Gröbner Bases

Exercise Sheet 7 for December 2, 2020

(1) (Automated Geometry Theorem Proving) Thales's Theorem states that in every triangle, the circumcentre lies on on one of the sides if and only if the triangle is right-angled. Give a formulation of this theorem in terms of polynomial equations, and use these equations to prove Thales's Theorem.

(2) (Automated Geometry Theorem Proving) We consider Desargues's Theorem:

Let $S, A, B, C, D, E, F, H, I, J$ points of the plane $\mathbb{R}^2$ such that

- (a) S, A, D are collinear.
- (b) S, B, E are collinear.
- (c) S, C, F are collinear.
- (d) A, B, H are collinear.
- (e) D, E, H are collinear.
- (f) A, C, J are collinear.
- (g) D, F, J are collinear.
- (h) B, C, I are collinear.
- (i) E, F, I are collinear.
- (j) E, A, D are not collinear.
- (k) F, A, D are not collinear
- (l) F, B, E are not collinear
- (m) C, A, D are not collinear

Then H, I, J are collinear.

(a) Make a good drawing to explain the content of the theorem.

(b) Prove the theorem using a Gröbner bases computation. *Remark:* Use a computer algebra system.

(3) Let $f, g, h, i \in \mathbb{C}[x_1, \ldots, x_n]$. For each of the following formulae, give a system of polynomial equations whose solvability/non-solvability is equivalent to the formula.

(a) $\forall x_1, \ldots, x_n \in \mathbb{C} : (f(x_1, \ldots, x_n) = 0 \wedge g(x_1, \ldots, x_n) \neq 0) \Rightarrow h(x_1, \ldots, x_n) = 0$.

(b) $\forall x_1, \ldots, x_n \in \mathbb{C} : (f(x_1, \ldots, x_n) = 0 \wedge g(x_1, \ldots, x_n)) = 0) \Rightarrow (h(x_1, \ldots, x_n) \neq 0 \wedge i(x_1, \ldots, x_n) \neq 0)$.

(c) $\forall x_1, \ldots, x_n \in \mathbb{C}$: $f(x_1, \ldots, x_n) = 0 \Rightarrow g(x_1, \ldots, x_n) \neq 0$.

(d) $\forall x_1, \ldots, x_n \in \mathbb{C}$: $f(x_1, \ldots, x_n) \neq 0 \Rightarrow g(x_1, \ldots, x_n) \neq 0$.

(4) Compute the greatest common divisor of $f = x^4y + x^3y^2 - 2x^2y^2 - 2xy^3 + x + y$ and $g = x^4y - x^3y^3 - 2x^2y^2 + 2xy^4 + x - y^2$ in $\mathbb{Q}[x, y]$ by computing the intersection of $\langle f \rangle_{\mathbb{Q}[x,y]} \cap \langle g \rangle_{\mathbb{Q}[x,y]}$. *Remark:* Use a computer algebra system.