Gröbner Bases

Exercise Sheet 9 for December 16, 2020

(1) (cf. [1])

(a) *Whitney's umbrella* is defined by the parametrization $x = uv, y = v, z = u^2$. Find an equation of the form $p(x, y, z) = 0$ with $p \neq 0$ that is satisfied by all points of this surface.

(b) Find a (nontrivial) equation satisfied by all points in the plane on the *Folium of Descartes* parametrized by $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$. *Hint:* Find p with $p(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}) = 0$.

(2) Let $I := (x_1^2 - 3, x_2^2 - 2)$, and let $R := \mathbb{Q}[x_1, x_2]/I$.

(a) Find an Ideal J of $\mathbb{Q}[t_1, t_2, t_3]$ such that $\mathbb{Q}[\boldsymbol{t}]/J$ is isomorphic to R, and an isomorphism φ with $\varphi(t_1+J) = (x_1+x_2)+I$, $\varphi(t_2+J) = x_1+I$, $\varphi(t_3+J) = x_2+I$.

(b) Find an ideal K of $\mathbb{Q}[s_1]$ such that $\mathbb{Q}[s_1]/K$ is isomorphic to the subring of $\mathbb{Q}[\boldsymbol{t}]/J$ generated by $t_1 + J$ via an isomorphism ψ with $\psi(t_1 + J) = s_1 + K$.

(c) Find a polynomial witnessing that $x_1 + x_2 + I$ is integral over $\mathbb{Q}$.

(3) Let $R = \mathbb{Q}[t^5, t^7]$. Find polynomials of minimal degrees that witness that t is algebraic and integral over R.

(4) Find a solution of $6a + 9b + 20c = 53$ in $\mathbb{N}_0^3$ by finding a polynomial $p(t_1, t_2, t_3, t_4) = t_1 - t_2^a t_3^b t_4^c$ with $p(x^{53}, x^6, x^9, x^{20}) = 0$.

(5) Find the gcd of 147 and 33 and the cofactors by finding a polynomial $p(t_1, t_2, t_3, t_4, t_5) = t_1^d - t_2^{u_1} t_3^{u_2} t_4^{v_1} t_5^{v_2}$ such that $p(x^1, x^{147}, \frac{1}{x^{147}}, x^{33}, \frac{1}{x^{33}}) = 0$ with minimal nonzero d.

References

[1] David Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992. An introduction to computational algebraic geometry and commutative algebra.