xrightarrow{g} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2m} & \xrightarrow{g} & b_2 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nm} & \xrightarrow{g} & b_n \\
\hline
f\downarrow & f\downarrow & \ldots & f\downarrow & & f\downarrow \\
c_1 & c_2 & \ldots & c_m & \xrightarrow{g} & d \\
\hline
\end{array}$$

Figure 1.5: Commuting operations

It is easy to see that the following statements are equivalent:

f commutes with $g \Leftrightarrow f \in \mathrm{Pol}\{g^\bullet\} \Leftrightarrow g \in \mathrm{Pol}\{f^\bullet\} \Leftrightarrow g$ commutes with f.

The binary relation "… commutes with …" on $\mathscr{O}_A$ generates a Galois connection where the operators $\overrightarrow{\rho}$ and $\overleftarrow{\rho}$ are the same and usually denoted by $(-)^*$. The closure operator is $F \mapsto F^{**}$ and the Galois closed sets are of the form F^*. They are usually referred to as *bicentralizers*, *bicentrally closed sets* or *primitive-positive clones* (the last name is due to Stanley Burris).

Proposition 1.43 *(a) Let $F \subseteq \mathscr{O}_A$ and let $F^\bullet = \{f^\bullet : f \in F\}$. Then $F^* = \mathrm{Pol}\,(F^\bullet)$.*

(b) For every primitive-positive clone C of operations on A there is an algebra $\mathbf{A}$ on A such that $C = \bigcup_{n \geqslant 1} \mathrm{hom}(\mathbf{A}^n, \mathbf{A})$.

Proof. (a) is just a reformulation of the definitions. As for (b), let C be a primitive-positive clone. Then $C = F^*$ for some $F \subseteq \mathscr{O}_A$. It is easy now to see that $C = \bigcup_{n \geqslant 1} \mathrm{hom}(\mathbf{A}^n, \mathbf{A})$ where $\mathbf{A} = (A, F)$. □

As we have seen from Theorems 1.7 and 1.27, both clones of operations and relational clones are subuniverses of algebras of the same type. In both cases we have a binary "composition", two operations that permute variables, one operation that identifies variables and a constant. So, starting from a set of operations F one can first produce a clone $\mathrm{Clo}(F)$ and then interprete it as a set of relations $\mathrm{Clo}(F)^\bullet$, or one can immediately treat F as a set of relations $F^\bullet$ and then produce the relational clone $\mathrm{Clr}(F^\bullet)$, Fig. 1.6. A question arises: what is the relationship

$$\begin{tikzcd}
F \arrow[rr, "(-)^\bullet"] \arrow[d, "\mathrm{Clo}"'] & & F^\bullet \arrow[d, "\mathrm{Clr} \cap \mathscr{O}_A^\bullet"] \\
\mathrm{Clo}(F) \arrow[r, "(-)^\bullet"] & \mathrm{Clo}(F)^\bullet \overset{?}{=} & \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet
\end{tikzcd}$$

Figure 1.6: $\mathrm{Clo}(F)^\bullet$ versus $\mathrm{Clr}(F^\bullet)$

between the two sets of relations? Clearly, $\mathrm{Clr}(F^\bullet)$ contains relations that are not graphs of operations, but what if we compare $\mathrm{Clo}(F)^\bullet$ and $\mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet$? The following easy lemma deals with the general case.

Proposition 1.44 *For every $F \subseteq \mathscr{O}_A$ we have $\mathrm{Clo}(F)^\bullet \subseteq \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet$.*

Proof. Having in mind Theorems 1.7 and 1.27, it suffices to note that if $\mathrm{ar}(f) \geqslant 2$ then $(f * g)^\bullet = (g^\bullet) \circ (f^\bullet)$, $\zeta(f)^\bullet = \mathrm{pr}_{n,1,2,\ldots,n-1,n+1}(f^\bullet)$, $\tau(f)^\bullet = \tau(f^\bullet)$, $\triangle(f)^\bullet = \triangle(f^\bullet)$ and $(\pi_1^2)^\bullet = \delta_3^{13|2}$. □

We are now going to show that $\mathrm{Clo}(F)^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet$ if and only if $\mathrm{Clo}(F)$ is a primitive-positive clone.

Lemma 1.45 *Let $F \subseteq \mathscr{O}_A$.*

(a) $(\mathrm{Pol}\,(F^\bullet))^\bullet = \mathrm{Inv}\,F \cap \mathscr{O}_A^\bullet$.

(b) $(F^{**})^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet$.

(c) $\mathrm{Clr}(\mathrm{Clo}(F)^\bullet) = \mathrm{Clr}(F^\bullet)$.

(d) *If $C = \mathrm{Clo}(F)$ then $F^{**} = C^{**}$.*

Proof. (a) Take any $g^\bullet \in (\mathrm{Pol}\,(F^\bullet))^\bullet$. Then $g \in \mathrm{Pol}\,(F^\bullet)$ if and only if g commutes with every $f \in F$ if and only if every $f \in F$ preserves $g^\bullet$ i.e., $g^\bullet \in \mathrm{Inv}\,F$.

(b) Since $F^* = \mathrm{Pol}\,(F^\bullet)$ we have $F^{**} = \mathrm{Pol}\,(\mathrm{Pol}\,(F^\bullet)^\bullet)$, so

$$(F^{**})^\bullet = \big(\mathrm{Pol}\,(\mathrm{Pol}\,(F^\bullet)^\bullet)\big)^\bullet.$$

Let $C = \mathrm{Pol}\,(F^\bullet)$. Then according to (a),

$$(F^{**})^\bullet = \big(\mathrm{Pol}\,(C^\bullet)\big)^\bullet = \mathrm{Inv}\,C \cap \mathscr{O}_A^\bullet = \mathrm{Inv}\,\mathrm{Pol}\,(F^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet.$$

(c) Inclusion $\supseteq$ is obvious and follows from $\mathrm{Clo}(F) \supseteq F$. For the other inclusion, note that $\mathrm{Clo}(F)^\bullet \subseteq \mathrm{Clr}(F^\bullet)$ according to Proposition 1.44, so $\mathrm{Clr}(\mathrm{Clo}(F)^\bullet) \subseteq \mathrm{Clr}(\mathrm{Clr}(F^\bullet)) = \mathrm{Clr}(F^\bullet)$.

(d) $(C^{**})^\bullet = \mathrm{Clr}(C^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(\mathrm{Clo}(F)^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet = (F^{**})^\bullet$. Therefore, $C^{**} = F^{**}$. $\square$

Theorem 1.46 *Let* $F \subseteq \mathscr{O}_A$. *Then* $\mathrm{Clo}(F)$ *is a primitive-positive clone if and only if* $\mathrm{Clo}(F)^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet$.

Proof. $(\Rightarrow)$ Let $C = \mathrm{Clo}(F)$ be a primitive-positive clone. Then $C = C^{**}$ and

$$\mathrm{Clo}(F)^\bullet = C^\bullet = (C^{**})^\bullet = \mathrm{Clr}(C^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(\mathrm{Clo}(F)^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet,$$

by (b) and (c) of Lemma 1.45.

$(\Leftarrow)$ Let $C = \mathrm{Clo}(F)$. According to Lemma 1.45 and the assumption:

$$(C^{**})^\bullet = \mathrm{Clr}(C^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(\mathrm{Clo}(F)^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clr}(F^\bullet) \cap \mathscr{O}_A^\bullet = \mathrm{Clo}(F)^\bullet = C^\bullet.$$

Therefore, $(C^{**})^\bullet = C^\bullet$, i.e., $C^{**} = C$ and hence C is a primitive-positive clone. $\square$

There is another description of primitive-positive clones that justifies the name. Recall that a *primitive-positive formula over a language* $\mathscr{F} \cup \mathscr{C} \cup \mathscr{Q}$, where $\mathscr{F}$ is a set of function symbols, $\mathscr{C}$ is a set of constant symbols and $\mathscr{Q}$ is a set of relation symbols, is a formula of the form

$$\exists x_1 \exists x_2 \ldots \exists x_m\, (\alpha_1 \wedge \ldots \wedge \alpha_k)$$

where α_i are *atomic formulas*, that is, either formulas of the form $Q(v_1,\ldots,v_n)$ for some $Q \in \mathscr{Q}$ and some variables v_j, or an equality $t(v_1,\ldots,v_n) = s(v_1,\ldots,v_n)$ where t and s are $(\mathscr{F} \cup \mathscr{C})$-terms.

Let F be a set of operations such that (A,F) is an $\mathscr{F}$-algebra. An operation $g \in \mathscr{O}_A^{(n)}$ is *primitive-positive definable over F* (or pp-definable, for short) if there is a primitive-positive formula $\varphi(x_1,\ldots,x_n,y)$ over $\mathscr{F}$ such that

$$g(a_1,\ldots,a_n) = b \quad \text{if and only if} \quad (A,F) \models \varphi[a_1,\ldots,a_n,b]$$

for every $a_1,\ldots,a_n,b \in A$. Then it is easy to show the following statement:

Proposition 1.47 *Let $F \subseteq \mathscr{O}_A$. The set of all operations that are pp-definable over F is a clone of operations on A.*

We are now ready to show that primitive-positive clones are precisely the clones of pp-definable operations.

Theorem 1.48 (Kuznecov) *Let $F \subseteq \mathscr{O}_A$. Then $F^{**} = \{g \in \mathscr{O}_A : g$ is pp-definable over $F\}$.*

Proof. ($\supseteq$) Suppose $h \in \mathscr{O}_A$ is pp-definable over F. Then there is a primitive positive formula φ over $\mathscr{F}$ that defines h, i.e.

$$h(x_1,\dots,x_n) = y \quad \text{if and only if} \quad (A,F) \models \varphi(x_1,\dots,x_n,y).$$

Let $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{A} = (A,F)$ and let $\varphi^{\mathbf{A}}$ denote the interpretation of φ in $\mathbf{A}$. Clearly, $\varphi^{\mathbf{A}} = h^\bullet$. Without loss of generality we may assume that

$$\varphi(\mathbf{x},y) = (\exists \mathbf{z}) \bigwedge_{i=1}^{s} (f_i(\mathbf{x},y,\mathbf{z}) = g_i(\mathbf{x},y,\mathbf{z}))$$

for appropriately chosen $f_i, g_i \in \mathrm{Clo}(F)$. Let us show that $\varphi^{\mathbf{A}} \in \mathrm{Clr}(C^\bullet)$. From $f_i, g_i \in C$ it follows that $f_i^\bullet, g_i^\bullet \in C^\bullet$, so $f_i^\bullet \times g_i^\bullet \in \mathrm{Clr}(C^\bullet)$ for all i. For appropriately chosen diagonals δ_i and mappings α_i we have

$$(f_i(\mathbf{x},y,\mathbf{z}) = g_i(\mathbf{x},y,\mathbf{z}))^{\mathbf{A}} = \mathrm{pr}_{\alpha_i}(\delta_i \cap (f_i^\bullet \times g_i^\bullet)) \in \mathrm{Clr}(C^\bullet).$$

Next, we have

$$\left(\bigwedge_{i=1}^{s} (f_i(\mathbf{x},y,\mathbf{z}) = g_i(\mathbf{x},y,\mathbf{z}))\right)^{\mathbf{A}} = \bigcap_{i=1}^{s} \mathrm{pr}_{\alpha_i}(\delta_i \cap (f_i^\bullet \times g_i^\bullet)) \in \mathrm{Clr}(C^\bullet),$$

and thus

$$\varphi^{\mathbf{A}} = \mathrm{pr}_{\xi_1\dots\xi_b}\left(\bigcap_{i=1}^{s} \mathrm{pr}_{\alpha_i}(\delta_i \cap (f_i^\bullet \times g_i^\bullet))\right) \in \mathrm{Clr}(C^\bullet)$$

for appropriately chosen $\xi_1,\dots,\xi_b$. This shows that $h^\bullet = \varphi^{\mathbf{A}} \in \mathrm{Clr}(C^\bullet)$. Therefore, $h \in C^{**} = F^{**}$.

($\subseteq$) Take any $h \in C^{**}$. Then $f^\bullet \in (C^{**})^\bullet = \mathrm{Clr}(C^\bullet) \cap \mathscr{O}_A^\bullet$. The proof now follows from Theorem 1.27 having in mind that operations $\circ$, ζ, τ, $\triangle$ and $\delta_3^{12|3}$ can easily be represented by primitive-positive formulas. $\square$

Therefore, only some very special clones are clones of pp-definable functions. Actually, we shall show that there are only finitely many such clones on a finite set. The situation with relational clones, however, is significantly different: every relational clone is a clone of pp-definable relations.

Let Q be a set of relations such that (A,Q) is a $\mathscr{Q}$-relational structure. A relation $\rho \in \mathscr{R}_A^{(n)}$ is *primitive-positive definable over* Q (or pp-definable, for short) if there is a primitive-positive formula $\varphi(x_1,\dots,x_n)$ over $\mathscr{Q}$ such that

$$(a_1,\dots,a_n) \in \rho \quad \text{if and only if} \quad (A,Q) \models \varphi[a_1,\dots,a_n]$$

for all $a_1,\dots,a_n \in A$.

Theorem 1.49 *Let $Q \subseteq \mathscr{R}_A$. Then* $\mathrm{Clr}(Q) = \{\rho \in \mathscr{R}_A : \rho$ *is pp-definable over* $Q\}$.

Theorem 1.50 (Burris, Willard 1987, [5]) *For any fintite set A there are only finitely many primitive-positive clones.*

Proof. Let (A,F) be an algebra of type $\mathscr{F}$. We say that a relation $\rho \in \mathscr{R}_A^{(4)}$ is definable by a principal congruence formula w.r.t (A,F) if there is a principal congruence formula ψ such that

$$(a,b,c,d) \in \rho \quad \text{if and only if} \quad (A,F) \models \psi[a,b,c,d],$$

for all $a,b,c,d \in A$.

Now let $\mathbf{A}_1 = (A,F_1)$ be an algebra of type $\mathscr{F}_1$ and $\mathbf{A}_2 = (A,F_2)$ an algebra of type $\mathscr{F}_2$, and assume that the following two conditions hold:

(1) $\rho \in \mathscr{R}_A^{(4)}$ is definable by a principal congruence formula w.r.t $\mathbf{A}_1$ if and only if ρ is definable by a principal congruence formula w.r.t $\mathbf{A}_2$; and

(2) $f \in \bigcup_{i=1}^{|A|} \mathscr{O}_A^{(i)}$ is a term function of algebra $\mathbf{A}_1$ if and only if f is a term function of algebra $\mathbf{A}_2$.

Then we are going to show that $\hom(\mathbf{A}_1^k,\mathbf{A}_1) = \hom(\mathbf{A}_2^k,\mathbf{A}_2)$, for every $k \geqslant 1$.

Let $k \geqslant 1$ be arbitrary and let us first show that (1) implies $\mathrm{Con}(\mathbf{A}_1^k) = \mathrm{Con}(\mathbf{A}_2^k)$. Clearly, it suffices to show that the principal congruences coincide. So, let $\Theta_1(\mathbf{c},\mathbf{d})$ be a principal congruence of $\mathbf{A}_1^k$ generated by $\mathbf{c},\mathbf{d} \in A^k$. Then there is a principal congruence formula ψ such that

$$(\mathbf{a},\mathbf{b}) \in \Theta_1(\mathbf{c},\mathbf{d}) \quad \text{if and only if} \quad \mathbf{A}_1^k \models \psi[\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}].$$

Since principal congruence formulas respect homomorphisms we have that

$$\mathbf{A}_1 \models \psi[a_1,b_1,c_1,d_1],\dots,\mathbf{A}_1 \models \psi[a_k,b_k,c_k,d_k],$$

where $\mathbf{a} = (a_1, \dots, a_k)$, …, $\mathbf{d} = (d_1, \dots, d_k)$, so the 4-ary relation

$$\rho = \{(a,b,c,d) \in A^4 : \mathbf{A}_1 \models \psi[a,b,c,d]\}$$

contains the quadruples (a_1,b_1,c_1,d_1), …, (a_k,b_k,c_k,d_k). But ρ is obviously definable by a principal congruence formual w.r.t. $\mathbf{A}_1$, so according to (1) ρ is definable by a principal congruence formual w.r.t. $\mathbf{A}_2$. Let $\widehat{\psi}$ be a principle congruence formula which defines ρ w.r.t. $\mathbf{A}_2$. Then

$$\mathbf{A}_2 \models \widehat{\psi}[a_1,b_1,c_1,d_1], \dots, \mathbf{A}_2 \models \widehat{\psi}[a_k,b_k,c_k,d_k],$$

whence follows that $\mathbf{A}_2^k \models \widehat{\psi}[\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}]$. Therefore, $(\mathbf{a},\mathbf{b}) \in \Theta_2(\mathbf{c},\mathbf{d})$. This concludes the proof that $\mathrm{Con}(\mathbf{A}_1^k) = \mathrm{Con}(\mathbf{A}_2^k)$.

Now, take any $\varphi \in \hom(\mathbf{A}_1^k, \mathbf{A}_1)$ and assume that $\varphi \notin \hom(\mathbf{A}_2^k, \mathbf{A}_2)$. From $\varphi \in \hom(\mathbf{A}_1^k, \mathbf{A}_1)$ it follows that $\ker\varphi \in \mathrm{Con}(\mathbf{A}_1^k) = \mathrm{Con}(\mathbf{A}_2^k)$, while $\varphi \notin \hom(\mathbf{A}_2^k, \mathbf{A}_2)$ means that there is a function symbol $f \in \mathscr{F}_2$ and some u_1, …, $u_s \in A^k$ such that

$$\varphi(f^{\mathbf{A}_2^k}(u_1,\dots,u_s)) \neq f^{\mathbf{A}_2}(\varphi(u_1),\dots,\varphi(u_s)). \tag{1.2}$$

Let g be a term obtained from f by identifying variables x_i and x_j if and only if $\varphi(u_i) = \varphi(u_j)$. In order to make it easier to follow the proof, we proceed by taking an example. Let $s = 5$ so that f depends on 5 variables $f(x,y,z,v,w)$ and assume that $\varphi(u_2) = \varphi(u_5)$ and $\varphi(u_3) = \varphi(u_4)$. Then

$$g(x,y,z) = f(x,y,z,z,y).$$

Then $f^{\mathbf{A}_2}(\varphi(u_1),\dots,\varphi(u_s)) = g^{\mathbf{A}_2}(\varphi(u_{i_1}),\dots,\varphi(u_{i_t}))$ for appropriately chosen indices i_1, …, i_t. In our example this means that $f^{\mathbf{A}_2}(\varphi(u_1),\varphi(u_2),\varphi(u_3),\varphi(u_4),$ $\varphi(u_5)) = g^{\mathbf{A}_2}(\varphi(u_1),\varphi(u_2),\varphi(u_3))$. Note that $\mathrm{ar}(g) = |\mathrm{im}(\varphi)| \leqslant |A|$.

Next let us show that

$$\varphi(f^{\mathbf{A}_2^k}(u_1,\dots,u_s)) = \varphi(g^{\mathbf{A}_2^k}(u_{i_1},\dots,u_{i_t})). \tag{1.3}$$

We again take a look at the example. Since $(u_1,u_1) \in \ker\varphi$, $(u_2,u_2) \in \ker\varphi$, $(u_3,u_3) \in \ker\varphi$, $(u_4,u_3) \in \ker\varphi$, $(u_5,u_2) \in \ker\varphi$ and since $\ker\varphi \in \mathrm{Con}(\mathbf{A}_2^k)$ we have that

$$(f^{\mathbf{A}_2^k}(u_1,u_2,u_3,u_4,u_5), f^{\mathbf{A}_2^k}(u_1,u_2,u_3,u_3,u_2)) \in \ker\varphi$$

i.e.

$$(f^{\mathbf{A}_2^k}(u_1,u_2,u_3,u_4,u_5), g^{\mathbf{A}_2^k}(u_1,u_2,u_3)) \in \ker\varphi,$$

so $\varphi(f^{\mathbf{A}_2^k}(u_1,\dots,u_s)) = \varphi(g^{\mathbf{A}_2^k}(u_{i_1},\dots,u_{i_t}))$. Equations (1.2) and (1.3) together with the definition of g imply that

$$\varphi(g^{\mathbf{A}_2^k}(u_{i_1},\dots,u_{i_t})) \neq g^{\mathbf{A}_2}(\varphi(u_{i_1}),\dots,\varphi(u_{i_t})). \tag{1.4}$$

Since $g^{\mathbf{A}_2}$ is a term function of $\mathbf{A}_2$ of arity $\leqslant |A|$, from assumption (2) it follows that there is an $\mathscr{F}_1$-term h such that $h^{\mathbf{A}_1} = g^{\mathbf{A}_2}$. From (1.4) if now follows that

$$\varphi(h^{\mathbf{A}_1^k}(u_{i_1},\ldots,u_{i_t})) \neq h^{\mathbf{A}_1}(\varphi(u_{i_1}),\ldots,\varphi(u_{i_t})),$$

which contradicts the fact that $\varphi \in \hom(\mathbf{A}_1^k, \mathbf{A}_1)$ and h is an $\mathscr{F}_1$-term. This completes the proof that assumptions (1) and (2) imply $\hom(\mathbf{A}_1^k,\mathbf{A}_1) = \hom(\mathbf{A}_2^k,\mathbf{A}_2)$.

For a primitive-positive clone C let γ_C be the set of all 4-ary relations on A definable by principal congruence formulas w.r.t. (A,C) and let τ_C be the set of all term functions of (A,C) of arity $\leqslant |A|$, and define the map Φ by $\Phi(C) = (\gamma_C, \tau_C)$. Since $C = \bigcup_{n\geqslant 1} \hom((A,C)^n,(A,C))$, the above discussion actually shows that Φ is an injective mapping. Since there are only finitely many possibilities to chose γ_C and τ_C, it follows that there are only finitely many primitive-positive clones on A. $\square$

1.8 Abstract clones

Each clone C can be understood as a structure with countably many layers $C^{(1)}$, $C^{(2)}$, $\ldots$, $C^{(n)}$, $\ldots$, substitution operations $S_k^n : C^{(n)} \times (C^{(k)})^n \to C^{(k)}$ and distinguished elements $\pi_1^n,\ldots,\pi_n^n$ of each layer $C^{(n)}$. An abstrast setting to express this point of view is that of multisorded algebras.

Let S be a nonempty set called the *set of sorts*. An *S-set* (or a *multisorted set*) is any family $(A_s)_{s\in S}$. A multisorted set $A = (A_s)_{s\in S}$ is a *subset of* a multisorted set $B = (B_s)_{s\in S}$, in symbols $A \subseteq B$, if $A_s \subseteq B_s$ for all $s \in S$. An *S-function* $f : A \to B$ between S-sets $A = (A_s)_{s\in S}$ and $B = (B_s)_{s\in S}$ is any family of maps $(f_s)_{s\in S}$ such that $f_s : A_s \to B_s$ is a (usual) mapping for all $s \in S$. An S-function $f : A \to B$ is *bijective* if all f_s's are bijective. An *S-equivalence relation* $\theta = (\theta_s)_{s\in S}$ on an S-set $A = (A_s)_{s\in S}$ is an S-set $(\theta_s)_{s\in S}$ such that θ_s is an equivalence relation on A_s for all $s \in S$. By A/θ we denote the S-set $(A_s/\theta_s)_{s\in S}$.

For a string of sorts $w = (s_1,\ldots,s_n) \in S^n$ and a sort $s_0 \in S$, a (w,s_0)*-operation* (or a *multisorted operation*) on $(A_s)_{s\in S}$ is any mapping $f : A_{s_1} \times \ldots \times A_{s_n} \to A_{s_0}$. In that case w is said to be the *arity* of f. If $w = ()$ is the empty string of sorts, then a $((),s_0)$-operation is just an element of A_{s_0}. Therefore, $((),s_0)$-operations correspond to constants of sort s_0.

An *S-signature* is a set Σ of pairs (w,s) where w is a string of sorts from S and $s \in S$. For a signature Σ, a Σ*-multisorted algebra* is a pair (A,F) where $A = (A_s)_{s\in S}$ is an S-set and F is a set of multisorted operations on A such that for every $\sigma = (w,s) \in \Sigma$ there is exactly one (w,s)-operation $f_\sigma \in F$ and there are no other multisorted operations in F.

Let (A,F^A) and (B,F^B) be Σ-multisorted algebras and let $h : A \to B$ be an S-function. Then *h is a Σ-homomorphism* if

$$h_{s_0}(f^A_\sigma(a_1,\dots,a_n)) = f^B_\sigma(h_{s_1}(a_1),\dots,h_{s_n}(a_n))$$

for every $\sigma = ((s_1,\dots,s_n),s_0) \in \Sigma$ and all $a_i \in A_{s_i}$, $1 \leqslant i \leqslant n$. A *$\Sigma$-isomorphism* is a bijective Σ-homomorphism. We say that (B,F^B) is a *Σ-subalgebra* of (A,F^A) if $B \subseteq A$ and

$$f^B_\sigma(b_1,\dots,b_n) = f^A_\sigma(b_1,\dots,b_n)$$

for every $\sigma = ((s_1,\dots,s_n),s_0) \in \Sigma$ and all $b_i \in B_{s_i}$, $1 \leqslant i \leqslant n$. An S-equivalence relation θ on A is a *Σ-congruence* of (A,F^A) if for all $\sigma = ((s_1,\dots,s_n),s_0) \in \Sigma$,

$$(a_1,b_1) \in \theta_{s_1},\dots,(a_n,b_n) \in \theta_{s_n}$$

implies

$$(f^A_\sigma(a_1,\dots,a_n), f^A_\sigma(b_1,\dots,b_n)) \in \theta_{s_0}.$$

If θ is a Σ-congruence of (A,F^A) then the S-set A/θ is the carrier of a Σ-algebra whose operations are defined by

$$f^{A/\theta}_\sigma(a_1/\theta_{s_1},\dots,a_n/\theta_{s_n}) = f^A_\sigma(a_1,\dots,a_n)/\theta_{s_0}$$

where $\sigma = ((s_1,\dots,s_n),s_0) \in \Sigma$. This algebra is referred to as the *factor algebra* and denoted also by A/θ. One can now show that all the facts from universal algebra easily carry over to multisorted algebras.

Definition 1.51 An *abstract clone* is a multisorted algebra whose carrier is an $\mathbb{N}$-set $(A_n)_{n\in\mathbb{N}}$, with operations S^n_k of signature $((n,\underbrace{k,\dots,k}_{n}),k)$, and with constants $e^n_i \in A_n$, $1 \leqslant i \leqslant n$, which satisfies the following identities for all reasonable choices of i, k, m and n:

(AC1) $S^n_n(f,e^n_1,\dots,e^n_n) = f$,

(AC2) $S^n_k(e^n_i,f_1,\dots,f_n) = f_i$, and

(AC3) $S^k_m(S^n_k(f,g_1,\dots,g_n),h_1,\dots,h_k) =$
$= S^n_m(f,S^k_m(g_1,h_1,\dots,h_k),\dots,S^k_m(g_n,h_1,\dots,h_k))$.

Example 1.52 (a) Clearly, every clone of operations C is an abstract clone. The carrier of the algebra is $(C^{(n)})_{n\in\mathbb{N}}$, the operations S^n_k are given by $S^n_k(f,g_1,\dots,g_n) = f(g_1,\dots,g_n)$ and the constants are $e^n_i = \pi^n_i$.

(b) There are abstract clones that are not clones of operations. We start with a straighforward example. Let $\mathscr{F}$ be an algebraic type, let $X_n = \{x_1,\dots,x_n\}$,

$n \in \mathbb{N}$, be an increasing chain of finite sets of variables and let $T_{\mathscr{F}}(X_n)$ denote the absolutely free $\mathscr{F}$-algebra over the set of variables X_n. Then $\mathrm{Term}_{\mathscr{F}} = (T_{\mathscr{F}}(X_n))_{n\in\mathbb{N}}$ is the carrier of an abstract clone whose constants are given by $e^n_i = x_i$ and the superposition operations are given by substituting terms for variables. Note that if $\mathbf{A} = (A,F)$ is an $\mathscr{F}$-algebra and $\iota : \mathscr{F} \to F$ is the interpretation of operation symbols, then ι extends to a clone homomorphism $\iota^\# : \mathrm{Term}_{\mathscr{F}} \to \mathrm{Clo}(\mathbf{A})$, where $\mathrm{Clo}(\mathbf{A})$ denotes the clone of term-operations of the algebra $\mathbf{A}$.

(c) Finally, there exist abstract clones which are not just clones of operations in a fancy robe. Let X be a nonempty set and let A_n, $n \in \mathbb{N}$, be the set of all mappings $f : X \to \{1,\dots,n\} \times X$. Let $e^n_i \in A_n$ be the following mapping: $e^n_i(x) = (i,x)$. For $g_1, \dots, g_n \in A_k$ define $[g_1,\dots,g_n] : \{1,\dots,n\} \times X \to \{1,\dots,k\} \times X$ by $[g_1,\dots,g_n](i,x) = g_i(x)$ and let $S^n_k(f,g_1,\dots,g_n) = [g_1,\dots,g_n] \circ f$. Then $\Big((A_n)_{n\in\mathbb{N}}, (S^n_k)_{n,k\in\mathbb{N}}, (e^n_i)_{\substack{n,i\in\mathbb{N}\\ 1\leqslant i\leqslant n}}\Big)$ is an abstract clone.

Theorem 1.53 *Every abstract clone is isomorphic to a clone of operations (not necessarily on a finite set).*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be an abstract clone. In order to make it easier to follow the proof we assume that all A_n's are pairwise disjoint and instead of $S^n_k(f,g_1,\dots,g_n)$ we shall simply write $f(g_1,\dots,g_n)$. For $a_k \in A_k$ let

$$\begin{aligned} L(a_k) &= \{a_k(e^n_1,\dots,e^n_k) : n \geqslant k\} \\ &= \{a_k, a_k(e^{k+1}_1,\dots,e^{k+1}_k), a_k(e^{k+2}_1,\dots,e^{k+2}_k),\dots\}. \end{aligned}$$

We say that $L(a_k)$ is a chain, or the chain that starts with a_k. Let $L_n(a_k) = L(a_k) \cap A_n$. Clearly, $L_n(a_k) = \varnothing$ if $n < k$ and $L_n(a_k) = \{a_k(e^n_1,\dots,e^n_k)\}$ for $n \geqslant k$.

Let us now show that if L and L' are chains such that $L_n = L'_n$ for some $n \in \mathbb{N}$ then $L \subseteq L'$ or $L' \subseteq L$. Assume that $L_n = L'_n = \{a_n\}$, let i be the least integer such that $L_i \neq \varnothing$ and let j be the least integer such that $L'_j \neq \varnothing$ so that $L = L(a_i)$ where $L_i = \{a_i\}$, and $L' = L(a'_j)$ where $L'_j = \{a'_j\}$. Clearly, $i \leqslant n$ and $j \leqslant n$ and without loss of generality we may assume that $i \leqslant j$. Let us show that $L' \subseteq L$.

First, we show that $L_j = L'_j$. Let $L_j = \{a_j\}$. Since $L'_n = \{a_n\}$ and $L' = L(a'_j)$ we have $a_n = a'_j(e^n_1,\dots,e^n_j)$ and similarly from $L = L(a_i)$ we have $a_n = a_i(e^n_1,\dots,e^n_i)$. Since $a_j = a_i(e^j_1,\dots,e^j_i)$ it is easy to see that $a_n = a_j(e^n_1,\dots,e^n_j)$:

$$\begin{aligned} a_j(e^n_1,\dots,e^n_j) &= a_i(e^j_1,\dots,e^j_i)(e^n_1,\dots,e^n_j) \qquad \text{[by the def. of } a_j\text{]} \\ &= a_i(e^j_1(e^n_1,\dots,e^n_j),\dots,e^j_i(e^n_1,\dots,e^n_j)) \qquad \text{[by (AC3)]} \\ &= a_i(e^n_1,\dots,e^n_i) \qquad \text{[by (AC2)]} \\ &= a_n. \end{aligned}$$

Therefore, $a_n = a'_j(e^n_1, \dots, e^n_j) = a_j(e^n_1, \dots, e^n_j)$, whence

$$a'_j(e^n_1, \dots, e^n_j)(\underbrace{e^j_1, \dots, e^j_j, e^j_j, \dots e^j_j}_{n}) = a_j(e^n_1, \dots, e^n_j)(\underbrace{e^j_1, \dots, e^j_j, e^j_j, \dots e^j_j}_{n})$$

so by (AC3) we have $a'_j(e^j_1, \dots, e^j_j) = a_j(e^j_1, \dots, e^j_j)$ and by $(AC1)$ we conclude $a'_j = a_j$, i.e., $L'_j = L_j$. Finally, we show that for all $k \geqslant j$ we have $L_k = L'_k$. Let $L_k = \{a_k\}$ and $L'_k = \{a'_k\}$. Then $a'_k = a'_j(e^k_1, \dots, e^k_j)$ and $a_k = a_i(e^k_1, \dots, e^k_i)$. As in the previous paragraph we can show that $a_k = a_j(e^k_1, \dots, e^k_j)$ so from $a_j = a'_j$ it follows that $a_k = a'_k$. Since $L'_k = \varnothing$ for $k < j$ and $L'_k = L_k$ for $k \geqslant j$ we conclude $L' \subseteq L$.

This shows that every $a \in \bigcup_{n\in\mathbb{N}} A_n$ is contained in finitely many chains $L^1, \dots, L^s$ and that all these chains are linearly ordered by inclusion, i.e., $L^1 \subseteq \dots \subseteq L^s$. Therefore, for every $a \in \bigcup_{n\in\mathbb{N}} A_n$ there is a maximal chain that contains it, and we shall denote it by $\overline{a}$. Let

$$\overline{A} = \{\overline{a} : a \in \bigcup_{n\in\mathbb{N}} A_n\}$$

be the set of all maximal chains. For $f \in A_n$ define the n-ary operation $\Omega_f : (\overline{A})^n \to \overline{A}$ on $\overline{A}$ as follows. Take any $\overline{a^1}, \dots, \overline{a^n} \in \overline{A}$ and find the least $t \geqslant 1$ such that $\overline{a^i} \cap A_t \neq \varnothing$ for all i. Let $\overline{a^i} \cap A_t = \{a^i_t\}$, $i \in \{1, \dots, n\}$, and set

$$\Omega_f(\overline{a^1}, \dots, \overline{a^n}) = \overline{f(a^1_t, \dots, a^n_t)}.$$

Note that for $s \geqslant t$ we have

$$\overline{f(a^1_t, \dots, a^n_t)} = \overline{f(a^1_s, \dots, a^n_s)} \tag{1.5}$$

where $\overline{a^i} \cap A_s = \{a^i_s\}$, $i \in \{1, \dots, n\}$. This follows easily since

$$f(a^1_s, \dots, a^n_s) = f(a^1_t(e^s_1, \dots, e^s_t), \dots, a^n_t(e^s_1, \dots, e^s_t)) = f(a^1_t, \dots, a^n_t)(e^s_1, \dots, e^s_t)$$

and thus $\overline{f(a^1_t, \dots, a^n_t)} \cap A_s = \{f(a^1_s, \dots, a^n_s)\}$. Therefore, $f(a^1_s, \dots, a^n_s)$ and $f(a^1_t, \dots, a^n_t)$ belong to the same chain and hance have the same maximal chain. To complete the proof, let

$$C = \{\Omega_f : f \in \bigcup_{n\in\mathbb{N}} A_n\}.$$

At the same time we show that C is a clone and that $\Omega : \bigcup_{n\in\mathbb{N}} A_n \to C : f \mapsto \Omega_f$ is a clone homomorphism. We start with projections:

$$\Omega_{e^n_i}(\overline{a^1}, \dots, \overline{a^n}) = \overline{e^n_i(a^1_t, \dots, a^n_t)} = \overline{a^i_t} = \overline{a^i},$$

where $t \geqslant 1$ is the least integer such that $\overline{a^i} \cap A_t \neq \varnothing$ for all i, and $\overline{a^i} \cap A_t = \{a_t^i\}$, $i \in \{1, \dots, n\}$. Therefore, $\Omega_{e_i^n} = \pi_i^n \in C$. Next, take any Ω_f, Ω_{g_1}, …, $\Omega_{g_k} \in C$ where $f \in A_k$ and g_1, …, $g_k \in A_n$. Then

$$\begin{aligned}\Omega_f(\Omega_{g_1}, \dots, \Omega_{g_k})(\overline{a^1}, \dots, \overline{a^n}) &= \Omega_f(\Omega_{g_1}(\overline{a^1}, \dots, \overline{a^n}), \dots, \Omega_{g_k}(\overline{a^1}, \dots, \overline{a^n}))\\ &= \Omega_f(\overline{g_1(a_t^1, \dots, a_t^n)}, \dots, \overline{g_k(a_t^1, \dots, a_t^n)}),\end{aligned}$$

where $t \geqslant 1$ is the least integer such that $\overline{a^i} \cap A_t \neq \varnothing$ for all i, and $\overline{a^i} \cap A_t = \{a_t^i\}$, $i \in \{1, \dots, n\}$. From (1.5) and (AC1)–(AC3) it now follows that

$$\begin{aligned}\Omega_f(\overline{g_1(a_t^1, \dots, a_t^n)}, \dots, \overline{g_k(a_t^1, \dots, a_t^n)}) &= \overline{f(g_1(a_t^1, \dots, a_t^n), \dots, g_k(a_t^1, \dots, a_t^n))}\\ &= \overline{f(g_1, \dots, g_k)(a_t^1, \dots, a_t^n)}\\ &= \Omega_{f(g_1, \dots, g_k)}(\overline{a^1}, \dots, \overline{a^n}).\end{aligned}$$

Therefore, $\Omega_f(\Omega_{g_1}, \dots, \Omega_{g_k}) = \Omega_{f(g_1, \dots, g_k)} \in C$. So, C is a clone and Ω is a clone homomorphism. Clearly, Ω is onto and in order to show that Ω is an isomorphism we still have to show that Ω is injective. Let $f, g \in A_n$ and let $\Omega_f = \Omega_g$. Then

$$\overline{f} = \overline{f(e_1^n, \dots e_n^n)} = \Omega_f(\overline{e_1^n}, \dots, \overline{e_n^n}) = \Omega_g(\overline{e_1^n}, \dots, \overline{e_n^n}) = \overline{g(e_1^n, \dots e_n^n)} = \overline{g}.$$

So, $\overline{f} = \overline{g}$ and from $f, g \in A_n$ and $|\overline{f} \cap A_n| = 1 = |\overline{g} \cap A_n|$ it follows $f = g$. This completes the proof. □

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