Gröbner Bases

Exercise Sheet 12 for January 27, 2021

(1) For $A \in R^{m\times n}$, let $\operatorname{col}(A) = \{Ax \mid x \in R^n\}$ be the column module of A and $\ker(A) = \{y \in R^n \mid Ay = 0\}$ be the module of solutions of $Ay = 0$. Let I_n be the $n \times n$ identity matrix. Prove the following statement:

Let R be a commutative ring with unit, let $l, m, n \in \mathbb{N}$, and let $F \in R^{l\times n}$, $G \in R^{l\times m}$, $A \in R^{n\times m}$, $B \in R^{m\times n}$ be such that $FA = G$ und $GB = F$. Then we have

$$\ker(F) = \{Ay \mid y \in \ker(G)\} + \operatorname{col}(I_n - AB).$$

Remark: This exercise is used when computing syzygies via the S-polynomial method.

(2) Let $A \in \mathbb{Q}[x, y, z]^{4\times 4}$ be defined by

$$A = \begin{pmatrix} x^2+y & z\,(x^2+y) & x^2 & 0 \\ y-z & z(y-z) & 0 & 0 \\ 0 & 0 & x^2z+2y & 0 \\ 0 & 0 & z & 1 \end{pmatrix}.$$

(a) Compute a matrix H in echelon normal form that has the same row module as A.

(b) Use this matrix H to compute $\operatorname{row}(A) \cap (\{0\}^r \times \mathbb{Q}[x, y, z]^{4-r})$ for $r \in \{1, 2, 3\}$.

(c) Compute module generators for $\ker(A)$.

(3) Let

$$A := \begin{pmatrix} x & 0 & xz+y & 1 \\ x^2 & z & y & 0 \end{pmatrix}.$$

(a) Compute generators for the solution module (over the ring $\mathbb{Q}[x, y, z]$)

$$\{(v_1, v_2, v_3, v_4) \in \mathbb{Q}[x, y, z]^4 \mid A \cdot (v_1, v_2, v_3, v_4)^T = 0\}.$$

(b) Compute a basis for the subvectorspace of $\mathbb{Q}(x, y, z)^4$ (as a vector space over $\mathbb{Q}(x, y, z)$) defined by

$$\{(v_1, v_2, v_3, v_4) \in \mathbb{Q}(x, y, z)^4 \mid A \cdot (v_1, v_2, v_3, v_4)^T = 0\}$$

with all basis vectors lying in $\mathbb{Q}[x, y, z]$.