(2) Let $I := (x_1^2 - 3, x_2^2 - 2)$, and let $R := \mathbb{Q}[x_1, x_2]/I$.

(a) Find an Ideal J of $\mathbb{Q}[t_1, t_2, t_3]$ such that $\mathbb{Q}[\boldsymbol{t}]/J$ is isomorphic to R, and an isomorphism φ with $\varphi(t_1+J) = (x_1+x_2)+I$, $\varphi(t_2+J) = x_1+I$, $\varphi(t_3+J) = x_2+I$.

(b) Find an ideal K of $\mathbb{Q}[s_1]$ such that $\mathbb{Q}[s_1]/K$ is isomorphic to the subring of $\mathbb{Q}[\boldsymbol{t}]/J$ generated by $t_1 + J$ via an isomorphism ψ with $\psi(t_1 + J) = s_1 + K$.

(c) Find a polynomial witnessing that $x_1 + x_2 + I$ is integral over $\mathbb{Q}$.

(a) Let $\alpha : \mathbb{Q}[t_1, t_2, t_3] \to \mathbb{Q}[x_1, x_2]$ be the $\mathbb{Q}$-homomorphism defined by

$\alpha(t_1) = x_1 + x_2$, $\alpha(t_2) = x_1$, $\alpha(t_3) = x_2$.

Clearly, α is surjective.

Let $\beta : \mathbb{Q}[x_1, x_2] \twoheadrightarrow \mathbb{Q}[x_1, x_2]/I$ be the canonical epimorphism into the quotient ring and $\gamma = \beta \circ \alpha$. Since γ is surjective, by the first isomorphism theorem of rings, there exists the isomorphism φ s.t. the following diagram commutes

$$\begin{tikzcd}
\mathbb{Q}[t_1, t_2, t_3] \arrow[r, two heads, "\alpha"] \arrow[rr, bend right, "\gamma"'] \arrow[dr, two heads, "\text{can. epimorph.}"'] & \mathbb{Q}[x_1, x_2] \arrow[r, two heads, "\beta"] & \mathbb{Q}[x_1, x_2]/I . \\
& \mathbb{Q}[t_1, t_2, t_3]/\ker\gamma \arrow[ur, "\varphi"', "\cong"] &
\end{tikzcd}$$

Now, let $J := \ker \gamma$, then clearly $\varphi(t_1 + J) = (x_1 + x_2) + I$, $\varphi(t_2 + J) = x_1 + I$ and $\varphi(t_3 + J) = x_2 + I$. It remains to compute J.

Satz 9.11 (Algebraische Abhängigkeit in $k[x_1, \ldots, x_n]/I$). *Sei k ein Körper, seien $r \in \mathbb{N}_0$, $n, s \in \mathbb{N}$, sei $I = \langle g_1, \ldots, g_r \rangle_{k[\boldsymbol{x}]}$, und seien $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. Sei $p \in k[t_1, \ldots, t_s]$, und sei $J := \langle g_1, \ldots, g_r, t_1 - f_1, \ldots, t_s - f_s \rangle_{k[\boldsymbol{t}, \boldsymbol{x}]}$. Dann sind äquivalent:*

(1) $p(f_1, \ldots, f_s) \in I$.

(2) $p \in J \cap k[t_1, \ldots, t_s]$.

```
Groebner:-Basis([x1^2 - 3, x2^2 - 2, t1 - (x1 + x2), t2 - x1, t3 - x2], plex(x1, x2, t1, t2, t3));
                           2           2
                       [t3  - 2, t2  - 3, t1 - t2 - t3, x2 - t3, x1 - t2]
```

↪ remaining poly's generate $\ker \alpha = J$

Hence, $J = \langle t_1 - t_2 - t_3,\ t_2^2 - 3,\ t_3^2 - 2 \rangle_{\mathbb{Q}[t_1,t_2,t_3]}$.

(b) Consider the diagram

$$\begin{tikzcd}
\mathbb{Q}[t_1] \arrow[r, hook, "\text{embed}"] \arrow[rr, bend right, "=:\mu"'] \arrow[dr, two heads, "\text{can. epi.}"'] & \mathbb{Q}[t_1,t_2,t_3] \arrow[r, two heads, "\text{can. epi.}"] & \mathbb{Q}[t_1,t_2,t_3]/J \\
& \mathbb{Q}[t_1]/\ker\mu \arrow[ur, hook, "=:\psi"'] &
\end{tikzcd}$$

(and $\mathbb{Q}[t_1]/\ker\mu \overset{\psi}{\cong} \operatorname{im}\mu \leq \mathbb{Q}[t_1,t_2,t_3]/J$)

Clearly, $\ker\mu = J \cap \mathbb{Q}[t_1]$ and μ is a $\mathbb{Q}$-hom. s.t. $\mu(t_1) = t_1 + J$.

Use GBs to compute $\ker\mu$:

```
Groebner:-Basis([x1^2 - 3, x2^2 - 2, t1 - (x1 + x2), t2 - x1, t3 - x2], plex(x1, x2, t3, t2, t1));
```

$$[t1^4 - 10\ t1^2 + 1,\ \cancel{t1^3 - 11\ t1 + 2\ t2},\ \cancel{-t1^3 + 9\ t1 + 2\ t3},\ \cancel{-t1^3 + 9\ t1 + 2\ x2},\ \cancel{t1^3 - 11\ t1 + 2\ x1}]$$

GB for J wrt. another monomial ordering

Thus, $\ker\varphi = \langle t_1^4 - 10t_1^2 + 1 \rangle_{\mathbb{Q}[t_1]}$. Now simply substitute t_1 by s_1, i.e. $K = \langle s_1^4 - 10s_1^2 + 1 \rangle_{\mathbb{Q}[s_1]}$. Clearly, $\mathbb{Q}[s_1]/K \cong \operatorname{im}\psi$ by mapping $s_1 + K$ to $t_1 + J$.

(c) Recall from (a) and (b):

$$\begin{tikzcd}
\mathbb{Q}[t_1] \arrow[r, hook, "\text{embed}"] \arrow[rrr, bend right, "=:\omega"'] & \mathbb{Q}[t_1,t_2,t_3] \arrow[r, two heads, "\text{can. epi.}"] & \mathbb{Q}[t_1,t_2,t_3]/J \arrow[r, "{\varphi,\ \text{cf. (a)}}", "\cong"'] & \mathbb{Q}[x_1,x_2]/I
\end{tikzcd}$$

with $\omega(t_1) = (x_1 + x_2) + I$ and $\ker\omega = \langle t_1^4 - 10t_1^2 + 1 \rangle_{\mathbb{Q}[t_1]}$.
Thus, $t_1^4 - 10t_1^2 + 1 \in \mathbb{Q}[t_1]$ is a witness that $(x_1 + x_2) + I$ is algebraic over $\mathbb{Q}$ (in $\mathbb{Q}[x_1,x_2]/I$):

```
with(PolynomialIdeals):
p := t1^4 - 10*t1^2 + 1:
q := eval(p, t1 = x1 + x2);
```

$$q := (x1 + x2)^4 - 10\ (x1 + x2)^2 + 1$$

```
q in PolynomialIdeal([x1^2 - 3, x2^2 - 2]);
```

true