Gröbner Bases
Exercise Sheet 7 for November 21, 2024

(1) Let I be the ideal of $\mathbb{Q}[x,y]$ generated by $\{x+y^2+2, xy+3\}$. Find generators of the ideals $I \cap \mathbb{Q}[x]$ and $I \cap \mathbb{Q}[y]$ of $\mathbb{Q}[x]$ and $\mathbb{Q}[y]$, respectively.

(2) Let $A = \{x_1+2x_2+2x_3+2x_4-15, -2x_1-4x_2+x_3+11x_4-20, -4x_1-8x_2+2x_3+22x_4-40, x_1+2x_2+5x_3+11x_4-45\}$.

(a) Compute a reduced Gröbner basis for the ideal of $\mathbb{Q}[x_1,x_2,x_3,x_4]$ that is generated by A. (Lexicographic ordering, $x_1 > x_2 > x_3 > x_4$.)

(b) Compute a basis for the linear subspace of $\mathbb{Q}^4$ that is generated by the rows of the matrix

$$B = \begin{pmatrix} 1 & 2 & 2 & 2 & -15 \\ -2 & -4 & 1 & 11 & -20 \\ -4 & -8 & 2 & 22 & -40 \\ 1 & 2 & 5 & 11 & -45 \end{pmatrix}.$$

(3) Compute the reduced Gröbnerbasis for the ideal of $\mathbb{Q}[x]$ that is generated by $\{x^2-x-2, x^3+x^2-6x\}$.

(4) Let $F \subseteq k[\boldsymbol{x}] \setminus \{0\}$, and let $f, g \in F$ be such that $f \neq g$, $\textsc{Lc}(f) = \textsc{Lc}(g) = 1$ and $\textsc{Lm}(g) \mid \textsc{Lm}(f)$. Which of the following statements must be true?

(a) If r is a remainder in a standard expression of f by $F \setminus \{f\}$, then r is a remainder in a standard expression of $S(f,g)$ by F.

(b) If r is a remainder in a standard expression of $S(f,g)$ by F, then r is a remainder in a standard expression of f by $F \setminus \{f\}$.

(5) Let F be a finite subset of $\mathbb{Q}[x_1,\dots,x_n]$, and let $f \in \mathbb{Q}[x_1,\dots,x_n]$ be an element of the ideal J that F generates in $\mathbb{C}[x_1,\dots,x_n]$. Show that f is also an element of the ideal I that F generates in $\mathbb{Q}[x_1,\dots,x_n]$.