Gröbner Bases

Exercise Sheet 6 for November 25, 2020

(1) Let $A = \{x_1 + 2x_2 + 2x_3 + 2x_4 - 15, -2x_1 - 4x_2 + x_3 + 11x_4 - 20, -4x_1 - 8x_2 + 2x_3 + 22x_4 - 40, x_1 + 2x_2 + 5x_3 + 11x_4 - 45\}$.

(a) Compute a reduced Gröbner basis for the ideal of $\mathbb{Q}[x_1, x_2, x_3, x_4]$ that is generated by A. (Lexicographic ordering, $x_1 > x_2 > x_3 > x_4$.)

(b) Compute a basis for the linear subspace of $\mathbb{Q}^4$ that is generated by the rows of the matrix

$$B = \begin{pmatrix} 1 & 2 & 2 & 2 & -15 \\ -2 & -4 & 1 & 11 & -20 \\ -4 & -8 & 2 & 22 & -40 \\ 1 & 2 & 5 & 11 & -45 \end{pmatrix}.$$

(2) Compute the reduced Gröbnerbasis for the ideal of $\mathbb{Q}[x]$ that is generated by $\{x^2 - x - 2, x^3 + x^2 - 6x\}$.

(3) Let G be a finite subset of $\mathbb{Q}[\boldsymbol{x}]$, let $f \in G$, and let r be the remainder in a standard expression of f by $G \setminus \{f\}$.

(a) Show that if G is a Gröbner basis of $\langle G \rangle_{\mathbb{Q}[\boldsymbol{x}]}$, then $(G \setminus \{f\}) \cup \{r\}$ is also a Gröbner basis of $\langle G \rangle_{\mathbb{Q}[\boldsymbol{x}]}$.

(b) Given an example where $(G \setminus \{f\}) \cup \{r\}$ is a Gröbner basis, but G is not a Gröbner basis.

(4) Let F be a finite subset of $\mathbb{Q}[x_1, \ldots, x_n]$, and let f be an element of the ideal J that F generates in $\mathbb{C}[x_1, \ldots, x_n]$. Show that f is also an element of the ideal I that F generates in $\mathbb{Q}[x_1, \ldots, x_n]$.

(5) Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$. Show that the system $f_1 = \cdots = f_s = 0$ has no solution in $\mathbb{C}^n$ if and only if the reduced Gröbner basis of $\langle f_1, \ldots, f_s \rangle_{\mathbb{C}[\boldsymbol{x}]}$ is $\{1\}$.