Gröbner Bases
Exercise Sheet 4 for November 11, 2020

We will also discuss Problems 2 and 3 from sheet 2 and Problem (2b) from sheet 3.

(1) Let k be a field , and let $F = \{f_1, \ldots, f_s\} \subseteq k[x_1, \ldots, x_n] \setminus \{0\}$.
  (a) Show the following statment: If F is a Groebner basis of $\langle F \rangle$ and
  $$\mathrm{LT}(f_i) \in \langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_{i-1}), \mathrm{LT}(f_{i+1}), \ldots, \mathrm{LT}(f_s) \rangle,$$
  then $F \setminus \{f_i\}$ is a Groebner basis of $\langle F \rangle$, too.
  (b) Is this assertion still valid if one replaces the words "Groebner basis" both times by "basis"?

(2) Use Buchberger's criterion to check whether the following sets F are Groebner bases for the ideals $\langle F \rangle_{k[\boldsymbol{x}]}$ they generate.
  (a) $F = \{x^2y + z, yz + 1\}$, lexicographic order, $x > y > z$.
  (b) $F = \{x^2y + z, yz + 1\}$, lexicographic order, $z > y > x$.
  (c) $F = \{x^5y^3, 3x + xy\}$, lexicographic order, $y > x$.
  (d) $F = \{x^2y^3, x^4y, x^3y^2, x^3y^3\}$, lexicographic order, $x > y$.

(3) Let k be a field, and let $f \in k[x_1, \ldots, x_n]$. Show that $\{f\}$ is a Groebner basis for the ideal $\langle f \rangle_{k[\boldsymbol{x}]}$. Can you give a proof that does not use the S-polynomial criterion?

(4) Let G be a finite set of monomials. Show that these monomials form a Groebner basis for the ideal they generate (with respect to every monomial order).