Forks, finitely related clones, and finitely generated varieties

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Results

Outline

In these lectures, we will present the proofs of:

Theorem (2009) [AMM14]

Every clone with edge operation on a finite set is finitely related.

Theorem (2014) [AM14]

Every subvariety of a finitely generated variety with edge term is finitely generated.

Classic Clone Theory

Clones

Operations

$O(A) := \bigcup_{k\in\mathbb{N}}\{f \mid f : A^k \to A\}$.

Clones

A subset C of $O(A)$ is a clone on A if

1. $\forall k, i \in \mathbb{N}$ with $i \leq k$: $\big((x_1, \ldots, x_k) \mapsto x_i\big) \in C$,
2. $\forall n \in \mathbb{N}, m \in \mathbb{N}, f \in C^{[n]}, g_1, \ldots, g_n \in C^{[m]}$:

$$f(g_1, \ldots, g_n) \in C^{[m]}.$$

$C^{[n]}$ ... the n-ary functions in C, $\quad C^{[n]} \subseteq A^{A^n}$.

Relational Description of Clones

Definition

I a finite set, $\rho \subseteq A^I$, $f : A^n \to A$. f preserves ρ ($f \rhd \rho$) if $\forall v_1, \ldots, v_n \in \rho$:

$$\langle f(v_1(i), \ldots, v_n(i)) \mid i \in I \rangle \in \rho.$$

In other words:

$$f^{A^I}(v_1, \ldots, v_n) \in \rho.$$

Remark

$f \rhd \rho \iff \rho$ is a subuniverse of $(A, f)^I$.

Definition (Polymorphisms)

Let R be a set of finitary relations on A, $\rho \in R$.

$$\begin{aligned} \mathrm{Pol}(\{\rho\}) &:= \{f \in O(A) \mid f \rhd \rho\}, \\ \mathrm{Pol}(R) &:= \bigcap_{\rho \in R} \mathrm{Pol}(\{\rho\}). \end{aligned}$$

Relational Description of Clones

Theorem

Let R be a set of finitary relations on A, and let $\rho_1, \rho_2 \in R$ with $\rho_1 \neq \emptyset$, $\rho_2 \neq \emptyset$. Then

1. $\mathrm{Pol}(R)$ is a clone.
2. $\mathrm{Pol}(\{\rho_1, \rho_2\}) = \mathrm{Pol}(\{\rho_1 \times \rho_2\})$.

Every clone can be described by relations

Theorem (see [PK79])

Let C be a clone on the finite set A. Then there is a set R of finitary relations such that $C = \mathrm{Pol}(R)$.

Proof:

- Observe $C^{[n]} \subseteq A^{A^n}$.
- Take $I_n := A^n$, $\rho_n := C^{[n]}$. Then $\rho_n \subseteq A^{I_n}$.
- Set $R := \{\rho_1, \rho_2, \ldots, \ldots\}$.
- Prove $C \subseteq \mathrm{Pol}(R)$: $f \in C^{[n]}$, $g_1, \ldots, g_n \in \rho_m$ implies $f(g_1, \ldots, g_n) \in \rho_m$ by the closure properties of clones.
- Prove $\mathrm{Pol}(R) \subseteq C$: Let $f : A^n \to A$ in $\mathrm{Pol}(R)$. Then $f \rhd \rho_n$, hence $f(\pi_1, \ldots, \pi_n) \in \rho_n$, thus $f \in C^{[n]}$.

Finitely related clones vs. DCC

Definition

A clone C is finitely related if there is a finite set of finitary relations R with $C = \mathrm{Pol}(R)$.

Theorem [PK79, 4.1.3]

Let C be a clone on the finite set A. TFAE:

1. C is not finitely related.
2. There is a strictly decreasing sequence $C_1 \supset C_2 \supset C_3 \supset \cdots$ with $C = \bigcap_{i \in \mathbb{N}} C_i$.

Finitely related clones vs. DCC

Theorem

Let C be a clone on the finite set A. TFAE:

1. C is not finitely related.
2. There is a strictly decreasing sequence $C_1 \supset C_2 \supset C_3 \supset \cdots$ with $C = \bigcap_{i \in \mathbb{N}} C_i$.

Proof of (1)⇒(2):

- We know $C = \mathrm{Pol}(\{\rho_1, \rho_2, \ldots\})$.
- Hence $\mathrm{Pol}(\{\rho_1\}) \supseteq \mathrm{Pol}(\{\rho_1, \rho_2\}) \supseteq \mathrm{Pol}(\{\rho_1, \rho_2, \rho_3\}) \supseteq \cdots$.

Proof of (2)⇒(1):

- Suppose $C = \mathrm{Pol}(\{\rho\})$, $|\rho| = N$.
- Then for some $n \in \mathbb{N}$, ${C_n}^{[N]} = C^{[N]}$.
- We show $\boxed{\forall f \in C_n : f \rhd \rho}$ on the next slide.
- Then $C_n \subseteq \mathrm{Pol}(\{\rho\}) = C \subseteq C_{n+1}$, a contradiction.

Finitely related clones vs. DCC

Proof of (2)⇒(1) (continued):

- Assumptions: $C = \mathrm{Pol}(\{\rho\})$, $\rho = \{b_1, \dots, b_N\}$, ${C_n}^{[N]} = C^{[N]}$.
- We want to show: $\boxed{\forall f \in C_n : f \rhd \rho}$.
- To this end, let $f \in C_n$, r-ary, and let $a_1, \dots, a_r \in \rho$.
- Goal: $f(a_1, \dots, a_r) \in \rho$.
- We have $f(a_1, \dots, a_r) = f(b_{i(1)}, \dots, b_{i(r)})$ with $i(k) \in \{1, \dots, N\}$ for all $k \in \{1, \dots, r\}$.
- Define $g(y_1, \dots, y_N) := f(y_{i(1)}, \dots, y_{i(r)})$ for all $\mathbf{y} \in A^N$.
- Then $f(b_{i(1)}, \dots, b_{i(r)}) = g(b_1, \dots, b_N)$.
- Now $g \in C_n^{[N]}$, hence $g \in C^{[N]}$. Thus $g(b_1, \dots, b_N) \in \rho$.

How to establish "finitely related"

Theorem

Let M be a clone on A. If

$$(\{C \mid C \text{ clone on A }, M \subseteq C\}, \subseteq)$$

satisfies the DCC, then every clone containing M is finitely related.

Definition

$(X, \leq)$ has the DCC $:\Leftrightarrow$ there is no $(x_i)_{i \in \mathbb{N}}$ with $x_1 > x_2 > x_3 > \cdots$.

Theorem

$(X, \leq)$ has the DCC $\Leftrightarrow$ Every nonempty subset Y of X has a minimal element.

Forks

Groups

Let G be a group, $n \in \mathbb{N}$.
Goal: represent subgroups of G^n. The following lemma will motivate the definition of forks and the formulation of the fork lemma.

Lemma

Let G be a group, $n \in \mathbb{N}$, $A \leq B \leq G^n$ subgroups. Assume

1. $A \subseteq B$
2. $\forall i \in \{1, \ldots, n\}$, $\forall g \in G$, $\forall r_{i+1}, \ldots, r_n \in G$:

$$(\underbrace{0, \ldots, 0}_{i-1}, g, r_{i+1}, \ldots, r_n) \in B \quad \Rightarrow$$

$$\exists s_{i+1}, \ldots, s_n \in G : \quad (0, \ldots, 0, g, s_{i+1}, \ldots, s_n) \in A.$$

Then $A = B$.

Mal'cev algebras I

$\mathbf{A}$ is a Mal'cev algebra $\Leftrightarrow \exists d \in \mathrm{Clo}_3\mathbf{A}\ \forall a, b \in A$:
$d(a, a, b) = d(b, a, a) = b$.

Definition of Forks

Let $\mathbf{A}$ be an algebra, let $m \in \mathbb{N}$, and let F be a subuniverse of $\mathbf{A}^m$. For $i \in \{1, \dots, m\}$, we define the relation $\varphi_i(F)$ on A by

$$\varphi_i(F) := \{(a_i, b_i) \mid (a_1, \dots, a_m) \in F, (b_1, \dots, b_m) \in F, \\ (a_1, \dots, a_{i-1}) = (b_1, \dots, b_{i-1})\}.$$

If $(c, d) \in \varphi_i(F)$, we call (c, d) a fork of F at i.
If

$$\begin{aligned} \mathbf{u} &= (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in F \text{ and} \\ \mathbf{v} &= (a_1, \dots, a_{i-1}, d, b_{i+1}, \dots, b_m) \in F, \end{aligned}$$

then $(\mathbf{u}, \mathbf{v})$ is a witness of the fork (c, d) at i.

Mal'cev algebras II

Forks have not been called forks, but are used, e.g., in:
[BD06, p.21], [BIM^{+}10], [Aic00, p.110]

The fork lemma

Lemma (cf. [BIM$^+$10, Cor. 3.9], [Aic10, Lemma 3.1])

Let $\mathbf{A}$ be an algebra with Mal'cev term d, and let $m \in \mathbb{N}$. Let F, G be subuniverses of $\mathbf{A}^m$ with $F \subseteq G$. We assume $\forall i \in \{1, \dots, m\}$: $\varphi_i(G) \subseteq \varphi_i(F)$. Then $F = G$.

Proof:

- For each $k \in \{1, \dots, m\}$, let

$$\begin{array}{rcl} F_k & := & \{(f_1, \dots, f_k) \mid (f_1, \dots, f_m) \in F\} \\ G_k & := & \{(g_1, \dots, g_k) \mid (g_1, \dots, g_m) \in G\}. \end{array}$$

- We prove $\forall k \in \{1, \dots, m\} : G_k \subseteq F_k$.
- $k = 1$: ✓

The fork lemma

- $k \geq 2$: Let $(g_1, \ldots, g_k) \in G_k$.
- Then $(g_1, \ldots, g_{k-1}) \in G_{k-1}$.
- By the induction hypothesis, $(g_1, \ldots, g_{k-1}) \in F_{k-1}$.
- Hence $\exists f_k$:
$$(g_1, \ldots, g_{k-1}, f_k) \in F_k.$$
- Since $(f_k, g_k) \in \varphi_k(G)$, we have $(f_k, g_k) \in \varphi_k(F)$.
- Thus $\exists$: $a_1, \ldots, a_{k-1} \in A$ such that
$$\begin{aligned}(a_1, \ldots, a_{k-1}, f_k) &\in F_k \\ (a_1, \ldots, a_{k-1}, g_k) &\in F_k.\end{aligned}$$
- By Mal'cev: $(g_1, \ldots, g_k) \in F_k$.

Limitation of forks

Fact

Let **A** be an algebra, $\alpha \in \mathrm{Aut}(\mathbf{A})$. Then

$$\begin{aligned} \mathbf{B} &= \{(a, a) \mid a \in A\} \\ \mathbf{C} &= \{(a, \alpha(a)) \mid a \in A\} \end{aligned}$$

have the same forks.

Fact [BD06, BIM+10]

(Forks + one witness per fork) represent subalgebras if we have a Mal'cev term. Can be modified to edge terms.

Representing Clones By Forks

Representing clones by forks

Let $C := \mathrm{Clo}((\mathbb{Z}_3, +))$; $A := \mathbb{Z}_3$. We represent the binary part $C^{[2]}$.

$$C^{[2]} = \{(x, y) \mapsto ax + by \mid a, b \in \mathbb{Z}_3\}.$$

- Order A: $0 < 1 < 2$.
- Order A^2 lexicographically: $00 < 01 < 02 < 10 < 11 < 12 < 20 < 21 < 22$.

Representing clones by forks

- For each $\mathbf{x} \in A^2$, compute $F(C, \mathbf{x}) := \{f(\mathbf{x}) \mid f \in C, \forall \mathbf{z} < \mathbf{x} : f(\mathbf{z}) = 0\}$.

$\mathbf{x}$	$F(C, \mathbf{x})$	Reason

Representing clones by forks

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00	$\{0\}$	

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00	$\{0\}$	
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02	0	$f(02) = f(01) + f(01)$

Representing clones by forks

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00	$\{0\}$	
01	A	$f(x, y) := y$ witnesses $1 \in F(C, 01)$
02	0	$f(02) = f(01) + f(01)$
10	A	$f(x, y) := x$ witnesses $1 \in F(C, 10)$

Representing clones by forks

- For each $\mathbf{x} \in A^2$, compute
  $F(C, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C, \forall \mathbf{z} < \mathbf{x} : f(\mathbf{z}) = 0\}$.

$\mathbf{x}$	$F(C, \mathbf{x})$	Reason
00	$\{0\}$	
01	A	$f(x, y) := y$ witnesses $1 \in F(C, 01)$
02	0	$f(02) = f(01) + f(01)$
10	A	$f(x, y) := x$ witnesses $1 \in F(C, 10)$
11	0	$f(11) = f(01) + f(10)$

Representing clones by forks

- For each $\mathbf{x} \in A^2$, compute $F(C, \mathbf{x}) := \{f(\mathbf{x}) \mid f \in C, \forall \mathbf{z} < \mathbf{x} : f(\mathbf{z}) = 0\}$.

$\mathbf{x}$	$F(C, \mathbf{x})$	Reason
00	$\{0\}$	
01	A	$f(x, y) := y$ witnesses $1 \in F(C, 01)$
02	0	$f(02) = f(01) + f(01)$
10	A	$f(x, y) := x$ witnesses $1 \in F(C, 10)$
11	0	$f(11) = f(01) + f(10)$
12	0	
20	0	
21	0	
22	0	

Representing clones by forks

- For each $\mathbf{x} \in A^2$, compute
  $F(C, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C, \forall \mathbf{z} < \mathbf{x} : f(\mathbf{z}) = 0\}$.

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11	0	$f(11) = f(01) + f(10)$
12	0	
20	0	
21	0	
22	0	

From groups to Mal'cev algebras

- $(A, +)$ group, C clone on A, $\mathbf{x} \in A^n$.

$$F(C, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C, \forall \mathbf{z} < \mathbf{x} : f(\mathbf{z}) = 0\}.$$

- A set with a Mal'cev operation, C clone on A, $\mathbf{x} \in A^n$.

$$\varphi(C, \mathbf{x}) := \{(f_1(\mathbf{x}), f_2(\mathbf{x})) \,|\, f_1, f_2 \in C, \forall \mathbf{z} < \mathbf{x} : f_1(\mathbf{z}) = f_2(\mathbf{z})\}.$$

Call $\varphi(C, \mathbf{x})$ the forks of C at $\mathbf{x}$.

Fork lemma for clones [Aic10]

Let C, D clones on A containing a Mal'cev operation. If $C \subseteq D$ and $\varphi(C, \mathbf{a}) = \varphi(D, \mathbf{a})$ for all $\mathbf{a} \in A^*$, then $C = D$.

Consequence

From a linearly ordered set of clones with the same Mal'cev term, the mapping

$$C \mapsto \langle \varphi(C, \mathbf{a}) \,|\, \mathbf{a} \in A^* \rangle$$

is injective.

Connections between forks at different places

Connections between forks

C . . . constantive clone on $\mathbb{Z}_2$. We observe $0110 \leq_e 0011101$.
Claim: $F(C, 0011101) \subseteq F(C, 0110)$.

Proof

- Let $a \in F(C, 0011101)$.
- Let $f \in C^{[7]}$ such that $f(0011101) = a$, $f(\mathbf{z}) = 0$ for all $\mathbf{z} \in \{0,1\}^7$ with $\mathbf{z} <_{\text{lex}} 0011101$.
- Define $g(x_1, x_2, x_3, x_4) := f(0, x_1, x_2, 1, x_3, x_4, 1)$.
- Then $g(0110) = f(0011101) = a$ and $g(\mathbf{z}) = 0$ for $\mathbf{z} \in \{0,1\}^4$ with $\mathbf{z} <_{\text{lex}} 0110$.
- Thus $a \in F(C, 0110)$.

The Embedded Forks Lemma

Abstract from $\mathbb{Z}_2$:
Clones on $A = \{0, \ldots, t-1\}$.

Word embedding

hen $\leq_e$ achievement,   austria $\leq_e$ australia

Embedded Forks Lemma (with constants) [Aic10]

Let C be a constantive clone on A. $\mathbf{a}, \mathbf{b} \in A^*$. Then

$$\mathbf{a} \leq_e \mathbf{b} \Rightarrow \varphi(C, \mathbf{b}) \subseteq \varphi(C, \mathbf{a}).$$

Connections between forks

Limitations of the Embedded Forks Lemma

In the proof of the Theorem, we used constants:

$$g(x_1, x_2, x_3, x_4) := f(0, x_1, x_2, 1, x_3, x_4, 1).$$

Without constants:

$$g(x_1, x_2, x_3, x_4) := f(x_4, x_1, x_2, x_2, x_3, x_4, x_2).$$

Then $g(0110) = f(0011101)$, but

$$0001 <_{\mathrm{lex}} 0110 \text{ and } 1000010 \not<_{\mathrm{lex}} 0011101.$$

Hence $g(0001) = 0$ not guaranteed.

Connections between forks

Connection between forks

C …clone on $\mathbb{Z}_2$. We observe $0110 \leq_E 0011101$.
Claim:

$$F(C, 0011101) \subseteq F(C, 0110).$$

Proof

Let $a \in F(C, 0011101)$,
$f \in C^{[7]}$ such that $f(0011101) = a$, $f(\mathbf{z}) = 0$ for all $\mathbf{z} \in \{0,1\}^7$ with $\mathbf{z} <_{\text{lex}} 0011101$.
Define

$$\begin{array}{rcl} g(x_1, x_2, x_3, x_4) & := & f(x_1, x_1, x_2, x_2, x_3, x_4, x_2) \\ & (\text{recall:} & f(x_4, x_1, x_2, x_2, x_3, x_4, x_2) \text{ was no help}) \end{array}$$

Then $g(0110) = f(0011101) = a$ and $g(\mathbf{z}) = 0$ for $\mathbf{z} \in \{0,1\}^4$ with $\mathbf{z} <_{\text{lex}} 0110$. Thus $a \in F(C, 0110)$.

The new embedding ordering: from $\leq_e$ to $\leq_E$

- $A^+ := \bigcup\{A^n \mid n \in \mathbb{N}\}$.
- For $\mathbf{a} = (a_1, \ldots, a_n) \in A^+$ and $b \in A$, we define the *index of the first occurrence of b in $\mathbf{a}$*, $\mathrm{firstOcc}\,(\mathbf{a}, b)$, by $\mathrm{firstOcc}\,(\mathbf{a}, b) := 0$ if $b \notin \{a_1, \ldots, a_n\}$, and $\mathrm{firstOcc}\,(\mathbf{a}, b) := \min\{i \in \{1, \ldots, n\} \mid a_i = b\}$ otherwise.

Definition

Let A be a finite set, and let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be elements of A^+. We say $\mathbf{a} \leq_E \mathbf{b}$ (read: $\mathbf{a}$ embeds into $\mathbf{b}$) if $\exists$ injective and increasing function $h : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that

1. for all $i \in \{1, \ldots, m\} : a_i = b_{h(i)}$,
2. $\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_n\}$,
3. for all $c \in \{a_1, \ldots, a_m\}$: $h(\mathrm{firstOcc}\,(\mathbf{a}, c)) = \mathrm{firstOcc}\,(\mathbf{b}, c)$.

We will call such an h a function *witnessing* $\mathbf{a} \leq_E \mathbf{b}$.

The new embedding ordering

Informal description

$\mathbf{a} \leq_E \mathbf{b}$ iff $\mathbf{b}$ can be obtained from $\mathbf{a}$ by inserting additional letters anywhere after their first occurrence in $\mathbf{a}$.

Embedded Forks Lemma

Clones on $A = \{0, \dots, t-1\}$.

Theorem (Embedded Forks Lemma without constants) [AMM14]

Let C be a clone on A, and let $\mathbf{a}, \mathbf{b} \in A^*$ with $\mathbf{a} \leq_E \mathbf{b}$.
Then $\varphi(C, \mathbf{b}) \subseteq \varphi(C, \mathbf{a})$.

Short representation of all forks

Facts on the embedding orderings

Let A be a finite set.

Facts on the embedding orderings

Let A be a finite set.

1. $(A^*, \leq_e)$ has no infinite descending chains.

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Facts on the embedding orderings

Let A be a finite set.

1. $(A^*, \leq_e)$ has no infinite descending chains.
2. $(A^*, \leq_E)$ has no infinite descending chains.
3. $(A^*, \leq_e)$ has no infinite antichains [Hig52].

Facts on the embedding orderings

Let A be a finite set.

1. $(A^*, \leq_e)$ has no infinite descending chains.
2. $(A^*, \leq_E)$ has no infinite descending chains.
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4. $(A^*, \leq_E)$ has no infinite antichains [AMM14].

Facts on the embedding orderings

Let A be a finite set.

1. $(A^*, \leq_e)$ has no infinite descending chains.
2. $(A^*, \leq_E)$ has no infinite descending chains.
3. $(A^*, \leq_e)$ has no infinite antichains [Hig52].
4. $(A^*, \leq_E)$ has no infinite antichains [AMM14].

Facts on the embedding orderings

Definition

Let $(X, \leq)$ be an ordered set, $Y \leq X$. Y is upward closed if $\forall y \in Y, x \in X : y \leq x \Rightarrow x \in Y$.

The set of upward closed subsets

Let $(X, \leq)$ be an ordered set. Let $U(X) := \{A | A \subseteq X, A \text{ upward closed}\}$.

Fact

$(X, \leq)$ has no infinite descending chain and no infinite anitchain (wpo) $\Longrightarrow$ $(U(X), \subseteq)$ has no infinite ascending chain.

Facts on the embedding orderings

Fact

$(X, \leq)$ has no infinite descending chains and no infinite anitchain (wpo) $\Rightarrow$ $(U(X), \subseteq)$ has no infinite ascending chain.

Proof:

1. Let $U_1 \subset U_2 \subset U_3 \subset$ be an ascending chain.
2. $U := \bigcup_{i \in \mathbb{N}} U_i$.
3. U has finitely many minimal elements (they form an antichain!).
4. There is j such that these minimal elements are in U_j.
5. Then $U \subseteq U_j$ because every element in U is $\geq$ some minimal element in U.

Facts on the embedding orderings

A a finite set.

Theorem
The set of upward closed subsets of $(A^*, \leq_e)$ has no infinite ascending chain with respect to $\subseteq$.

Theorem
The set of upward closed subsets of $(A^*, \leq_E)$ has no infinite ascending chain with respect to $\subseteq$.

Question
Is there an infinite antichain of upward closed subsets of $(A^*, \leq_e)$?

Forks of clones and upward closed sets

- Let $C_1 \supset C_2 \supset C_3 \supset \cdots$ be a chain of Mal'cev clones. Then we can determine i if we know

$$\varphi(C_i, \mathbf{a}) \text{ for every } a \in A^*.$$

- Let $S \subset A \times A$. Since $\mathbf{a} \leq_E \mathbf{b} \Rightarrow \varphi(C_i, \mathbf{b}) \subseteq \varphi(C_i, \mathbf{a})$,

$$\mathbf{\Psi}(C_i, S) := \{\mathbf{a} \in A^* \,|\, \varphi(C_i, \mathbf{a}) \subseteq S\}$$

  is an upward closed subset of $(A^*, \leq_E)$.
- Recover the forks from $\mathbf{\Psi}(C_i, S)$:

$$\begin{aligned} (c,d) \in \varphi(C_i, \mathbf{a}) \quad &\Leftrightarrow \quad \varphi(C_i, \mathbf{a}) \not\subseteq (A \times A) \setminus \{(c,d)\} \\ &\Leftrightarrow \quad \mathbf{a} \notin \mathbf{\Psi}(C_i, (A \times A) \setminus \{(c,d)\}). \end{aligned}$$

  Hence: if we know $\mathbf{\Psi}(C_i, S)$ for all $S \subseteq A \times A$, we can recover all forks.

Short representation of clones

1. Let $\mathcal{C}$ be a linearly order set of clones on A with the same Mal'cev operation.
2. We can "store" each $C \in \mathcal{C}$ by

$$\langle \mathbf{\Psi}(C, S) \,|\, S \subseteq A \times A \rangle.$$

3. Each $\mathbf{\Psi}(C, S)$ is an upward closed set of $(A^*, \leq_E)$ and has only *finitely many minimal elements*
4. Hence $\mathcal{C}$ is countable!

Chain conditions for sets of clones

DCC for Mal'cev clones

Lemma

Let $\mathbb{L}$ be a linearly order set of Mal'cev clones. Then the mapping

$$\begin{array}{rcccl} r & : & \mathbb{L} & \longrightarrow & (\mathcal{U}(A^*, \leq_E))^{2^A} \\ & & C & \longmapsto & \mathbf{\Psi}(C, S) = \langle\, \{\mathbf{x} \in A^* \mid \varphi(C, \mathbf{x}) \subseteq S\} \mid S \subseteq A \rangle \end{array}$$

is injective and inverts the ordering.

Consequence

Let A be a finite set, d a Mal'cev operation. There is no infinite descending chain of clones on A that contain d.

Proof: Such a chain produces an infinite ascending chain in $(\mathcal{U}(A^*, \leq_E))^{2^A}$, and hence in $\mathcal{U}(A^*, \leq_E)$. Contradiction.

Mal'cev clones on finite sets are finitely related

Theorem [AMM14]

Let A be a finite set, and let $\mathcal{M}$ be the set of all Mal'cev clones on A. Then we have:

1. There is no infinite descending chain in $(\mathcal{M}, \subseteq)$.
2. For every Mal'cev clone C, there is a finitary relation ρ on A such that $C = \mathrm{Pol}(\{\rho\})$.
3. The set $\mathcal{M}$ is finite or countably infinite.

Consequences

Mal'cev algebras

1. Up to term equivalence and renaming of elements, there are only countably many finite Mal'cev algebras.
2. Every finite Mal'cev algebra can be represented by a single finitary relation.

Corollary – The clone lattice above a Mal'cev clone

Let C be a Mal'cev clone on a finite set A.

1. The interval $\mathbb{I}[C, O(A)]$ has finitely many atoms.
2. every clone D with $C \subset D$ contains one of these atoms,
3. If $\mathbb{I}[C, O(A)]$ is infinite, it contains a clone that is not finitely generated (cf. König's Lemma).

From Mal’cev terms to edge terms

Edge terms

Edge operations

For $k \geq 3$, a $(k+1)$-ary operation is a *k-edge operation* on A if for all $a, b \in A$:

$$\begin{aligned} t(a,a,b,b,b,\dots,b) &= b \\ t(b,a,a,b,b,\dots,b) &= b \\ t(b,b,b,a,b,\dots,b) &= b \\ &\ddots \\ t(b,b,b,b,b,\dots,a) &= b \end{aligned}$$

(still wrong!)

Edge terms

Edge operations

For $k \geq 3$, a $(k+1)$-ary operation is a *k-edge operation* on A if for all $a, b \in A$:

$$\begin{array}{c} t(a,a,b,b,b,\ldots,b) = b \\ t(a,b,a,b,b,\ldots,b) = b \\ t(b,b,b,a,b,\ldots,b) = b \\ \ddots \\ t(b,b,b,b,b,\ldots,a) = b \end{array}$$

Examples of edge operations

Edge operation

$$\begin{array}{c} t(a,a,b,b,b,\dots,b) = b \\ t(a,b,a,b,b,\dots,b) = b \\ t(b,b,b,a,b,\dots,b) = b \\ \ddots \\ t(b,b,b,b,b,\dots,a) = b \end{array}$$

- d Mal'cev. Then $t(x,y,z) := d(y,x,z)$ is 2-edge.
- m majority. Then $t(x_1,x_2,x_3,x_4) := m(x_2,x_3,x_4)$ is 3-edge.
- f n-ary near-unanimity. Then $t(x_0,\dots,x_n) := f(x_1,\dots,x_n)$ is n-edge.

Edge terms and few subpowers

Theorem - Edge terms and few subpowers [BIM$^+$10]

Let $\mathbf{A}$ be a finite algebra. TFAE:

- $\mathbf{A}$ has an edge term.
- $\exists$ polynomial $p \in \mathbb{R}[t]$:

$$\forall n \in \mathbb{N} \ : \ |\mathrm{Sub}(\mathbf{A}^n)| \leq 2^{p(n)}.$$

The fork lemmas

$F \leq A^m$, $i \in \{1, \dots, m\}$.

$$\varphi_i(F) := \{(a_i, b_i) \mid \begin{array}{lcl} (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_m) & \in & F \text{ and} \\ (a_1, \dots, a_{i-1}, b_i, b_{i+1}, \dots, b_m) & \in & F \end{array} \}.$$

Fork Lemma - Mal'cev

Let $k, m \in \mathbb{N}, k \geq 2$, and let $\mathbf{A}$ be an algebra with a Mal'cev term. Let F, G be subuniverses of $\mathbf{A}^m$ with $F \subseteq G$. Assume

- $\varphi_i(G) = \varphi_i(F)$ for all $i \in \{1, \dots, m\}$.

Then $F = G$.

Fork Lemma - Edge [BIM+10, Cor. 3.9], [AM15, Lemma 4.2]

Let $k, m \in \mathbb{N}, k \geq 2$, and let $\mathbf{A}$ be an algebra with a k-edge term. Let F, G be subuniverses of $\mathbf{A}^m$ with $F \subseteq G$. Assume

- $\varphi_i(G) = \varphi_i(F)$ for all $i \in \{1, \dots, m\}$ and
- $\pi_T(F) = \pi_T(G)$ for all $T \subseteq \{1, \dots, m\}$ with $|T| \leq k - 1$.

Then $F = G$.

Fork lemma for clones

Fork lemma for clones with Mal'cev operation [AMM14]

Let C, D clones on A containing a Mal'cev operation. Assume:

- $C \subseteq D$,
- $\varphi(C, \mathbf{a}) = \varphi(D, \mathbf{a})$ for all $\mathbf{a} \in A^*$,

Then $C = D$.

Fork lemma for clones with edge operation [AM14]

Let C, D clones on A containing a k-edge operation, $t := |A|$. Assume:

- $C \subseteq D$,
- $\varphi(C, \mathbf{a}) = \varphi(D, \mathbf{a})$ for all $\mathbf{a} \in A^*$,
- $C^{[t^{k-1}]} \subseteq D^{[t^{k-1}]}$.

Then $C = D$.

DCC for clones with fixed edge operation

Consequence

A finite set, e edge operation. There is no $C_1 \supset C_2 \supset C_3 \supset \cdots$ of clones on A containing e.

Proof:

- There are only finitely many t^{k-1}-ary parts of clones on A.
- One of those appears infinitely often in $C_1 \supset C_2 \supset C_3 \supset \cdots$.
- Taking only those clones, we obtain a strictly ascending chain of upward closed subsets of $(A^*, \leq_E)$.
- Contradiction to order theory.

Clones on finite sets with edge terms

Theorem [AMM14]

Let A be a finite set, let $k \in \mathbb{N}$, $k > 1$, and let $\mathcal{M}_k$ be the set of all clones on A that contain a k-edge operation. Then we have:

1. For every clone C in $\mathcal{M}_k$, there is a finitary relation R on A such that $C = \mathrm{Pol}(A, \{R\})$.
2. There is no infinite descending chain in $(\mathcal{M}_k, \subseteq)$.
3. The set $\mathcal{M}_k$ is finite if $|A| \leq 3$ and countably infinite otherwise.

Varieties

Varieties

Question

Are subvarieties of finitely generated varieties again finitely generated?

Answer

- Sometimes yes.
- Sometimes no.

Classes of algebras

We will study:

- *classes of algebras* of with the same operation symbols (of the same *type*) $\mathcal{F}$.
- Example: $\mathcal{F} := \{\cdot, ^{-1}, 1\}$, $K :=$ class of all groups.
- *identities*: $s(x_1, \dots, x_k) \approx t(x_1, \dots, x_k)$.
- Example:
  $\Phi = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ 1 \cdot x \approx x,\ x^{-1} \cdot x \approx 1,\ x^6 \approx y^{15}\}$.
- *Validity of identities* in an algebra $\mathbf{A}$ of type $\mathcal{F}$.
- Example: $\mathbf{A} \models \Phi \Leftrightarrow \mathbf{A}$ is a group of exponent 1 or 3.

Varieties

Theorem [Bir35, Theorem 10]

Let K be a nonempty class of algebras of the same type $\mathcal{F}$. TFAE:

1. $\exists$ set of identities $\Phi : K = \{\mathbf{A} \,|\, \mathbf{A} \text{ is of type } \mathcal{F} \text{ and } \mathbf{A} \models \Phi\}$.
   (*Meaning:* K can defined using identities.)
2. K is closed under taking
   - $\mathbb{H}$ homomorphic images
   - $\mathbb{S}$ subalgebras
   - $\mathbb{P}$ cartesian products.

A class K of algebras that can be defined by a set of identities is called a *variety*.

Finitely generated varieties

Definition

$\mathbf{A}$ algebra. $\mathbb{V}(\mathbf{A}) :=$ the smallest variety that contains $\mathbf{A}$.

Theorem

$\mathbb{V}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A})$.

Theorem

$\mathbf{B} \in \mathbb{V}(\mathbf{A})$ if and only if $\forall s, t : \mathbf{A} \models s \approx t \Rightarrow \mathbf{B} \models s \approx t$.

Definition

A variety V is *finitely generated* $:\Leftrightarrow$ there is a finite algebra $\mathbf{A}$ with $V = \mathbb{V}(\mathbf{A})$.

Finite Generation of Subvarieties

Theorem [Jón67]

Let $\mathbf{L}$ be a finite lattice. Then every subvariety of $\mathbb{V}(\mathbf{L})$ is finitely generated.

Proof: $\mathbb{V}(\mathbf{L})$ contains, up to isomorphism, only finitely many subdirectly irreducible lattices (Jónsson's Lemma).

Theorem [OP64]

Let $\mathbf{G}$ be a finite group. Then every subvariety of $\mathbb{V}(\mathbf{G})$ is finitely generated.

Proof: $\mathbb{V}(\mathbf{G})$ contains, up to isomorphism, only finitely many groups $\mathbf{H}$ with $\mathbf{H} \notin \mathbb{V}(\{\mathbf{A} \mid \mathbf{A} \in \mathbb{V}(\mathbf{H}), |\mathbf{A}| < |\mathbf{H}|\})$. (Long proof using "critical groups".)

Note that both $\mathbb{V}(\mathbf{G})$ and $\mathbb{V}(\mathbf{L})$ contain only *finitely many* subvarieties.

Finite Generation of Subvarieties

Theorem [Bry82]

There is an expansion of a finite group with one constant operation such that the variety generated by this algebra has infinitely many subvarieties.

They might all be finitely generated, though.

Theorem [OMVL78]

There is a three-element algebra $\mathbf{M} = (M, *, c)$ such that $\mathbb{V}(\mathbf{M})$ has subvarieties that are not finitely generated

Recognizing finitely generated subvarieties

Lemma [OMVL78]

V finitely generated variety. TFAE:

1. The subvarieties of V, ordered by $\subseteq$, satisfy (ACC).
2. Every subvariety of V is finitely generated.

Proof of (1)$\Rightarrow$(2):

1. Let W be not finitely generated. Pick a finite $\mathbf{A}_1 \in W$.
2. Since $V(\mathbf{A}_1)$ is f.g., $V(\mathbf{A}_1) \subset W$.
3. Since W is generated by its finite members, there is a finite $\mathbf{A}_2 \in W$, $\mathbf{A}_2 \notin V(\mathbf{A}_1)$.
4. $V(\mathbf{A}_1) \subset V(\mathbf{A}_1 \times \mathbf{A}_2) \subset V(\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3) \subset \cdots$ is a failure of (ACC).

The equational theory of subvarieties

Equational theory of W in $\mathbf{A}$

Definition [AM14]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \dots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N}, \quad s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A} \text{ with } W \models s \approx t\}.$$

Examples

Equational theory of W in $\mathbf{A}$

Definition [AM14]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \dots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

$$s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A}$$

$$\text{with } W \models s \approx t\}.$$

Examples

1. $\mathrm{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})) = \{(t, t) \mid t \in \mathrm{Clo}(\mathbf{A})\}$.

Equational theory of W in $\mathbf{A}$

Definition [AM14]

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$$s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A}$$

$$\text{with } W \models s \approx t\}.$$

Examples

1. $\mathrm{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})) = \{(t, t) \mid t \in \mathrm{Clo}(\mathbf{A})\}$.
2. $\mathbf{A} := \mathbf{S}_3$, $W := \{\mathbf{G} \in \mathbb{V}(\mathbf{S}_3) \mid \mathbf{G} \text{ is abelian}\}$. Then $(\binom{\pi_1}{\pi_2} \mapsto \begin{pmatrix} \pi_1^{-1} \circ \pi_2 \circ \pi_1 \\ \pi_2 \end{pmatrix}) \in \mathrm{Th}_{\mathbf{S}_3}(W)$.

Definition [AM14]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$
$$s, t \text{ are } k\text{-variable terms in the language of } \mathbf{A}$$
$$\text{with } W \models s \approx t\}.$$

Examples

1. $\mathrm{Th}_{\mathbf{A}}(\mathbb{V}(\mathbf{A})) = \{(t, t) \mid t \in \mathrm{Clo}(\mathbf{A})\}$.
2. $\mathbf{A} := \mathbf{S}_3$, $W := \{\mathbf{G} \in \mathbb{V}(\mathbf{S}_3) \mid \mathbf{G} \text{ is abelian}\}$. Then $\left(\left(\begin{smallmatrix} \pi_1 \\ \pi_2 \end{smallmatrix}\right) \mapsto \begin{pmatrix} \pi_1^{-1} \circ \pi_2 \circ \pi_1 \\ \pi_2 \end{pmatrix}\right) \in \mathrm{Th}_{\mathbf{S}_3}(W)$.
3. $W :=$ class of one element algebras of type $\mathcal{F}$. Then $\mathrm{Th}_{\mathbf{A}}(W) = \{(s, t) \mid k \in \mathbb{N},\ s, t \in \mathrm{Clo}_k(\mathbf{A})\}$.

Kernels

Definition [AM14]

$\mathbf{A}$ algebra, W subvariety of $\mathbb{V}(\mathbf{A})$.

$$\mathrm{Th}_{\mathbf{A}}(W) := \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

s, t are k-variable terms in the language of $\mathbf{A}$

with $W \models s \approx t\}$.

Remark:

Let $\mathbf{F} := \mathbf{Free}_W(\aleph_0)$. $\mathrm{Th}_{\mathbf{A}}(W)$ is something like the "kernel" of the "homomorphism"

$$\begin{array}{cccc} \omega & : & \mathrm{Clo}(\mathbf{A}) & \longrightarrow & \mathbf{F} \\ & & s^{\mathbf{A}} & \longmapsto & [s]. \end{array}$$

Distinguishing subvarieties of $\mathbb{V}(\mathbf{A})$ inside A

Lemma

$\mathbf{A}$ be algebra, W_1 and W_2 subvarieties of $\mathbb{V}(\mathbf{A})$. Then we have:

$$W_1 \subseteq W_2 \text{ if and only if } \mathrm{Th}_{\mathbf{A}}(W_2) \subseteq \mathrm{Th}_{\mathbf{A}}(W_1).$$

Clonoids

What is

$$\mathrm{Th}_{\mathbf{A}}(W) = \{(a_1, \ldots, a_k) \mapsto \begin{pmatrix} s^{\mathbf{A}}(\mathbf{a}) \\ t^{\mathbf{A}}(\mathbf{a}) \end{pmatrix} \mid k \in \mathbb{N},$$

s, t are k-variable terms in the language of $\mathbf{A}$

with $W \models s \approx t\}$?

$\mathrm{Th}_{\mathbf{A}}(W)$ is a *clonoid with source set A and target algebra $\mathbf{A} \times \mathbf{A}$.*

Definition of Clonoids

A	$\dots$	set
$\mathbf{B}$	$\dots$	algebra
C	$\dots$	finitary functions from A to B, hence
C	$\subseteq$	$\bigcup_{n\in\mathbb{N}} B^{A^n}$
$C^{[k]}$	$:=$	$C \cap B^{A^k}$ (k-ary functions in C).

Definition of Clonoids

Definition

B algebra, A nonempty set, $C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}$. C is a *clonoid with source set A and target algebra* **B** if

1. for all $k \in \mathbb{N}$: $C^{[k]}$ is a subuniverse of $\mathbf{B}^{A^k}$, and
2. for all $k, n \in \mathbb{N}$, for all $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$, and for all $c \in C^{[k]}$, the function $c' : A^n \to B$ defined by

$$c'(a_1, \ldots, a_n) := c(a_{i_1}, \ldots, a_{i_k})$$

satisfies $c' \in C^{[n]}$.

Representation of Clonoids

We represent a clonoid C with source set $A = \{a_1, \ldots, a_t\}$ and target algebra $\mathbf{B}$ using forks.

Definition (forks of $\mathbf{B}^{A^n}$ at $\mathbf{a}$)

For $\mathbf{a} \in A^n$, let

$$\varphi(C, \mathbf{a}) := \{\big(f_1(\mathbf{a}), f_2(\mathbf{a})\big) \in B \times B \mid \\ f_1(\mathbf{z}) = f_2(\mathbf{z}) \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\mathrm{lex}} \mathbf{a}\}.$$

Representation of Clonoids by forks

Fork Lemma for Clonoids - Mal'cev term

A finite set, $\mathbf{B}$ finite algebra with Mal'cev term, C, D clonoids with source set A and target algebra $\mathbf{B}$. Assume

1. $C \subseteq D$,
2. $\varphi(C, \mathbf{a}) = \varphi(D, \mathbf{a})$ for all $\mathbf{a} \in A^*$.

Then $C = D$.

Fork Lemma for Clonoids - Edge term

A finite set, $\mathbf{B}$ finite algebra with k-edge term, C, D clonoids with source set A and target algebra $\mathbf{B}$. Assume

1. $C \subseteq D$,
2. $\varphi(C, \mathbf{a}) = \varphi(D, \mathbf{a})$ for all $\mathbf{a} \in A^*$,
3. $C^{[|A|^{k-1}]} = D^{[|A|^{k-1}]}$.

Then $C = D$.

Subvarieties of finitely generated varieties

Theorem [AM14]

A finite set, $\mathbf{B}$ finite algebra with edge term.
$\mathcal{C} := \{C \mid C$ is clonoid with source A and target $\mathbf{B}\}$.
Then $(\mathcal{C}, \subseteq)$ satisfies the (DCC).

Theorem [AM14]

$\mathbf{A}$ finite algebra with edge term, $\mathcal{W} :=$ subvarieties of $\mathbb{V}(\mathbf{A})$.
Then:

- $(\mathcal{W}, \subseteq)$ satisfies the (ACC).
- Every subvariety of $\mathbb{V}(\mathbf{A})$ is finitely generated.

Proof: From $W_1 \subset W_2 \subset \cdots$,
we obtain $\mathrm{Th}_{\mathbf{A}}(W_1) \supset \mathrm{Th}_{\mathbf{A}}(W_2) \supset \cdots$,
which is an infinite descending chains of clonoids with source A and target $\mathbf{B} := \mathbf{A} \times \mathbf{A}$. Contradiction.

(DCC) for subvarieties

Theorem [AM14]

A finite algebra with edge term. Then every subvariety of $\mathbb{V}(\mathbf{A})$ is finitely generated.

Corollary [AM14]

A finite algebra with an edge term. Then the following are equivalent:

1. There is no infinite descending chain of subvarieties of $\mathbb{V}(\mathbf{A})$.
2. Each $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ is finitely based relative to $\mathbb{V}(\mathbf{A})$.
3. $\mathbb{V}(\mathbf{A})$ has only finitely many subvarieties.
4. $\mathbb{V}(\mathbf{A})$ contains, up to isomorphism, only finitely many cardinality critical members.

B is *cardinality critical* $:\Leftrightarrow \mathbf{B} \notin \mathbb{V}(\{\mathbf{C} \mid \mathbf{C} \in \mathbb{V}(\mathbf{B}), |\mathbf{C}| < |\mathbf{B}|\})$.

Higher Commutators

Higher commutators

Lemma

Let $\mathbf{V} = (V, +, -, 0, F)$ be an expanded group. Then

- Every congruence α is determined by $0/\alpha$.
- 0-classes of congruences are called *ideals*.

Definition

$\mathrm{Pol}(\mathbf{V})$ is the clone generated by the (unary) constants and the fundamental operations of $\mathbf{V}$.

Higher commutators

Definition

Let $p \in \mathrm{Pol}_n \mathbf{V}$. p is absorbing $:\Leftrightarrow$ for all $x_1, \dots, x_n \in V$:

$$0 \in \{x_1, \dots, x_n\} \Rightarrow f(x_1, \dots, x_n) = 0.$$

Theorem [Hig67, BB87], cf. [Aic14]

Let $\mathbf{V}$ be a finite expanded group.

$$\begin{aligned} a_n(\mathbf{V}) &:= \log_2(|\{p \in \mathrm{Clo}_n(\mathbf{V}) \,|\, p \text{ is absorbing}\}|) \\ t_n(\mathbf{V}) &:= \log_2(|\mathrm{Clo}_n(\mathbf{V})|). \end{aligned}$$

Then for each $n \in \mathbb{N}_0$, we have

$$t_n(\mathbf{V}) = \sum_{i=0}^{n} a_i(\mathbf{V}) \binom{n}{i}.$$

Higher Commutators

Definition [Bul01, Mud09, AM10]

Let $A_1, \ldots, A_n$ be ideals of $\mathbf{V}$. Then $[A_1, \ldots, A_n]$ is the ideal generated by

$$\{p(a_1, \ldots, a_n) \mid p \text{ absorbing polynomial}, a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Properties of higher commutators

generators = $\{p(a_1, \ldots, a_n) \mid p$ absorbing, $\forall i : a_i \in A_i\}$

Higher Commutator Laws [Mud09, AM10]

- $[A_1, \ldots, A_n] \leq A_1 \cap A_2 \cap \cdots \cap A_n$,
- $(I_1, \ldots, I_n) \mapsto [I_1, \ldots, I_n]$ preserves $\subseteq$,
- $[A_1, A_2, \ldots, A_n] \leq [A_2, \ldots, A_n]$,
- $[A_1, \ldots, A_n] = [A_{\pi(1)}, \ldots, A_{\pi(n)}]$ for all $\pi \in S_n$,
- $[A_1 + B_1, A_2, \ldots, A_n] = [A_1, A_2, \ldots, A_n] + [B_1, A_2, \ldots, A_n]$,
- $[A_1, \ldots, A_k, [A_{k+1}, \ldots, A_n]] \leq [A_1, \ldots, A_n]$.

Higher Commutators and Forks: a connection to be explored

Lemma

Let $C = \mathrm{Clo}(\mathbf{V})$, $\mathbf{V}$ expanded group. Let $\mathbf{b} \in V^*$, and let $\mathbf{a} \in V^*$ be the word obtained from $\mathbf{b}$ by eliminating all 0 entries. Sort V such that 0 is smallest. Then

- $F(C, \mathbf{a}) = F(C, \mathbf{b})$,
- For every witness f of $x \in F(C, \mathbf{a})$, there is a function g in the clone generated by $\{f, 0\}$ that witnesses $x \in F(C, \mathbf{b})$.

Observation

If $\mathbf{a} \in (V \setminus \{0\})^*$ and $\mathbf{a} = (a_1, \ldots, a_n)$, then $F(C, \mathbf{a}) \in [V, \ldots, V]$ (n times).

Corollary

If $[V, \ldots, V] = 0$ (n times), then $\mathrm{Clo}(\mathbf{V})$ is finitely generated.

Other connections

- $F(C, a_1a_2a_3a_4a_5) \geq [F(C, a_1a_2a_4), F(C, a_2a_3a_5)]$.
- $\dots$

Where to continue

What could be added:

- Connections between witnesses of forks.

Open problems

- existence of infinite antichains of Mal'cev clones on a finite set,
- existence of a finite set with infinitely many not finitely generated Mal'cev clones (open as far as I know),
- bound of the size of the relation determining a Mal'cev clone.

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