Gröbner Bases

Exercise Sheet 4 for October 29, 2024

(1) The following algorithm computes a standard expression.

Input: $f \in k[\boldsymbol{x}] \setminus \{0\}$, $\{f_1, \ldots, f_s\} \subseteq k[\boldsymbol{x}] \setminus \{0\}$, an admissible order $\leq$ of ${\mathbb{N}_0}^n$.
Ouput: a standard expression $f = \sum_{i=1}^{s} a_i f_i + r$ of f by $(f_1, \ldots, f_s)$

```
r ← f; (a_1, ..., a_s) := 0.
while r ≠ 0 and ∃i ∈ s̲ : LM(f_i) divides a term a x^α that appears in r do
    r ← r − (a x^α / LT(f_i)) f_i.
    a_i := a_i + a x^α / LT(f_i).
Return (a_1, ..., a_s, r)
```

(a) Show that the algorithm terminates, regardless which monomial $a\boldsymbol{x}^\alpha$ is chosen at each step.

(b) Why does the algorithm produce a correct result? Which equation between $a_1, \ldots, a_s, r, f$ is true invariantly throughout the execution of the algorithm?

(2) Let $U \in \mathbb{R}^{n' \times n}$ be such that $\{\gamma \in \mathbb{Q}^n \mid U\gamma = 0\} = \{0\}$ and the first nonzero entry in every column of U is positive. Show that the order $\leq_U$ defined by

$$\alpha \leq_U \beta :\Leftrightarrow U\alpha \leq_{\text{lex}} U\beta$$

is admissible. *Remark:* The crucial point is to prove $U\alpha >_{\text{lex}} 0$ for each $\alpha \in {\mathbb{N}_0}^n$.

(3) Find the leading term ideal $\langle \mathrm{LT}(I) \rangle_{k[\boldsymbol{x}]}$ for the following ideals I.

(a) $I = \langle x^5 + 5x^4 - x^3 - 15x^2 - 6x,\ x^7 - 3x^5 \rangle_{\mathbb{Q}[x]}$ in the univariate polynomial ring $\mathbb{Q}[x]$.

(b) $I = \{f \in \mathbb{Q}[x, y] \mid f(-1, 2) = 0\}$, lexicographic ordering, $x > y$.

(4) Let k be a field , and let $F = \{f_1, \ldots, f_s\} \subseteq k[x_1, \ldots, x_n] \setminus \{0\}$.

(a) Show the following statment: If F is a Groebner basis of $\langle F \rangle$ and

$$\mathrm{LT}(f_i) \in \langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_{i-1}), \mathrm{LT}(f_{i+1}), \ldots, \mathrm{LT}(f_s) \rangle,$$

then $F \setminus \{f_i\}$ is a Groebner basis of $\langle F \rangle$, too.

(b) Is this assertion still valid if one replaces the words "Groebner basis" both times by "basis"?