Gröbner Bases
Exercise Sheet 5 for November 5, 2024

(1) Let $f, g \in k[x_1, \ldots, x_n] \setminus \{0\}$ with $\mathrm{LC}(f) = \mathrm{LC}(g) = 1$ and $\gcd(\mathrm{LM}(f), \mathrm{LM}(g)) = 1$. (The latter condition means that $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ have no common variables.)
   (a) Show $\mathrm{DEG}((g - \mathrm{LT}(g))\, f) \neq \mathrm{DEG}((f - \mathrm{LT}(f))\, g)$.
   (b) Compute $S(f, g)$ and show that $S(f, g)$ has a standard expression by $\{f, g\}$ with remainder 0.

(2) Let k be a field, and let S be a finite subset of k^n. We define $I(S) := \{f \in k[x_1, \ldots, x_n] \mid f(\boldsymbol{s}) = 0 \text{ for all } \boldsymbol{s} \in S\}$.
   (a) Show that $I(S)$ is an ideal of $k[x_1, \ldots, x_n]$.
   (b) Show that $\dim_k(k[x_1, \ldots, x_n]/I(S)) \leq |S|$. *Hint:* Consider $\Phi : k[\boldsymbol{x}] \to k^S, f \mapsto f|_S$.
   (c) Must $\dim_k(k[x_1, \ldots, x_n]/I(S))$ be equal to $|S|$?

(3) Exercise 11.10 (2).

(4) Exercise 11.10 (3).

(5) Exercise 11.10 (7).