Gröbner Bases

Exercise Sheet 10 for December 10^{th}, 2024

(1) Let $I := \langle x_1^3, x_2^2 \rangle$, and let $R := \mathbb{Q}[x_1, x_2]/I$.

(a) Find an ideal J of $\mathbb{Q}[t_1, t_2, t_3]$ such that $\mathbb{Q}[\boldsymbol{t}]/J$ is isomorphic to R, and an isomorphism φ with $\varphi(t_1{+}J) = (x_1{+}x_2){+}I$, $\varphi(t_2{+}J) = x_1{+}I$, $\varphi(t_3{+}J) = x_2{+}I$.

(b) Find a polynomial witnessing that $x_1 + x_2 + I$ is algebraic over $\mathbb{Q}$.

(2) Let $R := \mathbb{Q}[x_1, x_2]/I$ with $I := \langle x_1^6 - 2, x_1^4 - x_2 \rangle$. Show that $x_1 + x_1^2 + I$ is algebraic over $\mathbb{Q}' := \{q + I \mid q \in \mathbb{Q}\}$ by exhibiting a polynomial $f \in \mathbb{Q}[t] \setminus \{0\}$ with $f(x_1 + x_1^2 + I) = 0 + I$.

(3) Let

$$\begin{aligned} f &:= x^3 + 3 \\ g &:= x^6 + 6x^3 + 10 \\ h &:= x^9 + 9x^6 + 26x^3 + 26 \end{aligned}$$

Determine whether $f \in \mathbb{Q}[\![g, h]\!]$, $f \in \mathbb{Q}(\!(g, h)\!)$, $h \in \mathbb{Q}[\![f]\!]$.

(4) Let

$$f = \left(\left(x^2 + y\right)^2 + 2\right)^3,$$
$$g = \left(\left(x^2 + y\right)^2 + 1\right)^2$$

be polynomials over $\mathbb{Q}$. For each $h \in \{x, x^2 + y, (x^2 + y)^2\}$, find an ideal I such that $\mathbb{Q}[t_1, t_2, t_3]/I$ is isomorphic to $\mathbb{Q}[\![h, f, g]\!]$ and answer the following questions:

(a) Is h an element of $\mathbb{Q}[\![f, g]\!]$? How can h be expressed as $p(f, g)$?

(b) Is h integral[1] over $\mathbb{Q}[\![f, g]\!]$? In this case, find a polynomial in $\mathbb{Q}[\![f, g]\!][t]$ witnessing this fact.

(c) Is h algebraic over $\mathbb{Q}[\![f, g]\!]$? In this case, find a polynomial in $\mathbb{Q}[\![f, g]\!][t]$ witnessing this fact.

(d) Do we have $h \in \mathbb{Q}(\!(f, g)\!)$? In this case, find polynomials $p, q \in \mathbb{Q}[X, Y]$ with $h = p(f, g)/q(f, g)$.

(5) * [1] Let $a, b, q, r, f \in \mathbb{Q}[x]$ such that $\deg(f) > 0$, $b \neq 0$, $a = q \cdot b + r$ and $\deg(r) < \deg(b)$. Suppose that $a, b \in \mathbb{Q}[\![f]\!]$. Show that q and r are elements of $\mathbb{Q}[\![f]\!]$. *Hint:* Write $a = a'(f)$, $b = b'(f)$ and compare the divisions $a : b$ and $a' : b'$.

References

[1] H. T. Engstrom. Polynomial substitutions. *Amer. J. Math.*, 63:249–255, 1941.

[1] An element b is *integral* over A if there is $p \in A[t]$ with $\mathrm{LC}(p) = 1$ and $p(b) = 0$. Some authors prefer *entire*, and the German translation is “ganz”.