Gröbner Bases

Exercise Sheet 5 for November 18, 2020

(1) In this exercise, we verify the claim of Lemma 3.4 in a concrete example. Let

$$\begin{aligned} u &= xy + 4 \\ f &= x^3 + xy \\ v &= x^2y - 3x \\ g &= x^2 + xy^2. \end{aligned}$$

Find $a, b, c \in \mathbb{Q}[x, y]$ such that

$$uf = a\, S(f, g) + bf + cg,$$

$\textsc{Deg}(aS(f, g)) < \textsc{Deg}(uf)$, $\textsc{Deg}(bf) < \textsc{Deg}(uf)$, and $\textsc{Deg}(cg) = \textsc{Deg}(uf)$.

(2) Compute a Gröbner basis of the following ideal I of $\mathbb{Q}[x, y]$ with respect to the lexicographic ordering, $x > y$, where

$$I = \langle -1 - x\,y + y^2 + x\,y^2,\ -1 + y^2 \rangle.$$

(3) Compute a Gröbner basis of $\langle a^2b + c + 1, a^2c + b \rangle$ in $\mathbb{Q}[a, b, c]$ with respect to the lexicographic ordering, $a > b > c$.

(4) A *binomial* is a polynomial which contains exactly two monomials, such as $xy^2 + 5xy^3z$. Show that an ideal that is generated by monomials and binomials has a Gröbner basis containing only monomials and binomials.

(5) Let I be the ideal of $\mathbb{Q}[x, y]$ generated by $\{x + y^2 + 2, xy + 3\}$. Find generators of the ideals $I \cap \mathbb{Q}[x]$ and $I \cap \mathbb{Q}[y]$ of $\mathbb{Q}[x]$ and $\mathbb{Q}[y]$, respectively.