Gröbner Bases

Exercise Sheet 10 for January 13, 2021

(1) Let

$$f = \left(\left(x^2+y\right)^2+2\right)^3,$$
$$g = \left(\left(x^2+y\right)^2+1\right)^2$$

be polynomials over $\mathbb{Q}$. For each $h \in \{x, x^2+y, (x^2+y)^2\}$, find an ideal I such that $\mathbb{Q}[t_1, t_2, t_3]/I$ is isomorphic to $\mathbb{Q}[\![h, f, g]\!]$ and answer the following questions:

(a) Is h an element of $\mathbb{Q}[\![f, g]\!]$? How can h be expressed as $p(f, g)$?

(b) Is h integral over $\mathbb{Q}[\![f, g]\!]$? In this case, find a polynomial in $\mathbb{Q}[\![f, g]\!][t]$ witnessing this fact.

(c) Is h algebraic over $\mathbb{Q}[\![f, g]\!]$? In this case, find a polynomial in $\mathbb{Q}[\![f, g]\!][t]$ witnessing this fact.

(d) Do we have $h \in \mathbb{Q}(f, g)$? In this case, find polynomials $p, q \in \mathbb{Q}[X, Y]$ with $h = p(f, g)/q(f, g)$.

(2) Let k be a field, and let R be a subring of k containing 1. Suppose that every $\alpha \in k$ is integral over R. Show that R is a field. *Hint:* What can you say about $\frac{1}{r}$ for $r \in R$?

(3) * [Eng41] Let $a, b, q, r, f \in \mathbb{Q}[x]$ such that $\deg(f) > 0$, $b \neq 0$, $a = q \cdot b + r$ and $\deg(r) < \deg(b)$. Suppose that $a, b \in \mathbb{Q}[\![f]\!]$. Show that q and r are elements of $\mathbb{Q}[\![f]\!]$.

(4) (a) Let k be a field and let M be a $k[x, y]$-submodule of $k[x, y] \times k[x, y]$. Let J be the ideal of $k[x, y, e_1, e_2]$ generated by $\{e_1^2, e_1e_2, e_2^2\}$. Show that $I := \{(m_1e_1 + m_2e_2) + J \mid (m_1, m_2) \in M\}$ is an ideal of $k[x, y, e_1, e_2]/J$.

(b) By solving an ideal membership question, determine which of the vectors $(0, x^2y - xy^3)$ and $(x^2 - xy^2, 0)$ lie in the submodule of $\mathbb{Q}[x, y] \times \mathbb{Q}[x, y]$ that is generated by $(x^2 + 1, y)$ and $(xy^2 + 1, y)$

References

[Eng41] H. T. Engstrom. Polynomial substitutions. *Amer. J. Math.*, 63:249–255, 1941.