Gröbner Bases

Exercise Sheet 13 for January 21st, 2025

We fix an admissible ordering $\leq$ on ${\mathbb{N}_0}^n$.

(1) Let R be a commutative ring with 1, and let I be an ideal of $R[\boldsymbol{x}]$. We say that G is a *strong Gröbner basis* of I if for every $f \in I$, there is a $g \in G$ with $\mathrm{LT}(g) \mid \mathrm{LT}(f)$. (Here we say that $r\boldsymbol{x}^\alpha$ *divides* $s\boldsymbol{x}^\beta$ if there are $t \in R, \gamma \in {\mathbb{N}_0}^n$ with $r\boldsymbol{x}^\alpha \cdot t\boldsymbol{x}^\gamma = s\boldsymbol{x}^\beta$.) Show that the ideal of R that is generated by G is equal to I.

(2) Let R be a commutative ring with 1, and suppose that every ideal of $R[\boldsymbol{x}]$ has a finite strong Gröbner basis.

  (a) Show that every ideal J of R is a finite union of principal ideals of R, i.e., there are $c_1, \ldots, c_m \in R$ with $\bigcup_{i=1}^m (c_i) = J$.

  (b) Give an example of a commutative ring R with 1 in which every ideal is a finite union of principal ideals, but R contains an ideal that is not principal.

  (c) * (optional and maybe hard; I do not know the answer) Is there a domain R in which every ideal is a finite union of principal ideals, but R contains an ideal that is not principal?

For the next problems, let R be a commutative ring with 1, and let I be an ideal of $R[\boldsymbol{x}]$. We define a mapping $C : {\mathbb{N}_0}^n \to \mathcal{P}(R)$ (the power set of R) by

$$C(\alpha) := \{r \in R \mid r = 0 \vee \exists p \in I : \mathrm{LT}(p) = r\boldsymbol{x}^\alpha\}.$$

(3) (a) Show that $C(\alpha)$ is an ideal of R.

  (b) Show that $\alpha \sqsubseteq \beta$ implies that $C(\alpha) \subseteq C(\beta)$.

We now define an ordering $\sqsubseteq_C$ on ${\mathbb{N}_0}^n$ by

$$\alpha \sqsubseteq_C \beta :\Leftrightarrow \alpha \sqsubseteq \beta \text{ and } C(\alpha) = C(\beta),$$

and we assume that R is noetherian (i.e., has no infinite ascending chain of ideals).

(4) (a) Show that there is no infinite descending chain $\alpha_1 \sqsupset_C \alpha_2 \sqsupset_C \cdots$.

  (b) Show that there is no infinite antichain in ${\mathbb{N}_0}^n$ with respect to $\sqsubseteq_C$.

  (c) Conclude that every subset of ${\mathbb{N}_0}^n$ has only finitely many minimal elements with respect to $\sqsubseteq_C$.

(5) Suppose that every ideal of R is the union of finitely many principal ideals. For each minimal element α in $({\mathbb{N}_0}^n, \sqsubseteq_C)$, choose $c_1, \ldots, c_{r_\alpha} \in R \setminus \{0\}$ with $C(\alpha) = \bigcup_{i=1}^{r_\alpha} (c_i)$, and then choose $p_i \in I$ such that $\mathrm{LT}(p_i) = c_i\boldsymbol{x}^\alpha$. Show that the union of these $\{p_1, \ldots, p_{r_\alpha}\}$ is a strong Gröbner basis of I.