Discrete Mathematics
368.115
Exercise sheet 9 for December 2, 2016

Solve 3 out of 6!

(1) (Source: Applications of Hall's Marriage Theorem. `Brilliant.org`. Retrieved 19:56, November 29, 2016, from `https://brilliant.org/wiki/applications-of-hall-marriage-theorem/`) In a $2n \times 2n$ board, there are n rooks in each row and each column of the board. Show that there exist $2n$ rooks that belong to pairwise distinct rows and pairwise distinct columns.

(2) (Source: `https://www.quora.com/What-are-some-interesting-applications-of-Halls-marriage-theorem`) We call an $n \times n$ matrix A *doubly stochastic* if all entries are nonnegative, and every row and every column sum is 1. Let A be an $n \times n$ doubly stochastic matrix with rational entries. Use Birkhoff's Theorem to prove that there are $m \in \mathbb{N}$, permutation matrices $P_1, \ldots, P_m$ and there are rational $\alpha_1, \ldots, \alpha_m \in [0, 1]$ such that $A = \sum_{i=1}^{m} \alpha_i P_i$ and $\sum_{i=1}^{m} \alpha_i = 1$.

(3) (For aficionados of analysis) Use your knowledge of analysis (for example the Bolzano Weierstrass Theorem in $\mathbb{R}^{n!}$) to show that the statement of Problem (2) also holds if *rational* is replaced with *real*.

(4) (Source: `https://brilliant.org/wiki/hall-marriage-theorem/`) (Putnam 2012 B3) A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

(5) (Source: [1], Exercise 3.17) Turn the following partial Latin square into a Latin square: $\begin{pmatrix} 2 & 1 & & & \\ & & & 3 & 4 \\ & & & & 3 \\ 1 & & & & \end{pmatrix}$

(6) (Source: [1], Exercise 3.17) Turn the following partial Latin square into a Latin square: $\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 2 \end{pmatrix}$

Literatur

[1] D. Mašulović. The discrete charm of discrete mathematics. Lecture notes for a course at JKU Linz, Austria, 2006; available at `http://www.algebra.uni-linz.ac.at/Students/DiskreteMathematik/ws16/`, 2006.