Gröbner Bases

Exercise Sheet 8 for December 9, 2020

(1) For an ideal I of $k[x_1,\dots,x_n]$ and $f \in k[x_1,\dots,x_n]$, we define the *ideal quotient* $(I:f)$ by

$$(I:f) = \{g \in k[\boldsymbol{x}] \mid gf \in I\}.$$

   (a) Show that $(I:f) = \{\frac{h}{f} \mid h \in I \cap \langle f\rangle_{k[\boldsymbol{x}]}\}$.

   (b) Let $f = x^2$ and $I = \langle x^7y^2, x^9y + 2x^8y\rangle_{k[\boldsymbol{x}]}$. Compute $(I:(f^n))$ for each $n \in \mathbb{N}$.

(2) For an ideal I of $k[x_1,\dots,x_n]$ and $f \in k[x_1,\dots,x_n]$, the *saturation* $(I:f^\infty)$ of I with respect to f is defined by

$$(I:f^\infty) = \bigcup_{n\in\mathbb{N}} (I:f^n).$$

   (a) Show that $(I:f^\infty) = \langle I \cup \{fy-1\}\rangle_{k[\boldsymbol{x},y]} \cap k[\boldsymbol{x}]$.

   (b) Use this fact to compute $(I:f^\infty)$ for $f = x^2$ and $I = \langle x^7y^2, x^9y + 2x^8y\rangle_{k[\boldsymbol{x}]}$.

(3) Let $R = \mathbb{Q}[\![x^2+1, x^4+2]\!] = \{p(x^2+1, x^4+2) \mid p \in \mathbb{Q}[t_1,t_2]\}$.

   (a) Show that R is isomorphic to $\mathbb{Q}[t_1,t_2]/I$, where $I = \{p \in \mathbb{Q}[t_1,t_2] \mid p(x^2+1, x^4+2) = 0\}$.

   (b) Compute this ideal I, and find an isomorphism φ from $\mathbb{Q}[t_1,t_2]/I$ to R.

(4) We consider the ring $\mathbb{Q}[x,y,z]/I$ with $I = \langle xz, yz\rangle$.

   (a) Find $f \in \mathbb{Q}[t_1,t_2,t_3]$ such that $f \neq 0$ and $f(x,y,z^3+x+1) \in \langle x,y\rangle$.

   (b) Find $g \in \mathbb{Q}[t_1,t_2,t_3]$ with $g \neq 0$ und $g(x,y,z^3+x+1) \in \langle z\rangle$.

   (c) Find $h \in \mathbb{Q}[t_1,t_2,t_3]$ mit $h \neq 0$ und $h(x,y,z^3+x+1) \in I_1 = \langle z\rangle \cap \langle x,y\rangle$.

(5) Find $f \in \mathbb{Q}[t_1,t_2]$ with $f \neq 0$ such that

   (a) $f(x^3, \frac{1}{x^9-1}) = 0$.

   (b) $f(\frac{y^3z^7+3y^2z^5+3yz^3+z^4+z}{(yz^2+1)^4}, \frac{z^2}{(yz^2+1)^2}) = 0$.