Repeatedly Applying the Combinatorial Nullstellensatz for Zero-Sum Grids to Martin Gardner's Minimum No-3-in-a-line Problem

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Dear John: 2 June 75

Luckily you had sent me several pages, many years ago, about analysis of the heptominoes that tile and do not tile, so I've been able to summarize. You once said it was okay for me to use things mentioned in these early letters, so I've I called you procedure for checking the "Conway criterion," pictured the heptominoes to which it doesn't apply, identified the tilers, including the one you dubbed the "most interesting," and given as a problem for readers to finding af a tiling for one of the nontilers, combined with mmmmm 3 x 3 squares. My other problem is finding a tiling for one of Penrose's poly-mmmmmmm iamonds, that I call the loaded wheelbarrow (he gave me permission to use." It calls for 8 different orientations to form a fundamental region that tiles by translation.

I am postponing the column on "nonperiodic tiling" until next year, hoping that Penrose will give me permission by then to picture his two tiles. At that time I would very much wish to include your observations of this pattern -- and will do my best to alert you at least two months ahead of my deadline. So far, Penrose has no yet revealed to me the shapes of the tiles

Any chance of mmmmmmm getting a look now at your two maps: the one for the Century puzzle, and for Dad's? I once did a column on sliding blocks, but one of these days I'll do another one, and I'd like to have these items on file with permission to use sometime. Does Knuth have them? I mention it because I think his next volume (on combinatoric algorithms) is going to include some sliding block material.

And OF COURSE I would like to get from you or somebody the new news about the angel problem!

I have no memory at this point of which of my column collections I've sent you, but there is a new paperback edition just out of the last one, The Sixth Book of etc., which contains my old column on sliding blocks. I have today put one in ship mail for you. Incidentally, I hope you don't mind that I have dedicated the seventh volume, Mathematical Carnival, to you, which is coming out this fall by Knopf. I have a few lines in the dedication, which I think you will like, but let me hold this off so there'll be some surprise left.

I enclose a piece about me in Time, occasioned by my April Fool hoax column, which produced about 2,000 letters from readers who didn't know it was a joke.

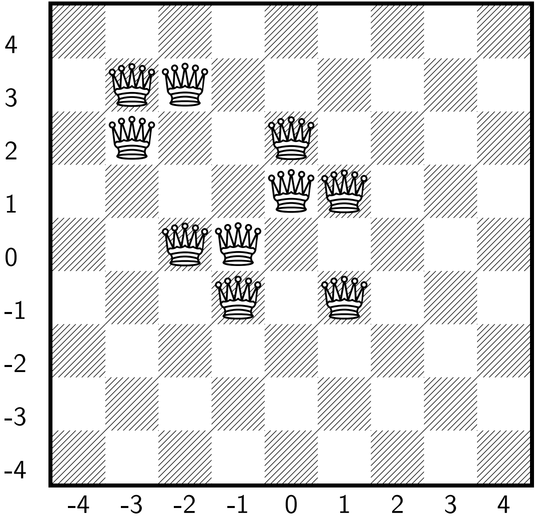
The current (June) column mentions Ulam's game of taking turns putting a counter on an n x n until one person wins by getting 3 in line, orthogonally or diagonally. This suggested to me the following problem, which I believe is new. What is the minimum number of counters that can be put on a square, no 3 in line, such that one more counter produces 3 in line?

I will enclose my best results through n = 12. I haven't been able to prove that my n = 8 (10 counters) is minimal, or to prove to at least n counters are required mmmahh for all n, though I suspect it might not be hard to settle the latter conjecture one way or the other. It would be nice if the minimum for all odd n were N + 1, and for all egen n, Mmmm n or n + 2. If "line" is taken to any straight line on the field, the general problem becomes much more difficult. I note an article on the traditional "no-3-in-line" problem in the May J. of Comb. Theory, on which Guy has done some work. (I recently wrote him about all of this, asking for one of his papers on it, but haven't received a reply yet). He let me see a copy of his paper (forthcoming) on your Sylver coinage problem, about which I also have some comments from you in letters.

If you ever type up anything on Football, such as definitions of terms, etc., please keep me in mind for a copy. As I've said before, I'm holding off on this for a future column, for which SA would make a substantial payment (whether you like this or not), but anything you record about it I'd like to have on file, on a confidential basis. By fall, SA should know whether they've lost money of the six filmstrips I did (my work now completed, but strips not yet on market), which will determine how they will feel about trying a game.

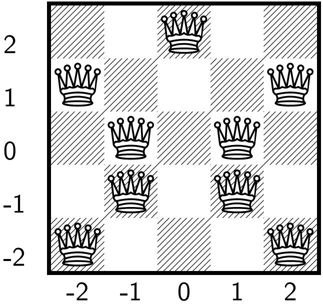
I mistakenly told Penrose in a letter that the game agent here who handled Piet Hein, Tom Atwater, had retired from the game business. It is true that he quite his job with a lwa firm, to become heamdmmf dean of a business school, but I recently learned that he is continuing to act as agent for "select customers." Anyway, he can still be reached at his home, 15 Walden St., fmmmm Concord, Mass 01742. In fact, I'll write him a letter today about Football, and ask him to contact you if he thinks it has market possibilities.

Best,



A maximal placement $\mathcal{Q}$ of minimum size on $\overline{B}_9$

Some queens are lonely



A lonely queen at $(0,2)$ on $\overline{B}_5$

Theorem (Cooper, Pikhurko, S., Warrington - 2014)

For $n \geq 1$, the answer to Gardner's no-3-in-a-line queens version problem is at least n, except in the case when n is congruent to 3 modulo 4, in which case one less may suffice.

- Proof 1 - Combinatorial Nullstellensatz
- Proof 2 - combinatorial for n even (and a bound of $n-1$ for n odd)

OEIS - A219760

n	1	2	3	4	5	6	7	8	9	10
$m_3(n)$	1	4	4	4	6	6	8	9	10	10
n	11	12	13	14	15	16	17	18	19	20
$m_3(n)$	12	12	14	15	16	17	18	18	20	21
n	21	22	23	24	25	26	27	28	29	30
$m_3(n)$	22	23	24	25	26	26	28	29	30	?

Table: $m_3(n)$ for all values of n where it is known precisely

- Cooper, Pikhurko, S., Warrington - and brute-force search for $n = 8, \ldots, 11$
- Rob Pratt - and Integer Linear Programming for $n = 13, \ldots, 25, 27$
- Don Knuth - and SAT solver for $n = 26$
- Andy Huchala - $n = 28, 29$

Combinatorial Nullstellensatz

Theorem (Alon - 1999)

Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.

Combinatorial Nullstellensatz for Zero-sum Grids

Theorem (B. Nica - 2023)

Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $1 + \sum_{i=1}^{n} t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are zero-sum subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.

Theorem (Oh, S., Wang - 2025)

For n congruent to 1 modulo 4 and $n \geq 5$, we have $m_3(n) \geq n+1$.

Proof:

Let $n = 4k + 1$ and $k \geq 1$.
Let $\mathcal{Q}$ be a maximal no-3-in-a-line placement on $\overline{B}_n$ with size $q = |\mathcal{Q}| \leq 4k + 1$.
Let $\mathcal{Q}'$ denote the (possibly empty) subset of lonely queens in $\mathcal{Q}$.
Let $|\mathcal{Q}'| = q'$.
We work towards a contradiction.
Collinear queens will define a line.

Case 1: all cases where the placement defines at most $8k$ lines or all cases where Alon's original CN suffices

Case 1: $q' \neq 1$, $q < 4k+1$, or $q' = 1$ and some queen(s) that is not lonely participates in defining fewer than 4 lines.

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$$f(x,y) = \prod_{j=1}^{2k} (x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

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- $f(x,y) = 0$ on $\overline{B}_n$

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Apply Alon's CN $t_1 = t_2 = 4k$ and $S_1 = S_2 = \{-2k, \ldots, 2k\}$, we find $s_1 \in S_1, s_2 \in S_2$ for which $f(s_1, s_2) \neq 0$, a contradiction.

Case 2: all cases where the placement defines strictly more than $8k$ lines or all cases where Alon's CN does NOT suffice

Case 2: $q = 4k + 1$, $q' = 1$ and all other queens participate in defining a line of each of the four possible slopes.

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A vertical line through the lonely queen who has coordinates (α_0, β_0):

$$f_1(x, y) = (x - \alpha_0) \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

$$f_1|_{x^{4k}y^{4k}} \quad = \quad (-1)^k \binom{2k}{k} \sum_{j=0}^{2k} -\alpha_j + (-1)^k \binom{2k-1}{k-1} \sum_{j=1}^{2k} -\gamma_j + (-1)^k \binom{2k-1}{k-1} \sum_{j=1}^{2k} -\delta_j.$$

If this coefficient is non-zero, then by CNZS we are done. Thus, assume it is zero.

Case 2: $q = 4k + 1$, $q' = 1$ and all other queens participate in defining a line of each of the four possible slopes.

A horizontal line through the lonely queen who has coordinates (α_0, β_0):

$$f_2(x, y) = (y - \beta_0) \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

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If this coefficient is non-zero, then by CNZS we are done. Thus, assume it is zero.

Case 2: $q = 4k+1$, $q' = 1$ and all other queens participate in defining a line of each of the four possible slopes.

A line of slope $+1$ through the lonely queen who has coordinates (α_0, β_0):
Let

$$f_3(x,y) = (x - y - \gamma_0)\prod_{j=1}^{2k}(x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

$$f_3|_{x^{4k}y^{4k}} \quad = \quad (-1)^k\binom{2k}{k}\sum_{j=1}^{2k} -\alpha_j + (-1)^{k+1}\binom{2k}{k}\sum_{j=1}^{2k} -\beta_j + (-1)^k\binom{2k}{k}\sum_{j=0}^{2k} -\gamma_j.$$

If this coefficient is non-zero, then by CNZS we are done. Thus, assume it is zero.

Case 2: $q = 4k+1$, $q' = 1$ and all other queens participate in defining a line of each of the four possible slopes.

A line of slope -1 through the lonely queen who has coordinates (α_0, β_0):

$$f_4(x,y) = (x+y-\delta_0)\prod_{j=1}^{2k}(x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

$$f_4|_{x^{4k}y^{4k}} = (-1)^k\binom{2k}{k}\sum_{j=1}^{2k}-\alpha_j + (-1)^k\binom{2k}{k}\sum_{j=1}^{2k}-\beta_j + (-1)^k\binom{2k}{k}\sum_{j=0}^{2k}-\delta_j.$$

If this coefficient is non-zero, then by CNZS we are done. Thus, assume it is zero.

Four more equations from simple geometry

Consider the lonely queen located on square (α_0, β_0): the values of α_0 and β_0 determine the values of γ_0 and δ_0 as follows. The line of slope $+1$ that goes through the square (α_0, β_0) has equation

$$y - \beta_0 = 1(x - \alpha_0), \quad x - y - (\alpha_0 - \beta_0) = 0$$

and the line of slope -1 that goes through the square (α_0, β_0) has equation

$$y - \beta_0 = -1(x - \alpha_0), \quad x + y - (\alpha_0 + \beta_0) = 0.$$

As a result, we have

$$\alpha_0 - \beta_0 - \gamma_0 = 0,$$
$$\alpha_0 + \beta_0 - \delta_0 = 0.$$

Four more equations from simple geometry

For any non-lonely queen, there exist an $\alpha \in \{\alpha_1, \ldots, \alpha_{2k}\}$ and $\beta \in \{\beta_1, \ldots, \beta_{2k}\}$ that give her coordinates. The line of slope $+1$ that goes through the square (α, β) has equation

$$y - \beta = 1(x - \alpha), \quad x - y - (\alpha - \beta) = 0$$

and the line of slope -1 that goes through the square (α, β) has equation

$$y - \beta = -1(x - \alpha), \quad x + y - (\alpha + \beta) = 0.$$

Four more equations from simple geometry

As a result, we have $\gamma = \alpha - \beta$ for some $\gamma \in \{\gamma_1, \ldots, \gamma_{2k}\}$ and $\delta = \alpha + \beta$ for some $\delta \in \{\delta_1, \ldots, \delta_{2k}\}$. As each such diagonal line is defined by two queens, upon considering the $4k$ equations deriving from the -1-slope lines each element of $\{\gamma_1, \ldots, \gamma_{2k}\}$ occurs twice in this set of equations; similarly, in the $4k$ equations deriving from the $+1$-slope lines each element of $\{\delta_1, \ldots, \delta_{2k}\}$ occurs twice. Thus, we can write the following:

Four more equations from simple geometry

$$\sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}'} (\alpha - \beta) = 2 \sum_{i=1}^{2k} \gamma_i, \tag{1}$$

and

$$\sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}'} (\alpha + \beta) = 2 \sum_{i=1}^{2k} \delta_i. \tag{2}$$

The restrictions of Case 2 give that each queen in $\mathcal{Q} \setminus \mathcal{Q}'$ is contained in both a vertical line and a horizontal line. As a result, each $\alpha \in \{\alpha_1, \ldots, \alpha_{2k}\}$ and each $\beta \in \{\beta_1, \ldots, \beta_{2k}\}$ appears twice on the left side of each of Equation 1 and Equation 2. This enables us to rewrite the left side of Equations 1 and 2 to obtain

Four more equations from simple geometry

$$\sum_{i=1}^{2k} \alpha_i - \sum_{i=1}^{2k} \beta_i - \sum_{i=1}^{2k} \gamma_i = 0,$$

and

$$\sum_{i=1}^{2k} \alpha_i + \sum_{i=1}^{2k} \beta_i - \sum_{i=1}^{2k} \delta_i = 0.$$

A linear system

A homogeneous system of eight linear equations in the variables $\alpha_0, \beta_0, \gamma_0, \delta_0, \sum_{j=1}^{2k} \alpha_j, \sum_{j=1}^{2k} \beta_j, \sum_{j=1}^{2k} \gamma_j, \sum_{j=1}^{2k} \delta_j$. For notation's convenience we set $\omega := (-1)^k \binom{2k}{k}$ and express these equations using the following augmented matrix.

$$
\begin{array}{c}
\qquad\qquad \alpha_0 \quad \beta_0 \quad \gamma_0 \quad \delta_0 \quad \sum_{j=1}^{2k}\alpha_j \quad \sum_{j=1}^{2k}\beta_j \quad \sum_{j=1}^{2k}\gamma_j \quad \sum_{j=1}^{2k}\delta_j \\
[\mathbf{A} \;\; \mathbf{0}] = \begin{bmatrix}
\omega & 0 & 0 & 0 & \omega & 0 & \omega/2 & \omega/2 & 0 \\
0 & \omega & 0 & 0 & 0 & \omega & \omega/2 & \omega/2 & 0 \\
0 & 0 & \omega & 0 & \omega & -\omega & \omega & 0 & 0 \\
0 & 0 & 0 & \omega & \omega & \omega & 0 & \omega & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

Thus,

$$\alpha_0 = 1s, \beta_0 = 1s, \gamma_0 = 0, \delta_0 = 2s$$

$$\sum_{j=1}^{2k} \alpha_j = -s/2, \sum_{j=1}^{2k} \beta_j = -s/2, \sum_{j=1}^{2k} \gamma_j = 0, \sum_{j=1}^{2k} \delta_j = -1s,$$

where s is an integer.

Thus, the lonely queen is on the 'back diagonal'.

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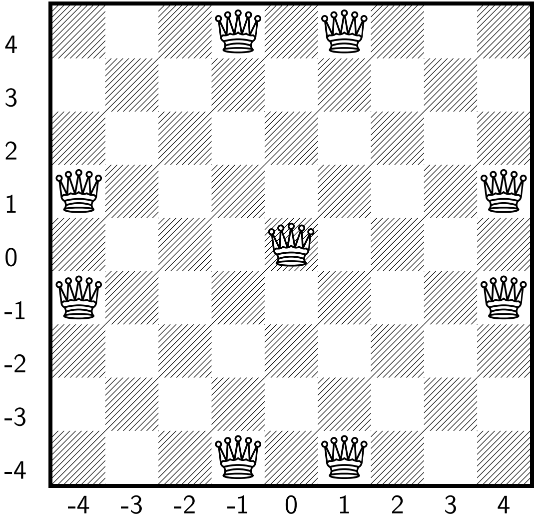
where s is an integer.

Thus, the lonely queen is on the 'back diagonal'. Rotate the placement 90-degrees and repeat the argument: the lonely queen is centered on the board, i.e. $s = 0$.

$$\alpha_0, \beta_0, \gamma_0, \delta_0, \sum_{j=1}^{2k} \alpha_j, \sum_{j=1}^{2k} \beta_j, \sum_{j=1}^{2k} \gamma_j, \sum_{j=1}^{2k} \delta_j = 0.$$

A placement in the null space of A



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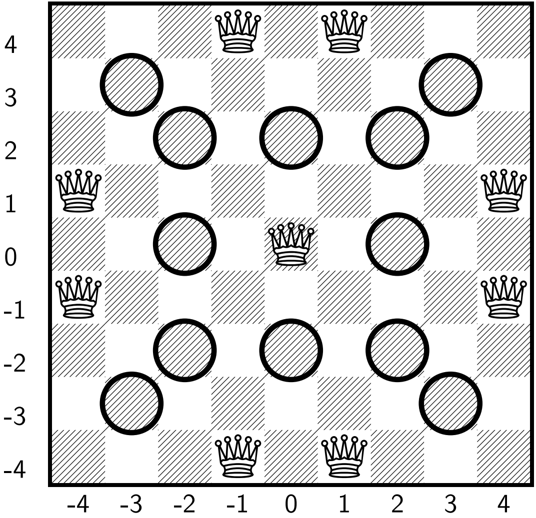
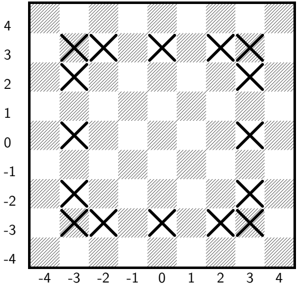
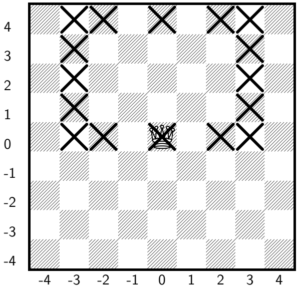


Consider the perimeter of box uncovered by orthogonals



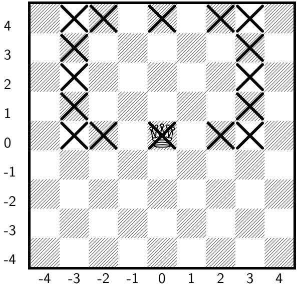
$2k + 1$ squares in each of left-most, right-most columns, and top-most and bottom-most rows - a total of $8k$ squares, each to be covered by a diagonal of which there are $4k$

Consider the perimeter of box uncovered by orthogonals



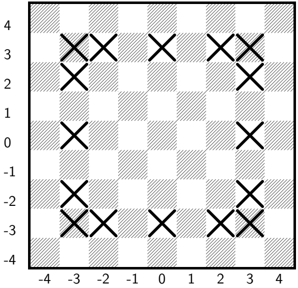
Perimeter can't contain lonely queen: we have $\sum_{j=1}^{2k} \alpha_j = 0$ and $\sum_{j=1}^{2k} \beta_j = 0$

Consider the perimeter of box uncovered by orthogonals



Perimeter must be square: we have $4k + 2$ squares on left and right to cover with only $4k$ diagonals and if 'far apart' each diagonal only covers one of these squares

Consider the perimeter of box uncovered by orthogonals



Finally, with $8k$ squares to cover each of $4k$ diagonals must cover two. Long diagonals exist. -1-diagonals leave an even number of uncovered squares above forward-diagonal. Implies number of $+1$-diagonals is 1 plus an even, which is an odd number of $+1$-diagonals - a contradiction.

Open Problems

1. Prove that $m_3(n) \geq n+1$ when $n \equiv 3 \pmod 4)$
2. Give a non-trivial upper bound on $m_3(n)$ for all n
   1. Idea 1: Ideas from Don Knuth
      I do believe there must be a way to place n+1 queens on an n-by-n board in all cases, and there probably is an explicit construction that will do it in infinitely many cases.
   2. Idea 2: Add queens to placements in *NulA* - I can give an explicit description of these
3. The problem in general, not just queen slopes.
4. Other applications of Nica's Structured Nullstellensatz?

Combinatorial Nullstellensatz for λ-null sets

Theorem (B. Nica - 2023)

Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $\lambda + \sum_{i=1}^n t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are λ-null finite subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.

For S a finite subset of a field, the *characteristic polynomial* of S is given by

$$\prod_S(t) := \prod_{s \in S}(t - s).$$

Let $\lambda \in \{0, \ldots, |S|\}$. We say that S is λ-**null** if, in the characteristic polynomial, the coefficients of $t^{|S|-1}, \ldots, t^{|S|-\lambda}$ vanish.

Thank you!!