Discrete Mathematics
368.115
Exercise sheet 12 for January 20, 2017

Solve 4 out of the 6 new problems!

(1) Let $f : \mathbb{N}_0 \to \mathbb{R}$. Show that the following are equivalent:
   (a) $\exists k \in \mathbb{N}_0 \exists n_0 \in \mathbb{N}_0 \forall n \in \mathbb{N}_0 : n \geq n_0 \Rightarrow f(n) \leq n^k$.
   (b) $\exists p \in \mathbb{R}[t] \forall n \in \mathbb{N}_0 : f(n) \leq p(n)$.
   (c) $\exists k \in N_0 \exists c \in \mathbb{R} \forall n \in \mathbb{N}_0 : n \geq 1 \Rightarrow f(n) \leq cn^k$.

(2) For each of the following languages, describe a program that runs in polynomial time and decides the language.
   (a) $L_1 = \mathrm{Lang}((aa)^*b(aaa)^*)$.
   (b) $L_2 = \{a^n b^n \mid n \in \mathbb{N}\}$.

(3) Show that $L_3 = \{a^p \mid p \text{ is a prime number}\}$ is an element of **P**. Why does this not show that primality testing is solvable in polynomial time?

(4) For each of the following languages, give a proof that the language is in **NP** by providing a witness and a verification algorithm for that language.
   (a) $L_1 = \{x \in \{0,1\}^* \mid x \text{ is the binary representation of some natural number} \geq 4 \text{ which is not prime}\}$.
   (b) $L_2 = \{\langle D \rangle \mid D \text{ is a directed graph with a directed Hamiltonian cycle}\}$.

(5) (Source: `https://www.cs.umd.edu/~gasarch/COURSES/452/F14/poly.pdf`)
   Prove that the class **P** is closed under intersection, complement and concatenation.

(6) Prove that the class **NP** is closed under intersection. Hence show: if $L_1, L_2 \subseteq \Sigma^*$ are both in **NP**, then $L_1 \cap L_2 \in \mathbf{NP}$.

And we still want to solve:

> Let (V, E) be a connected graph, and let $c : E \to \mathbb{R}^+$ be the cost function for the edges. We assume that all edge costs are distinct (i.e., c is injective). Prove that (V, E) contains exactly one minimal spanning tree.