

# A LECTURE ON THE COMBINATORIAL NULLSTELLENSATZ

## 1. ALON'S COMBINATORIAL NULLSTELLENSATZ

Throughout,  $\mathbb{K}$  will denote a field. The degree of a unary polynomial  $f \in \mathbb{K}[x]$  is denoted by  $\deg(f)$ , and we set  $\deg(0) = -\infty$ .

**Theorem 1.1.** *Let  $S$  be a finite subset of  $\mathbb{K}$  and let  $f \in \mathbb{K}[x] \setminus \{0\}$  with  $\deg(f) < |S|$ . Then there is  $s \in S$  with  $f(s) \neq 0$ .*

*Proof 1.* Suppose that for all  $s \in S$ , we have  $f(s) = 0$ . Then for all  $s \in S$ , we have  $x - s \mid f$ , and since for  $s_1, s_2 \in S$  with  $s_1 \neq s_2$ , the polynomials  $x - s_1$  and  $x - s_2$  are coprime, we have  $\prod_{s \in S} (x - s) \mid f$ , and therefore  $f = 0$  or  $\deg(f) \geq |S|$ .  $\square$

*Proof 2.* Induction on  $|S|$ . If  $|S| = 1$  and  $\deg(f) = 0$ , then  $f = cx^0$  for some  $c \in \mathbb{K}$  with  $c \neq 0$ , and hence  $f(s) = c$  for the element  $s \in S$ .

For the induction step, we assume that  $|S| \geq 2$ . Suppose that  $f(s) = 0$  for all  $s \in S$  and  $f \neq 0$  and let  $s_1 \in S$ . Then  $x - s_1 \mid f$ . Let  $g \in \mathbb{K}[x]$  be such that  $(x - s_1) \cdot g = f$ . Now  $g(s) = 0$  for all  $s \in S \setminus \{s_1\}$ , and therefore by the induction hypothesis,  $\deg(g) \geq |S| - 1$ , and thus  $\deg(f) \geq |S|$ .  $\square$

**Theorem 1.2.** *Let  $S_1, \dots, S_n$  be finite subsets of  $\mathbb{K}$ , and let  $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ . We assume that for all  $i \in \underline{n}$ , we have  $\deg_{x_i}(f) < |S_i|$ . Then there is  $\mathbf{s} \in S := \times_{i=1}^n S_i$  with  $f(\mathbf{s}) \neq 0$ .*

*Proof.* Induction by  $n$ : If  $n = 1$ , then the result is Theorem 1.1. For the induction step, let  $n \geq 2$ .

**Case 1:**  $\deg_{x_i}(f) = 0$  for all  $i \in \underline{n}$ . Then  $f$  is constant and not 0. By the assumptions,  $S \neq \emptyset$ , and every element of  $S$  is a nonzero of  $f$ .

**Case 2:** There is  $i \in \underline{n}$  with  $\deg_{x_i}(f) > 0$ . Let  $t := \deg_{x_i}(f)$ . We write

$$f = \sum_{j=0}^t f_j(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \cdot x_i^j.$$

By the induction hypothesis, there is  $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in \times_{i \in \underline{n} \setminus \{i\}} S_i$  with

$$f_t(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \neq 0.$$

Then  $g(x) = f(s_1, \dots, s_{i-1}, x, s_{i+1}, \dots, s_n)$  is a univariate polynomial with  $\deg(g) = \deg_{x_i}(f)$ . Since  $\deg(g) < |S_i|$ , Theorem 1.1 yields an  $s_i \in S_i$  with  $g(s_i) \neq 0$ , and therefore  $f(s_1, \dots, s_n) \neq 0$ .  $\square$

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Notes for EA's lecture in the course "The Polynomial Method for Combinatorial Problems" by J. Schmitt, O. Roche-Newton and E. Aichinger.

**Theorem 1.3** (cf. [Alo99, Theorem 1.1]). *Let  $S_1, \dots, S_n$  be finite subsets of  $\mathbb{K}$ , and let  $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ . For  $i \in \underline{n}$ , let  $g_i := \prod_{a \in S_i} (x_i - a)$ . If  $f(\mathbf{s}) = 0$  for all  $\mathbf{s} \in S := \times_{i=1}^n S_i$ , then  $f \in \langle g_1, \dots, g_n \rangle$ .*

*Proof.* By multivariate polynomial division, we compute  $h_1, \dots, h_n$  such that  $f = \sum_{i=1}^n h_i g_i + r$  and no monomial of  $r$  is divisible by any  $x_i^{|S_i|}$ . In the case  $r = 0$ , we are done. If  $r \neq 0$ , then  $r$  satisfies the assumptions of Theorem 1.2, and thus there is  $\mathbf{s} \in S$  with  $r(\mathbf{s}) \neq 0$ , and thus  $f(\mathbf{s}) \neq 0$ . This is excluded by the assumption that  $f$  vanishes on  $S$ .  $\square$

For a polynomial  $f \in \mathbb{K}[x_1, \dots, x_n]$ , we denote its total degree by  $\deg(f)$ .

**Theorem 1.4** (Alon's Combinatorial Nullstellensatz [Alo99, Theorem 1.2]). *Let  $S = \times_{i=1}^n S_i$  be a grid over  $\mathbb{K}$ , let  $f \in \mathbb{K}[x_1, \dots, x_n]$  and let  $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  with  $\alpha_i < |S_i|$  for all  $i \in \underline{n}$  be such that  $f$  contains the monomial  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ . Then if*

$$(1.1) \quad \sum_{i=1}^n \alpha_i = \deg(f),$$

*there is  $\mathbf{s} \in S$  such that  $f(\mathbf{s}) \neq 0$ .*

*Proof.* Let  $g_i := \prod_{a \in S_i} (x_i - a)$ . By multivariate polynomial division, we obtain  $h_1, \dots, h_n, r \in \mathbb{K}[x_1, \dots, x_n]$  such that

$$f = \sum_{i=1}^n h_i g_i + r,$$

and for all  $i \in \underline{n}$ , we have  $\deg(h_i g_i) \leq \deg f$  and  $\deg_{x_i}(r) < \deg(g_i)$ . Seeking a contradiction, we suppose that  $f$  vanishes on  $S$ . Then  $r$  vanishes on  $S$ , and therefore  $r = 0$  by Theorem 1.2. The *leading term* of  $g_i$  is  $x_i^{|S_i|}$ , which we abbreviate by  $\text{LT}(g_i)$ .

$$f = \sum_{i=1}^n h_i g_i = \sum_{i=1}^n h_i \text{LT}(g_i) + \sum_{i=1}^n h_i (g_i - \text{LT}(g_i)).$$

All monomials in the last sum have total degree at most  $\deg(h_i) + \deg(g_i) - 1 < \deg(f)$ , and therefore the monomial  $\mathbf{x}^\alpha$  appears in  $\sum_{i=1}^n h_i \text{LT}(g_i)$  and is therefore divisible by  $\text{LT}(g_i)$  for some  $i \in \underline{n}$ . But then  $\alpha_i \geq \deg_{x_i}(g_i) = |S_i|$ , contradicting the assumption  $\alpha_i < |S_i|$ .  $\square$

## 2. GENERALISATIONS BY LASOŃ AND NICA

For ensuring the existence of a nonzero on a grid, Alon's Theorem requires that  $f$  contains a monomial of maximal total degree such that the degree in each variable is smaller than the corresponding side length of the grid. Lasoń's result [Las10, Theorem 2] relaxes the condition (1.1) on such a monomial. For  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ , we write  $\alpha \sqsubseteq \beta$  if  $\alpha_i \leq \beta_i$  for all  $i \in \underline{n}$ . The monomial  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$  is also written as  $\mathbf{x}^\alpha$ . Clearly, a monomial  $\mathbf{x}^\alpha$  divides a monomial  $\mathbf{x}^\beta$  if and only if  $\alpha \sqsubseteq \beta$ . For a polynomial  $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \mathbf{x}^\alpha$ , we let  $\text{Mon}(f) := \{\mathbf{x}^\alpha \mid \alpha \in \mathbb{N}_0^n, c_\alpha \neq 0\}$  be the set of monomials that appear in  $f$ , and  $\text{Supp}(f) = \{\alpha \in \mathbb{N}_0^n \mid c_\alpha \neq 0\}$  be the set of exponents of these monomials, called the *support* of  $f$ .

Lason's Theorem tells that Theorem 1.4 still holds if we replace (1.1) by the weaker condition

$$(2.1) \quad (\alpha_1, \dots, \alpha_n) \text{ is maximal in } \text{Supp}(f) \text{ with respect to } \sqsubseteq .$$

Stated differently, (2.1) requires that  $\text{Supp}(f)$  does not contain a  $\gamma \in \mathbb{N}_0^n$  with  $\alpha \sqsubseteq \gamma$  and  $\alpha \neq \gamma$ .

Nica [Nic23, Theorem 3.1] generalizes Theorem 1.4 in a different direction, taking into account a possible special structure of the grid. For  $\lambda \in \mathbb{N}_0$ , we call a univariate polynomial  $f \in \mathbb{K}[x]$  of degree  $\nu \in \mathbb{N}_0$   $\lambda$ -lacunary if in  $f$ , all coefficients of  $x^\alpha$  with  $\nu - \lambda \leq \alpha < \nu$  vanish. Then [Nic23] defines a finite set  $A \subseteq \mathbb{K}$  to be  $\lambda$ -null if the polynomial  $\prod_{a \in A} (x - a)$  is  $\lambda$ -lacunary. Nica's Theorem now is:

**Theorem 2.1** ([Nic23, Theorem 3.1]). *Let  $S = \times_{i=1}^n S_i$  be a grid over the field  $\mathbb{K}$ , and let  $\lambda \in \mathbb{N}_0$  be such that each  $S_i$  is  $\lambda$ -null. Let  $f \in \mathbb{K}[x_1, \dots, x_n]$  and let  $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  with  $\alpha_i < |S_i|$  for all  $i \in \underline{n}$  be such that  $f$  contains the monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . Then if*

$$(2.2) \quad \sum_{i=1}^n \alpha_i \geq \deg(f) - \lambda$$

there is  $\mathbf{s} \in S$  such that  $f(\mathbf{s}) \neq 0$ .

We will prove the following common generalization:

**Theorem 2.2.** *Let  $n \in \mathbb{N}$  and let  $\lambda_1, \dots, \lambda_n \in \mathbb{N}_0$ . For each  $i \in \underline{n}$ , let  $S_i$  be a  $\lambda_i$ -null subset of the field  $\mathbb{K}$ , and let  $S := \times_{i=1}^n S_i$ . Let  $f \in \mathbb{K}[x_1, \dots, x_n]$  and let  $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  with  $\alpha_i < |S_i|$  for all  $i \in \underline{n}$  be such that  $f$  contains the monomial  $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  and*

$$(2.3) \quad \text{there is no pair } (i, \mathbf{x}^\gamma) \in \underline{n} \times \text{Mon}(f) \text{ such that } x_i^{\lambda_i+1} \mathbf{x}^\alpha \text{ divides } \mathbf{x}^\gamma.$$

Then there is  $\mathbf{s} \in S$  such that  $f(\mathbf{s}) \neq 0$ .

Let  $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$  be the  $i$ th unit vector in  $\mathbb{N}_0^n$ , and let  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Then Condition (2.3) is equivalent to:

$$(2.4) \quad \text{There do not exist } \gamma \in \text{Supp}(f) \text{ and } i \in \underline{n} \text{ such that } \alpha + (\lambda_i + 1)\mathbf{e}_i \sqsubseteq \gamma.$$

The idea of the proof is rather simple: By Theorem 1.3, the ideal  $I$  of polynomials vanishing on  $S$  is generated by  $\{g_i \mid i \in \underline{n}\}$  with  $g_i := \prod_{a \in S_i} (x_i - a)$ . The polynomial  $g_i$  has leading term  $x_i^{|S_i|}$ . For proving  $f \notin I$ , we show that its remainder  $r$  modulo  $\{g_i \mid i \in \underline{n}\}$  after multivariate polynomial division by  $F$  is nonzero. The conditions on  $f$  and the  $g_i$ 's ensure that the monomial  $\mathbf{x}^\alpha$  can never be reduced in the course of the division, and that all other monomials in  $f$  have too small exponents to be able to produce a term  $c_\alpha \mathbf{x}^\alpha$  that would allow to cancel  $\mathbf{x}^\alpha$  before it remains in the remainder. In order to state this idea precisely, we will make use of some notions from the arithmetic of multivariate polynomials, in particular of multivariate polynomial division, which is one of the basics of the theory *Gröbner bases* [Buc70, Buc85]; here, we will use some terminology from [Eis95].

## 3. LACUNARY MULTIVARIATE POLYNOMIALS

We first extend the definition of lacunary polynomials to multivariate polynomials. Let  $\mathbb{K}$  be a field, let  $f \in \mathbb{K}[x_1, \dots, x_n]$ , let  $M \subseteq \mathbb{N}_0^n$ . We will say that a monomial  $\mathbf{x}^\nu \in \text{Mon}(f)$  is an  $M$ -dominating monomial in  $f$  if for all  $\varepsilon \in \text{Supp}(f) \setminus \{\nu\}$  and all  $\mu \in M$ , we have

$$\varepsilon + \mu \sqsubseteq \nu.$$

The polynomial  $f$  will be defined to be  $M$ -lacunary if it contains an  $M$ -dominating monomial. Hence if  $g_1, g_2, \dots, g_n$  are  $\lambda$ -lacunary (as defined in Section 1) elements of  $\mathbb{K}[x]$  then for each  $j \in \underline{n}$ ,  $g_j(x_j)$  is an  $M$ -lacunary polynomial in  $\mathbb{K}[x_1, \dots, x_n]$  for  $M = \{(\lambda + 1) \mathbf{e}_i \mid i \in \underline{n}\}$ .

We sort monomials using *admissible orderings*: A linear order  $\leq_a$  on  $\mathbb{N}_0^n$  is *admissible* if it is total, and for all  $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ , we have  $(\alpha \sqsubseteq \beta \Rightarrow \alpha \leq_a \beta)$  and  $(\alpha \leq_a \beta \Rightarrow \alpha + \gamma \leq_a \beta + \gamma)$ . If  $\alpha \in \mathbb{N}_0^n$  is maximal in  $\text{Supp}(f)$  with respect to  $\leq_a$ , then  $\mathbf{x}^\alpha$  is called the *leading monomial of  $f$*  and abbreviated by  $\text{LM}(f)$ , and  $\alpha$  is the *leading exponent* or *multidegree* of  $f$ , abbreviated by  $\text{LEXP}(f)$  and  $\text{mdeg}(f)$ . The coefficient  $c_\alpha$  of the leading monomial  $\mathbf{x}^\alpha$  is the *leading coefficient*, abbreviated as  $\text{LC}(f)$ , and  $c_\alpha \mathbf{x}^\alpha = \text{LC}(f) \cdot \text{LM}(f) = \text{LC}(f) \mathbf{x}^{\text{LEXP}(f)}$  is the *leading term* of  $f$ , denoted by  $\text{LT}(f)$ . Every admissible ordering is a well ordering, i.e., it is total and has no infinite descending chains; a proof can be found, e.g., in the survey [Aic24] (Lemma 9.2).

It is easy to see that for  $M \neq \emptyset$ , an  $M$ -dominating monomial in  $f$  is the leading monomial with respect to every admissible ordering of monomials. We summarize:

**Definition 3.1.** Let  $\mathbb{K}$  be a field,  $g \in \mathbb{K}[x_1, \dots, x_n]$ , and let  $M \subseteq \mathbb{N}_0^n$ . We say that  $g$  is  $M$ -lacunary if for every  $\varepsilon \in \text{Supp}(g) \setminus \{\text{LEXP}(g)\}$ , there is  $\mu \in M$  such that  $\varepsilon + \mu \sqsubseteq \text{LEXP}(g)$ .

## 4. MULTIVARIATE POLYNOMIAL DIVISION

Multivariate polynomial division can be expressed by *natural standard expressions*: Let  $G \subseteq \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ . A *natural standard expression* of  $f$  by  $G$  with remainder  $r$  with respect to the admissible ordering  $\leq_a$  is an equality

$$(4.1) \quad f = \sum_{j=1}^t c_j \mathbf{x}^{\delta_j} g_j + r,$$

where  $t \in \mathbb{N}_0$ ,  $c_1, \dots, c_t \in \mathbb{K} \setminus \{0\}$ ,  $\delta_1, \dots, \delta_t \in \mathbb{N}_0^n$ ,  $g_1, \dots, g_t \in G$ ,  $r \in \mathbb{K}[x_1, \dots, x_n]$ , for each  $j \in \underline{t}$ , we have

$$(4.2) \quad \text{LM}(c_j \mathbf{x}^{\delta_j} g_j) \in \text{Mon}\left(f - \sum_{i=1}^{j-1} c_i \mathbf{x}^{\delta_i} g_i\right)$$

and

$$(4.3) \quad \text{LM}(c_j \mathbf{x}^{\delta_j} g_j) \notin \text{Mon}\left(f - \sum_{i=1}^j c_i \mathbf{x}^{\delta_i} g_i\right),$$

and  $r$  does not contain a monomial that is divisible by any monomial in  $\{\text{LM}(g) \mid g \in G\}$ .

There are two differences to standard expressions as used in the literature: first (cf. [Eis95, p.334]), standard expressions are usually written in a collected form  $\sum_{i=1}^s h_i g_i + r$  with  $h_i \in$

$\mathbb{K}[x_1, \dots, x_n]$ . The second main difference is that a *natural* standard expression should come from reducing the polynomial  $f$  by  $G$ , reflected in (4.2) and (4.3). Usual polynomial reduction of  $f$  by  $G$  yields a natural standard expression; [CLO15] provides an explicit algorithm for multivariate polynomial division that produces such a natural standard expression.

In this reduction process, certain monomials of  $f$  can never be reduced and will therefore end up in the remainder  $r$ .

**Definition 4.1.** Let  $G \subseteq \mathbb{K}[x_1, \dots, x_n]$ ,  $M \subseteq \mathbb{N}_0^n$ ,  $f \in \mathbb{K}[x_1, \dots, x_n]$ ,  $\alpha \in \mathbb{N}_0^n$ , and let  $\leq_a$  be an admissible monomial ordering. We say that  $\mathbf{x}^\alpha$  is a  $(G, M)$ -stable monomial in  $f$  with respect to  $\leq_a$  if the following conditions hold:

- (1)  $\alpha \in \text{Supp}(f)$ ,
- (2) there is no  $g \in G$  with  $\text{LM}(g) \mid \mathbf{x}^\alpha$ , and
- (3) there are no  $\gamma \in \text{Supp}(f)$  and  $\mu \in M$  such that  $\alpha + \mu \sqsubseteq \gamma$ .

**Theorem 4.2.** Let  $M \subseteq \mathbb{N}_0^n$ , and let  $G \subseteq \mathbb{K}[x_1, \dots, x_n]$  such that every  $g' \in G$  is  $M$ -lacunary, and let  $\leq_a$  be an admissible monomial ordering. Let  $f \in \mathbb{K}[x_1, \dots, x_n]$ ,  $g \in G$ ,  $\delta \in \mathbb{N}_0^n$  be such that  $\text{LM}(\mathbf{x}^\delta g) \in \text{Mon}(f)$ . Let  $\mathbf{x}^\alpha$  be a  $(G, M)$ -stable monomial in  $f$ , let  $c \in \mathbb{K} \setminus \{0\}$ , and let

$$h = f - c \mathbf{x}^\delta g.$$

Then  $\mathbf{x}^\alpha$  is a  $(G, M)$ -stable monomial in  $h$ .

*Proof.* We first show

$$(4.4) \quad \alpha \in \text{Supp}(h).$$

Seeking a contradiction, we suppose  $\alpha \notin \text{Supp}(h)$ . Then  $\alpha \in \text{Supp}(\mathbf{x}^\delta g)$ . Thus there is  $\mathbf{x}^\varepsilon \in \text{Mon}(g)$  with  $\mathbf{x}^\alpha = \mathbf{x}^\delta \mathbf{x}^\varepsilon$ . If  $\mathbf{x}^\varepsilon = \text{LM}(g)$ , then  $\text{LM}(g) \mid \mathbf{x}^\alpha$ , contradicting the assumption that  $\mathbf{x}^\alpha$  is  $(G, M)$ -stable in  $f$ . If  $\mathbf{x}^\varepsilon \neq \text{LM}(g)$ , then by lacunarity, there is  $\mu \in M$  such that  $\varepsilon + \mu \sqsubseteq \text{LEXP}(g)$ , and therefore since  $\alpha = \varepsilon + \delta$ , we have  $\alpha + \mu \sqsubseteq \text{LEXP}(g) + \delta$ . We have assumed that  $\text{LEXP}(g) + \delta \in \text{Supp}(f)$ . This implies that  $\alpha$  is not  $(G, M)$ -stable in  $f$ , contradicting the assumptions. This completes the proof of (4.4).

We will now show that there are no  $\gamma \in \text{Supp}(h)$  and  $\mu \in M$  with  $\alpha + \mu \sqsubseteq \gamma$ . Seeking a contradiction, we suppose that there are  $\gamma \in \text{Supp}(h)$  and  $\mu \in M$  with  $\alpha + \mu \sqsubseteq \gamma$ . Since  $\alpha$  is  $(G, M)$ -stable in  $f$ , we know that

$$(4.5) \quad \gamma \notin \text{Supp}(f).$$

Therefore,  $\gamma \in \text{Supp}(\mathbf{x}^\delta g)$ . Thus there is  $\mathbf{x}^\rho \in \text{Mon}(g)$  such that  $\mathbf{x}^\gamma = \mathbf{x}^\delta \mathbf{x}^\rho$ . If  $\mathbf{x}^\rho = \text{LM}(g)$ , then  $\mathbf{x}^\gamma = \mathbf{x}^\delta \text{LM}(g)$ . Since by assumption,  $\text{LM}(\mathbf{x}^\delta g) \in \text{Mon}(f)$ , we have  $\mathbf{x}^\gamma \in \text{Supp}(f)$ , contradicting (4.5).

Hence  $\mathbf{x}^\rho \neq \text{LM}(g)$ . Then by lacunarity, there is  $\nu \in M$  such that we have  $\rho + \nu \sqsubseteq \text{LEXP}(g)$ . Thus  $\delta + \rho + \nu \sqsubseteq \delta + \text{LEXP}(g)$ , and therefore  $\gamma + \nu \sqsubseteq \text{LEXP}(\mathbf{x}^\delta g)$ . Since  $\alpha + \mu \sqsubseteq \gamma$ , we have  $\alpha \sqsubseteq \gamma$ . Altogether  $\alpha + \nu \sqsubseteq \text{LEXP}(\mathbf{x}^\delta g)$ . Since  $\text{LM}(\mathbf{x}^\delta g) \in \text{Mon}(f)$ , we obtain that  $\alpha$  is not  $(G, M)$ -stable in  $f$ , a contradiction. Hence such  $\gamma$  and  $\mu$  cannot exist, completing the proof that  $\alpha$  is  $(G, M)$ -stable in  $h$ .  $\square$

**Corollary 4.3.** *Let  $M \subseteq \mathbb{N}_0^n$ , and let  $G \subseteq \mathbb{K}[x_1, \dots, x_n]$  such that every  $g \in G$  is  $M$ -lacunary, and let  $\leq_a$  be an admissible ordering of  $\mathbb{N}_0^n$ . Let*

$$(4.6) \quad f = \sum_{j=1}^t c_j \mathbf{x}^{\delta_j} g_j + r$$

*be a natural standard expression of  $f$  by  $G$ . Then all  $(G, M)$ -stable elements of  $\text{Mon}(f)$  are also elements of  $\text{Mon}(r)$ .*

*Proof.* Let  $\mathbf{x}^\alpha$  be a  $(G, M)$ -stable monomial in  $f$ . By induction on  $s$ , we show that  $\mathbf{x}^\alpha$  is also  $(G, M)$ -stable in

$$f - \sum_{j=1}^s c_j \mathbf{x}^{\delta_j} g_j.$$

For  $s = 0$ , there is nothing to prove. Now assume that  $s \in \{0, \dots, t-1\}$ . As inductive hypothesis, we assume that  $\mathbf{x}^\alpha$  is  $(G, M)$ -stable in  $f - \sum_{j=1}^s c_j \mathbf{x}^{\delta_j} g_j$ . Then by Theorem 4.2,  $\mathbf{x}^\alpha$  is also  $(G, M)$ -stable in  $f - \sum_{j=1}^s c_j \mathbf{x}^{\delta_j} g_j - c_{s+1} \mathbf{x}^{\delta_{s+1}} g_{s+1}$ , completing the induction step.  $\square$

## 5. PROOF OF THEOREM 2.2

For  $i \in \underline{n}$ , we let  $f_i(x) := \prod_{a \in S_i} x - a$ , and we let  $g_i := f_i(x_i)$ . If each  $f_i$  is a univariate  $\lambda_i$ -lacunary polynomial in  $\mathbb{K}[x]$ , then for each  $i \in \underline{n}$ ,  $g_i$  is an  $M$ -lacunary polynomial in  $\mathbb{K}[x_1, \dots, x_n]$  for  $M := \{(\lambda_j + 1)\mathbf{e}_j \mid j \in \underline{n}\}$ .

With this observation in mind, we can apply Corollary 4.3 to prove the main result:

*Proof of Theorem 2.2.* For each  $i \in \underline{n}$ , let  $g_i := \prod_{a \in S_i} (x_i - a)$ . Let  $I$  be the ideal of  $\mathbb{K}[x_1, \dots, x_n]$  generated by  $G = \{g_1, \dots, g_n\}$ . By Theorem 1.3 a polynomial  $f$  vanishes on  $S_1 \times \dots \times S_n$  if and only if it lies in  $I$ . Now we seek to apply Corollary 4.3. To this end, let  $M := \{x_j^{\lambda_j+1} \mid j \in \underline{n}\}$  and let  $i \in \underline{n}$ . Since  $S_i$  is  $\lambda_i$ -null, the polynomial  $g_i$  is  $M$ -lacunary. We fix an admissible ordering  $\leq_a$  of monomials. We will now show that  $\alpha$  is an  $(G, M)$ -stable monomial in  $f$  with respect to  $\leq_a$ . First, since  $\alpha_i < |S_i|$ , the monomial  $\text{LM}(g_i)$  does not divide  $\mathbf{x}^\alpha$ . Second, suppose that  $f$  contains a monomial  $\mathbf{x}^\gamma$  for which there is  $\mu \in M$  with

$$\alpha + \mu \sqsubseteq \gamma.$$

Then there exists  $i \in \underline{n}$  such that  $\alpha + (\lambda_i + 1)\mathbf{e}_i \sqsubseteq \gamma$ , and therefore  $x_i^{\lambda_i+1} \mathbf{x}^\alpha$  divides  $\mathbf{x}^\gamma$ . This is excluded by assumption (2.3). Let  $f = \sum_{j=1}^t c_j \mathbf{x}^{\delta_j} g_j + r$  be a natural standard expression of  $f$  by  $G$ . Since  $\alpha$  is a  $(G, M)$ -stable monomial in  $f$ , Corollary 4.3 yields that  $\mathbf{x}^\alpha \in \text{Mon}(r)$ . Since the leading monomials of the polynomials in  $G$  are coprime, the set  $G$  is a Gröbner basis of  $I$  (cf. [CLO15, p.89, Exercise 11]). Since then all elements of  $I$  have zero remainder in every standard expression by  $G$ , we obtain  $f \notin I$ , and therefore,  $f$  does not vanish on all points in  $S_1 \times \dots \times S_n$ .  $\square$

As an example, consider the sets  $S_1 = S_2 = S_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$ , let  $S := S_1 \times S_2 \times S_3$ , and let

$$\begin{aligned} M_A &:= \{x^\alpha y^\beta z^\gamma \mid \alpha + \beta + \gamma \leq 6, (\alpha, \beta, \gamma) \neq (2, 2, 2)\}, \\ M_L &:= \{x^\alpha y^\beta z^\gamma \mid x \leq 1 \text{ or } y \leq 1 \text{ or } z \leq 1\}, \\ M_N &:= \{x^\alpha y^\beta z^\gamma \mid \alpha + \beta + \gamma \leq 8, (\alpha, \beta, \gamma) \neq (2, 2, 2)\}, \\ M_T &:= \{x^\alpha y^\beta z^\gamma \mid x^\alpha y^\beta z^\gamma \text{ is not a multiple of any of} \\ &\quad x^5 y^2 z^2, x^2 y^5 z^2, x^2 y^2 z^5\}. \end{aligned}$$

Now [Alo99] implies that every polynomial that is a sum of  $x^2 y^2 z^2$  and a linear combination of monomials in  $M_A$  has a nonzero in  $S$ ; [Las10] allows the same conclusion for sums of  $x^2 y^2 z^2$  and a linear combination of monomials in  $M_L$ , [Nic23] for sums of  $x^2 y^2 z^2$  and a linear combination of monomials in  $M_N$ , and Theorem 2.2 for sums of  $x^2 y^2 z^2$  and a linear combination of monomials in  $M_T$ . We note that  $M_A \subseteq M_L$ ,  $M_A \subseteq M_N$ ,  $M_L \subseteq M_T$ ,  $M_N \subseteq M_T$ , and the inclusion  $M_L \cup M_N \subseteq M_T$  is proper since, e.g.,  $x^4 y^4 z^4 \in M_T \setminus (M_L \cup M_N)$ .

Finally, assuming  $\lambda_1 = \dots = \lambda_n = \lambda$  and observing that  $\deg(x_i^{\lambda_i+1} \mathbf{x}^\alpha) > \lambda + \sum_{i=1}^n \alpha_i$ , Theorem 2.2 allows us to recover Theorem 2.1, and setting  $\lambda_1 = \dots = \lambda_n = 0$ , we recover [Las10, Theorem 2].

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