Introduction to category theory

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December 10, 2003

Contents

*These notes are a compilation of sources [1]-[5], not an original work by V.L.

1 Categories

1.1 On set-theoretical foundations

It is well-known that the set of all sets does not exist. However, category theory would like to deal with collections of all sets, all groups, all topological spaces etc. A way to handle this is to assume the axiom of the existence of so-called universes.

Definition 1.1 $(\mathcal{U}, \in)$ is a **universe**, if it satisfies the following properties:

1. $x \in y$ and $y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$,
2. $I \in \mathcal{U}$ and $(\forall i \in I)(x_i \in \mathcal{U}) \Rightarrow \bigcup_{i \in I} x_i \in \mathcal{U}$,
3. $x \in \mathcal{U} \Rightarrow \mathcal{P}(x) \in \mathcal{U}$,
4. $x \in \mathcal{U}, y \subseteq \mathcal{U}$, $f : x \to y$ is a surjective function $\Rightarrow y \in \mathcal{U}$,
5. $\omega \in \mathcal{U}$,

where ω is the set of all finite ordinals and $\mathcal{P}(x)$ is the set of all subsets of x.

Some of the consequences of this definition are the following (cf. [1], Proposition 1.1.3).

Proposition 1.2 *1. $x \in \mathcal{U}, y \subseteq \mathcal{U}$ and $y \subseteq x \Rightarrow y \in \mathcal{U}$,*

2. $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$,

3. $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow x \times y \in \mathcal{U}$,

4. $x \in \mathcal{U}$ and $y \in \mathcal{U} \Rightarrow x^y \in \mathcal{U}$.

The properties of $\mathcal{U}$ ensure that any of the standard operations of set theory applied to elements of $\mathcal{U}$ will always produce elements of $\mathcal{U}$; in particular $\omega \in \mathcal{U}$ provides that $\mathcal{U}$ also contains the usual sets of real numbers and related infinite sets. We can then regard "ordinary" mathematics as carried out extensively within $\mathcal{U}$, i.e. on elements of $\mathcal{U}$.

Now let the universe $\mathcal{U}$ be fixed and call its elements **sets**. Thus $\mathcal{U}$ is the collection of all sets and we denote it by Set. We also call subsets of $\mathcal{U}$ **classes**. Since $x \in y \in \mathcal{U}$ implies $x \in \mathcal{U}$, every element y of $\mathcal{U}$ is also a subset of $\mathcal{U}$, that is, every set is a class. Conversely, the class $\mathcal{U}$ itself is not a set because $\mathcal{U} \in \mathcal{U}$ would contradict the axiom of regularity, which asserts that there are no infinite chains $\ldots x_n \in x_{n-1} \in x_{n-2} \in \ldots \in x_0$. The classes that are not sets are called **proper classes**.

In this way we can consider for example proper classes of all groups, all topological spaces etc. as subclasses of Set.

1.2 Definition of category

Definition 1.3 A **category** $\mathcal{C}$ consists of the following:

1. a class $\mathcal{C}_0$, whose elements will be called "objects of the category";
2. for every pair A, B of objects, a set $\mathcal{C}(A, B)$, whose elements will be called "morphisms" or "arrows" from A to B;
3. for every triple A, B, C of objects, a composition law
$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C);$$
the composite of the pair (f, g) will be written $g \circ f$ or just gf;
4. for every object A, a morphism $1_A \in \mathcal{C}(A, A)$, called the identity morphism of A.

These data are subject to the following axioms.

1. Associativity axiom: given morphisms $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(C, D)$ the following equality holds:
$$h(gf) = (hg)f.$$

2. Identity axiom: given morphisms $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, the following equalities hold: $1_B f = f,\ \ g1_B = g$.

A morphism $f \in \mathcal{C}(A,B)$ is often written as $f : A \to B$, A is called the **domain** (notation: $\operatorname{dom} f$) or the source of f and B is called the **codomain** (notation: $\operatorname{cod} f$) or the target of f. A morphism $f : A \to A$ is called an **endomorphism** of A.

Remarks 1.4 1. It turns out that 1_A is the only identity morphism of A, because if $i_A \in \mathcal{C}(A,A)$ is another morphism satisfying the identity axiom, then $1_A = 1_A i_A = i_A$.

2. Very much in the same way as in the case of semigroups, the associativity axion implies that parentheses in any composite of finite number of morphisms can be placed in arbitrary way and therefore actually omitted.

Definition 1.5 A category is **small** if its objects constitute a set, otherwise it is **large** .

Example 1.6 The next table gives some examples of categories.

Notation	Objects	Morphisms
Set	sets	mappings
Rel	sets	binary relations between sets
Mon	monoids	monoid homomorphisms
Gr	groups	group homomorphisms
Ab	abelian groups	group homomorphisms
Rng	rings with unit	ring homomorphisms
$\mathsf{Vec}_{\mathbb{R}}$	real vector spaces	linear mappings
Mod_R	right modules over ring R	module homomorphisms
Ban_∞	real Banach spaces	bounded linear mappings
Ban_1	real Banach spaces	linear contractions
Top	topological spaces	continuous mappings
Pos	posets	order-preserving mappings
Lat	lattices	lattice homomorphisms
Graph	graphs	graph homomorphisms
Sgraph	graphs	strong graph homomorphisms
0	none	none
1	A	1_A
2	A, B	$A \to B, 1_A, 1_B$

In most cases, the composition of morphisms is just the composition of mappings and identity morphisms are identity transformations. In Rel, the composition of relations is their product and the identity morphism is the equality relation. The category **0** is called the **empty category** .

Example 1.7 1. Objects: natural numbers, morphisms from m to n are all matrices (over a fixed field) with m rows and n columns, the composition of morphisms is the usual multiplication of matrices.

2. A poset $(P, \leq)$ can be regarded as a category $\mathcal{P}$ with object set P. If $x, y \in P$ then $\mathcal{P}(x,y)$ consists of exactly one morphism if $x \leq y$, and is empty otherwise.

3. Every set can be viewed as a **discrete category**, i.e. a category where the only morphisms are the identity morphisms.

4. Every monoid $(M, \cdot)$ gives rise to a category $\mathcal{M}$ with a single object $*$, $\mathcal{M}_0 = \{*\}$, and $\mathcal{M}(*,*) = M$; the composition of morphisms is the multiplication $\cdot$ of M. Also conversely, the set of all morphisms of every one-object category is a monoid.

1.3 Functional programming languages as categories

A (pure) functional programming language has

1. Primitive data types, given in the language.

2. Constants of each type.

3. Operations, which are functions between the types.

4. Constructors, which can be applied to data types and operations to produce derived data types and operations of the language.

The language consists of the set of all operations and types derivable from the primitive data types and primitive operations.

To see that a functional programming language L corresponds in a canonical way to a category $\mathcal{C}(L)$, we have to make two assumptions and one small change.

1. We assume that for each type A (both primitive and constructed) there is a do-nothing operation 1_A. When applied, it does nothing to the data.

2. We add to the language an additional type called $\mathbf{1}$, which has the property that from every type A there is a unique operation to $\mathbf{1}$. We interpret each constant c of type A as a morphism $c : \mathbf{1} \to A$.

3. We assume that the language has a comptosition constructor: if f is an operation with inputs of type A and outputs of type B and another operation g has inputs of type B and output of type C, then doing one after the other is a derived operation (or program) typically denoted $f;g$, which has input of type A and output of type C.

Under these conditions, a functional programming language L induces a category $\mathcal{C}(L)$, for which

1. the objects of $\mathcal{C}(L)$ are the types of L,

2. the morphisms of $\mathcal{C}(L)$ are the operations (primitive and derived) of L,

3. domain and codomain of a morphism are the input and output types of the corresponding operation,

4. composition of morphisms is given by the composition constructor, written in the reverse order,

5. the identity morphisms are the do-nothing operations.

Example 1.8 Consider a language with three data types NAT (natural numbers and 0), BOOLEAN (true or false) and CHAR (characters). We give a description of its operations in categorical style.

1. NAT should have a constant $0 : \mathbf{1} \to \text{NAT}$ and an operation $\texttt{succ} : \text{NAT} \to \text{NAT}$.

2. There should be two constants $\texttt{true}, \texttt{false} : \mathbf{1} \to \text{BOOLEAN}$ and an operation $\neg : \text{BOOLEAN} \to \text{BOOLEAN}$ subject to the equalities $\neg \circ \texttt{true} = \texttt{false}$ and $\neg \circ \texttt{false} = \texttt{true}$.

3. CHAR should have one constant $c : \mathbf{1} \to \text{CHAR}$ for each character c.

4. There should be two type conversion operators $\texttt{ord} : \text{CHAR} \to \text{NAT}$ and $\texttt{chr} : \text{NAT} \to \text{CHAR}$. These must satisfy the equality $\texttt{chr} \circ \texttt{ord} = 1_{\text{CHAR}}$. (One can think of chr as operating modulo the number of characters, so that it is defined on all natural numbers.)

An example program is the morphism '`next`' defined to be the composite $\texttt{chr} \circ \texttt{succ} \circ \texttt{ord} : \text{CHAR} \to \text{CHAR}$, which calculates the next character. Note that two morphisms (programs) in $\mathcal{C}(L)$ are identified if they must be the same because of the equalities. For example, the morphisms $\texttt{chr} \circ \texttt{succ} \circ \texttt{ord}$ and $\texttt{chr} \circ \texttt{succ} \circ \texttt{ord} \circ \texttt{chr} \circ \texttt{ord}$ must be the same.

Observe that NAT has constants $\texttt{succ} \circ \ldots \circ \texttt{succ} \circ 0 : \mathbf{1} \to \text{NAT}$ where succ occures zero or more times.

1.4 Some constructions

Definition 1.9 A category $\mathcal{B}$ is called a **subcategory** of a category $\mathcal{A}$, if

1. $\mathcal{B}_0 \subseteq \mathcal{A}_0$;

2. $\mathcal{B}(B, B') \subseteq \mathcal{A}(B, B')$ for every pair $B, B' \in \mathcal{B}_0$, so that

    (a) $f \in \mathcal{B}(B, B')$ and $g \in \mathcal{B}(B', B'')$ implies $gf \in \mathcal{B}(B, B'')$,

(b) $1_B \in \mathcal{B}(B, B)$ for every $B \in \mathcal{B}_0$.

Definition 1.10 A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is called a **full subcategory** if

$$B, B' \in \mathcal{B}_0 \Rightarrow \mathcal{B}(B, B') = \mathcal{A}(B, B').$$

Example 1.11 The category Ab is a full subcategory of Gr, Gr is a full subcategory of Mon, Mon is a subcategory of the category Sgr of semigroups, which is not full. The category Ban_∞ is a subcategory of $\mathsf{Vec}_\mathbb{R}$, but not a full subcategory.

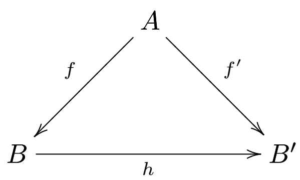
Definition 1.12 The **product** of two categories $\mathcal{A}$ and $\mathcal{B}$ is the category $\mathcal{A} \times \mathcal{B}$ defined as follows.

1. $(\mathcal{A} \times \mathcal{B})_0 = \mathcal{A}_0 \times \mathcal{B}_0$.
2. $(\mathcal{A} \times \mathcal{B})((A, B), (A', B')) = \{(a, b) \mid a : A \to A' \text{ in } \mathcal{A}, b : B \to B' \text{ in } \mathcal{B}\}$.
3. The compostion of $\mathcal{A} \times \mathcal{B}$ is induced by the compositions of $\mathcal{A}$ and $\mathcal{B}$, namely

$$(a', b')(a, b) = (a'a, b'b).$$

Definition 1.13 If $A \in \mathcal{A}_0$ is a fixed object of a category $\mathcal{A}$ then there is a **category** $(A \downarrow \mathcal{A})$ **of objects under** A, defined as follows.

1. Objects are pairs (B, f) where $f : A \to B$.
2. A morphism $h : (B, f) \to (B', f')$ is a morphism $h : B \to B'$ of $\mathcal{A}$ such that $hf = f'$.
3. The composition in $(A \downarrow \mathcal{A})$ is induced by the composition in $\mathcal{A}$.



Similarly one can construct the category $(\mathcal{A} \downarrow A)$ of objects over A.

Example 1.14 If $(P, \leq)$ is a poset considered as a category $\mathcal{P}$ and $a \in P$, then $(a \downarrow \mathcal{P})$ is the set of all elements, greater than or equal to a, i.e. the upper cone induced by a.

Exercises 1.15
1. Choose some objects and morphisms between them in such a way that they form a category. Explain why they form a category. In what follows, let us call this category "your favourite category".
2. Give some example of a full and non-full subcategory of your favourite category.

2 Properties of morphisms and objects

2.1 Properties of morphisms

Definition 2.1 *A morphism $f : A \to B$ in a category $\mathcal{C}$ is called*

1. *a* ***monomorphism*** *if it is left cancellable, i.e.*

$$fg = fh \Rightarrow g = h$$

for every pair of morphisms $g, h : C \to A$;

2. *a* ***split monomorphism*** *(or section or coretraction) if it is left invertible, i.e. there exists a morphism $g : B \to A$ such that $gf = 1_A$. In that case A is called a* ***retract*** *of B.*

Proposition 2.2 *In a category $\mathcal{C}$,*

1. *every split monomorphism is a monomorphism;*

2. *every identity morphism is a split monomorphism;*

3. *the composite of two (split) monomorphisms is a (split) monomorphism;*

4. *if the composite kf of two morphisms is a (split) monomorphism then f is a (split) monomorphism.*

Proof. 1. Suppose that $kf = 1_A$ and $fg = fh$ where $f : A \to B$, $k : B \to A$ and $g, h : C \to A$. Then

$$g = 1_A g = (kf)g = k(fg) = k(fh) = (kf)h = 1_A h = h.$$

2. For every $A \in \mathcal{C}_0$, $1_A = 1_A 1_A$.
3. Suppose $k : B \to D$ and $f : A \to B$ are monomorphisms and $(kf)g = (kf)h$, where $g, h : C \to A$.

$$C \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B \xrightarrow{k} D$$

Then $k(fg) = k(fh)$ and hence $fg = fh$, because k is a monomorphism. Since f is a monomorphism, $g = h$. Thus we have shown that kf is a monomorphism.

If $sk = 1_B$ and $tf = 1_A$ for some $s : D \to B$ and $t : B \to A$ then

$$(ts)(kf) = t(sk)f = t1_B f = tf = 1_A$$

shows that kf is a split monomorphism.

4. Suppose that kf is a monomorphism and $fg = fh$, where $f : A \to B$, $g, h : C \to A$, $k : B \to D$. Then

$$(kf)g = k(fg) = k(fh) = (kf)h$$

implies $g = h$. Hence f is a monomorphism.

If kf is a split monomorphism, i.e. $s(kf) = 1_A$ for some $s : D \to A$ then $(sk)f = 1_A$ means that also f is a split monomorphism. ■

Definition 2.3 A morphism $f : A \to B$ in a category $\mathcal{C}$ is called

1. an **epimorphism** if it is right cancellable, i.e.

$$gf = hf \Rightarrow g = h$$

for every pair of morphisms $g, h : B \to C$;

2. a **split epimorphism** (or retraction) if it is right invertible, i.e. there exists a morphism $g : B \to A$ such that $fg = 1_B$.

Proposition 2.4 *In a category $\mathcal{C}$,*

1. *every split epimorphism is an epimorphism;*

2. *every identity morphism is a split epimorphism;*

3. *the composite of two (split) epimorphisms is a (split) epimorphism;*

4. *if the composite kf of two morphisms is a (split) epimorphism*
then k is a (split) epimorphism.

Definition 2.5 A category is called a **concrete category** if its objects are sets (usually with some structure), morphisms are mappings (preserving that structure), composition of morphisms is composition of mappings and identity morphisms are identity mappings.

Set, Gr, Rng, Top, Ban_∞, Pos and many others are examples of concrete categories. Rel is not a concrete category. In concrete categories, more can be said about relationships between different types of morphisms.

Proposition 2.6 *In a concrete category, the following implications are valid for a morphism:*

$$\text{split monomorphism} \overset{(smi)}{\Longrightarrow} \text{injective} \overset{(im)}{\Longrightarrow} \text{monomorphism},$$
$$\text{split epimorphism} \overset{(ses)}{\Longrightarrow} \text{surjective} \overset{(se)}{\Longrightarrow} \text{epimorphism}.$$

Proof. (smi). Suppose that for $f : A \to B$ there is $g : B \to A$ such that $gf = 1_A$. If $f(a_1) = f(a_2)$, $a_1, a_2 \in A$, then also $a_1 = (gf)(a_1) = g(f(a_1)) = g(f(a_2)) = (gf)(a_2) = a_2$, i.e. f is injective.

(im). Suppose that $f : A \to B$ is injective and $fg = fh$ for $g, h : C \to A$. Then for every $c \in C$, $f(g(c)) = f(h(c))$, which implies $g(c) = h(c)$. Thus $g = h$ and we have proven that f is a monomorphism.

It is not difficult to see that also (ses) and (se) hold. ∎

Example 2.7 In the category Set, the monomorphisms are exactly the injective mappings. By Proposition 2.6 we already know that injective mappings are monomorphisms, so let us prove the converse. Let $f : A \to B$ be a monomorphism and for every $a \in A$, let $g_a : \{*\} \to A$ be a mapping from a singleton to A, that maps $*$ to a. If $f(a) = f(a')$, $a, a' \in A$, then $fg_a(*) = fg_{a'}(*)$, or $fg_a = fg_{a'}$. By the assumption, $g_a = g_{a'}$, which is equivalent to $a = a'$. Hence f is injective.

Example 2.8 In the category Top, the monomorphisms are exactly the injective continuous mappings. Indeed, considering the singleton topological space $\{*\}$ and any other topological space A we notice that the mappings $g_a : \{*\} \to A$, $a \in A$, defined in Example 2.7, are continuous. Hence the proof of Example 2.7 can be carried over.

Example 2.9 In the categories Gr and Ab, the monomorphisms are exactly the injective group homomorphisms. Again, we only need to show that monomorphisms are injective. Let $f : G \to H$ be a monomorphism of groups and for every $a \in G$, let $g_a : \mathbb{Z} \to G$ be a mapping

$$g_a(z) := a^z.$$

Clearly g_a is a group homomorphism. If $f(a) = f(a')$, then $fg_a(1) = fg_{a'}(1)$ and it follows that $fg_a = fg_{a'}$. By the assumption, $g_a = g_{a'}$ and so $a = a'$.

Example 2.10 In the category Ban_1, the monomorphisms are exactly the injective linear contractions. The elements of the unit ball of a Banach space B are in bijective correspondence with the linear contractions $\mathbb{R} \to B$. Indeed, if $||b|| \le 1$ then $k_b : \mathbb{R} \to B$, defined by

$$k_b(r) := rb,$$

$r \in \mathbb{R}$, is a linear contraction. Conversely, if $k : \mathbb{R} \to B$ is a linear contraction then $||k(1)|| \le |1| = 1$, that is $k(1)$ belongs to the unit ball of B. Clealry these constructions are inverse to each other.

Suppose now that $f : B \to C$ is a monomorphism of Banach spaces and $f(b) = f(b')$. If $||b||, ||b'|| \le 1$ then $fk_b(1) = fk_{b'}(1)$, which implies $fk_b = fk_{b'}$. Since f is a monomorphism, $k_b = k_{b'}$ and hence $b = b'$. Thus f is injective on unit ball. By the linearity of f this extends to the whole B.

Example 2.11 The implication (im) is not reversible, i.e. not all monomorphisms in concrete categories are injective. Consider the category Div of divisible abelian groups. The natural surjection $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ on the quotient group of a divisible group $\mathbb{Q}$ by $\mathbb{Z}$ is obviously not injective. Let us show that it is a monomorphism. Suppose that A is a divisible abelian group and $f, g : A \to \mathbb{Q}$ are group homomorphisms such that $\pi f = \pi g$. Setting $h := f - g$ we have $\pi h = 0$ and the claim becomes $h = 0$. For an element $a \in A$, $\overline{0} = \pi h(a) = \overline{h(a)}$ means that $h(a) \in \mathbb{Z}$. If $h(a) \neq 0$ then

$$h\left(\frac{a}{2h(a)}\right) = \frac{1}{2}$$

and therefore $\pi h\left(\frac{a}{2h(a)}\right) = \overline{\left(\frac{1}{2}\right)} \neq \overline{0}$, a contradiction.

Example 2.12 Also the implication (smi) is not reversible. In Set, the empty mapping $\emptyset \to A$ is injective but has no left inverse if $A \neq \emptyset$. However, every injective mapping $B \to A$ in Set, where $B \neq \emptyset$, is a split monomorphism.

Example 2.13 In Set, the epimorphisms are exactly the surjective mappings. Let $f : A \to B$ be an epimorphism, consider the two-element set $\{0,1\}$ with the mappings $g, h : B \to \{0,1\}$ defined by

$$\begin{aligned} g(b) &= \begin{cases} 1, & \text{if } b \in f(A), \\ 0, & \text{if } b \notin f(A), \end{cases} \\ h(b) &= 1 \text{ for every } b \in B. \end{aligned}$$

Clearly $gf = hf$ is the constant mapping on 1. Hence $g = h$, which means that $f(A) = B$. The converse follows from Proposition 2.6.

Moreover, in Set every surjective mapping is a split epimorphism. Let $f : A \to B$ be a surjective mapping. We define a mapping $g : B \to A$ by choosing for every $b \in B$ an element $a \in A$ such that $f(a) = b$ and setting $g(b) := a$. Then clearly $fg(b) = f(a) = b$ for every $b \in B$, so $fg = 1_B$. (Note that the axiom of choice is exactly what is needed to make all these generally infinitely many choices.)

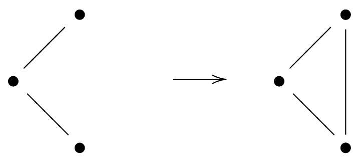
Example 2.14 In the category Top, the epimorphisms are exactly the surjective continuous mappings. The proof of Example 2.13 applies if we use the topological space $\{0,1\}$ with the topology $\{\emptyset, \{0,1\}\}$ and the same g and h.

Example 2.15 In the category Ban_1, the epimorphisms are the linear contractions with the dense image. (A linear mapping $f : A \to B$ in Ban_1 has dense image if $\overline{f(A)} = B$, that is, for every $b \in B$ and every $\varepsilon > 0$ there exists $a \in A$ such that $||f(a) - b|| < \varepsilon$.)

Suppose that $f : A \to B$ is an epimorphism. Then $\overline{f(A)}$ is a closed subspace of B and $B/\overline{f(A)}$ is a Banach space. Both the canonical surjection $\pi : B \to B/\overline{f(A)}$ and the zero mapping $0 : B \to B/\overline{f(A)}$ are linear contractions. From the equalities $\pi f = 0 = 0f$ we conclude $\pi = 0$, which means $B = \overline{f(A)}$.

Conversely, suppose that $\overline{f(A)} = B$ and $gf = hf$, $g, h : B \to C$. Since g and h agree on $f(A)$, by continuity g, h agree also on $\overline{f(A)} = B$. Hence $g = h$.

Example 2.16 The implication (ses) is not reversible. In the category Graph there is a surjective homomorphism



which has no right inverse.

Example 2.17 The implication (se) is not reversible. Consider the category Mon of monoids and monoid homomorphisms and the inclusion i of the monoid $(\mathbb{N} \cup \{0\}, +)$ into $(\mathbb{Z}, +)$, which is definitely not surjective. However it turns out to be an epimorphism.

Suppose that $f : (\mathbb{Z}, +) \to (S, \cdot)$ is a monoid homomorphism. For every $n \in \mathbb{N}$,

$$f(n) = f(1 + \cdots + 1) = f(1)^n,$$

i.e. $f(n)$ is determined by $f(1)$. Also, $f(1)$ is the inverse of $f(-1)$, because

$$f(1)f(-1) = f(-1)f(1) = f(-1 + 1) = f(0) = 1.$$

Since every element of S can have only one inverse, $f(-1)$ is determined by $f(1)$. Also the value of f at every other negative integer is determined by $f(1)$. Hence the homomorphism f is completely determined by $f(1)$.

Now suppose that $g, h : \mathbb{Z} \to S$ are such that $gi = hi$. Then $g(1) = gi(1) = hi(1) = h(1)$ and hence $g = h$. Thus i is an epimorphism.

Definition 2.18 A morphism is called a **bimorphism**, if it is both monomorphism and epimorphism, i.e. if it is cancellable.

Definition 2.19 A morphism is called an **isomorphism**, if it is both split monomorphism and split epimorphism, i.e. if it is invertible. Two objects A and B of a category $\mathcal{C}$ are **isomorphic** if there exists an isomorphism $f : A \to B$. We write $A \cong B$ to denote isomorphism.

Proposition 2.20 *In a category $\mathcal{C}$,*

1. *every isomorphism is a bimorphism;*
2. *every identity morphism is an isomorphism;*
3. *the composite of two bimorphisms (isomorphisms) is a bimorphism (isomorphism).*

Proof. Follows from Proposition 2.2 and Proposition 2.4. ■

Proposition 2.21 *In a category, if an epimorphism is a split monomorphism, it is an isomorphism.*

Proof. Exercise. ■

Example 2.22 In Set the isomorphisms are exactly the bijective mappings.

Example 2.23 In the categories Gr, Ab and Rng, the isomorphisms are exactly the bijective homomorphisms.

Example 2.24 In the category Top, the isomorphisms are exactly the homeomorphisms. Since continuous bijections need not be homeomorphisms, this provides an example of a category where bimorphisms need not be isomorphisms.

Example 2.25 In the category $\mathsf{Vec}_{\mathbb{R}}$, the isomorphisms are exactly the bijective linear mappings.

Example 2.26 In the category Ban_1, the isomorphisms are precisely the isometric bijections. Every isometric bijection $f : A \to B$ is an isomorphism, because $||b|| = ||ff^{-1}(b)|| = ||f^{-1}(b)||$ for every $b \in B$ and hence f^{-1} is a linear contraction. Conversely, if the linear contraction $f : A \to B$ has an inverse mapping $f^{-1} : B \to A$ which is also a linear contraction, then f^{-1} is isometric, because

$$||a|| = ||f^{-1}f(a)|| \leq ||f(a)||$$

and $||f(a)|| \leq ||a||$ for every $a \in A$ by the contracting properties of f^{-1} and f respectively.

Example 2.27 Every group can be viewed as a one-object category where all morphisms are isomorphisms.

Example 2.28 Recall that every poset can be regarded as a category. In such category every morphism is a bimorphism because between any two objects there is at most one morphism. But isomorphisms are only the identity morphisms.

2.2 Properties of objects

Definition 2.29 An object $\mathbf{1}$ of a category $\mathcal{C}$ is called a **terminal object** if from every object C of $\mathcal{C}$ there is exactly one morphism from C to $\mathbf{1}$. An object $\mathbf{0}$ of $\mathcal{C}$ is **initial** if there is exactly one morphism from $\mathbf{0}$ to every object of $\mathcal{C}$. An object is a **zero object** if it is both terminal and initial.

Proposition 2.30 *Every two terminal (initial, zero) objects of a category are isomorphic.*

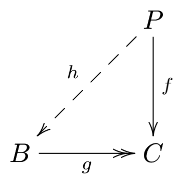
Proof. If $C, C' \in \mathcal{C}_0$ are terminal objects then $\mathcal{C}(C, C) = \{1_C\}$ and $\mathcal{C}(C', C') = \{1_{C'}\}$. Also there exist morphisms $f : C \to C'$ and $g : C' \to C$. Since $gf : C \to C$, $gf = 1_C$, and similarly $fg = 1_{C'}$. ■

Example 2.31 In the category Set, the empty set is the initial object and singletons are terminal objects. The same is true for the category Top.

Example 2.32 In the categories Ab, $\mathsf{Vec}_{\mathbb{R}}$, Ban_1 etc., $\{0\}$ is both the initial and terminal object, hence the zero object.

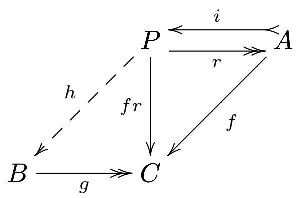
Example 2.33 In the category Rng of rings with unit and homomorphisms that preserve unit, $\{0\}$ is the terminal object and $\mathbb{Z}$ is the initial object.

Definition 2.34 An object P of a category $\mathcal{C}$ is called **projective** if for every epimorphism $g : B \to C$ and every morphism $f : P \to C$ there exists a morphism $h : P \to B$ such that $gh = f$.



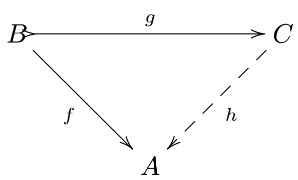
Proposition 2.35 *A retract of a projective object is projective.*

Proof. In the following diagram, let P be projective and let A with $ri = 1_A$ be its retract (see Definition 2.1).



Given $f : A \to C$, by projectivity of P there exists $h : P \to B$ such that $gh = fr$. Hence $ghi = fri = f$ ■

Definition 2.36 An object A of a category $\mathcal{C}$ is called **injective** if for every monomorphism $g : B \to C$ and every morphism $f : B \to A$ there exists a morphism $h : C \to A$ such that $hg = f$.



Example 2.37 Every object of Set is projective. Using the fact that every epimorphism splits in Set and the notation of Definition 2.34, we can find $p : C \to B$ with $gp = 1_C$. Setting $h := pf$ we obtain $gh = gpf = f$.

Example 2.38 A classical result, the so-called Baer criterion for injectivity, states that an abelian group is injective if and only if it is divisible.

Example 2.39 Projective and injective objects play an important role in the theory of module over rings. A classical result is that a module is projective if and only if it is a direct summand of a free module.

Exercises 2.40

1. Prove that if an epimorphism is a split monomorphism then it is an isomorphism.
2. Prove that in a concrete category, every split epimorphism is surjective (see Proposition 2.6).
3. Prove that in a concrete category, every surjective morphism is an epimorphism (see Proposition 2.6).
4. What are the monomorphisms, epimorphisms, bimorphisms and isomorphisms in your favourite category?
5. Does your favourite category have terminal, initial or zero object?
6. Prove directly, that every object of Set is projective (see Example 2.37).
7. What are the projective objects of your favourite category?
8. Prove that an object of Set is injective if and only if it is nonempty.

3 Functors

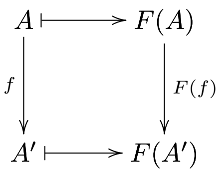
3.1 Covariant and contravariant functors

Definition 3.1 A (**covariant**) **functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of

1. a mapping $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ between the classes of objects of $\mathcal{A}$ and $\mathcal{B}$; the image of $A \in \mathcal{A}_0$ is written $F(A)$;
2. for every pair of objects A, A' of $\mathcal{A}$, a mapping $F_1^{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$; the image of $f \in \mathcal{A}(A, A')$ is written $F(f)$;

such that the following axioms are satisfied:

1. for every pair of morphisms $f \in \mathcal{A}(A, A')$, $g \in \mathcal{A}(A', A'')$,
$$F(gf) = F(g)F(f);$$
2. for every object $A \in \mathcal{A}_0$,
$$F(1_A) = 1_{F(A)}.$$

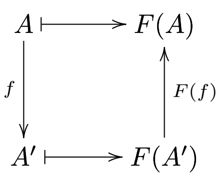


Definition 3.2 A **contravariant functor** F from a category $\mathcal{A}$ to a category $\mathcal{B}$ consists of:

1. a mapping $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ between the classes of objects of $\mathcal{A}$ and $\mathcal{B}$; the image of $A \in \mathcal{A}_0$ is written $F(A)$;
2. for every pair of objects A, A' of $\mathcal{A}$, a mapping $F_1^{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(F(A'), F(A))$; the image of $f \in \mathcal{A}(A, A')$ is written $F(f)$;

such that the following axioms are satisfied:

1. for every pair of morphisms $f \in \mathcal{A}(A, A')$, $g \in \mathcal{A}(A', A'')$,
$$F(gf) = F(f)F(g);$$
2. for every object $A \in \mathcal{A}$,
$$F(1_A) = 1_{F(A)}.$$



Examples 3.3 Let us list some examples of (covariant) functors.

1. For every category $\mathcal{A}$, there is the **identity functor** $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, defined by $1_{\mathcal{A}}(A) = A$ on objects and by $1_{\mathcal{A}}(f) = f$ on morphisms.
2. Every subcategory $\mathcal{B}$ of a category $\mathcal{A}$ induces in a natural way the **inclusion functor** $\mathcal{B} \to \mathcal{A}$, which is just the restriction of the identity functor $1_{\mathcal{A}}$ to $\mathcal{B}$.

3. If $\mathcal{A}$ and $\mathcal{B}$ are categories and $B \in \mathcal{B}_0$ is a fixed object then there is the **constant functor** on B, $\Delta_B^{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ (or simply Δ_B), defined by
$$\Delta_B(A) := B, \quad \Delta_B(f) := 1_B$$
for every $A \in \mathcal{A}_0$ and every morphism f of $\mathcal{A}$.

4. The **forgetful functor** $U : \mathsf{Gr} \to \mathsf{Set}$ maps a group $(A, \cdot)$ to its underlying set A and group homomorphisms to the corresponding mappings. Also, one can for example consider forgetful functors $\mathsf{Rng} \to \mathsf{Ab}$ (forgets multiplication), $\mathsf{Rng} \to \mathsf{Mon}$ (forgets addition) or $\mathsf{Ban}_1 \to \mathsf{Vec}_{\mathbb{R}}$ (forgets norm).

5. The free group functor $F : \mathsf{Set} \to \mathsf{Gr}$ assigns to every set A the free group with a set of generators A, and to each mapping f of sets the induced homomorphism that coincides with f on the generators.

6. For every group A, let A' be the commutator subgroup of A, i.e. the subgroup generated by all elements of the form $aba^{-1}b^{-1}$. Define $F : \mathsf{Gr} \to \mathsf{Ab}$ by
$$F(A) := A/A',$$
$$F(h)(aA') := h(a)B',$$
for all groups A, B and group homomorphisms $h : A \to B$. Then F is a functor which is called the **abelianization functor**.

7. If R is a ring and M_R is a fixed right R-module then there is a functor $M_R \otimes - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ of tensor multiplication by M_R (from the category of left R-modules to the category Ab of abelian groups) defined by
$$(M_R \otimes -)({}_RN) := M \otimes_R N, \quad M_R(f) := 1_M \otimes f$$
for every left R-module ${}_RN$ and every homomorphism f of left R-modules.

Examples 3.4 There are also many examples of contravariant functors.

1. There is a functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, which takes every set A to its powerset $\mathcal{P}(A)$ and a mapping $f : A \to B$ to the inverse image mapping $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$, which maps a subset C of B to $f^{-1}(C) \subseteq A$.

2. If (X, τ) is a topological space then the set τ of all open subsets U of X is a partially ordered set with respect to inclusion relation and hence gives rise to a category; there is a morphism $i_U^V : V \to U$ if and only if $V \subseteq U$. Let $\overline{C}(U) = \{h : U \to \mathbb{R} \mid h \text{ is continuous }\}$. Then $\overline{C} : \tau \to \mathsf{Set}$ becomes a functor if we define a mapping $\overline{C}(i_U^V) : \overline{C}(U) \to \overline{C}(V)$ by
$$\overline{C}(i_U^V)(h) := h|_V.$$

Next we introduce functors which play a fundamental role in the theory of categories.

Example 3.5 For a fixed object C of a category $\mathcal{C}$, the assignment

$$\begin{array}{ccc} A & \longmapsto & \mathcal{C}(C, A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\circ -} \\ B & \longmapsto & \mathcal{C}(C, B) \end{array}$$

defines a covariant functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathsf{Set}$, which is called a **covariant representable functor** (the functor is represented by C) or a **covariant hom-functor**. (Note that $\mathcal{C}(C, f)$ is often written instead of $f \circ -$.)

Example 3.6 For a fixed object C of a category $\mathcal{C}$, the assignment

$$\begin{array}{ccc} A & \longmapsto & \mathcal{C}(A, C) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle -\circ f} \\ B & \longmapsto & \mathcal{C}(B, C) \end{array}$$

defines a contravariant functor $\mathcal{C}(-, C) : \mathcal{C} \to \mathsf{Set}$, which is called a **contravariant representable functor** or a **contravariant hom-functor**.

For functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ we can form their composite $GF : \mathcal{A} \to \mathcal{C}$ (which again will be a functor) by defining

$$(GF)(A) := G(F(A)), \quad GF(f) := G(F(f))$$

for every object $A \in \mathcal{A}_0$ and every morphism f of $\mathcal{A}$. Such composition will be associative and identity functors will act as identities with respect to this composition. This can lead to an idea of forming the "category of all categories" where functors would play the role of morphisms. Unfortunately, the collection of all categories is not a class anymore and also the collection of all functors from one category to another needs not to be a set. A way out is to modify the definition of category by allowing bigger collections (sometimes called conglomerates) for objects and morphisms. We shall call such things **quasi-categories** (following Herrlich and Strecker). Thus the collection of all categories and functors between them equipped with the composition of functors is a quasicategory which is denoted by CAT. However, if $\mathcal{A}$ is a small category and $\mathcal{B}$ is arbitrary category then the collection $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ of all functors from $\mathcal{A}$ to $\mathcal{B}$ is a class, and if $\mathcal{B}$ is also small then $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ is a set. Hence we can speak about the **category of all small categories**, which is denoted by Cat.

3.2 On duality

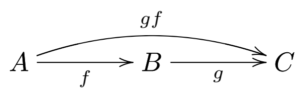
Roughly speaking, categorical duality is the process "Reverse all morphisms".

Definition 3.7 For a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{\mathrm{op}}$ is defined as follows.

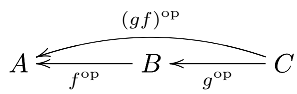
1. $\mathcal{C}_0^{\mathrm{op}} = \mathcal{C}_0$ ($\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ have the same objects).
2. For all $A, B \in \mathcal{C}_0^{\mathrm{op}}$, $\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A)$, so the morphisms of $\mathcal{C}^{\mathrm{op}}$ are the morphisms of $\mathcal{C}$ "written in the reverse direction". To avoid confusion, we write $f^{\mathrm{op}} : A \to B$ if we consider a morphism $f : B \to A$ of $\mathcal{C}$ as a morphism of $\mathcal{C}^{\mathrm{op}}$. Note that $(f^{\mathrm{op}})^{\mathrm{op}} = f$ and the morphisms 1_A and 1_A^{op} are identified.
3. The composition in $\mathcal{C}^{\mathrm{op}}$ is defined by

$$f^{\mathrm{op}} g^{\mathrm{op}} := (gf)^{\mathrm{op}}.$$

In $\mathcal{C}$:



In $\mathcal{C}^{\mathrm{op}}$:



For every notion defined in a category, there is a dual notion (usually named using the prefix 'co-') which is obtained by reversing the direction of all morphisms involved and replacing every composite fg by the composite gf. Similarly, every statement has a dual statement.
The duality principle asserts that if some statement is true in every category then also its dual statement is true in every category.

So we see that projectivity is the dual notion of injectivity, terminal objects are dual to initial objects etc. Also, for example, if we know that terminal objects are determined uniquely up to isomorphism then the duality principle implies that also initial objects are determined uniquely up to isomorphism.

If $F : \mathcal{A} \to \mathcal{B}$ is a contravariant functor, we define a functor $\hat{F} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ by

$$\hat{F}(A) := F(A), \quad \hat{F}(f^{\mathrm{op}}) := F(f).$$

Then

$$\hat{F}(f^{\mathrm{op}} g^{\mathrm{op}}) = \hat{F}((gf)^{\mathrm{op}}) = F(gf) = F(f)F(g) = \hat{F}(f^{\mathrm{op}})\hat{F}(g^{\mathrm{op}})$$

and $\hat{F}(1_A^{\mathrm{op}}) = F(1_A) = 1_{F(A)} = 1_{\hat{F}(A)}^{\mathrm{op}}$, which makes $\hat{F}$ a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$.

Conversely, if $G : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is a covariant functor, we define $\overline{G} : \mathcal{A} \to \mathcal{B}$ by

$$\overline{G}(A) := G(A), \quad \overline{G}(f) := G(f^{\mathrm{op}}).$$

Then

$$\overline{G}(gf) = G((gf)^{\rm op}) = G(f^{\rm op}g^{\rm op}) = G(f^{\rm op})G(g^{\rm op}) = \overline{G}(f)\overline{G}(g),$$

$\overline{G}(1_A) = G(1_A^{\rm op}) = 1_{G(A)} = 1_{\overline{G}(A)}$, so $\overline{G}$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$.

Moreover, if $F : \mathcal{A} \to \mathcal{B}$ is a contravariant functor and f a morphism of $\mathcal{A}$ then $\overline{\hat{F}}(f) = \hat{F}(f^{\rm op}) = F(f)$, and if $G : \mathcal{A}^{\rm op} \to \mathcal{B}$ is a covariant functor and $f^{\rm op}$ a morphism of $\mathcal{A}^{\rm op}$ then $\hat{\overline{G}}(f^{\rm op}) = \overline{G}(f) = G(f^{\rm op})$. Thus we have proven the following

Proposition 3.8 *For categories $\mathcal{A}$ and $\mathcal{B}$, there is one-to-one correspondence between contravariant functors $\mathcal{A} \to \mathcal{B}$ and covariant functors $\mathcal{A}^{\rm op} \to \mathcal{B}$.*

Having this result in mind, in what follows, by default we always speak about covariant functors (either from $\mathcal{A}$ or $\mathcal{A}^{\rm op}$). Note also that sometimes it may be convenient to regard contravariant functors $\mathcal{A} \to \mathcal{B}$ as covariant functors $\mathcal{A} \to \mathcal{B}^{\rm op}$.

3.3 Some properties of functors

Definition 3.9 Consider a functor $F : \mathcal{A} \to \mathcal{B}$ and for every pair of objects $A, A' \in \mathcal{A}_0$ the mapping

$$F_1^{A,A'} : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A')), \quad f \mapsto F(f).$$

The functor F is

1. **faithful** if the mapping $F_1^{A,A'}$ is injective for every $A, A' \in \mathcal{A}_0$;
2. **full** if the mapping $F_1^{A,A'}$ is surjective for every $A, A' \in \mathcal{A}_0$;
3. **full and faithful** if the mapping $F_1^{A,A'}$ is bijective for every $A, A' \in \mathcal{A}_0$;
4. an **isomorphism of categories** if it is full and faithful and induces a bijection $\mathcal{A}_0 \to \mathcal{B}_0$ between the classes of objects.

For example, the forgetful functor $U : \mathsf{Gr} \to \mathsf{Set}$ is faithful but not full and not a bijection between objects.

Definition 3.10 Consider a functor $F : \mathcal{A} \to \mathcal{B}$.

1. F **preserves monomorphisms** if, for every morphism f of $\mathcal{A}$,

$$f \text{ monomorphism } \Rightarrow F(f) \text{ monomorphism }.$$

2. F **reflects monomorphisms** if, for every morphism f of $\mathcal{A}$,

$$F(f) \text{ monomorphism } \Rightarrow f \text{ monomorphism }.$$

Similar definitions can, of course, be given for all other types of morphisms.

Proposition 3.11 *A faithful functor reflects monomorphisms and epimorphisms.*

Proof. Consider a faithful functor $F : \mathcal{A} \to \mathcal{B}$ and a morphism $f : A \to A'$ in $\mathcal{A}$ such that $F(f) : F(A) \to F(A')$ is a monomorphism in $\mathcal{B}$. Suppose that $fg = fh$ for some $g, h : A'' \to A$. Then

$$fg = fh \Rightarrow F(f)F(g) = F(f)F(h) \Rightarrow F(g) = F(h) \Rightarrow g = h,$$

where the second implication holds because $F(f)$ is a monomorphism and the last implication holds because F is faithful. Similarly if $F(f)$ is an epimorphism and $gf = hf$ for some $g, h : A' \to A''$, then

$$gf = hf \Rightarrow F(g)F(f) = F(h)F(f) \Rightarrow F(g) = F(h) \Rightarrow g = h.$$

■

Proposition 3.12 *Every functor preserves isomorphisms.*

Proof. If $kf = 1_A$ for $f : A \to B$ and $k : B \to A$ then $F(k)F(f) = F(1_A) = 1_{F(A)}$. This proves that F preserves both split monomorphisms and split epimorphisms, and hence isomorphisms. ■

Proposition 3.13 *A full and faithful functor reflects isomorphisms.*

Proof. Suppose $F : \mathcal{A} \to \mathcal{B}$ is a full and faithful functor, $f : A \to A'$ in $\mathcal{A}$ and $F(f) : F(A) \to F(A')$ is an isomorphism in $\mathcal{B}$. Then $F(f)s = 1_{F(A')}$ and $sF(f) = 1_{F(A)}$ for some $s : F(A') \to F(A)$. Since F is full, $s = F(g)$ for some $g : A' \to A$. Hence $F(fg) = F(1_{A'})$ and $F(gf) = F(1_A)$ imply $fg = 1_{A'}$, $gf = 1_A$ by faithfulness of F. Thus f is an isomorphism. ■

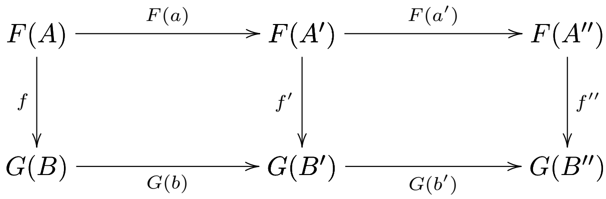
3.4 Comma categories

Using a given functor (or functors), it is possible to construct new categories. We present some of such constructions.

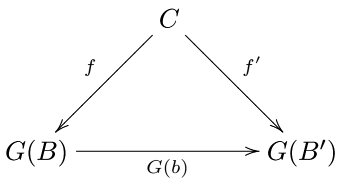
Definition 3.14 Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. The **comma category** $(F \downarrow G)$ (also written (F, G)) is defined in the following way.

1. The objects of $(F \downarrow G)$ are the triples (A, f, B) where $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ and $f : F(A) \to G(B)$ is a morphism of $\mathcal{C}$.
2. A morphism from (A, f, B) to (A', f', B') in $(F \downarrow G)$ is a pair (a, b) where $a : A \to A'$ in $\mathcal{A}$, $b : B \to B'$ in $\mathcal{B}$ and $f'F(a) = G(b)f$.
3. The composition law of $(F \downarrow G)$ is induced by the composition laws of $\mathcal{A}$ and $\mathcal{B}$, that is,

$$(a', b')(a, b) = (a'a, b'b).$$



Example 3.15 In the previous definition, let $\mathcal{A} = \mathbf{1}$ be the discrete category with a single object $\star$ and let $C \in \mathcal{C}_0$. Then C may be regarded as a functor $F_C : \mathbf{1} \to \mathcal{C}$, $\star \mapsto C$. Taking $F = F_C$ we obtain a category $(F_C \downarrow G)$ where objects are triples $(\star, f, B)$ where $B \in \mathcal{B}_0$ and $f : C \to G(B)$, and morhisms $(\star, f, B) \longrightarrow (\star, f', B')$ are pairs $(1_\star, b)$ where $b : B \to B'$ is such that $f' = f'F(1_\star) = G(b)f$. Clearly this category is isomorphic to a category with objects (f, B), $B \in \mathcal{B}_0$, $f : C \to G(B)$, where morphisms $(f, B) \to (f', B')$ are morphisms $b : B \to B'$ such that $f' = G(b)f$. We denote the last category by $(C \downarrow G)$ and call it the **category of objects G-under** C.

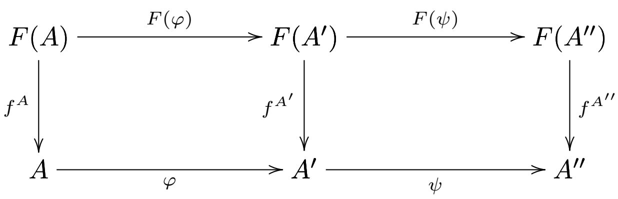


If we further take $G = 1_\mathcal{C}$ the identity functor of $\mathcal{C}$, we obtain precisely the category $(C \downarrow 1_\mathcal{C}) = (C \downarrow \mathcal{C})$ of objects under $\mathcal{C}$ (cf. Def. 1.13). Similarly one obtains the category of objects F-over C and the category of objects over C.

Another construction, which is very similar to the construction of comma categories, is the construction of algebras of an endofunctor. (A functor $\mathcal{A} \to \mathcal{A}$ is called an **endofunctor** of $\mathcal{A}$.)

Definition 3.16 The **category** $\mathsf{Alg}(F)$ **of algebras of an endofunctor** $F : \mathcal{A} \to \mathcal{A}$ (shortly: F-algebras) is constructed as follows.

1. Objects are pairs (A, f^A) where $f^A : F(A) \to A$.
2. A morphism $\varphi : (A, f^A) \to (A', f^{A'})$ is a morphism $\varphi : A \to A'$ of $\mathcal{A}$ such that $f^{A'}F(\varphi) = \varphi f^A$.
3. The composition law of $\mathsf{Alg}(F)$ is induced by the composition law of $\mathcal{A}$.



Example 3.17 Traditionally, a universal algebra $(A, (f_i)_{i \in I})$ is given by a set A and a collection of operations $f_i^A : A^{n_i} \to A$. The operations may be combined into a single mapping $f^A : \bigsqcup_{i \in I} A^{n_i} \to A$, so that a universal algebra is given by a set A and a mapping $f^A : F(A) \to A$, where the endofunctor F of Set, $F(X) = \bigsqcup_{i \in I} X^{n_i}$, determines the type of that algebra. (For example, a group can be regarded as an F-algebra of the Set endofunctor $F(X) = X^0 \sqcup X^1 \sqcup X^2$.) If (A, f^A) and (B, f^B) are two algebras of the same type, then a mapping $\varphi : A \to B$ induces canonically a mapping $F(\varphi) : F(A) \to F(B)$. It can be shown that φ is a homomorhism if and only if the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\varphi)} & F(B) \\ {\scriptstyle f^A}\downarrow & & \downarrow{\scriptstyle f^B} \\ A & \xrightarrow[\varphi]{} & B \end{array} .$$

In computer science, so-called coalgebras have turned out to be very useful.

Definition 3.18 The **category** $\mathsf{Coalg}(F)$ **of coalgebras of an endofunctor** $F : \mathcal{A} \to \mathcal{A}$ (shortly: F-coalgebras) is constructed as follows.

1. Objects are pairs (A, α_A) where $\alpha_A : A \to F(A)$. (If $\mathcal{A} = \mathsf{Set}$, this α_A is called the structure mapping of the coalgebra (A, α_A).)
2. A morphism $\varphi : (A, \alpha_A) \to (A', \alpha_{A'})$ is a morphism $\varphi : A \to A'$ of $\mathcal{A}$ such that $F(\varphi)\alpha_A = \alpha_{A'}\varphi$.
3. The composition law of $\mathsf{Coalg}(F)$ is induced by the composition law of $\mathcal{A}$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ F(A) & \xrightarrow[F(\varphi)]{} & F(A') \end{array}$$

Example 3.19 Finite and infinite Σ-lists can be modelled as a coalgebra of the functor $F : \mathsf{Set} \to \mathsf{Set}$, defined by

$$F(X) := \{*\} \sqcup \Sigma \times X.$$

On the set Σ^∞ of all finite and infinite sequences of elements of Σ, the structure mapping $\alpha_{\Sigma^\infty} : \Sigma^\infty \to \{*\} \sqcup \Sigma \times \Sigma^\infty$ is defined by

$$\alpha_{\Sigma^\infty}(\sigma) := \begin{cases} *, & \text{if } \sigma \text{ is the empty sequence,} \\ (\mathrm{head}(\sigma), \mathrm{tail}(\sigma)), & \text{if } \sigma \text{ is a nonempty sequence.} \end{cases}$$

Example 3.20 Nonempty binary trees with leaves of type Σ can be modelled as a coalgebra of the functor $F : \mathsf{Set} \to \mathsf{Set}$, defined by

$$F(X) := \Sigma \sqcup X \times X.$$

On the set T of all such trees, the structure mapping $\alpha_T : T \to \Sigma \sqcup T \times T$ is defined by

$$\alpha_T(t) := \begin{cases} t \in \Sigma, & \text{if } t \text{ is a leaf,} \\ (\mathrm{left}(t), \mathrm{right}(t)), & \text{if } t \text{ is not a leaf.} \end{cases}$$

Example 3.21 Assume that the operations `deposit`(*amount*) and `showbalance` are defined for a data type A = BANK ACCOUNT:

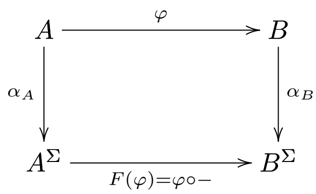
$$\texttt{deposit} : A \times \mathbb{R} \to A,$$
$$\texttt{showbalance} : A \to \mathbb{R}.$$

A bank account type with these operations is a coalgebra for the functor $F(X) = X^{\mathbb{R}} \times \mathbb{R}$. The structure mapping $\alpha_A : A \to A^{\mathbb{R}} \times \mathbb{R}$ has the form

$$\alpha_A(a) = (\texttt{deposit}(a, -), \texttt{showbalance}(a)).$$

Example 3.22 Deterministic automata can be considered as coalgebras. For a fixed set Σ, consider the covariant hom-functor $F = \mathsf{Set}(\Sigma, -) = (-)^\Sigma : \mathsf{Set} \to \mathsf{Set}$. An F-coalgebra is given by a mapping $\alpha_A : A \to A^\Sigma$. Such mappings are in one-to-one correspondense with mappings $\delta_A : A \times \Sigma \to A$, that is, with deterministic automata with input alphabet Σ, set of states A and transition function δ_A, if we define

$$\delta_A(a, \sigma) := \alpha_A(a)(\sigma).$$



A mapping $\varphi : A \to B$ is a homomorphism of F-coalgebras if $\varphi\alpha_A(a) = \alpha_B(\varphi(a))$, that is, if

$$\varphi(\delta_A(a, \sigma)) = \varphi(\alpha_A(a)(\sigma)) = (\varphi\alpha_A(a))(\sigma) = (\alpha_B(\varphi(a)))(\sigma) = \delta_B(\varphi(a), \sigma),$$

for every $a \in A$ and $\sigma \in \Sigma$. The equality $\varphi(\delta_A(a, \sigma)) = \delta_B(\varphi(a), \sigma)$ is exactly the definition of a homomorphism of automata.

Exercises 3.23

1. Construct a functor from your favourite category to some other category of from some category to your favourite category. Does it have any nice properties?
2. Prove that a functor $F : \mathcal{A} \to \mathcal{B}$ is an isomorphism of categories if and only if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that $FG = 1_\mathcal{A}$ and $GF = 1_\mathcal{B}$.
3. Prove that an isomorphism of categories preserves and reflects monomorphisms and epimorphisms.
4. Show that the product $\mathcal{A} \times \mathcal{B}$ of categories $\mathcal{A}$ and $\mathcal{B}$ (see 1.12) can be considered as a comma category.

4 Natural transformations

4.1 Definition and examples of natural transformations

Natural transformations can be regarded as morphisms between functors.

Definition 4.1 Let $F, G : \mathcal{A} \to \mathcal{B}$ be two functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. A **natural transformation** $\alpha : F \Rightarrow G$ from F to G is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}_0}$ of morphisms of $\mathcal{B}$ indexed by the objects of $\mathcal{A}$, such that $\alpha_{A'} F(f) = G(f)\alpha_A$ for every morphism $f : A \to A'$ in $\mathcal{A}$, i.e. the following square is commutative:

$$\begin{array}{cccc} A & F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle f}\downarrow & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ A' & F(A') & \xrightarrow[\alpha_{A'}]{} & G(A') \end{array} .$$

The morphism α_A for an object $A \in \mathcal{A}_0$ is called the **component** of the natural transformation α at A.

Definition 4.2 Let $F, G : \mathcal{A} \to \mathcal{B}$ be two contravariant functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. A **natural transformation** $\alpha : F \Rightarrow G$ from F to G is a family $(\alpha_A : F(A) \to G(A))_{A \in \mathcal{A}_0}$ of morphisms of $\mathcal{B}$ indexed by the objects of $\mathcal{A}$, such that $G(f)\alpha_{A'} = \alpha_A F(f)$ for every morphism $f : A \to A'$ in $\mathcal{A}$, i.e. the following square is commutative:

$$\begin{array}{cccc} A & F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle f}\downarrow & {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ A' & F(A') & \xrightarrow[\alpha_{A'}]{} & G(A') \end{array} .$$

Definition 4.3 A natural transformation $\alpha : F \Rightarrow G$, where $F, G : \mathcal{A} \to \mathcal{B}$, is called a **natural isomorphism** if α_A is an isomorphism in $\mathcal{B}$ for every $A \in \mathcal{A}_0$. Functors F and G are called **naturally isomorphic** (notation: $F \cong G$) if there exists a natural isomorphism $F \Rightarrow G$.

For functors $F, G : \mathcal{A} \to \mathcal{B}$, we denote the conglomerate of all natural transformations from F to G by $\mathsf{Nat}(F, G)$.

Example 4.4 For every functor $F : \mathcal{A} \to \mathcal{B}$ there is the **identity natural transformation** $1_F : F \Rightarrow F$ defined by

$$(1_F)_A := 1_{F(A)}.$$

Example 4.5 Let $f : B \to A$ be a fixed morphism of a category $\mathcal{A}$. The definition

$$\mathcal{A}(f, -)_C(g) := gf,$$

$g : A \to C$, gives a natural transformation $\mathcal{A}(f, -) : \mathcal{A}(A, -) \Rightarrow \mathcal{A}(B, -)$ between the functors represented by A and B because of the commutativity of the following square:

$$\begin{array}{cccc} C & \mathcal{A}(A, C) & \xrightarrow{\mathcal{A}(f,-)_C} & \mathcal{A}(B, C) \\ {\scriptstyle h}\downarrow & {\scriptstyle \mathcal{A}(A,h)=h\circ-}\downarrow & & \downarrow{\scriptstyle \mathcal{A}(B,h)=h\circ-} \\ C' & \mathcal{A}(A, C') & \xrightarrow[\mathcal{A}(f,-)_{C'}]{} & \mathcal{A}(B, C') \end{array} .$$

Dually, for a fixed $f : A \to B$ in $\mathcal{A}$, the definition

$$\mathcal{A}(-, f)_C(g) := fg,$$

$g : C \to A$, gives a natural transformation $\mathcal{A}(-, f) : \mathcal{A}(-, A) \Rightarrow \mathcal{A}(-, B)$ between contravariant representable functors.

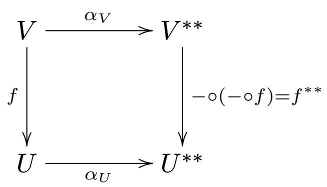
Example 4.6 For a fixed field K we can consider the contravariant hom-functor $\mathsf{Vec}_K(-,K) : \mathsf{Vec}_K \to \mathsf{Set}$. It is well-known that for a vector space V over K, the set $\mathsf{Vec}_K(V,K) = \mathrm{Hom}(V,K) = V^*$ of linear mappings from V to K can be considered as a vector space over K with pointwise operations. Moreover, if $f : V \to U$ is a linear mapping in Vec_K then $\mathsf{Vec}_K(-,K)(f) = - \circ f : U^* \to V^*$ is also a linear mapping. Hence the functor $\mathsf{Vec}_K(-,K)$ can be considered as a contravariant functor $\mathsf{Vec}_K \to \mathsf{Vec}_K$. We denote this functor shortly by $(-)^*$. The composite of $(-)^*$ with itself will be a covariant functor $\mathsf{Vec}_K \to \mathsf{Vec}_K$. We denote it by $(-)^{**}$ and call it the **second dual functor**. For every vector space V we define $\alpha_V : V \to V^{**} = \mathrm{Hom}(\mathrm{Hom}(V,K),K)$ by

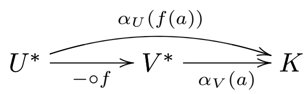
$$(\alpha_V(a))(g) := g(a),$$

$a \in V$, $g \in \mathrm{Hom}(V,K)$. From linear algebra we know that α_V is a linear mapping of vector spaces. For every $f \in \mathrm{Hom}(V,U)$, every $a \in V$ and every $h \in \mathrm{Hom}(U,K) = U^*$,

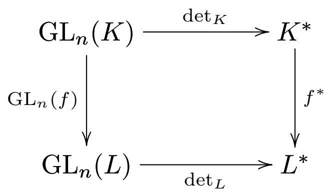
$$(f^{**}\alpha_V)(a)(h) = (\alpha_V(a)(- \circ f))(h) = \alpha_V(a)(hf) = hf(a) = h(f(a)) = \alpha_U(f(a))(h) = (\alpha_U f)(a)(h),$$

which proves that $\alpha = (\alpha_V)_{V \in (\mathsf{Vec}_K)_0} : 1_{\mathsf{Vec}_K} \Rightarrow (-)^{**}$ is a natural transformation.





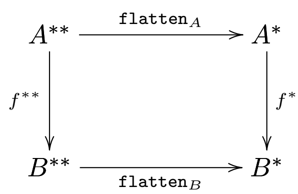
Example 4.7 If K^* is the multiplicative group of a field K then $\det_K : \mathrm{GL}_n(K) \to K^*$ is a homomorphism of groups. Moreover, the square



commutes for every homomorphism $f : K \to L$ of fields (here $\mathrm{GL}_n(f)$ applies f to all entries of a matrix and f^* is the restriction of f to K^*). This means that determinant is a natural transformation $\det : \mathrm{GL}_n \Rightarrow (-)^*$ between two functors $\mathrm{GL}_n, (-)^* : \mathsf{Field} \to \mathsf{Gr}$.

Example 4.8 Suppose that some programming language L (for instance the language from Example 1.8) has a constructor $(-)^*$ which allows for every data type A (i.e. a set of constants) to construct a new type A^* which is the set of all finite lists with elements of type A (shortly: A-lists). For example, if $A = \{a,b\}$, then the lists $[a,b,a], [\,], [b,b,b,b]$ will belong to A^*. If $f : A \to B$ is an operation in language L then f^* will denote the (derived) operation $A^* \to B^*$ which applies f to all elements of a given A-list (e.g. $f^*([a,b,a]) = [f(a), f(b), f(a)] \in B^*$). It is easy to see that $(-)^* : \mathcal{C}(L) \to \mathcal{C}(L)$ is an endofunctor of the category $\mathcal{C}(L)$ induced by the language L.

For a type A, one can also consider the set $(A^*)^* = A^{**}$, consisting of all lists of A-lists, obtained by applying the functor $(-)^*$ twice. For such type A^{**}, one may want to have the operation of **flattening**, which simply concatenates the lists in the list. For example, $\texttt{flatten}([[a,b,a],[\,],[b,b,b,b]]) = [a,b,a,b,b,b,b]$. The collection of all these flattening operations is a natural transformation $(-)^{**} \Rightarrow (-)^*$, because the following square commutes for every operation $f : A \to B$:



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4.2 Categories of functors

Lemma 4.9 *If $F, G, H : \mathcal{A} \to \mathcal{B}$ are functors from $\mathcal{A}$ to $\mathcal{B}$ and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ are natural transformations then the formula*

$$(\beta \circ \alpha)_A := \beta_A \alpha_A,$$

$A \in \mathcal{A}_0$, defines a natural transformation $\beta \circ \alpha : F \Rightarrow H$.

Proof. For every morphism $f : A \to A'$ in $\mathcal{A}$,

$$H(f)(\beta \circ \alpha)_A = H(f)\beta_A\alpha_A = \beta_{A'}G(f)\alpha_A = \beta_{A'}\alpha_{A'}F(f) = (\beta \circ \alpha)_{A'}F(f).$$

$$\begin{array}{ccccc} F(A) & \xrightarrow{\alpha_A} & G(A) & \xrightarrow{\beta_A} & H(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\ F(A') & \xrightarrow[\alpha_{A'}]{} & G(A') & \xrightarrow[\beta_{A'}]{} & H(A') \end{array}$$

■

It easily follows from the definition that the composition $\circ$ of natural transformations is associative and identity natural transformations act as units with repect to this composition. Thus we may form the **quasicategory** $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ **of all functors from $\mathcal{A}$ to $\mathcal{B}$** with morphisms the natural transformations between such functors. If $\mathcal{A}$ is small, this quasicategory becomes a category.

4.3 The Yoneda Lemma

Theorem 4.10 (The Yoneda Lemma) *Let $\mathcal{A}$ be a category, $F : \mathcal{A} \to \mathsf{Set}$ a functor, $A \in \mathcal{A}_0$ and $\mathcal{A}(A, -) : \mathcal{A} \to \mathsf{Set}$ the corresponding representable functor.*

1. *There exists a bijective correspondence*

   $$\theta_{F,A} : \mathsf{Nat}(\mathcal{A}(A, -), F) \longrightarrow F(A)$$

   between the natural transformations from $\mathcal{A}(A, -)$ to F and the elements of the set $F(A)$; in particular those natural transformations constitute a set.

2. *The bijections $\theta_{F,A}$ constitute a natural transformation in the variable A.*

3. *If $\mathcal{A}$ is a small category, the bijections $\theta_{F,A}$ also constitute a natural transformation in the variable F.*

Proof. 1. We define mappings

$$\mathsf{Nat}(\mathcal{A}(A, -), F) \underset{\tau}{\overset{\theta_{F,A}}{\rightleftarrows}} F(A)$$

by

$$\theta_{F,A}(\alpha) := \alpha_A(1_A)$$

for every natural transformation $\alpha : \mathcal{A}(A, -) \Rightarrow F$, and

$$\tau(a)_B(f) := F(f)(a)$$

for every $a \in F(A), B \in \mathcal{A}_0$ and $f : A \to B$. We have to check that $\tau(a) = (\tau(a)_B)_{B \in \mathcal{A}_0}$ is a natural transformation $\mathcal{A}(A, -) \Rightarrow F$. Indeed, for all morphisms $g : B \to C$ and $f : A \to B$ in $\mathcal{A}$,

$$F(g)\tau(a)_B(f) = F(g)F(f)(a) = F(gf)(a) = \tau(a)_C(gf) = \tau(a)_C(g \circ -)(f).$$

$$\begin{array}{ccc} \mathcal{A}(A, B) & \xrightarrow{\tau(a)_B} & F(B) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathcal{A}(A, C) & \xrightarrow[\tau(a)_C]{} & F(C) \end{array}$$

Next, for every $a \in F(A)$ we have

$$\theta_{F,A}\tau(a) = \tau(a)_A(1_A) = F(1_A)(a) = 1_{F(A)}(a) = a,$$

proving that $\theta_{F,A}\tau = 1_{F(A)}$. On the other hand, if $\alpha : \mathcal{A}(A,-) \Rightarrow F$ and $f : A \to B$ in $\mathcal{A}$, then

$$\begin{array}{rcl} (\tau\theta_{F,A}(\alpha))_B(f) & = & (\tau(\alpha_A(1_A)))_B(f) = F(f)(\alpha_A(1_A)) = (F(f)\alpha_A)(1_A) \\ & = & (\alpha_B(f \circ -))(1_A) = \alpha_B(f1_A) = \alpha_B(f). \end{array}$$

Here the fourth equality follows from the naturality of α, more precisely from the commutativity of diagram

$$\begin{array}{ccc} \mathcal{A}(A,A) & \xrightarrow{\alpha_A} & F(A) \\ {\scriptstyle f\circ -}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathcal{A}(A,B) & \xrightarrow[\alpha_B]{} & F(B) \end{array} .$$

Hence $\tau\theta_{F,A}(\alpha) = \alpha$ for every α, which proves that $\tau\theta_{F,A} = 1_{\mathsf{Nat}(\mathcal{A}(A,-),F)}$.

2. To prove the naturality of the bijections for a fixed functor F, let us consider the functor $N : \mathcal{A} \to \mathsf{Set}$, defined on objects by

$$N(A) := \mathsf{Nat}(\mathcal{A}(A,-), F),$$

and on morphism $f : A \to B$ of $\mathcal{A}$, the mapping $N(f) : \mathsf{Nat}(\mathcal{A}(A,-), F) \longrightarrow \mathsf{Nat}(\mathcal{A}(B,-), F)$ is defined by

$$N(f)(\alpha) := \alpha \circ \mathcal{A}(f,-),$$

where $\mathcal{A}(f,-) : \mathcal{A}(B,-) \Rightarrow \mathcal{A}(A,-)$ (see Example 4.5) and $\alpha : \mathcal{A}(A,-) \Rightarrow F$. We are claiming that the definition $\nu_A := \theta_{F,A}$, $A \in \mathcal{A}_0$, gives a natural transformation $\nu : N \Rightarrow F$.

$$\begin{array}{ccc} \mathsf{Nat}(\mathcal{A}(A,-),F) & \xrightarrow{\theta_{F,A}} & F(A) \\ {\scriptstyle N(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathsf{Nat}(\mathcal{A}(B,-),F) & \xrightarrow[\theta_{F,B}]{} & F(B) \end{array}$$

Indeed, for every $\alpha : \mathcal{A}(A,-) \Rightarrow F$,

$$\begin{array}{rcl} F(f)\theta_{F,A}(\alpha) & = & F(f)\alpha_A(1_A) = \alpha_B(f \circ -)(1_A) = \alpha_B(f), \\ \theta_{F,B}N(f)(\alpha) & = & \theta_{F,B}(\alpha \circ \mathcal{A}(f,-)) = (\alpha \circ \mathcal{A}(f,-))_B(1_B) \\ & = & \alpha_B\mathcal{A}(f,-)_B(1_B) = \alpha_B(1_B f) = \alpha_B(f). \end{array}$$

3. If $\mathcal{A}$ is a small category, then we can consider the category $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ of all functors from $\mathcal{A}$ to Set. For a fixed object $A \in \mathcal{A}_0$ we this time consider the functor $M : \mathsf{Fun}(\mathcal{A}, \mathsf{Set}) \to \mathsf{Set}$ defined on objects $F : \mathcal{A} \to \mathsf{Set}$ by

$$M(F) := \mathsf{Nat}(\mathcal{A}(A,-), F);$$

and if $\gamma : F \Rightarrow G$ is a morphism of $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$, then the mapping $M(\gamma) : \mathsf{Nat}(\mathcal{A}(A,-), F) \longrightarrow \mathsf{Nat}(\mathcal{A}(A,-), G)$ is defined by

$$M(\gamma)(\alpha) := \gamma \circ \alpha.$$

On the other hand, there is a functor of "evaluation in A" $\mathsf{ev}_A : \mathsf{Fun}(\mathcal{A}, \mathsf{Set}) \to \mathsf{Set}$ defined by

$$\begin{array}{rcl} \mathsf{ev}_A(F) & := & F(A); \\ \mathsf{ev}_A(\gamma) & := & \gamma_A. \end{array}$$

The assertion is that the definition $\mu_F := \theta_{F,A}$, $F : \mathcal{A} \to \mathsf{Set}$, gives a natural transformation $\mu : M \Rightarrow \mathsf{ev}_A$.

$$\begin{array}{ccc} \mathsf{Nat}(\mathcal{A}(A,-),F) & \xrightarrow{\theta_{F,A}} & F(A) \\ {\scriptstyle \gamma\circ - = M(\gamma)}\downarrow & & \downarrow{\scriptstyle \gamma_A = \mathsf{ev}_A(\gamma)} \\ \mathsf{Nat}(\mathcal{A}(A,-),G) & \xrightarrow[\theta_{G,A}]{} & G(A) \end{array}$$

Indeed, for every $\gamma : F \Rightarrow G$ and $\alpha : \mathcal{A}(A, -) \Rightarrow F$,

$$\gamma_A \theta_{F,A}(\alpha) = \gamma_A \alpha_A(1_A) = (\gamma \circ \alpha)_A(1_A) = \theta_{G,A}(\gamma \circ \alpha) = \theta_{G,A}(\gamma \circ -)(\alpha).$$

■

Corollary 4.11 *If $A, B \in \mathcal{A}_0$ then each natural transformation $\mathcal{A}(A, -) \Rightarrow \mathcal{A}(B, -)$ has the form $\mathcal{A}(f, -)$ for a unique morphism $f : B \to A$.*

Proof. Let $\alpha : \mathcal{A}(A, -) \Rightarrow \mathcal{A}(B, -)$; we shall apply the Yoneda Lemma for $F = \mathcal{A}(B, -)$. Since the mapping $\tau : \mathcal{A}(B, A) \longrightarrow \mathsf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -))$ is bijective, there is a unique morphism $f : B \to A$ such that $\alpha = \tau(f)$. Hence, for every $C \in A_0$ and $g : A \to C$,

$$\alpha_C(g) = \tau(f)_C(g) = (\mathcal{A}(B, -)(g))(f) = (- \circ g)(f) = gf = \mathcal{A}(f, -)_C(g),$$

which means that $\alpha = \mathcal{A}(f, -)$. ■

For a category $\mathcal{A}$, define a mapping $Y^{\mathrm{op}} : \mathcal{A} \to \mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ by

$$\begin{array}{rcl} Y^{\mathrm{op}}(A) & := & \mathcal{A}(A, -), \\ Y^{\mathrm{op}}(f) & := & \mathcal{A}(f, -) : \mathcal{A}(A, -) \Rightarrow \mathcal{A}(B, -) \end{array}$$

where $f : B \to A$ in $\mathcal{A}$. It can be seen that for morphisms $f : A \to B$, $h : B \to C$ in $\mathcal{A}$,

$$Y^{\mathrm{op}}(hf) = Y^{\mathrm{op}}(f)Y^{\mathrm{op}}(h), \quad Y^{\mathrm{op}}(1_B) = 1_{Y^{\mathrm{op}}(B)}. \tag{1}$$

So Y^{op} is a contravariant functor. It is called the **contravariant Yoneda embedding**.

Proposition 4.12 *The Yoneda embedding functors are full and faithful.*

Proof. We have to show that for all $A, B \in \mathcal{A}_0$, the mapping

$$(Y^{\mathrm{op}})_1^{B,A} : \mathcal{A}(B, A) \longrightarrow \mathsf{Fun}(\mathcal{A}, \mathsf{Set})(Y^{\mathrm{op}}(A), Y^{\mathrm{op}}(B)) = \mathsf{Nat}(\mathcal{A}(A, -), \mathcal{A}(B, -))$$

is bijective. But as we saw in Corollary 4.11, $\tau(f) = \mathcal{A}(f, -) = Y^{\mathrm{op}}(f)$ for every $f : B \to A$. Hence Y^{op} is bijective on morphisms because τ is. ■

4.4 The Godement product of natural transformations

There exists another way of composing natural transformations.

Proposition 4.13 *Consider the following situation:*

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{K}{\overset{H}{\rightrightarrows}} \mathcal{C}, \qquad (\Downarrow\alpha,\ \Downarrow\beta)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are categories, F, G, H, K are functors and α, β are natural transformations. The formula

$$(\beta * \alpha)_A := \beta_{G(A)} H(\alpha_A) = K(\alpha_A)\beta_{F(A)} : HF(A) \to KG(A)$$

defines a natural transformation $\beta * \alpha : HF \Rightarrow KG$.

$$\mathcal{A} \underset{KG}{\overset{HF}{\rightrightarrows}} \mathcal{C} \qquad (\Downarrow \beta * \alpha)$$

Proof. First we note that, indeed, $\beta_{G(A)} H(\alpha_A) = K(\alpha_A)\beta_{F(A)}$, because β is a natural transformation:

$$\begin{array}{ccc} HF(A) & \xrightarrow{H(\alpha_A)} & HG(A) \\ {\scriptstyle \beta_{F(A)}}\downarrow & & \downarrow{\scriptstyle \beta_{G(A)}} \\ KF(A) & \xrightarrow[K(\alpha_A)]{} & KG(A) \end{array} .$$

The claim is that, for every morphism $f : A \to A'$ in $\mathcal{A}$, the outer rectangle in the diagram

$$\begin{array}{ccccc}
HF(A) & \xrightarrow{H(\alpha_A)} & HG(A) & \xrightarrow{\beta_{G(A)}} & KG(A) \\
\Big\downarrow{\scriptstyle HF(f)} & & \Big\downarrow{\scriptstyle HG(f)} & & \Big\downarrow{\scriptstyle KG(f)} \\
HF(A') & \xrightarrow[H(\alpha_{A'})]{} & HG(A') & \xrightarrow[\beta_{G(A')}]{} & KG(A')
\end{array}$$

is commutative. This is true since the left-hand square commutes by naturality of α and functoriality of H and the right-hand square commutes by naturality of β. ■

The composition $*$, defined in Proposition 4.13 is called the **Godement product** of the natural transformations α and β. Also, sometimes the composition $*$ is called the **horizontal composition** of natural transformations and $\circ$ is called the **vertical composition** of natural transformations.

Very often, instead of writing 1_F for the identity natural transformation of a functor F, the symbol F itself is used. Hence, for example

$$(\beta * F)_A = \beta_{F(A)}, \tag{2}$$
$$(H * \alpha)_A = H(\alpha_A), \tag{3}$$

where $\beta * F : HF \Rightarrow KF$ and $H * \alpha : HF \Rightarrow HG$.

Exercises 4.14 1. Recall that every monoid can be considered as a category (see Example 1.7). What are functors between such categories? Suppose that A, B are groups, considered as one-object categories, and $f, g : A \to B$ two functors between these categories. Show that there is a natural transformation $f \Rightarrow g$ if and only if f and g are conjugate, that is,

$$(\exists b \in B)(\forall a \in A)(f(a) = b^{-1}g(a)b).$$

2. Prove that a natural transformation $\alpha : F \Rightarrow G$ is a natural isomorphism if and only if there exists a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. In other words, natural isomorphisms are isomorphisms in quasicategories $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ of functors.

3. For a fixed object $A \in \mathcal{A}_0$, define $\mathsf{ev}_A : \mathsf{Fun}(\mathcal{A}, \mathsf{Set}) \to \mathsf{Set}$ by

$$\mathsf{ev}_A(F) := F(A),$$
$$\mathsf{ev}_A(\gamma) := \gamma_A,$$

$F, G : \mathcal{A} \to \mathsf{Set}, \gamma : F \to G$. Prove that ev_A is a functor.

4. Prove the equalities (1).

5. For a category $\mathcal{A}$, consider the quasicategory $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$. Prove that a morphism α of $\mathsf{Fun}(\mathcal{A}, \mathsf{Set})$ (i.e. a natural transformation) is a monomorphism if and only if each component α_A, $A \in \mathcal{A}_0$, is a monomorphism in Set (i.e. an injective mapping). (Hint: use the Yoneda Lemma.)

6. Prove the **interchange law** for compositions $*$ and $\circ$, i.e. that the equality

$$(\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha)$$

holds whenever all composites on both sides are defined.

5 Limits and colimits

5.1 Products and coproducts

If A and B are sets then their cartesian product is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

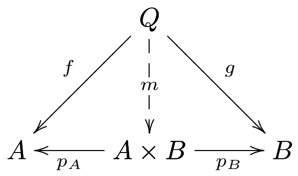
It is equipped with two canonical projections

$$p_A : A \times B \to A, \quad (a, b) \mapsto a,$$
$$p_B : A \times B \to B, \quad (a, b) \mapsto b.$$

Moreover if Q is a set and $f : Q \to A$, $g : Q \to B$ are arbitrary mappings then there is a unique mapping

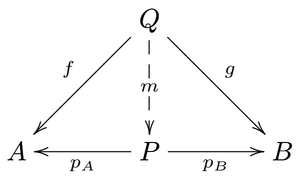
$$m : Q \to A \times B, \qquad q \mapsto (f(q), g(q)),$$

which makes the following diagram commutative:



This motivates the following definition.

Definition 5.1 A **product** of objects $A, B \in \mathcal{C}_0$ is a triple (P, p_A, p_B) where $P \in \mathcal{C}_0$ and $p_A : P \to A$, $p_B : P \to B$ are morphisms in $\mathcal{C}$ (called **projections**) with the property that if $Q \in \mathcal{C}_0$ is any other object and $f : Q \to A$, $g : Q \to B$ are morphisms, then there exists a unique morphism $m : Q \to P$ such that the diagram

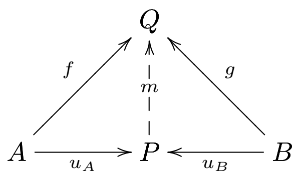


commutes.

Usually, $A \times B$ is written instead of P. The property of existing the unique m is often called the **universal property** of products. Instead of m, $\langle f, g\rangle$ is sometimes written.

By dualizing the definition of product we obtain the notion of coproduct which generalizes the construction of disjoint union of sets.

Definition 5.2 A **coproduct** (or **sum**) of objects $A, B \in \mathcal{C}_0$ is a triple (P, u_A, u_B) where $P \in \mathcal{C}_0$ and $u_A : A \to P$, $u_B : B \to P$ are morphisms in $\mathcal{C}$ (called **injections**) with the property that if $Q \in \mathcal{C}_0$ is any other object and $f : A \to Q$, $g : B \to Q$ are morphisms, then there exists a unique morphism $m : P \to Q$ such that the diagram



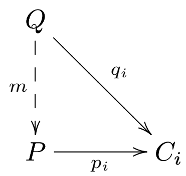
commutes.

Usually, $A \coprod B$ is written instead of P.

Definitions of products and coproducts can be generalized to any number of objects.

Definition 5.3 Let I be a set and $(C_i)_{i \in I}$ a family of objects in a category $\mathcal{C}$. A **product** of that family is a pair $(P, (p_i)_{i \in I})$ with the following properties:

1. P is an object of $\mathcal{C}$;
2. for every $i \in I$, $p_i : P \to C_i$ is a morphism in $\mathcal{C}$, called the **projection** of P on C_i;
3. for each pair $(Q, (q_i)_{i \in I})$ (where $Q \in \mathcal{C}_0$ and $q_i : Q \to C_i$ for every $i \in I$) there exists a unique morphism $m : Q \to P$ such that the triangle

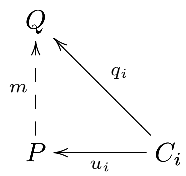


commutes for every $i \in I$.

Usually, $\prod_{i \in I} C_i$ is written instead of P.

Definition 5.4 Let I be a set and $(C_i)_{i \in I}$ a family of objects in a category $\mathcal{C}$. A **coproduct** of that family is a pair $(P, (u_i)_{i \in I})$ with the following properties:

1. P is an object of $\mathcal{C}$;
2. for every $i \in I$, $u_i : C_i \to P$ is a morphism in $\mathcal{C}$, called the **injection** of C_i into P;
3. for each pair $(Q, (q_i)_{i \in I})$ (where $Q \in \mathcal{C}_0$ and $q_i : C_i \to Q$ for every $i \in I$) there exists a unique morphism $m : P \to Q$ such that the triangle

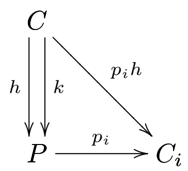


commutes for every $i \in I$.

Usually, $\coprod_{i \in I} C_i$ is written instead of P.
Let us prove some properties of products.

Proposition 5.5 *If $(P, (p_i)_{i \in I})$ is a product of $(C_i)_{i \in I}$ and $h, k : C \to P$ are morphisms with the property that for each $i \in I$, $p_i h = p_i k$, then $h = k$.*

Proof. Both h and k make all triangles

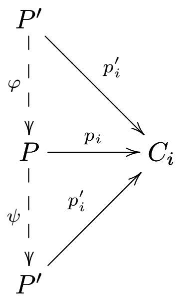


commutative, hence they must be equal by Definition 5.3. ■

The claim of Proposition 5.5 can also be expressed by saying that "projections of products are simultaneously left cancellable".

Proposition 5.6 *If both $(P, (p_i)_{i \in I})$ and $(P', (p'_i)_{i \in I})$ are products of $(C_i)_{i \in I}$ then P and P' are isomorphic.*

Proof. Since P and P' are products of C_i, $i \in I$, there exist φ and ψ making the triangles



commutative for every $i \in I$. Now

$$\begin{aligned} p'_i &= p_i\varphi = p'_i\psi\varphi, \\ p_i &= p'_i\psi = p_i\varphi\psi, \end{aligned}$$

for every $i \in I$ imply $\psi\varphi = 1_{P'}$ and $\varphi\psi = 1_P$ by Proposition 5.5. Hence $P \cong P'$. ∎

Of course, the dual statements are true for coproducts.

We say that a category $\mathcal{C}$ **has (co)products** if every family $(C_i)_{i\in I}$ of objects of $\mathcal{C}$ has a (co)product. We say that $\mathcal{C}$ **has finite (co)products** if every finite family of objects has a (co)product.

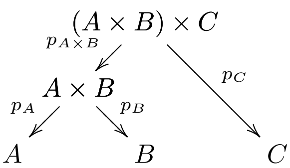
Proposition 5.7 *If a category $\mathcal{C}$ has binary products (of all pairs of objects) then it has ternary products. Moreover, for every $A, B, C \in \mathcal{C}_0$,*

$$(A \times B) \times C \cong A \times (B \times C),$$
$$A \times B \cong B \times A,$$

and if $\mathcal{C}$ has a terminal object $\mathbf{1}$ then

$$A \times \mathbf{1} \cong A, \quad \mathbf{1} \times A \cong A.$$

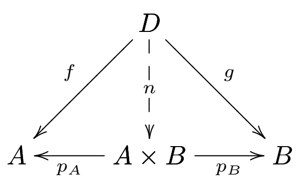
Proof. For $A, B, C \in \mathcal{C}_0$ let $(A \times B, p_A, p_B)$ be the product of A and B, and let $((A \times B) \times C, p_{A\times B}, p_C)$ be the product of $A \times B$ and C.



We claim that $((A \times B) \times C, p_A p_{A\times B}, p_B p_{A\times B}, p_C)$ is the product of A, B and C in the sense of Definition 5.3. To verify the universal property, suppose that for some object D there exist morphisms $f : D \to A$, $g : D \to B$ and $h : D \to C$. We must prove that there exists a unique morphism $m : D \to (A \times B) \times C$ such that

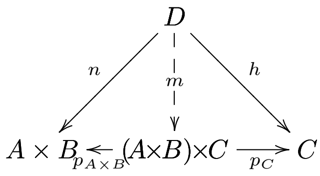
1. $p_A p_{A\times B} m = f$,
2. $p_B p_{A\times B} m = g$,
3. $p_C m = h$.

First we can find a unique morphism $n : D \to A \times B$ such that the diagram



(4)

commutes. This in turn induces a unique morphism $m : D \to (A \times B) \times C$ such that



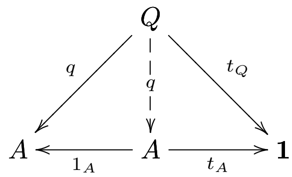
(5)

is commutative. Hence $p_A p_{A\times B} m = p_A n = f$, $p_B p_{A\times B} m = p_B n = g$ and $p_C m = h$.

If $m' : D \to (A\times B)\times C$ is another morphism such that $p_A p_{A\times B} m' = f$, $p_B p_{A\times B} m' = g$ and $p_C m' = h$ then by the uniqueness of n in (4), $n = p_{A\times B} m'$. Therefore also $m = m'$ because m in (5) is unique.

Similarly one can prove that also $A\times(B\times C)$ is a ternary product of A, B and C. Hence $(A\times B)\times C \cong A \times (B \times C)$ by Proposition 5.6. It is also easy to prove that $A \times B \cong B \times A$.

Suppose now that $\mathcal{C}$ has a terminal object $\mathbf{1}$ and for every $C \in \mathcal{C}_0$ let t_C be the unique morphism $C \to \mathbf{1}$. We shall show that $(A, 1_A, t_A)$ is a product of A and $\mathbf{1}$. Indeed, in the diagram



the vertical q is the unique morphism that makes both triangles commute. ■

A generalization of this proof shows that if a category has binary products then it has products of n objects for every $n \geq 2$. It is also easy to see that a product of the empty family of objects is a terminal object and $(A, 1_A)$ is a product of a family consisting of a single object A. Hence the following result holds.

Proposition 5.8 *If a category has binary products and a terminal object then it has all finite products.*

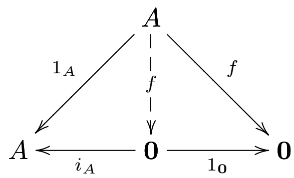
Proposition 5.9 *Let $\mathcal{C}$ be a category with finite products and an initial object $\mathbf{0}$. The following assertions are equivalent.*

1. *For every $A \in \mathcal{C}_0$, if there is a morphism $f : A \to \mathbf{0}$ then $A \cong \mathbf{0}$.*
2. *For every object $A \in \mathcal{C}_0$, $A \times \mathbf{0} \cong \mathbf{0}$.*

Proof. For every $C \in \mathcal{C}_0$ denote the unique morphism $\mathbf{0} \to C$ by i_C.

If 1 holds then $p_{\mathbf{0}} : A \times \mathbf{0} \to \mathbf{0}$ implies $A \times \mathbf{0} \cong \mathbf{0}$.

Conversely, suppose that $A \times \mathbf{0} \cong \mathbf{0}$ and $f : A \to \mathbf{0}$ for an object $A \in \mathcal{C}_0$. Then also $\mathbf{0}$ is a product of A and $\mathbf{0}$, and the projections must be i_A and $1_{\mathbf{0}} = i_{\mathbf{0}}$. Hence the diagram



must commute. In particular, f is a split monomorphism. Since its codomain is an initial object, it is also an epimorphism. Hence f is an isomorphism. ■

If an initial object $\mathbf{0}$ of a category $\mathcal{C}$ has the property mentioned in Proposition 5.9(1) then it is called a **strict initial object**. In a poset (considered as a category), the smallest element (if it exists) is a strict initial object. In Mon, the one-element monoid $\{1\}$ is initial but not strictly initial, because $S \times \{1\} \cong S$.

Let us list some examples of products.

Example 5.10 In the category Set, the product of a family $(C_i)_{i\in I}$ is the cartesian product

$$\prod_{i\in I} C_i = \{(x_i)_{i\in I} \mid x_i \in C_i\}$$

with projections $p_k\left((x_i)_{i\in I}\right) = x_k$, $k \in I$.

Example 5.11 Binary products of objects of the quasicategory CAT are defined in Definition 1.12. The projecions are defined by

$$\begin{array}{ll} P_{\mathcal{A}} : \mathcal{A} \times \mathcal{B} \to \mathcal{A}, & (A, B) \mapsto A, \quad (a, b) \mapsto a, \\ P_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathcal{B}, & (A, B) \mapsto B, \quad (a, b) \mapsto b, \end{array}$$

which obviously will be functors. Similarly the product of an arbitrary small family of categories can be defined.

Example 5.12 In the categories of algebraic structures (e.g. groups, abelian groups, rings, modules, vector spaces, boolean algebras etc.) the product of a family of objects is their cartesian product equipped with pointwise operations. For example if $C_i, i \in I$, are abelian groups then $\prod_{i \in I} C_i = \{(x_i)_{i \in I} \mid x_i \in C_i\}$ and addition on $\prod_{i \in I} C_i$ is defined by

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}.$$

Example 5.13 In the category Ban_1, the product of a family $(C_i)_{i \in I}$ is given by

$$\begin{array}{c} \prod_{i \in I} C_i = \{(x_i)_{i \in I} \mid x_i \in C_i, \ \sup_{i \in I} ||x_i|| \leq \infty\}, \\ ||(x_i)_{i \in I}|| := \sup_{i \in I} ||x_i|| \end{array}$$

and pointwise operations. The projections $p_k : \prod_{i \in I} C_i \to C_k$ are linear contractions because, for every $k \in I$, $||p_k\left((x_i)_{i \in I}\right)|| = ||x_k|| \leq \sup_{i \in I} ||x_i|| = ||(x_i)_{i \in I}||$. Let $q_i : Q \to C_i$, $i \in I$, be also a family of linear contractions and consider an element $x \in Q$. Then $||q_i(x)|| \leq ||x||$ for every $i \in I$. Hence $\sup_{i \in I} ||q_i(x)|| \leq ||x||$ and we can define a mapping $m : Q \to \prod_{i \in I} C_i$ by $m(x) := (q_i(x))_{i \in I} \in \prod_{i \in I} C_i$ which is unique and a linear contraction.

Example 5.14 In the category Top, the product of a family $(X_i, \tau_i)_{i \in I}$ is (X, τ), where $X = \prod_{i \in I} X_i$ is just the cartesian product. For τ, we choose as basic open subsets the subsets of the form

$$\prod_{i \in I} U_i = \{(x_i)_{i \in I} \mid x_i \in U_i\} \subseteq X$$

where $U_i \in \tau_i$ for every $i \in I$ and the set $\{i \in I \mid U_i \neq X_i\}$ is finite. The topology τ consists of all unions of basic open subsets. The projections $p_k : X \to X_k$, $(x_i)_{i \in I} \mapsto x_k$, are continuous because if $U \in \tau_k$ then

$$p_k^{-1}(U) = \prod_{i \in I} V_i$$

where $V_k = U$ and $V_i = X_i$ for every $i \in I \setminus \{k\}$.

To check the universal property, let

$$q_i : (Y, \sigma) \to (X_i, \tau_i), \ i \in I,$$

be a family of continuous mappings. We have to show that the mapping

$$m : Y \to X, \quad y \mapsto (q_i(y))_{i \in I}$$

is continuous. If $\prod_{i \in I} U_i \in \tau$ is a basic open subset then

$$m^{-1}\left(\prod_{i \in I} U_i\right) = \{y \in Y \mid (\forall i \in I)(q_i(y) \in U_i)\} = \bigcap_{i \in I} q_i^{-1}(U_i). \tag{6}$$

For every $i \in I$, $q_i^{-1}(U_i) \in \sigma$, since q_i is continuous and $U_i \in \tau_i$. Moreover, if $U_i = X_i$ then $q_i^{-1}(U_i) = Y$ and this term does not play any role in the intersection (6). Hence

$$m^{-1}\left(\prod_{i \in I} U_i\right) = \bigcup_{i \in I, U_i \neq X_i} q^{-1}(U_i)$$

which is a finite intersection of open subset of Y and hence an open subset.

Example 5.15 If we consider a poset $(P, \le)$ as a category (see 1.7) then products (if they exist) are precisely the greatest lower bounds.

Example 5.16 To allow operations depending on several variables in a functional programming language L (as considered in subsection 1.3), it is reasonable to assume that for any types A and B the language has a record type P with two field selectors $P.A : P \to A$ and $P.B : P \to B$ which satisfies the universal property. For example a record type PERSON could have fields NAME and AGE. Thus having the record type constructor in a language L would mean that the corresponding category $\mathcal{C}(L)$ has all finite products.

Consider some examples of coproducts.

Example 5.17 In the category Set, the coproduct of a family $(C_i)_{i\in I}$ is its disjoint union. A convenient way to construct it is to set

$$\coprod_{i\in I} C_i := \bigcup_{i\in I} (C_i \times \{i\}) = \{(x,i) \mid i \in I,\ x \in C_i\}.$$

The injections $u_i : C_i \to \coprod_{i\in I} C_i$ are defined by $u_i(x) := (x,i)$, $x \in C_i$.

Example 5.18 In the quasicategory CAT, the coproduct of family of categories just their disjoint union.

Example 5.19 In the category Ab, the coproduct of a family $(A_i)_{i\in I}$ is their direct sum

$$\coprod_{i\in I} A_i := \{(x_i)_{i\in I} \mid x_i \in A_i,\ \{i \in I \mid x_i \neq 0\} \text{ is finite } \} \le \prod_{i\in I} A_i,$$

where addition is componentwise. The injections $u_k : A_k \to \coprod_{i\in I} A_i$ are defined by $u_k(x) := (x_i)_{i\in I}$ where $x_k = x$ and the other components are all zeros. If B is another abelian group and $q_i : A_i \to B$, $i \in I$, is a family of group homomorphisms then the unique mapping $m : \coprod_{i\in I} A_i \to B$ is defined by $m((x_i)_{i\in I}) := \sum_{i\in I} q_i(x_i)$ where the last sum is actually the sum of (finitely many) nonzero elements.

Example 5.20 In the category Gr, the coproduct of a family $(G_i)_{i\in I}$ is constructed as follows. Let V be the disjoint union of the sets G_i and let V^* be the free monoid with the basis V, i.e. the set of all finite sequences ("words") of elements of V together with the concatenation operation. Consider on V^* the equivalence relation σ generated by the following rules:

- the unit element of every group G_i is equivalent to the empty sequence,
- a sequence containing two consequtive elements belonging to the same G_i is equivalent to the sequence obtained from it by replacing these two elements by their product in G_i.

This σ will be a congruence and the quotient monoid

$$\coprod_{i\in I} G_i := V^*/\sigma$$

will be a group where $[v_1 \dots v_n]_\sigma^{-1} = [v_n^{-1} \dots v_1^{-1}]_\sigma$, $v_1, \dots, v_n \in V$. The mappings

$$u_i : G_i \to \coprod_{i\in I} G_i, \quad x \mapsto [x]_\sigma,$$

are group homomorphisms and it can be shown that $\coprod_{i\in I} G_i$ is indeed the coproduct of groups $G_i, i \in I$. In group theory, $\coprod_{i\in I}$ is usually called the **free product of groups** $G_i, i \in I$.

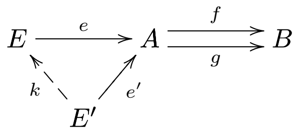
Example 5.21 In the category Top, the coproduct of a family $(X_i, \tau_i)_{i\in I}$ is (X, τ) where X is the disjoint union of the sets X_i and τ is the topology on X generated by the disjoint union of topologies τ_i.

Example 5.22 If we consider a poset $(P, \le)$ as a category (see 1.7) then coproducts (if they exist) are precisely the least upper bounds.

5.2 Equalizers and coequalizers

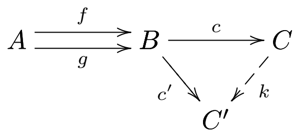
Definition 5.23 Let $f, g : A \to B$ be morphisms in a category $\mathcal{C}$. An **equalizer** of f and g is a pair (E, e) with the following properties:

1. $e : E \to A$ is a morphism in $\mathcal{C}$;
2. $fe = ge$;
3. for any other morphism $e' : E' \to A$ in $\mathcal{C}$ such that $fe' = ge'$, there exists a unique morphism $k : E' \to E$ such that $ek = e'$.



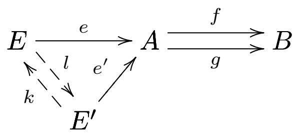
Definition 5.24 Let $f, g : A \to B$ be morphisms in a category $\mathcal{C}$. A **coequalizer** of f and g is a pair (C, c) with the following properties:

1. $c : B \to C$ is a morphism in $\mathcal{C}$;
2. $cf = cg$;
3. for any other morphism $c' : B \to C'$ in $\mathcal{C}$ such that $c'f = c'g$, there exists a unique morphism $k : C \to C'$ such that $kc = c'$.

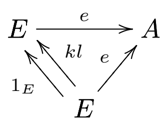


Proposition 5.25 *Equalizer of two morphisms, if it exists, is unique up to isomorphism.*

Proof. Suppose that both (E, e) and (E', e') are equalizers of $f, g : A \to B$. Then there exist $k : E' \to E$ and $l : E \to E'$ such that $ek = e'$ and $e'l = e$.



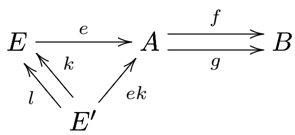
Now both 1_E and kl make the triangle



commutative and therefore have to be equal. Similarly $lk = 1_{E'}$, which proves that $E \cong E'$. ■

Proposition 5.26 *If (E, e) is an equalizer of $f, g : A \to B$ in a category $\mathcal{C}$ then e is a monomorphism.*

Proof. Suppose that $ek = el$ for $k, l : E' \to E$. Then $k = l$ because there is exactly one morphism making the triangle below commutative.



■

Example 5.27 In most concrete categories (Set, Top, Gr, Ab, Ban_1, ...) an equalizer of two morphisms $f, g : A \to B$ is given by

$$E = \{a \in A \mid f(a) = g(a)\}, \tag{7}$$

provided with the structure induced by the structure of A, and the inclusion mapping $e : E \to A$.

For instance, in Set an equalizer of the mappings $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = x^2 + y^2$, $g(x, y) = 1$, is the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

In Ab, an equalizer of $f, g : A \to B$, defined by (7), is the kernel of the difference homomorphism $f - g : A \to B$.

Example 5.28 In the category of all nonempty sets (or nonempty semigroups) there exist parallel morphisms which do not have an equalizer.

Example 5.29 In the category Set, a coequalizer of mappings $f, g : A \to B$ is the quotient B/σ where σ is the equivalence relation generated by the set of pairs $\{(f(a), g(a)) \mid a \in A\}$.

Example 5.30 In the category Ab, a coequalizer of a homomorphism $f : A \to B$ and the zero homomorphism is the natural surjection $\pi : B \to B/f(A)$ on the quotient group $B/f(A)$. More generally, a coequalizer of homomorphisms $f, g : A \to B$ is a coequalizer of $f - g : A \to B$ and the zero homomorphism, that is, the natural surjection $B \to B/(f - g)(A)$. Descriptions of coequalizers in $\mathsf{Vec}_{\mathbb{R}}$ and Mod_R are analogous.

Example 5.31 In many categories of algebraic structures (e.g. groups, rings) the situation is more complicated. The general procedure for constructing a coequalizer of homomorphisms $f, g : A \to B$ is to factor B by the congruence generated by the set $\{(f(a), g(a)) \mid a \in A\}$.

Example 5.32 If $f, g : (X, \tau) \to (Y, \theta)$ are continuous mappings in the category Top then their coequalizer is $(Y/\sigma, \theta')$ where σ is the equivalence relation generated by the set $\{(f(x), g(x)) \mid x \in X\}$ and θ' is the quotient topology, i.e.

$$\theta' = \{V \subseteq Y/\sigma \mid \pi^{-1}(V) \in \theta\}$$

where $\pi : Y \to Y/\sigma$ is the natural surjection.

One way of thinking about an equalizer is as the largest subobject on which an equation or a set of equations is true. A coequalizer is the least identification necessary to force an equation to be true on the equivalence classes. The following example illustrates this.

Example 5.33 There are two ways to define the rational numbers. In the first construction we identify $\frac{a}{b}$ with $\frac{c}{d}$ if and only if $a \cdot d = b \cdot c$. This can be described as a coequalizer

$$T \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{Z} \times \mathbb{N} \xrightarrow{\pi} (\mathbb{Z} \times \mathbb{N})/\sigma = \mathbb{Q}$$

where

$$T = \{(a, b, c, d) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{N} \mid ad = bc\},$$

$f(a, b, c, d) = (a, b)$, $g(a, b, c, d) = (c, d)$, σ is constructed canonically as in Example 5.29 and we denote $\frac{a}{b} := [(a, b)]_\sigma$.

The second way is to define that rational numbers are pairs $(a, b) \in \mathbb{Z} \times \mathbb{N}$ where a and b are relatively prime. The set of such pairs is an equalizer

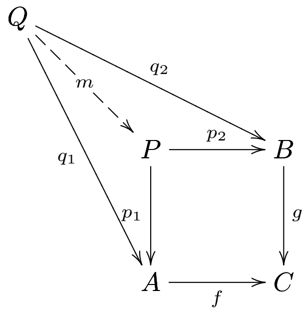
$$\mathbb{Q} = \{(a, b) \in \mathbb{Z} \times \mathbb{N} \mid \gcd(a, b) = 1\} \xrightarrow{e} \mathbb{Z} \times \mathbb{N} \underset{\mathrm{const}_1}{\overset{\gcd}{\rightrightarrows}} \mathbb{N}$$

where e is the inclusion and const_1 is the constant mapping on 1.

5.3 Pullbacks and pushouts

Definition 5.34 Let $f : A \to C$ and $g : B \to C$ be two morphisms in a category $\mathcal{C}$. A **pullback** of f and g is a triple (P, p_1, p_2) with the following properties:

1. $p_1 : P \to A$ and $p_2 : P \to B$ are morphisms in $\mathcal{C}$;
2. $fp_1 = gp_2$;
3. for any other morphisms $q_1 : Q \to A$ and $q_2 : Q \to B$ in $\mathcal{C}$ such that $fq_1 = gq_2$, there exists a unique morphism $m : Q \to P$ such that $p_1 m = q_1$ and $p_2 m = q_2$.

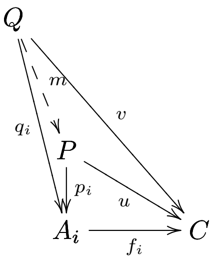


The square in the previous diagram is called a **pullback square**.

The dual notion of "pullback" is "**pushout**".

Definition 5.35 Let I be a set and $f_i : A_i \to C$, $i \in I$, a family of morphisms in a category $\mathcal{C}$. A **multiple pullback** of morphisms f_i is a pair $(P, (p_i)_{i \in I})$ with the following properties:

1. $p_i : P \to A_i$ is a morphism in $\mathcal{C}$ for each $i \in I$;
2. there is a morphism $u : P \to C$ such that $f_i p_i = u$ for each $i \in I$;
3. if $q_i : Q \to A_i$, $i \in I$, and $v : Q \to C$ are morphisms in $\mathcal{C}$ such that $f_i q_i = v$ for each $i \in I$ then there exists a unique morphism $m : Q \to P$ in $\mathcal{C}$ such that $p_i m = q_i$ for each $i \in I$.



Example 5.36 In Set, a pullback of mappings $f : A \to C$ and $g : B \to C$ is (P, p_1, p_2) where

$$P = \{(a, b) \mid a \in A, b \in B, f(a) = g(b)\} \subseteq A \times B$$

and $p_1(a, b) = a$, $p_2(a, b) = b$ for every $(a, b) \in P$.

If A and B are subsets of C and f, g inclusions then P is isomorphic to the intersection $A \cap B$.

A similar construction of pullbacks works in many categories (e.g. Gr, Mod_R, Rng, Top) if P is equipped with the structure induced by the structure of the product $A \times B$.

Definition 5.37 A pullback of a morphism f with itself is called a **kernel pair** of f.

Example 5.38 From Example 5.36 it follows that the kernel of a morphism f in Set is a kernel pair of f. The same holds in the category Mon of all monoids.

5.4 Limits and colimits

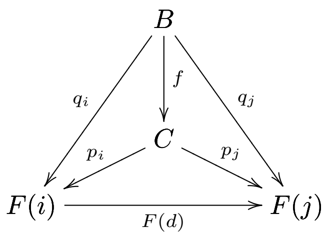
It turns out that all the constructions considered in this section are special cases of a more general construction. From this point on to the end of this section $\mathcal{D}$ will always stand for a small category and $I = \mathcal{D}_0$ for its set of objects.

Definition 5.39 Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. A **cone** on F is a pair $(C, (p_i)_{i \in I})$ with the following properties:

1. $C \in \mathcal{C}_0$;
2. for every object $i \in I$, $p_i : C \to F(i)$;
3. for every morphism $d : i \to j$ in $\mathcal{D}$, $p_j = F(d)p_i$.

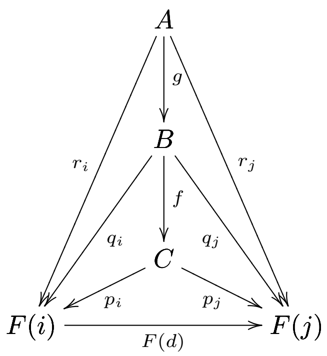
Alternatively, a pair $(C, (p_i)_{i \in I})$ is a cone on F if and only if $p = (p_i)_{i \in I} : \Delta_C \Rightarrow F$, where $\Delta_C : \mathcal{D} \to \mathcal{C}$ is the constant functor on C (see Example 3.3(3)). That is, cones on F are natural transformations from constant functors to F.

Definition 5.40 Let $F : \mathcal{D} \to \mathcal{C}$ be a functor and $(B, (q_i)_{i \in I})$, $(C, (p_i)_{i \in I})$ two cones on F. A morphism $f : B \to C$ of $\mathcal{C}$ is called a **morphism from cone** $(B, (q_i)_{i \in I})$ **to cone** $(C, (p_i)_{i \in I})$ if $p_i f = q_i$ for every $i \in I$.



Proposition 5.41 *For a fixed functor $F : \mathcal{D} \to \mathcal{C}$, the cones on F and their morphisms form a category.*

Proof. The morphisms in this new category will be composed as in category $\mathcal{C}$.



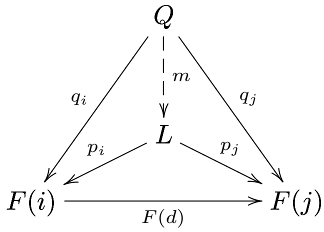
Suppose that $f : (B, (q_i)) \to (C, (p_i))$ and $g : (A, (r_i)) \to (B, (q_i))$ are two morphisms of cones on F. Then

$$p_i(fg) = (p_i f)g = q_i g = r_i$$

and hence indeed $fg : (A, (r_i)) \to (C, (p_i))$ is a morphism of cones. The identity morphism of a cone $(C, (p_i))$ will, of course, be 1_C. The associativity of composition follows from the corresponding property of $\mathcal{C}$. ■

The category constructed in Proposition 5.41 is denoted by $\mathsf{cone}(F)$ and called the **category of cones on** F.

Definition 5.42 A **limit** of a functor $F : \mathcal{D} \to \mathcal{C}$ is a terminal object of the category $\mathsf{cone}(F)$ of cones on F. Thus a cone $(L, (p_i)_{i \in I})$ on F is a limit of F if for every cone $(Q, (q_i)_{I \in I})$ on F, there exists a unique morphism $m : Q \to L$ such that for every object $i \in I$, $q_i = p_i m$. One often writes $(L, (p_i)_{i \in I}) = \lim F$ or even $L = \lim F$.

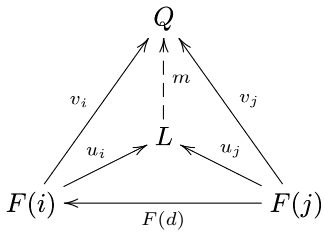


Definition 5.43 Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. A **cocone** on F is a pair $(C, (u_i)_{i\in I})$ with the following properties:

1. $C \in \mathcal{C}_0$;
2. for every object $i \in I$, $u_i : F(i) \to C$;
3. for every morphism $d : j \to i$ in $\mathcal{D}$, $u_j = u_i F(d)$.

Dually to the case of cones, one can define the category $\mathsf{cocone}(F)$ of cocones on a functor F.

Definition 5.44 A **colimit** of a functor $F : \mathcal{D} \to \mathcal{C}$ is an initial object of the category $\mathsf{cocone}(F)$ of cocones on F. Thus a cocone $(L, (u_i)_{i\in I})$ on F is a colimit of F if for every cone $(Q, (v_i)_{i\in I})$ on F, there exists a unique morphism $m : L \to Q$ such that for every object $i \in I$, $v_i = m u_i$. One often writes $(L, (u_i)_{i\in I}) = \mathrm{colim} F$



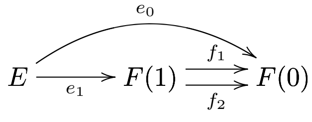
Examples 5.45

1. Consider a discrete category $\mathcal{D}$ with two objects, $\mathcal{D}_0 = \{1, 2\}$ (see Example 1.7(3)). A functor $F : \mathcal{D} \to \mathcal{C}$ is determined by a pair $(F(1), F(2))$ of objects of $\mathcal{C}$. A cone on F is a diagram
$$F(1) \xleftarrow{p_1} P \xrightarrow{p_2} F(2)$$
and a cocone on F is a diagram
$$F(1) \xrightarrow{u_1} P \xleftarrow{u_2} F(2).$$
Hence (P, p_1, p_2) is a limit of F if and only if it is a product of $F(1)$ and $F(2)$ and (P, u_1, u_2) is a colimit of F if and only if it is a coproduct of $F(1)$ and $F(2)$.

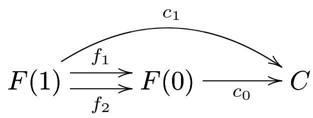
2. Let $\mathcal{D} = \mathbf{0}$ be the empty category. Then a cone or a cocone on a functor $F : \mathcal{D} \to \mathcal{C}$ is just an object of C. Hence C is a limit (colimit) of F if and only if it is a terminal (resp. initial) object of $\mathcal{C}$.

3. If I is a set and $\mathcal{D}$ is a discrete category with $\mathcal{D}_0 = I$ then a functor $F : \mathcal{D} \to \mathcal{C}$ is determined by a family $(F(i))_{i\in I}$ of objects of $\mathcal{C}$. A cone on F is a pair $(C, (p_i)_{i\in I})$ where $p_i : C \to F(i)$. This cone is a limit of F if and only if it is a product of the family $(F(i))_{i\in I}$. Dually a colimit of F is a coproduct of $(F(i))_{i\in I}$.

4. Consider a category $\mathcal{D}$ with $\mathcal{D}_0 = \{1, 0\}$ and two nonidentity morphisms $d_1, d_2 : 1 \to 0$. A functor $F : \mathcal{D} \to \mathcal{C}$ is determined by a pair $f_1, f_2 : F(1) \to F(0)$ of parallel arrows in $\mathcal{C}$. A cone on F is a commutative diagram

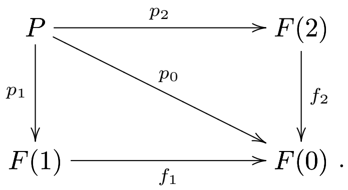


and a cocone on F is a diagram



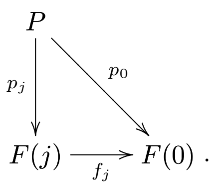
Hence (E, e_0, e_1) is a limit of F if and only if (E, e_1) is an equalizer of f_1 and f_2 and (C, c_0, c_1) is a limit of F if and only if (C, c_0) is a coequalizer of f_1 and f_2.

5. Consider a category $\mathcal{D}$ with $\mathcal{D}_0 = \{0, 1, 2\}$ and two nonidentity morphisms $d_1 : 1 \to 0$ and $d_2 : 2 \to 0$. A functor $F : \mathcal{D} \to \mathcal{C}$ is determined by a pair $f_1 : F(1) \to F(0)$, $f_2 : F(2) \to F(0)$ of morphisms of $\mathcal{C}$. A cone on F is a commutative diagram



Hence (P, p_0, p_1, p_2) is a limit of F if and only if (P, p_1, p_2) is a pullback of f_1 and f_2.

6. Let $\mathcal{D}$ be a category with $\mathcal{D}_0 = J \sqcup \{0\}$ and nonidentity morphisms $d_j : j \to 0, j \in J$. A functor $F : \mathcal{D} \to \mathcal{C}$ is determined by a family $f_j : F(j) \to F(0)$ of morphisms of $\mathcal{C}$. A cone on F is a commutatve diagram



Hence $(P, (p_i)_{i \in J \sqcup \{0\}})$ is a limit of F if and only if $(P, (p_j)_{j \in J})$ is a multiple pullback of $f_j, j \in J$.

The following table summarizes the previous example. For the sake of completeness we have included here also multiple equalizers, coequalizers and pushouts, although we have not given their explicit definitions because we shall not need these constructions in what follows. We believe that an interested reader can work out the details if needed.

Object set of $\mathcal{D}$	Nonidentity morphisms of $\mathcal{D}$	$\mathcal{C}$ has $\mathcal{D}$-limits means that $\mathcal{C}$ ...	$\mathcal{C}$ has $\mathcal{D}^{\mathrm{op}}$-colimits means that $\mathcal{C}$ ...
none	none	has a terminal object	has an initial object
$\{1, 2\}$	none	has binary products	has binary coproducts
$\{0, 1\}$	$d_1, d_2 : 1 \rightrightarrows 0$	has equalizers	has coequalizers
$\{0, 1, 2\}$	$d_1 : 1 \to 0$, $d_2 : 2 \to 0$	has pullbacks	has pushouts
Object set of $\mathcal{D}$	**Nonidentity morphisms of $\mathcal{D}$**	**$\mathcal{C}$ has $\mathcal{D}$-limits for all categories $\mathcal{D}$ of this form means that $\mathcal{C}$ ...**	**$\mathcal{C}$ has $\mathcal{D}^{\mathrm{op}}$-colimits for all categories $\mathcal{D}$ of this form means that $\mathcal{C}$ ...**
I set	none	has products	has coproducts
$\{1, 2\}$	$d_i : 1 \to 2, i \in I$	has multiple equalizers	multiple coequalizers
$J \sqcup \{0\}$, J set	$d_j : j \to 0, j \in J$	has multiple pullbacks	has multiple pushouts

It turns out that all limits (but also colimits) exist in Set and can be constructed in a canonical way.

Theorem 5.46 *Let $F : \mathcal{D} \to \mathsf{Set}$ be a functor and*

$$L = \{(x_i)_{i \in I} \mid x_i \in F(i),\ (\forall d : i \to j \text{ in } \mathcal{D})(F(d)(x_i) = x_j)\} \subseteq \prod_{i \in I} F(i).$$

Then $(L, (p_i)_{i \in I})$, where $p_i : L \to F(i)$, $i \in I$, are the restrictions of the projections of the cartesian product $\prod_{i \in I} F(i)$, is a limit of F.

Proof. Obviously $(L, (p_i)_{i\in I})$ is a cone on F. If $(Q, (q_i)_{i\in I})$ is another cone on F then the required unique mapping $m : Q \to L$ is for $x \in Q$ defined by

$$m(x) := (q_i(x))_{i\in I}.$$

■

Limits in many other concrete categories may be constructed in a similar way. For example if $F : \mathcal{D} \to \mathsf{Gr}$ then L is a subset of the direct product $\prod_{i\in I} F(i)$ of groups $F(i)$, which is a group with respect to componentwise operations. Since every $F(d)$ is a homomorphism of groups,

$$F(d)(x_i y_i) = F(d)(x_i) \cdot F(d)(y_i) = x_j y_j,$$
$$F(d)(x_i^{-1}) = (F(d)(x_i))^{-1} = x_j^{-1},$$

and hence L is a subgroup of $\prod_{i\in I} F(i)$. Clearly the mappings p_i are group homomorphisms and also m is a group homomorphism if all q_i's are.

5.5 Complete categories

Definition 5.47 A category $\mathcal{C}$ is

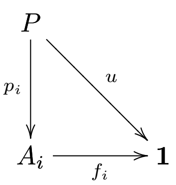
1. **$\mathcal{D}$-complete** (or **has $\mathcal{D}$-limits**), where $\mathcal{D}$ is a category, if every functor $D : \mathcal{D} \to \mathcal{C}$ has a limit;
2. **finitely complete** (or **has finite limits**) if $\mathcal{C}$ is $\mathcal{D}$-complete for every finite category $\mathcal{D}$;
3. **complete** (or **has limits**) if $\mathcal{C}$ is $\mathcal{D}$-complete for every small category $\mathcal{D}$.

Theorem 5.48 *For every category $\mathcal{C}$, the following assertions are equivalent:*

1. *$\mathcal{C}$ is complete.*
2. *$\mathcal{C}$ has multiple pullbacks and a terminal object.*
3. *$\mathcal{C}$ has products and pullbacks.*
4. *$\mathcal{C}$ has products and equalizers.*

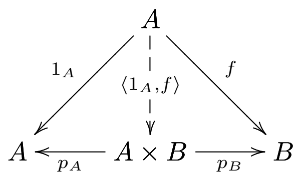
Proof. $1 \Rightarrow 2$ is obvious.

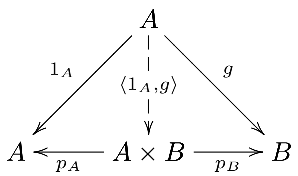
$2 \Rightarrow 3$. Let $\mathbf{1}$ be a terminal object of $\mathcal{C}$ and let us show that the product of objects $A_i, i \in I$, of $\mathcal{C}$ exists, where I is a set. For every $i \in I$, there is a unique morphism $f_i : A_i \to \mathbf{1}$. Let $(P, (p_i)_{i\in I})$ be the multiple pullback of f_i, $i \in I$.



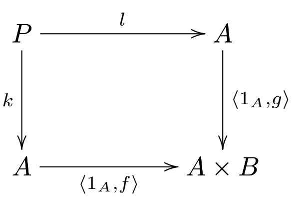
It easily follows from the terminality of $\mathbf{1}$ that then $(P, (p_i)_{i\in I})$ is also a product of A_i, $i \in I$.

$3 \Rightarrow 4$. We have to prove that an equalizer of a pair $f, g : A \to B$ exists. By the universal property of products we obtain unique morphisms $\langle 1_A, f\rangle$, $\langle 1_A, g\rangle$ such that the diagrams



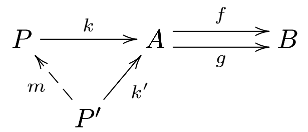


commute. Construct a pullback



of $\langle 1_A, f\rangle$ and $\langle 1_A, g\rangle$, and observe that

$$k = 1_A k = p_A\langle 1_A, f\rangle k = p_A\langle 1_A, g\rangle l = 1_A l = l,$$
$$fk = p_B\langle 1_A, f\rangle k = p_B\langle 1_A, g\rangle l = gl = gk.$$



If there is another $k' : P' \to A$ such that $fk' = gk'$ then also

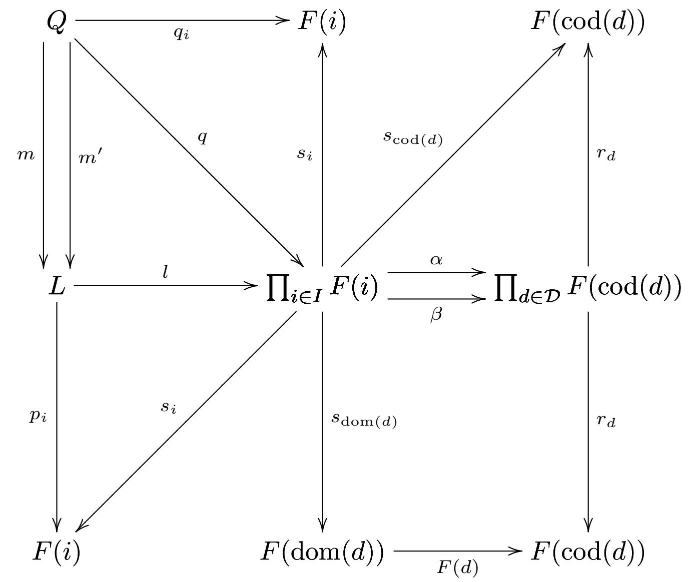
$$\begin{aligned} p_A\langle 1_A, f\rangle k' &= k' = p_A\langle 1_A, g\rangle k' \\ p_B\langle 1_A, f\rangle k' &= p_B\langle 1_A, g\rangle k', \end{aligned}$$

which by Proposition 5.5 imply $\langle 1_A, f\rangle k' = \langle 1_A, g\rangle k'$. Since P is a pullback, there is a unique morphism $m : P' \to P$ such that $km = k' = lm$.

4 ⇒ 1. Consider a small category $\mathcal{D}$ and a functor $F : \mathcal{D} \to \mathcal{C}$. First we construct the products

$$\left(\prod_{i\in I} F(i), (s_i)_{i\in I}\right) \quad \text{and} \quad \left(\prod_{d\in\mathcal{D}} F(\mathrm{cod}(d)), (r_d)_{d\in\mathcal{D}}\right).$$

By the universal property of the second product, there exist a unique morphism α such that $r_d\alpha = s_{\mathrm{cod}(d)} = s_j$ for every $d : i \to j \in \mathcal{D}$ and a unique morphism β such that $r_d\beta = F(d)s_{\mathrm{dom}(d)} = F(d)s_i$ for every $d : i \to j \in \mathcal{D}$. Let (L, l) be the equalizer of the pair (α, β), so in particular $\alpha l = \beta l$. We define $p_i := s_i l$ and we shall prove that $(L, (p_i)_{i\in I})$ is the limit of the functor F.



For every morphism $d : i \to j$ in $\mathcal{D}$ we have

$$F(d)p_i = F(d)s_i l = r_d\beta l = r_d\alpha l = s_j l = p_j,$$

which means that $(L, (p_i)_{i\in I})$ is a cone on F. Suppose that $(Q, (q_i)_{i\in I})$ is another cone on F. By the universal property of product $\prod_{i\in I} F(i)$, there is a unique morphism q such that $s_i q = q_i$ for every $i \in I$. Now, for every $d : i \to j$ in $\mathcal{D}$,

$$r_d\alpha q = s_j q = q_j = F(d)q_i = F(d)s_i q = r_d\beta q.$$

By Proposition 5.5 we obtain $\alpha q = \beta q$. This implies, by the universal property of equalizer (L, l), the existence of a unique $m : Q \to L$ such that $lm = q$. Consequently

$$p_i m = s_i l m = s_i q = q_i$$

for every $i \in I$. It remains to prove that m is unique. Suppose that $m' : Q \to L$ is another morphism such that $p_i m' = q_i$ for every $i \in I$. Then

$$s_i l m = s_i q = q_i = p_i m' = s_i l m'$$

for every $i \in I$, which again by Proposition 5.5 implies $lm = lm'$. This implies $m = m'$ by Proposition 5.26, because l is an equalizer. ■

Similarly one can prove the followig result.

Theorem 5.49 *For every category $\mathcal{C}$, the following assertions are equivalent:*

1. *$\mathcal{C}$ is finitely complete.*
2. *$\mathcal{C}$ has pullbacks and a terminal object.*
3. *$\mathcal{C}$ has finite products and pullbacks.*
4. *$\mathcal{C}$ has finite products and equalizers.*

Examples 5.50
1. The category of finite sets and the category of finite topological spaces are both finitely complete and finitely cocomplete but neither of them is complete or cocomplete.
2. The category of finite groups is finitely complete but not finitely cocomplete.
3. The categories Set, Gr, Ab and Top are complete and cocomplete.
4. A poset considered as a category is complete as a category precisely when it is complete as a poset.

5.6 Limit preserving functors

Definition 5.51 Let $\mathcal{D}$ be a small category and denote again $I = \mathcal{D}_0$. A functor $G : \mathcal{A} \to \mathcal{B}$

1. **preserves $\mathcal{D}$-limits** if for every functor $F : \mathcal{D} \to \mathcal{A}$, if $(L, (p_i)_{i\in I})$ is a limit of F then $(G(L), (G(p_i))_{i\in I})$ is a limit of $GF : \mathcal{D} \to \mathcal{B}$;
2. **reflects $\mathcal{D}$-limits** if for every functor $F : \mathcal{D} \to \mathcal{A}$, if $(L, (p_i)_{i\in I})$ is a cone on F in $\mathcal{A}$ and $(G(L), (G(p_i))_{i\in I})$ is a limit of $GF : \mathcal{D} \to \mathcal{B}$, then $(L, (p_i)_{i\in I})$ is a limit of F.

Definition 5.52 A functor $G : \mathcal{A} \to \mathcal{B}$

1. **preserves limits** (resp. **preserves finite limits**) if G preserves $\mathcal{D}$-limits for every small category (resp. finite category) $\mathcal{D}$;
2. **reflects limits** (resp. **reflects finite limits**) if G reflects $\mathcal{D}$-limits for every small category (resp. finite category) $\mathcal{D}$.

Examples 5.53
1. The forgetful functors from Gr, Mod_R, Rng to Set preserve and reflect limits but none of them preserves or reflects arbitrary colimits.
2. The forgetful functor from Top to Set preserves limits and colimits but does not reflect either.
3. If $\mathcal{A}$ has finite products and $A \in \mathcal{A}_0$ is a fixed object then the functor $(A \times -) : \mathcal{A} \to \mathcal{A}$ preserves limits.

Using the proofs of Theorem 5.48 and Theorem 5.49, one can prove the following results.

Theorem 5.54 *If $\mathcal{A}$ is a finitley complete category and $F : \mathcal{A} \to \mathcal{B}$ is a functor then the following assertions are equivalent:*

1. *G preserves finite limits.*
2. *G preserves pullbacks and terminal objects.*
3. *G preserves finite products and pullbacks.*
4. *G preserves finite products and equalizers.*

Theorem 5.55 *If $\mathcal{A}$ is a complete category and $G : \mathcal{A} \to \mathcal{B}$ is a functor then the following assertions are equivalent:*

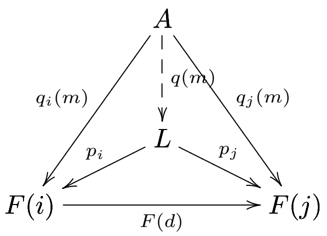
1. *G preserves limits.*
2. *G preserves multiple pullbacks and terminal objects.*
3. *G preserves products and pullbacks.*
4. *G preserves products and equalizers.*

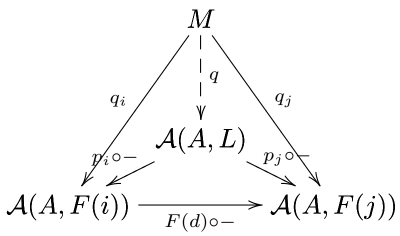
Proposition 5.56 *Consider a category $\mathcal{A}$ and an object $A \in \mathcal{A}_0$. The covariant representable functor $\mathcal{A}(A, -) : \mathcal{A} \to \mathsf{Set}$ preserves all existing limits, including large ones.*

Proof. Consider an arbitrary functor $F : \mathcal{D} \to \mathcal{A}$ and its limit $(L, (p_i)_{i \in I})$, where $I = \mathcal{D}_0$. Then of course $(\mathcal{A}(A, L), (p_i \circ -)_{i \in I})$ is a cone on the composite functor $\mathcal{A}(A, F(-)) = \mathcal{A}(A, -) \circ F$ in Set. Let $(q_i : M \to \mathcal{A}(A, F(i)))_{i \in I}$ be another cone on the functor $\mathcal{A}(A, F(-))$. That is, for every $d : i \to j$ in $\mathcal{D}$,

$$q_j = \mathcal{A}(A, F(d)) \circ q_i = (F(d) \circ -) \circ q_i,$$

so for every element $m \in M$, $q_j(m) = F(d) \circ q_i(m)$. The last means that $(A, (q_i(m))_{i \in I})$ is a cone on F. Hence there exists a unique morphism $q(m) : A \to L$ such that $p_i \circ q(m) = q_i(m)$ for every $i \in I$.





This defines a mapping $q : M \to \mathcal{A}(A, L)$ in Set with the property that $(p_i \circ -) \circ q = q_i$ for each $i \in I$.

It remains to prove that q is unique. Suppose that also $r : M \to \mathcal{A}(A, L)$ is such that $(p_i \circ -)r = q_i$ for each $i \in I$. Then $p_i \circ r(m) = q_i(m)$ for every $m \in M$ and $i \in I$. Since L is a limit, $q(m) = r(m)$ for every $m \in M$. Hence the mappings q and r are equal. ■

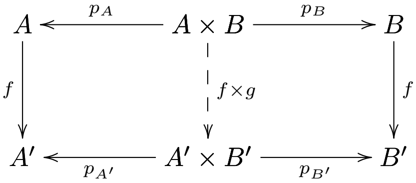
5.7 Exercises

Exercises 5.57

1. Does your favourite category have (co)products, (co)equalizers or pullbacks? Is it (co)complete?
2. Let $\mathcal{C}$ be a category with finite products. Show that there is a functor $- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by assignments

$$(- \times -)(A, B) := A \times B,$$
$$(- \times -)(f, g) := f \times g : A \times B \to A' \times B',$$

   where $f : A \to A'$, $g : B \to B'$ in $\mathcal{C}$ and $f \times g$ is the unique morphism which makes the diagram



   commute. (A functor from a product of two categories to a third category is usually called a **bifunctor**.)
3. Show that any nontrivial group considered as a one-object category does not have binary products.
4. Let (E, e) be an equlizer of $f, g : A \to B$. Prove that e is an isomorphism if and only if $f = g$.

5. Give an example of a category $\mathcal{C}$ where there are two morphisms $f, g : A \to B$, which do not have a coequalizer in $\mathcal{C}$. (Hint: one can consider subcategories of Set.)

6. Formulate the definition of pushout as the dual of the definition of pullback.

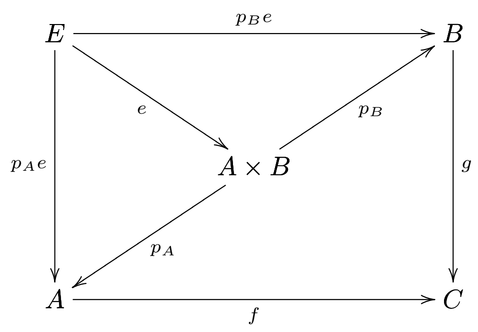
7. Prove that the projections of kernel pairs are split epimorphisms.

8. Define the notion of a morphism of cones in terms of natural transformations.

9. Prove that Proposition 5.5 generalizes to arbitrary limits, i.e. projections of limits are simultaneously left cancellable.

10. Using the previous exercise prove that if f is a monomorphism and (P, p_1, p_2) is a pullback of f and g then p_2 is a monomorphism.

11. Prove directly that pullbacks can be (canonically) constructed using products and equalizers. In more detail, let $f : A \to C$ and $g : B \to C$ be two morphisms of $\mathcal{C}$. Prove that if $(A \times B, p_A, p_B)$ is a product of A and B and if (E, e) is an equalizer of fp_A and gp_B then the outer square in the diagram



is a pullback square.

12. Prove that if $G : \mathcal{A} \to \mathcal{B}$ and $H : \mathcal{B} \to \mathcal{C}$ preserve all limits then also $HG : \mathcal{A} \to \mathcal{C}$ preserves all limits.

13. For a category $\mathcal{A}$ and a fixed object $A \in \mathcal{A}_0$ give the definition of a functor $(A \times -) : \mathcal{A} \to \mathcal{A}$ analogously to exercise 2. Prove that if $\mathcal{A}$ is finitely complete then $A \times -$ preserves equalizers.

6 Adjunctions

6.1 Motivating examples

Consider the category Vec_K of all vector spaces over a field K with linear mappings as morphisms. The forgetful functor $U : \mathsf{Vec}_K \to \mathsf{Set}$ sends every vector space V to its set of elements. For any set X there is a vector space V_X with X as a set of basis vectors; it consists of all formal finite K-linear combinations $k_1x_1 + \ldots + k_nx_n$ of the elements of X with the evident operations. Every mapping $f : X \to Y$ can in an obvious way be extended to a linear mapping $V_X \to V_Y$, so that a functor $F : \mathsf{Set} \to \mathsf{Vec}_K$ with $F_0(X) = V_X$, for every set X, results. For every set X and a vector space W there is a bijective mapping

$$\begin{array}{c} \varphi_{X,W} : \mathsf{Vec}_K(F(X), W) \longrightarrow \mathsf{Set}(X, U(W)), \\ f \mapsto f|_X. \end{array} \tag{8}$$

Its inverse $\psi_{X,W} : \mathsf{Set}(X, U(W)) \longrightarrow \mathsf{Vec}_K(F(X), W)$ extends every mapping $g : X \to U(W)$ to a unique linear mapping $f_g : F(X) \to W$, which is explicitly given by

$$f_g(k_1x_1 + \ldots + k_nx_n) = k_1g(x_1) + \ldots + k_ng(x_n) \tag{9}$$

(so f_g takes formal linear combinations in $F(X)$ to actual linear combinations in W). The mappings $\varphi_{X,W}$ are the components of a natural transformation φ, if both sides of (8) are considered as functors of X and W. It suffices to verify naturality in X and W separately (see Exercise 6.12 (1)). Naturality in X means that for each morphism $k : X' \to X$ the diagram

$$\begin{array}{ccc} \mathsf{Vec}_K(F(X), W) & \xrightarrow{\varphi_{X,W}} & \mathsf{Set}(X, U(W)) \\ {\scriptstyle -\circ F(k)}\big\downarrow & & \big\downarrow{\scriptstyle -\circ k} \\ \mathsf{Vec}_K(F(X'), W) & \xrightarrow[\varphi_{X',W}]{} & \mathsf{Set}(X', U(W)) \end{array}$$

is commutative. Indeed, for every linear mapping $f : F(X) \to W$ and every $x \in X'$,

$$\begin{aligned} [(-\circ k) \circ \varphi_{X,W}](f)(x) &= (\varphi_{X,W}(f) \circ k)(x) = f|_X(k(x)) = fk(x) = f(F(k)(x)) = (f \circ F(k))|_{X'}(x) \\ &= [\varphi_{X',W}(f \circ F(k))](x) = [\varphi_{X',W} \circ (-\circ F(k))](f)(x). \end{aligned}$$

A similar calculation shows that φ is natural in W. Moreover, the mapping that sends every $x \in X$ into the same x regarded as a vector of V_X is a morphism $\iota_X : X \to U(V_X) = UF(X)$ in Set. For any other vector space W and a mapping $g : X \to U(W)$, the linear mapping $f_g : V_X \to W$ is the unique linear mapping extending g, i.e. making the triangle

$$\begin{array}{ccccc} X & \xrightarrow{\iota_X} & U(V_X) & \qquad & V_X \\ & {\scriptstyle g}\searrow & \big\downarrow{\scriptstyle U(f_g)} & & {\scriptstyle f_g}\big\downarrow \\ & & U(W) & & W \end{array}$$

commutative.

There are some other similar examples.

There is a bijective mapping

$$\varphi_{S,R} : \mathsf{Set}(S \times T, R) \longrightarrow \mathsf{Set}(S, \mathsf{Set}(T, R))$$

given by $[\varphi_{S,R}(f)(s)](t) := f(s,t)$ for every mapping $f : S \times T \to R$ and elements $s \in S$, $t \in T, r \in R$. Such φ is natural in S and R (but also T). If a set T is fixed and we define $F, G : \mathsf{Set} \to \mathsf{Set}$ by $F := - \times T$ and $G := \mathsf{Set}(T, -)$, this bijection takes the form

$$\varphi_{S,R} : \mathsf{Set}(F(S), R) \longrightarrow \mathsf{Set}(S, G(R)).$$

For modules A, B over a commutative ring R and an abelian group C there is a similar isomorphism

$$\varphi_{B,C} : \mathsf{Ab}(A \otimes_R B, C) \longrightarrow \mathsf{Mod}_R(A, \mathsf{Mod}_R(B, C)).$$

6.2 Adjoint functors

Definition 6.1 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An **adjunction** from $\mathcal{A}$ to $\mathcal{B}$ is a triple $\langle F, G, \varphi\rangle : \mathcal{A} \to \mathcal{B}$, where $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors and φ is a mapping which assigns to each pair of objects $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ a bijective mapping

$$\varphi_{A,B} : \mathcal{B}(F(A), B) \to \mathcal{A}(A, G(B)) \tag{10}$$

which is natural in A and B. If $\langle F, G, \varphi\rangle : \mathcal{A} \to \mathcal{B}$ is an adjunction, we say that F is **left adjoint** for G and G is **right adjoint** for F, and write $F \dashv G$.

The functor $\mathcal{B}(F(-), -)$ on the left hand side of (10) is the bifunctor

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{F\times 1_{\mathcal{B}}} \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{\text{hom}} \mathsf{Set}$$

which sends each pair of objects (A, B) to the morphism-set $\mathcal{B}(F(A), B)$, and a morphism $(k^{\mathrm{op}}, h) : (A, B) \to (A', B')$ of the product category $\mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ to the mapping $h \circ - \circ F(k) : \mathcal{B}(F(A), B) \to \mathcal{B}(F(A'), B')$ in Set, and the right hand side is a similar bifunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$. Therefore the naturality of the bijection φ means that for all $k : A' \to A$ and $h : B \to B'$ both the diagrams

$$\begin{array}{ccc} \mathcal{B}(F(A),B) & \xrightarrow{\varphi_{A,B}} & \mathcal{A}(A,G(B)) \\ {\scriptstyle -\circ F(k)}\downarrow & & \downarrow{\scriptstyle -\circ k} \\ \mathcal{B}(F(A'),B) & \xrightarrow[\varphi_{A',B}]{} & \mathcal{A}(A',G(B)) \end{array} \qquad \begin{array}{ccc} \mathcal{B}(F(A),B) & \xrightarrow{\varphi_{A,B}} & \mathcal{A}(A,G(B)) \\ {\scriptstyle h\circ -}\downarrow & & \downarrow{\scriptstyle G(h)\circ -} \\ \mathcal{B}(F(A),B') & \xrightarrow[\varphi_{A,B'}]{} & \mathcal{A}(A,G(B')) \end{array}$$

will commute (see Exercise 6.12 (1)). That is, for all $k : A' \to A$ in $\mathcal{A}$, $h : B \to B'$ and $f : F(A) \to B$,

$$\varphi_{A',B}(fF(k)) = \varphi_{A,B}(f)k, \tag{11}$$
$$\varphi_{A,B'}(hf) = G(h)\varphi_{A,B}(f). \tag{12}$$

Definition 6.2 Let $G : \mathcal{B} \to \mathcal{A}$ be a functor and $A \in \mathcal{A}_0$. A **universal morphism** from A to G is an initial object of the category $(A \downarrow G)$ of objects G-under A (see Example 3.15).

Thus a universal morphism from A to G is a pair (u, B), consisting of an object $B \in \mathcal{B}_0$ and a morphism $u : A \to G(B)$ such that for every pair (g, B') with $B' \in \mathcal{B}_0$ and $g : A \to G(B')$ a morphism of $\mathcal{A}$, there is a unique morphism $f : B \to B'$ in $\mathcal{B}$ such that $G(f)u = g$. In other words, every morphism g to G factors uniquely through the universal morphism u.

$$\begin{array}{ccccc} A & \xrightarrow{u} & G(B) & \qquad & B \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle f} \\ & & G(B') & & B' \end{array}$$

Dually one can define universal morphisms from F to B. Since initial and terminal objects are unique up to isomorphism, also universal morphisms are unique up to isomorphism.

Theorem 6.3 *An adjunction $\langle F, G, \varphi\rangle : \mathcal{A} \to \mathcal{B}$ determines*

1. *a natural transformation $\eta : 1_{\mathcal{A}} \Rightarrow GF$ such that for each object $A \in \mathcal{A}_0$ the pair $(\eta_A, F(A))$ is a universal morphism from A to G, and for each $f : F(A) \to B$,*

$$\varphi_{A,B}(f) = G(f)\eta_A : A \to G(B); \tag{13}$$

2. *a natural transformation $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ such that for each object $B \in \mathcal{B}_0$ the pair $(G(B), \varepsilon_B)$ is a universal morphism from F to B, and for each $g : A \to G(B)$,*

$$\varphi^{-1}_{A,B}(g) = \varepsilon_B F(g) : F(A) \to B. \tag{14}$$

Moreover, both the composites $(G*\varepsilon)\circ(\eta*G)$ and $(\varepsilon*F)\circ(F*\eta)$ are the identity natural transformations (of G, resp. F).

$$G \xrightarrow{\eta*G} GFG \xrightarrow{G*\varepsilon} G, \quad F \xrightarrow{F*\eta} FGF \xrightarrow{\varepsilon*F} F. \tag{15}$$

We call η the **unit** and ε the **counit** of the adjunction.

Note that by (2) and (3) the condition (15) translates to so-called **triangular identities**

$$G(\varepsilon_B)\eta_{G(B)} = 1_{G(B)}, \quad \varepsilon_{F(A)}F(\eta_A) = 1_{F(A)}, \tag{16}$$

$A \in \mathcal{A}_0, B \in \mathcal{B}_0$, i.e. the commutativity of triangles

$$\begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\ & {\scriptstyle 1_{G(B)}} \searrow & \downarrow {\scriptstyle G(\varepsilon_B)} \\ & & G(B) \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{F(\eta_A)} & FGF(A) \\ & {\scriptstyle 1_{F(A)}} \searrow & \downarrow {\scriptstyle \varepsilon_{F(A)}} \\ & & F(A) \end{array} \quad . \tag{17}$$

Proof. 1. For every $A \in \mathcal{A}_0$, take $B := F(A)$ in (10) and define

$$\eta_A : A \to GF(A), \quad \eta_A := \varphi_{A,F(A)}(1_{F(A)}). \tag{18}$$

First we note that (12) implies that for every $f : F(A) \to B$,

$$\varphi_{A,B}(f) = \varphi_{A,B}(f1_{F(A)}) = G(f)\varphi_{A,F(A)}(1_{F(A)}) = G(f)\eta_A. \tag{19}$$

To prove the universality of $(\eta_A, F(A))$, suppose that $g : A \to G(B)$ is a morphism from A to G. Then $\varphi^{-1}_{A,B}(g) : F(A) \to B$ and we obtain

$$G(\varphi^{-1}_{A,B}(g))\eta_A = \varphi_{A,B}(\varphi^{-1}_{A,B}(g)) = g.$$

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & GF(A) & & F(A) \\ & {\scriptstyle g} \searrow & \downarrow {\scriptstyle G(\varphi^{-1}_{A,B}(g))} & & \downarrow {\scriptstyle \varphi^{-1}_{A,B}(g)} \\ & & G(B) & & B \end{array}$$

If also $G(h)\eta_A = g$ for a morphism $h : F(A) \to B$ then $g = \varphi_{A,B}(h)$ implies $\varphi^{-1}_{A,B}(g) = h$. Thus η_A is a universal morphism from A to G.

To prove that $\eta = (\eta_A)_{A\in\mathcal{A}_0}$ is a natural transformation, consider for $k : A' \to A$ a diagram

$$\begin{array}{ccc} A' & \xrightarrow{\eta_{A'}} & GF(A') \\ {\scriptstyle k} \downarrow & & \downarrow {\scriptstyle GF(k)} \\ A & \xrightarrow[\eta_A]{} & GF(A) \end{array}$$

Its commutativity follows from the calculation

$$\begin{aligned} GF(k)\eta_{A'} &= GF(k)\varphi_{A',F(A')}(1_{F(A')}) = \varphi_{A',F(A)}(F(k)1_{F(A')}) = \varphi_{A',F(A)}(1_{F(A)}F(k)) \\ &= \varphi_{A,F(A)}(1_{F(A)})k = \eta_A k \end{aligned}$$

where we used (18), (11) and (12). This calculation may also be illustrated by the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{B}(F(A'), F(A')) & \xrightarrow{F(k)\circ -} & \mathcal{B}(F(A'), F(A)) & \xleftarrow{-\circ F(k)} & \mathcal{B}(F(A), F(A)) \\ {\scriptstyle \varphi_{A',F(A')}} \downarrow & & \downarrow {\scriptstyle \varphi_{A',F(A)}} & & \downarrow {\scriptstyle \varphi_{A,F(A)}} \\ \mathcal{A}(A', GF(A')) & \xrightarrow[GF(k)\circ -]{} & \mathcal{A}(A', GF(A)) & \xleftarrow[-\circ k]{} & \mathcal{A}(A, GF(A)) \end{array}$$

2. For every $B \in \mathcal{B}_0$, set $A := G(B)$ in (10) and define a morphism

$$\varepsilon_B : FG(B) \to B, \quad \varepsilon_B := \varphi^{-1}_{G(B),B}(1_{G(B)}). \tag{20}$$

It turns out to be a universal morphism from F to B and $\varphi^{-1}_{A,B}(g) = \varepsilon_B F(g)$ for every $g : A \to G(B)$.

Finally, the triangles (17) commute because

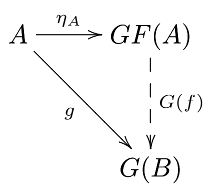
$$\begin{aligned} 1_{G(B)} &= \varphi_{G(B),B}(\varepsilon_B) = G(\varepsilon_B)\eta_{G(B)}, \\ 1_{F(A)} &= \varphi^{-1}_{A,F(A)}(\eta_A) = \varepsilon_{F(A)}F(\eta_A). \end{aligned}$$

■

Theorem 6.4 *Each adjunction $\langle F, G, \varphi\rangle : \mathcal{A} \to \mathcal{B}$ is completely determined by the items in any one of the following lists:*

1. *Functors F, G, and a natural transformation $\eta : 1_{\mathcal{A}} \Rightarrow GF$ such that each $\eta_A : A \to GF(A)$ is universal from A to G. Then φ is defined by (13).*
2. *Functors F, G, and a natural transformation $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ such that each $\varepsilon_B : FG(B) \to B$ is universal from F to B. Here φ^{-1} is defined by (14).*
3. *Functors F, G and natural transformations $\eta : 1_{\mathcal{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ such that both composites (15) are the identity transformations. Here φ is defined by (13) and φ^{-1} by (14).*

Proof. 1. The universality of $\eta_A : A \to GF(A)$ means that for each $g : A \to G(B)$ there is exactly one $f : F(A) \to B$ making the triangle



commutative. This states precisely that

$$\psi_{A,B}(f) := G(f)\eta_A$$

defines a bijection $\psi_{A,B} : \mathcal{B}(F(A), B) \to \mathcal{A}(A, G(B))$.

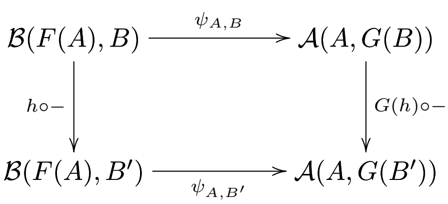
For every $k : A' \to A$ in $\mathcal{A}$ and $f : F(A) \to B$ in $\mathcal{B}$,

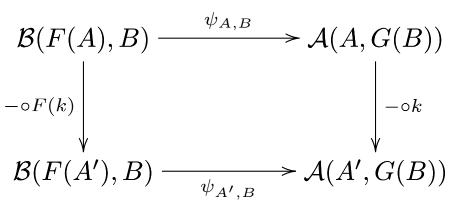
$$\psi_{A,B}(f)k = G(f)\eta_A k = G(f)GF(k)\eta_{A'} = G(fF(k))\eta_{A'} = \psi_{A',B}(fF(k)),$$

hence ψ is natural in A. For every $h : B \to B'$ and $f : F(A) \to B$ in $\mathcal{B}$,

$$G(h)\psi_{A,B}(f) = G(h)G(f)\eta_A = G(hf)\eta_A = \psi_{A,B'}(hf),$$

so ψ is also natural in B.





Thus we have an adjunction $\langle F, G, \psi\rangle$. In case η was the unit obtained from an adjunction $\langle F, G, \varphi\rangle$, then $\psi = \varphi$, because, by (13), $\psi_{A,B}(f) = G(f)\eta_A = \varphi_{A,B}(f)$ for every $f : F(A) \to B$.

2 is dual to 1.

3. For every $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$ we define two mappings $\mathcal{B}(F(A), B) \underset{\theta_{A,B}}{\overset{\varphi_{A,B}}{\rightleftarrows}} \mathcal{A}(A, G(B))$ by

$$\begin{aligned} \varphi_{A,B}(f) &:= G(f)\eta_A, \\ \theta_{A,B}(g) &:= \varepsilon_B F(g), \end{aligned}$$

$f : F(A) \to B$, $g : A \to G(B)$. Using functoriality of G, naturality of η and the first triangular identity (16), we obtain

$$\begin{aligned} \varphi_{A,B}\theta_{A,B}(g) &= \varphi_{A,B}(\varepsilon_B F(g)) = G(\varepsilon_B F(g))\eta_A = G(\varepsilon_B)GF(g)\eta_A \\ &= G(\varepsilon_B)\eta_{G(B)}g = 1_{G(B)}g = g. \end{aligned}$$

Hence $\varphi_{A,B}\theta_{A,B} = 1_{\mathcal{A}(A,G(B))}$. Dually $\theta_{A,B}\varphi_{A,B} = 1_{\mathcal{B}(F(A),B)}$, and so $\varphi_{A,B}$ is a bijection. Naturality of φ, that is, equalities (11) and (12) for arbitrary morphisms $k : A' \to A, h : B \to B'$ and $f : F(A) \to B$, is proven by calculations

$$\begin{aligned} \varphi_{A',B}(fF(k)) &= G(fF(k))\eta_{A'} = G(f)GF(k)\eta_{A'} = G(f)\eta_A k = \varphi_{A,B}(f)k, \\ \varphi_{A,B'}(hf) &= G(hf)\eta_A = G(h)G(f)\eta_A = G(h)\varphi_{A,B}(f). \end{aligned}$$

Consequently we have obtained an adjunction (and if we started with an adjunction, it is the one from which we started). ■

As an example, consider two partially ordered sets A and B as categories (see Example 1.7(2)). Order-preserving mappings between A and B are covariant functors and order-reversing mappings are contravariant functors. Consider two order-reversing mappings $f : A \to B$, $g : B \to A$ as covariant functors

$$A \underset{g}{\overset{f}{\rightleftarrows}} B^{\mathrm{op}}$$

(see Proposition 3.8). Then $f \dashv g$ if

$$a \le gf(a) \text{ in } A \text{ and } fg(b) \le b \text{ in } B^{\mathrm{op}} \text{ (or } b \le fg(b) \text{ in } B) \tag{21}$$

for every $a \in A$ and $b \in B$. This follows from Theorem 6.4 (3): the naturality conditions and the triangular identities are automatically satisfied since in a poset, between two objects there is at most one morphism. Thus condition (21) is equivalent to the existence of a bijection $B^{\mathrm{op}}(f(a), b) \to A(a, g(b))$, which reduces to

$$b \le f(a) \text{ in } B \Longleftrightarrow a \le g(b) \text{ in } A$$

for every $a \in A$ an $b \in B$. Such a situation between posets is called a **Galois connection**.

6.3 Examples of adjuncions

The following table lists some examples of adjoints.

	$\mathcal{A}$	$\mathcal{B}$	$F : \mathcal{A} \to \mathcal{B}$	$G : \mathcal{B} \to \mathcal{A}$	Unit of adjunction
1	Set	Vec_K	$X \mapsto V_X$, vector space on basis X	forgetful functor U	$\iota_X : X \to U(V_X)$, insertion of generators (cf. 6.1)
2	Set	Gr	$X \mapsto F(X)$, free group with generators $x \in X$	forgetful functor U	$X \to UF(X)$, insertion of generators
3	Gr	Ab	$A \mapsto A/A'$, abelianization functor (cf. Example 3.3 (6))	forgetful functor U	$A \to A/A'$, projection on the quotient
4	Dom_m	Field	$D \mapsto Q(D)$, field of quotients	forgetful functor U	$\iota_D : D \to UQ(D)$, inclusion of D: $a \mapsto \frac{a}{1}$
5	Met	Cmet	completion of metric space	inclusion functor	$X \to \overline{X}$, inclusion of X into its completion
6	Set	Top	$X \mapsto (X, \tau)$, τ discrete (discrete space functor)	forgetful functor U	$1_X : X \to X$
7	Top	Haus	$(X, \tau) \mapsto (X, \tau)/\overline{\Delta_X}$, quotient by the closure of diagonal	inclusion functor	$(X, \tau) \to (X, \tau)/\overline{\Delta_X}$, projection on the quotient
8	Set	Set	$- \times T$, T is a fixed set	$\mathsf{Set}(T, -) = (-)^T$, T is a fixed set	$S \to \mathsf{Set}(T, S \times T)$, $s \mapsto f_s$ where $f_s(t) = (s, t)$
9	Mod_R	Ab	$- \otimes_R B$, B is a fixed R-module	$\mathsf{Mod}_R(B, -)$, B is a fixed R-module	$A \to \mathsf{Mod}_R(B, A \otimes B)$, $a \mapsto f_a$ where $f_a(b) = a \otimes b$
10	$\mathcal{B}^2$	$\mathcal{B}$	$\coprod : (B, B') \mapsto B \coprod B'$, coproduct	$B \mapsto (B, B)$, diagonal functor	pair of injections $u_B : B \to B \coprod B'$, $u_{B'} : B' \to B \coprod B'$
11	$\mathcal{A}$	$\mathcal{A}^2$	$A \mapsto (A, A)$, diagonal functor	$\prod : (A, A') \mapsto A \times A'$, product functor	$\delta_A : A \to A \times A$

There are similar descriptions for counits. For example in the vector space V_X on basis X, if $\iota_X(x) = \langle x \rangle \in V_X$ for every $x \in X$, then the elements of V_X are the finite vector sums $k_1\langle x_1\rangle + \ldots + k_n\langle x_n\rangle$, $k_i \in K$, $x_i \in X$. Then for every vector space A the counit $\varepsilon_A : V_{U(A)} \to A$ is defined by $\varepsilon_A = \varphi^{-1}_{U(A),A}(1_{U(A)})$ where $\varphi^{-1}_{U(A),A} = \psi_{U(A),A}$ is given by (9). Hence $\varepsilon_A = \psi_{U(A),A}(1_{U(A)}) = f_{1_{U(A)}}$ (see (20) and (6.1)) and

$$\varepsilon_A(k_1\langle a_1\rangle + \ldots + k_n\langle a_n\rangle) = k_1 1_{U(A)}(a_1) + \ldots + k_n 1_{U(A)}(a_n) = k_1 a_1 + \ldots + k_n a_n.$$

In the adjunction 4, Dom_m is the category of all integral domains with arrows all monomorphisms of integral domains (note that a homomorphism of fields is necessarily a monomorphism). For every integral domain D, a familiar construction gives a field $Q(D)$ of quotients of D together with a monomorphisn $\iota_D : D \to Q(D)$, $a \mapsto \frac{a}{1}$. If K is any field and $g : D \to U(K)$ a monomorphism then there is a unique homomorphism $f : Q(D) \to K$ such that $U(f)\iota_D = g$. Thus ι_D is a universal morphism from D to $UQ(D)$. However, in the larger category Dom where morphisms are all homomorphisms of integral domains there is no universal homomorphism from (for example) $\mathbb{Z}$ to forgetful functor U. This follows from the fact that for every prime p there is a surjective homomorphism $\mathbb{Z} \to \mathbb{Z}_p$.

In the adjunction 5, Met is the category of all metric spaces with distance preserving mappings as morphisms (these must be necessarily injective). Cmet is the full subcategory of Met where objects are complete metric spaces. The completion procedure of metric spaces induces a functor Met $\to$ Cmet which is a left adjoint to the inclusion functor Cmet $\to$ Met. The usual inclusion $X \to \overline{X}$ of a metric space into its completion is the unit of this adjunction.

In the adjunction 6, a left adjoint of the forgetful functor Top $\to$ Set takes every set X to a topological space (X, τ) where τ is the discrete topology on X.

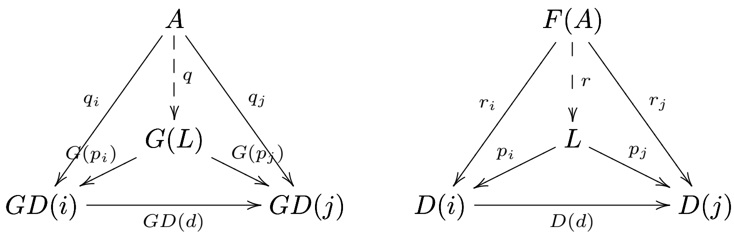
In the adjunction 7, Haus is the full subcategory of Top, consisting of all Hausdorff spaces. A topological space (X, τ) is a Hausdorff space if and only if its diagonal $\Delta_X \subseteq X \times X$ is closed. The inclusion functor Haus $\to$ Top has a left adjoint $F :$ Top $\to$ Haus where $F(X, \tau) = (X, \tau)/\overline{\Delta_X}$ is the quotient of the topological space (X, τ) by the closure of its diagonal $\overline{\Delta_X} \subseteq X \times X$, which is indeed an equivalence relation.

6.4 The Adjoint Functor Theorem

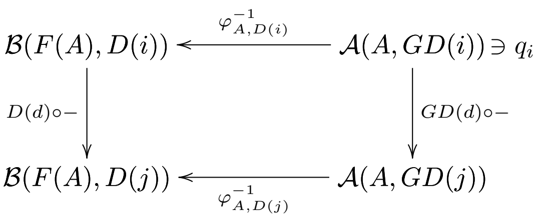
Proposition 6.5 *If the functor $G : \mathcal{B} \to \mathcal{A}$ has a left adjoint, G preserves all limits which exist in $\mathcal{B}$.*

Proof. Let $\langle F, G, \varphi \rangle : \mathcal{A} \to \mathcal{B}$ be an adjunction. Consider a category $\mathcal{D}$ and write $I = \mathcal{D}_0$. Suppose $(L, (p_i)_{i \in I})$ is a limit of a functor $D : \mathcal{D} \to \mathcal{B}$. We must prove that $(G(L), G(p_i)_{i \in I})$ is a limit of GD. Clearly $(G(L), G(p_i)_{i \in I})$ is a cone on GD, so it suffices to prove the universal property.

Consider a cone $(A, (q_i)_{i \in I})$ on GD.



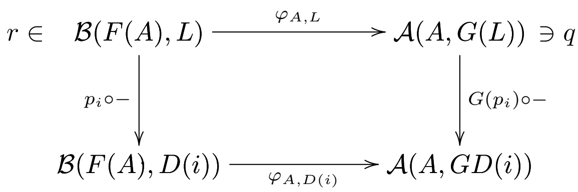
For every $i \in I$, set $r_i := \varphi^{-1}_{A,D(i)}(q_i) : F(A) \to D(i)$. Then for any morphism $d : i \to j$ in $\mathcal{D}$, the commutativity of the square



implies

$$r_j \quad = \quad \varphi^{-1}_{A,D(j)}(q_j) = \varphi^{-1}_{A,D(j)}(GD(d)q_i) = D(d)\varphi^{-1}_{A,D(i)}(q_i) = D(d)r_i,$$

and hence $(F(A), (r_i)_{i \in I})$ is a cone on D. Therefore there exists a unique morphism $r : F(A) \to L$ such that $p_i r = r_i$ for every $i \in I$. Denoting $q := \varphi_{A,L}(r) : A \to G(L)$ and using the commutativity of



we obtain

$$G(p_i)q = G(p_i)\varphi_{A,L}(r) = \varphi_{A,D(i)}(p_i r) = \varphi_{A,D(i)}(r_i) = q_i.$$

If also $q' : A \to G(L)$ is such that $G(p_i)q' = q_i$ for every $i \in I$, then

$$\varphi_{A,D(i)}(p_i \varphi^{-1}_{A,L}(q')) = G(p_i)q' = q_i = \varphi_{A,D(i)}(r_i)$$

implies $p_i \varphi^{-1}_{A,L}(q') = r_i$ for every $i \in I$ by the injectivity of $\varphi_{A,D(i)}$. By the uniqueness of r, we conclude $\varphi^{-1}_{A,L}(q') = r$, which means $q' = \varphi_{A,L}(r) = q$. ■

The dual of Proposition 6.5 states that a functor that has a right adjoint preserves all colimits. Thus all functors F in the fourth column of the table of examples above preserve all colimits.

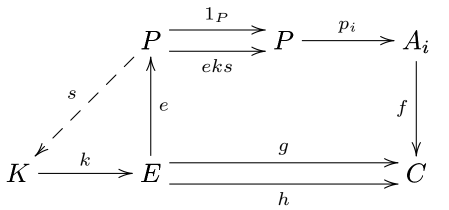
Theorem 6.6 *Let $\mathcal{C}$ be a complete category. Then $\mathcal{C}$ has an initial object if and only if it satisfies the following*

Solution Set Condition. *There is a set $S \subseteq \mathcal{C}_0$ such that for every $C \in \mathcal{C}_0$ there is a morphism $A \to C$ with $A \in S$.*

We shall use the abbreviation SSC for the Solution Set Condition. Every small category $\mathcal{C}$ has a solution set $S = \mathcal{C}_0$.

Proof. Necessity. If $\mathbf{0}$ is an initial object of $\mathcal{C}$, we may take $S = \{\mathbf{0}\}$.

Sufficiency. Let $S = \{A_i \mid i \in I\}$ be a solution set for $\mathcal{C}$. Let $(P, (p_i)_{i\in I})$ be the product of objects A_i, $i \in I$.



Construct the multiple equalizer (E, e) of the set of endomorphisms of P. We claim that E is the initial object of $\mathcal{C}$. If $C \in \mathcal{C}_0$, then there is a morphism $f : A_i \to C$ for some $i \in I$. Hence we have at least one morphism $fp_ie : E \to C$ form E to C. Suppose that there are two morphisms $g, h : E \to C$ and consider their equalizer (K, k). By the assumption and the construction of P, there is a morphism $s : P \to K$. So both 1_P and eks are endomorphisms of P, which implies $ekse = 1_P e = e1_E$. Since e is a monomorphism by a generalization of Proposition 5.26 we conclude $kse = 1_E$. Thus k is a split epimorphism, and hence $gk = hk$ implies $g = h$. ■

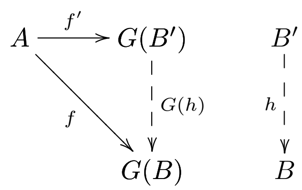
The proofs of the next two results have been omitted.

Lemma 6.7 *If $\mathcal{B}$ is complete and a functor $G : \mathcal{B} \to \mathcal{A}$ preserves all products (resp. equalizers) then for every $A \in \mathcal{A}_0$, the category $(A \downarrow G)$ of all objects G-under A has all small products (resp. equalizers).*

Theorem 6.8 (The Adjoint Functor Theorem) *Let $\mathcal{B}$ be a complete category. A functor $G : \mathcal{B} \to \mathcal{A}$ has a left adjoint functor if and only if G preserves limits and the category $(A \downarrow G)$ satisfies the SSC for every $A \in \mathcal{A}_0$.*

We note that explicitly the SSC for $(A \downarrow G)$ means that there exists a set $S_A \subseteq \mathcal{B}_0$ of objects such that

$$(\forall B \in \mathcal{B}_0)(\forall f : A \to G(B))(\exists B' \in S_A)(\exists f' : A \to G(B'))(\exists h : B' \to B)(G(h)f' = f).$$



If $F \dashv G$ then we can set $S_A := \{F(A)\}$ and use the universal morphism $\eta_A : A \to GF(A)$.

Let A and B be posets considered as categories and let B be complete, that is, B has arbitrary greatest lower bounds. A covariant functor (i.e. an order-preserving mapping) $g : B \to A$ has a left adjoint if and only if g preserves all greatest lower bounds. The SSC is satisfied because the category $(a \downarrow g)$ is small for every $a \in A$.

6.5 Equivalence of categories

Definition 6.9 A functor $G : \mathcal{B} \to \mathcal{A}$ is called an **equivalence of categories** if there is a functor $F : \mathcal{A} \to \mathcal{B}$ and natural isomorphisms $\eta : 1_{\mathcal{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$. In such a case the categories $\mathcal{A}$ and $\mathcal{B}$ are called **equivalent**.

Theorem 6.10 *For a functor $G : \mathcal{B} \to \mathcal{A}$ the following assertions are equivalent.*

1. *G is an equivalence of categories.*
2. *G has a left adjoint $F : \mathcal{A} \to \mathcal{B}$ and the unit $\eta : 1_{\mathcal{A}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ are isomorphisms.*
3. *G is full and faithful and for every $A \in \mathcal{A}_0$ there exists $B \in \mathcal{B}_0$ such that $G(B) \cong A$.*

Thus for a functor $G : \mathcal{B} \to \mathcal{A}$ the following implications hold:

$$G \text{ isomorphism } \Rightarrow G \text{ equivalence } \Rightarrow G \text{ has left adjoint.}$$

Examples 6.11 1. The category of all finite sets is equivalent to its full subcategory, which has as objects all sets $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, and the empty set.

2. For any field K, the category of all finite dimensional vector spaces over K is equivalent to the category of Example 1.7 (1).

6.6 Exercises

Exercises 6.12 1. Prove that if $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ are bifunctors and $\varphi = (\varphi_{A,B})_{(A,B) \in (\mathcal{A} \times \mathcal{B})_0} : F \Rightarrow G$ is natural both in A and B then it is natural in (A, B).

2. Choose some concrete adjunction and write out explicitly its unit, counit and bijection φ.

3. Prove that equivalence of categories is an equivalence relation (i.e. reflexive, symmetric and transitive).

Index

References

[1] F. Borceux, *Handbook of Categorical Algebra 1: Basic Category Theory* (Cambridge University Press, Cambridge, 1994).

[2] S. Mac Lane, *Categories for the Working Mathematician* (Springer, New York, 1971).

[3] Ch. Wells, *Electronic supplement to Category Theory for Computing Science*.

[4] H. Herrlich, G. Strecker, *Category Theory* (Allyn and Bacon, Boston, 1973).

[5] H.P. Gumm, *Elements of the general theory of coalgebras* (1999?), www.mathematik.uni-marburg.de/~gumm/Papers/Luatcs.ps