Gröbner Bases
Exercise Sheet 1 for October 13 and 20, 2020

(1) Using resultants, find a polynomial f in $\{ap + bq \mid a, b \in \mathbb{Q}[x, y]\} \cap \mathbb{Q}[y]$ with $f \neq 0$.

$$p = x^2y + 2xy, \;\; q = 2x^2 + xy + 1.$$

For the following problems, we use the notation of the lecture notes.

$$\begin{aligned} B &:= ((x^{n-1}, 0), \ldots, (x^0, 0), (0, x^{m-1}), \ldots, (0, x^0)), \\ C &:= (x^{m+n-1}, \ldots, x^0). \end{aligned}$$

(2) (a) Let k be a field of characteristic 0, let $f \in k[x]$ with $\deg(f) = m > 1$, and let K be a field in which f splits into linear factors. Show that f has a double root $\alpha \in K$ (meaning that $(x - \alpha)^2 \mid f$) if and only if $\operatorname{res}^{[m,m-1]}(f, f') = 0$. *Remark:* The result also holds in positive characteristic, but then requires one more step in the proof.

(b) Give a criterion when $x^2 + px + q \in \mathbb{Q}[x]$ has a double root in $\mathbb{C}$.

(3) Give a polynomial $p \in \mathbb{C}[a, b, c, d]$ with $p \neq 0$ such that every matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ with $p(\alpha, \beta, \gamma, \delta) \neq 0$ is diagonalisable over $\mathbb{C}$. *Hint:* It is sufficient that all eigenvalues are distinct.

(4) Let k be a field, $m, n \in \mathbb{N}$, $f, g \in k[x]$ with $\deg(f) = m$, $\deg(g) = n$, and let d be the gcd of f and g in $k[x]$. Show that the row space of $\operatorname{Syl}^{[m,n]}(f, g)$ is equal to

$$\{(p)_C \,|\, p \in k_{<m+n}[x], p \text{ lies in the ideal of } k[x] \text{ generated by } f, g\},$$

and also equal to

$$\{(p)_C \,|\, p \in k_{<m+n}[x],\; d \mid p\}.$$

(5) Let k be a field, $m, n \in \mathbb{N}$, $f, g \in k[x]$ with $\deg(f) = m$, $\deg(g) = n$, and let d be the gcd of f and g in $k[x]$. Show that the rank of $\operatorname{Syl}^{[m,n]}(f, g)$ is $m + n - \deg(d)$.

(6) (Gcd-computation via linear algebra) Let k be a field, $m, n \in \mathbb{N}$, $f, g \in k[x]$ with $\deg(f) = m$, $\deg(g) = n$, and let d be the gcd of f and g in $k[x]$. Let H be a matrix in echelon form such that the row space of H is equal to the row space of $\operatorname{Syl}^{[m,n]}(f, g)$. (For example, H could be the Hermite normal form of $\operatorname{Syl}^{[m,n]}(f, g)$.) Show that the last nonzero row r of H contains the polynomial d (in the sense $r^T = (d)_C$).

(7) Compute $\gcd(x^5 - 2x^3, x^4 - x^2 - 2)$ in $\mathbb{Q}[x]$ by finding a matrix in echelon form with the same row space as the Sylvester matrix of these two polynomials.

(8) Compute $\gcd(2x^3 + 5x^2 - 4x - 3, x^4 + 2x^3 - x^2 + 4x - 6)$ in $\mathbb{Q}[x]$ by finding a matrix in echelon form with the same row space as the Sylvester matrix of these two polynomials.

(9) In $\mathbb{Q}[x, y]$, the polynomials x and y have only constant common divisors. Nevertheless, there are no $a, b \in \mathbb{Q}[x, y]$ with $1 = ax + by$. Show the following statement:

If $f, g \in \mathbb{Q}[x, y]$ have no nonconstant common divisor in $\mathbb{Q}[x, y]$, then there are $a, b \in \mathbb{Q}[x, y]$ with $af + bg \in \mathbb{Q}[x] \setminus \{0\}$.