Gröbner Bases

Exercise Sheet 11 for January 20, 2021

(1) For an $m \times n$-matrix with entries in $\mathbb{Q}[\boldsymbol{x}]$, we let $\mathrm{row}(A) = \{vA \mid v \in \mathbb{Q}[\boldsymbol{x}]^m\}$ be its row module. Compute generators of the intersection

$$\mathrm{row}(\begin{pmatrix} xy^2+1 & -4 & x \\ 0 & x & 0 \end{pmatrix}) \cap \mathrm{row}(\begin{pmatrix} 1 & -4 & x \\ 0 & 4y & -xy \end{pmatrix}).$$

Hint: Intersect the corresponding ideals of $\mathbb{Q}[x, y, e_1, e_2, e_3]$.

(2) Compute a set of generators for the module

$$\{(f_1, f_2) \in \mathbb{Q}[x,y]^2 \,|\, (x^2y^3 + xy^3 + 3x + 3)f_1 + (xy^5 + xy^3 + 3y^2 + 3)f_2 = 0\}$$

and explain how the solution relates to the gcd of the given polynomials.

(3) Let $n \in \mathbb{N}$ and let A and B be submodules of the $\mathbb{Q}[\boldsymbol{x}]$-module $\mathbb{Q}[\boldsymbol{x}]^n$. Assume:

(a) $A \subseteq B$

(b) $\forall i \in \{1, \ldots, n\}$, $\forall g \in \mathbb{Q}[\boldsymbol{x}]$, $\forall r_{i+1}, \ldots, r_n \in \mathbb{Q}[\boldsymbol{x}]$:

$$\Big((\underbrace{0, \ldots, 0}_{i-1}, g, r_{i+1}, \ldots, r_n) \in B \Longrightarrow$$
$$\exists s_{i+1}, \ldots, s_n \in \mathbb{Q}[\boldsymbol{x}] \ : \ (0, \ldots, 0, g, s_{i+1}, \ldots, s_n) \in A \Big).$$

Show that then $A = B$.

(4) Compute generators for the solution module of the linear system

$$\begin{pmatrix} x^2y & x & y^3 - x & -y^2 \\ x^2y & -x & y^3 + x & -y^2 \\ x^3 & y & xy^2 - y & -xy \end{pmatrix} \cdot \boldsymbol{v} = 0.$$

Hint: Let

$$M := \begin{pmatrix} x^2y & x^2y & x^3 & 1 & 0 & 0 & 0 \\ x & -x & y & 0 & 1 & 0 & 0 \\ y^3-x & x+y^3 & xy^2-y & 0 & 0 & 1 & 0 \\ -y^2 & -y^2 & -xy & 0 & 0 & 0 & 1 \end{pmatrix},$$

and compute $\mathrm{row}(M) \cap (\{0\}^3 \times \mathbb{Q}[x,y]^4)$.