Discrete Mathematics
368.115
Exercise sheet 8 for November 25, 2016

Solve 4 out of 7!

(1) Let $\mathcal{A} = (A_1, A_2, A_3, A_4) = (\{F, H\}, \{F, H\}, \{F, G\}, \{E, I\})$.
  (a) Find a nonempty subset J of $\{1, 2, 3, 4\}$ such that $|\mathcal{A}(J)| = |J|$.
  (b) Find a system of distinct representatives for $\mathcal{B} = (A_j)_{j\in J}$.
  (c) Find a system of distinct representatives for $\mathcal{A}$ that extends the system found for $\mathcal{B}$.

(2) A *Latin rectangle* of size $r \times n$ is a $r \times n$-matrix with entries in $\{1, \ldots, n\}$ such that in each row and in each column, every number appears at most once. For every $n \in \mathbb{N}$, give an example of a $1 \times n$, a $2 \times n$, and an $n \times n$ Latin rectangle. *Hint:* Use addition modulo n.

(3) Show that every $(n-1) \times n$ Latin rectangle can be completed to an $n \times n$ Latin rectangle.

(4) [1, Exercise 3.13] For each $n \geq 3$ find n subsets $A_1, \ldots, A_n$ of $\{1, 2, \ldots, n\}$ such that $|A_1| = \ldots = |A_n|$ and $\mathrm{SDR}(A_1, \ldots, A_n) = 2$. *Remark:* Here, $\mathrm{SDR}(A_1, \ldots, A_n)$ denotes the number of systems of distinct representatives for $(A_1, \ldots, A_n)$.

(5) In a Wikipedia article on Hall's Theorem [2], we find:

> The theorem has many other interesting "non-marital" applications. For example, take a standard deck of cards, and deal them out into 13 piles of 4 cards each. Then, using the marriage theorem, we can show that it is always possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (Ace, 2, 3, ..., Queen, King).

Prove this! To this end, let $C = \{1, \ldots, 13\} \times \{1, \ldots, 4\}$ be the set of cards, and let $(P_1, \ldots, P_{13})$ be the sequence of 13 piles. For each $i \in \{1, \ldots, 13\}$, let A_i be the list of indices of those piles in which we find a card of rank i. Formally,

$$A_i = \{j \in \{1, \ldots, 13\} \mid \exists c : (i, c) \in P_j\}.$$

  (a) Show that $\mathcal{A} = (A_1, \ldots, A_{13})$ satisfies the conditions of Hall's marriage theorem.
  (b) Explain how to use an SDR $(e_1, \ldots, e_{13})$ for $\mathcal{A}$ to select the cards from the piles.

Hint: This is not the only way to solve this problem; others are also welcome.

(6) Let $k, m, n \in \mathbb{N}$ be such that $n = km$, let X be a set with n elements, and let α and β be two equivalence relations on X such that each of these relations has exactly k classes, and each class contains exactly m elements. Show that there is a subset R of X with $|R| = k$ such that R contains exactly one element of each α-class and exactly one element of each β-class. In other words, find a set R that is a transversal for both α and β.

(7) Let $n \in \mathbb{N}$, and let $A_1, \ldots, A_n$ be subsets of $X = \{1, \ldots, n\}$. Let M be the $n \times n$-matrix defined by $M(i, j) = 1$ if $j \in A_i$, and $M(i, j) = 0$ if $j \notin A_i$.

(a) Prove that the number of SDR's of $\mathcal{A} = (A_1, \ldots, A_n)$ is equal to

$$\sum_{f \in S_n} \prod_{i=1}^{n} M(i, f(i)).$$

(This number is called the *permanent* of M.)

(b) Show that if M is invertible (as a matrix over the real numbers), then $\mathcal{A}$ has an SDR.

References

[1] D. Mašulović. The discrete charm of discrete mathematics. Lecture notes for a course at JKU Linz, Austria, 2006; available at `http://www.algebra.uni-linz.ac.at/Students/DiskreteMathematik/ws16/`, 2006.

[2] Wikipedia. Hall's marriage theorem — wikipedia, the free encyclopedia, 2016. [Online; accessed 8-August-2016].