

(2) Let $I := (x_1^2 - 3, x_2^2 - 2)$, and let $R := \mathbb{Q}[x_1, x_2]/I$.

(a) Find an Ideal J of $\mathbb{Q}[t_1, t_2, t_3]$ such that $\mathbb{Q}[t]/J$ is isomorphic to R , and an isomorphism φ with $\varphi(t_1 + J) = (x_1 + x_2) + I$, $\varphi(t_2 + J) = x_1 + I$, $\varphi(t_3 + J) = x_2 + I$.

(b) Find an ideal K of $\mathbb{Q}[s_1]$ such that $\mathbb{Q}[s_1]/K$ is isomorphic to the subring of $\mathbb{Q}[t]/J$ generated by $t_1 + J$ via an isomorphism ψ with $\psi(t_1 + J) = s_1 + K$.

(c) Find a polynomial witnessing that $x_1 + x_2 + I$ is integral over $\mathbb{Q}$.

(a) Let $\alpha: \mathbb{Q}[t_1, t_2, t_3] \rightarrow \mathbb{Q}[x_1, x_2]$ be the $\mathbb{Q}$ -homomorphism defined by

$$\alpha(t_1) = x_1 + x_2, \alpha(t_2) = x_1, \alpha(t_3) = x_2.$$

Clearly, α is surjective.

Let $\beta: \mathbb{Q}[x_1, x_2] \twoheadrightarrow \mathbb{Q}[x_1, x_2]/I$ be the canonical epimorphism into the quotient ring and $\gamma = \beta \circ \alpha$. Since γ is surjective, by the first isomorphism theorem of rings, there exists the isomorphism φ s.t.

the following diagram commutes

$$\begin{array}{ccccc} \mathbb{Q}[t_1, t_2, t_3] & \xrightarrow{\alpha} & \mathbb{Q}[x_1, x_2] & \xrightarrow{\beta} & \mathbb{Q}[x_1, x_2]/I \\ & \searrow \text{can. epimorph.} & \searrow \gamma & \nearrow \cong & \nearrow \varphi \\ & & \mathbb{Q}[t_1, t_2, t_3]/\ker \gamma & & \end{array}$$

Now, let $J := \ker \gamma$, then clearly $\varphi(t_1 + J) = (x_1 + x_2) + I$, $\varphi(t_2 + J) = x_1 + I$ and $\varphi(t_3 + J) = x_2 + I$. It remains to compute J .

SATZ 9.11 (Algebraische Abhängigkeit in $k[x_1, \dots, x_n]/I$). Sei k ein Körper, seien $r \in \mathbb{N}_0$, $n, s \in \mathbb{N}$, sei $I = \langle g_1, \dots, g_r \rangle_{k[x]}$, und seien $f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Sei $p \in k[t_1, \dots, t_s]$, und sei $J := \langle g_1, \dots, g_r, t_1 - f_1, \dots, t_s - f_s \rangle_{k[t, x]}$. Dann sind äquivalent:

(1) $p(f_1, \dots, f_s) \in I$.

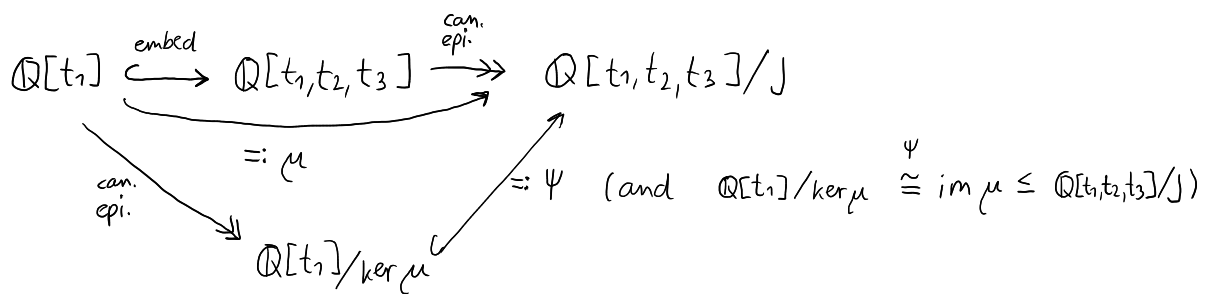
(2) $p \in J \cap k[t_1, \dots, t_s]$.

Groebner:-Basis($[x_1^2 - 3, x_2^2 - 2, t_1 - (x_1 + x_2), t_2 - x_1, t_3 - x_2]$, plex(x_1, x_2, t_1, t_2, t_3));
 $[t_3^2 - 2, t_2^2 - 3, t_1 - t_2 - t_3, x_2 - t_3, x_1 - t_2]$

↳ remaining poly's generate $\ker \alpha = J$

Hence, $J = \langle t_1 - t_2 - t_3, t_2^2 - 3, t_3^2 - 2 \rangle_{\mathbb{Q}[t_1, t_2, t_3]}$.

(b) Consider the diagram



Clearly, $\ker \mu = J \cap \mathbb{Q}[t_1]$ and μ is a $\mathbb{Q}$ -hom. s.t. $\mu(t_1) = t_1 + J$.

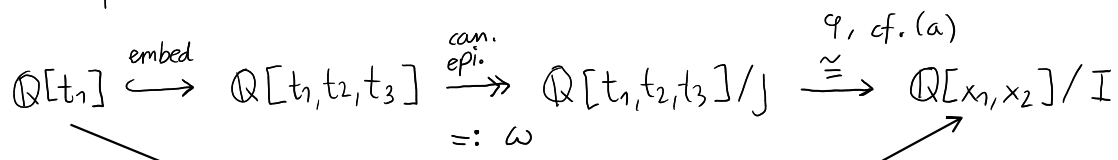
Use GBs to compute $\ker \mu$:

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Groebner:-Basis([x1^2 - 3, x2^2 - 2, t1 - (x1 + x2), t2 - x1, t3 - x2], plex(x1, x2, t3, t2, t1));
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GB for J wrt. another monomial ordering

Thus, $\ker \varphi = \langle t_1^4 - 10t_1^2 + 1 \rangle_{\mathbb{Q}[t_1]}$. Now simply substitute t_1 by s_1 , i.e. $K = \langle s_1^4 - 10s_1^2 + 1 \rangle_{\mathbb{Q}[s_1]}$. Clearly, $\mathbb{Q}[s_1]/K \cong \text{im } \psi$ by mapping $s_1 + K$ to $t_1 + J$.

(c) Recall from (a) and (b):



with $\omega(t_1) = (x_1 + x_2) + I$ and $\ker \omega = \langle t_1^4 - 10t_1^2 + 1 \rangle_{\mathbb{Q}[t_1]}$.

Thus, $t_1^4 - 10t_1^2 + 1 \in \mathbb{Q}[t_1]$ is a witness that $(x_1 + x_2) + I$ is algebraic over $\mathbb{Q}$ (in $\mathbb{Q}[x_1, x_2]/I$):

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with(PolynomialIdeals):
p := t1^4 - 10*t1^2 + 1;
q := eval(p, t1 = x1 + x2);

q := (x1 + x2)^4 - 10 (x1 + x2)^2 + 1

q in PolynomialIdeal([x1^2 - 3, x2^2 - 2]);
true
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