Forks, Clones, Varieties
Exercises[1]

(1) Let $A := \mathbb{Z}_3$, let $C_1 := \mathsf{Clo}((\mathbb{Z}_3, +))$, and let $n \in \mathbb{N}$. We order $0 < 1 < 2$, and we order the elements of A^n lexicographically. For $\mathbf{x} \in A^n$, we define

$$F(C_1, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C_1, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\text{lex}} \mathbf{x}\}.$$

Complete the following tables for unary, binary, and 3-ary functions in C_1.

$\mathbf{x}$	$F(C_1, \mathbf{x})$	Reason
0		
1		
2		

$\mathbf{x}$	$F(C_1, \mathbf{x})$	Reason
00		
01		
02		
10		
11		
12		
20		
21		
22		

$\mathbf{x}$	$F(C_1, \mathbf{x})$	Reason
000		
001		
002		
010		
011		
012		
020		
021		
022		
100		
101		
102		
110		
$\vdots$		

(2) Let $A := \mathbb{Z}_3$, let $C_2 := \mathsf{Pol}((\mathbb{Z}_3, +)) = \mathsf{Clo}((\mathbb{Z}_3, +, 1))$, and let $n \in \mathbb{N}$. We order $0 < 1 < 2$, and we order the elements of A^n lexicographically. For $\mathbf{x} \in A^n$, we define

$$F(C_2, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C_2, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\text{lex}} \mathbf{x}\}.$$

[1] Exercises for the course at SSAOS 2015 in Srní. The solutions to all of this exercises are known, many are standard problems in universal algebra. The aim is to provide some material for those participants who prefer to discover by their own.

Complete the following tables for unary, binary, and 3-ary functions in C_2.

$\mathbf{x}$	$F(C_2, \mathbf{x})$	Reason
0		
1		
2		

$\mathbf{x}$	$F(C_2, \mathbf{x})$	Reason
00		
01		
02		
10		
11		
12		
20		
21		
22		

$\mathbf{x}$	$F(C_2, \mathbf{x})$	Reason
000		
001		
002		
010		
011		
012		
020		
021		
022		
100		
101		
102		
110		
$\vdots$		

(3) Let $A := \mathbb{Z}_3$, let $C_3 := \mathsf{Pol}((\mathbb{Z}_3, +, \cdot))$, and let $n \in \mathbb{N}$. We order $0 < 1 < 2$, and we order the elements of A^n lexicographically. For $\mathbf{x} \in A^n$, we define

$$F(C_3, \mathbf{x}) := \{f(\mathbf{x}) \,|\, f \in C_3, f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in A^n \text{ with } \mathbf{z} <_{\text{lex}} \mathbf{x}\}.$$

Complete the following tables for unary, binary, and 3-ary functions in C_3.

$\mathbf{x}$	$F(C_3, \mathbf{x})$	Reason
0		
1		
2		

$\mathbf{x}$	$F(C_3, \mathbf{x})$	Reason
00		
01		
02		
10		
11		
12		
20		
21		
22		

$\mathbf{x}$	$F(C_3, \mathbf{x})$	Reason
000		
001		
002		
010		
011		
012		
020		
021		
022		
100		
101		
102		
110		
$\vdots$		

(4) (Connections between forks of different arities) Let C be a clone on $\mathbb{Z}_3$ that contains all unary constant functions. Prove:
   (a) $F(C,12) \subseteq F(C,2)$.
   (b) $F(C,2102) = F(C,212)$.
(5) (Connections between forks of different arities) Let C be a clone on $\mathbb{Z}_3$. Prove:
   (a) $F(C,121) \subseteq F(C,12)$.
   (b) $F(C,1112020) \subseteq F(C,1120)$.
(6) (Representation of subpowers) Let G be a group, $n \in \mathbb{N}$, $A \leq B \leq G^n$ subgroups. Assume:
   (a) $A \subseteq B$
   (b) $\forall i \in \{1,\dots,n\}$, $\forall g \in G$, $\forall r_{i+1},\dots,r_n \in G$:

$$\begin{aligned} &(\underbrace{0,\dots,0}_{i-1}, g, r_{i+1},\dots,r_n) \in B \Rightarrow \\ \exists s_{i+1},\dots,s_n \in G \quad : \quad &(0,\dots,0,g,s_{i+1},\dots,s_n) \in A, \end{aligned}$$

Show that then $A = B$.
(7) (Generation of subpowers and the fork lemma) Let $\mathbf{A}$ be a finite algebra with a Mal'cev term, and let $n \in \mathbb{N}$. Prove that every subalgebra of $\mathbf{A}^n$ can be generated by at most $n \cdot |A|^2$ elements. From this, derive that there is a real number c such that for all $n \in \mathbb{N}$, $\mathbf{A}^n$ has at most 2^{cn^2} subalgebras.
(8) (Varieties) Let $\mathcal{F}$ be a type of algebras, let V be a variety of algebras of type $\mathcal{F}$, let $k \in \mathbb{N}$, and let $\varphi := (s \approx t)$ be an equation over $\mathcal{F}$ that uses at most k variables. Prove:
   $V \models \varphi$ if and only if every k-generated algebra in V satisfies φ.
(9) (Varieties) Let $\mathcal{F}$ be a type of algebras, let V be a variety of algebras of type $\mathcal{F}$, let $k \in \mathbb{N}$, and let $\mathbf{A}$ be a k-generated algebra of type $\mathcal{F}$. Prove:
   $\mathbf{A} \in V$ if and only if $\mathbf{A}$ satisfies every identity of V with at most k variables.
(10) (Chain conditions) Let V be a locally finite variety of arbitrary type $\mathcal{F}$, and let W be a subvariety of V. Prove that the following are equivalent:
   (a) There exists no infinite strictly ascending chain of varieties $V_1 \subset V_2 \subset V_3 \subset \cdots$ with $W := \mathbb{V}(\bigcup_{i\in\mathbb{N}} V_i)$.
   (b) W is finitely generated.

   Remark: This exercise will be frustrating if you are not familiar with the following notions: *equational theory of a variety, free algebras, Galois connection between varieties and equational theories, locally finite varieties*. In the lecture, we will need only the implication (10a)⇒(10b).
(11) Let V be a finitely generated variety. Prove that the following are equivalent:
   (a) The subvarieties of V, ordered by $\subseteq$, satisfy (ACC).

(b) Every subvariety of V is finitely generated

(12) Let $V_1 \supset V_2 \supset V_3 \supset \cdots$ be an infinite strictly descending chain of varieties, and assume that V_1 is locally finite. Prove that there is no $k \in \mathbb{N}$ such that $W := \bigcap_{i \in \mathbb{N}} V_i$ is the model of a set of equations such that each of these has at most k variables; in particular W is not finitely based.

(13) Let V be a finitely generated variety. Prove that the following are equivalent.

(a) The subvarieties of V, ordered by $\subseteq$, satisfy (DCC).

(b) For every subvariety W of V there is a finite set of identities Φ with $W = \{\mathbf{A} \in V \mid \mathbf{A} \models \Phi\}$. ($W$ is finitely based relative to V.)

(14) (Ideals and expanded groups) Let $\mathbf{V}$ be an expanded group. Prove that I is the 0-class of a congruence of $\mathbf{V}$ if and only if I is a normal subgroup of $(V, +, -, 0)$ and for all n, for all n-ary fundamental operations f, and for all vectors $\mathbf{a} \in V^n$ and $\mathbf{i} \in I^n$, we have $f(\mathbf{a} + \mathbf{i}) - f(\mathbf{a}) \in I$.

(15) (Ideals and expanded groups) Let $\mathbf{V}$ be an expanded group, and let $B \subseteq V$. Show that the ideal generated by B is given by

$$I = \{\sum_{i=1}^{n} p_i(b_i) \mid n \in \mathbb{N}_0, p_i \in \mathsf{Pol}_1(\mathbf{V}), p_i(0) = 0, b_i \in B \text{ for all } i \in \{1, \ldots, n\}\}.$$

(16) (Properties of commutators for expanded groups) Let $\mathbf{V}$ be an expanded group, and let I, A, B be ideals of V such that $I \leq A$, $B \leq I$. Prove that in $\mathbf{V}/I$, we have $[A/I, B/I] = ([A, B] + I)/I$.

(17) (Properties of higher commutators for expanded groups) Let $A_1, \ldots, A_n$ be ideals of the expanded group $\mathbf{V}$. Show

$$[A_1 + B_1, A_2, \ldots, A_n] = [A_1, A_2, \ldots, A_n] + [B_1, A_2, \ldots, A_n].$$