Gröbner Bases

Exercise Sheet 9 for December 3rd, 2024

(1) Compute the greatest common divisor of $f = x^4y + x^3y^2 - 2x^2y^2 - 2xy^3 + x + y$ and $g = x^4y - x^3y^3 - 2x^2y^2 + 2xy^4 + x - y^2$ in $\mathbb{Q}[x, y]$ by computing the intersection of $\langle f\rangle_{\mathbb{Q}[x,y]} \cap \langle g\rangle_{\mathbb{Q}[x,y]}$. *Remark:* Use a computer algebra system. Note that in a UFD, we have $(a) \cap (b) = (\operatorname{lcm}(a, b))$ and $\gcd(a, b) = ab/\operatorname{lcm}(a, b)$ for $a, b \neq 0$.

(2) For an ideal I of $k[x_1, \ldots, x_n]$ and $f \in k[x_1, \ldots, x_n]$, we define the *ideal quotient* $(I : f)$ by

$$(I : f) = \{g \in k[\boldsymbol{x}] \mid gf \in I\}.$$

(a) Show that $(I : f) = \{\frac{h}{f} \mid h \in I \cap \langle f\rangle_{k[\boldsymbol{x}]}\}$.

(b) Let $f = x^2$ and $I = \langle x^7y^2, x^9y + 2x^8y\rangle_{k[\boldsymbol{x}]}$. Compute $(I : (f^n))$ for each $n \in \mathbb{N}$.

(3) Find $f \in \mathbb{Q}[t_1, t_2]$ with $f \neq 0$ such that

(a) $f(x^3, \frac{1}{x^9-1}) = 0$.

(b) $f(\frac{y^3z^7+3y^2z^5+3yz^3+z^4+z}{(yz^2+1)^4}, \frac{z^2}{(yz^2+1)^2}) = 0$.

(4) Let $R = \mathbb{Q}[\![x^2 + 1, x^4 + 2]\!] = \{p(x^2 + 1, x^4 + 2) \mid p \in \mathbb{Q}[t_1, t_2]\}$.

(a) Show that R is isomorphic to $\mathbb{Q}[t_1, t_2]/I$, where $I = \{p \in \mathbb{Q}[t_1, t_2] \mid p(x^2 + 1, x^4 + 2) = 0\}$.

(b) Compute this ideal I, and find an isomorphism φ from $\mathbb{Q}[t_1, t_2]/I$ to R.

(5) (a) Find a solution of $6a+9b+20c = 53$ in $\mathbb{N}_0^3$ by finding a polynomial $p(t_1, t_2, t_3, t_4) = t_1 - t_2^a t_3^b t_4^c$ with $p(x^{53}, x^6, x^9, x^{20}) = 0$.

(b) Find the gcd of 147 and 33 and the cofactors by finding a polynomial $p(t_1, t_2, t_3, t_4, t_5) = t_1^d - t_2^{u_1} t_3^{u_2} t_4^{v_1} t_5^{v_2}$ such that $p(x^1, x^{147}, \frac{1}{x^{147}}, x^{33}, \frac{1}{x^{33}}) = 0$ with minimal nonzero d.