THE COMPLEXITY OF CHECKING QUASI-IDENTITIES OVER FINITE ALGEBRAS WITH A MAL'CEV TERM

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Der Wissenschaftsfonds.

THE PROBLEM

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We decide the validity of these quasi-identities in finite algebras.

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- Not an algebra: $(\mathbb{R}, +, -, \cdot, /)$ (because $/$ is not total)

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Problem is in coNP.

Question: For which $\mathbf{A}$ is QuasiIdVal($\mathbf{A}$) in P or coNP-complete?

RELATION TO STUDIED PROBLEMS

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Hence $\textsc{TermEqv}(\mathbf{A}) \leq_m^{\mathrm{P}} \textsc{QuasiIdVal}(\mathbf{A})$.

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Let $\mathbf{G}$ be a finite group, let $\mathbf{R}$ be a finite ring.

- $\textsc{TermEqv}(\mathbf{R}) \in \mathrm{coNPC}$ if $\mathbf{R}$ is nonnilpotent (Burris, Lawrence 1993).

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Our contribution: coNP-complete in both open cases.

COMPLEXITY FOR MAL'CEV ALGEBRAS

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Main Result

Theorem [Aichinger, Grünbacher]

Let $\mathbf{A}$ be a finite Mal'cev algebra. Then $\textsc{QuasiIdVal}(\mathbf{A})$ is in P if $\mathbf{A}$ is abelian and coNP-complete otherwise.

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Let $\mathbf{R} = (R, +, -, \cdot)$ be a finite ring. Then $\textsc{QuasiIdVal}(\mathbf{R})$ is in P if $a \cdot b = 0$ for all $a, b \in R$, and coNP-complete otherwise.

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- Let $G = (V, E)$ be any graph.
- Let $\Phi(E, z)$ denote the formula $\bigwedge_{(u,v)\in E} y_{(u,v)} \cdot (x_u - x_v) = z$.

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- Let $\Phi(E, z)$ denote the formula $\bigwedge_{(u,v)\in E} y_{(u,v)} \cdot (x_u - x_v) = z$.
- Then $c(v) := x_v$ is a homomorphism $G \to H_z$ iff $\Phi(E, z)$ is satisfiable.

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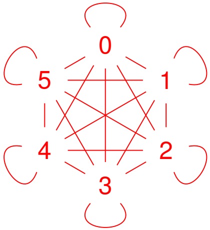
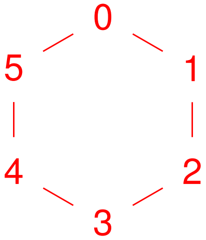
Theorem [Hell, Nešetřil 1990]

Let H be an undirected, loopless non-bipartite graph. Then H-COLORING is NP-complete.

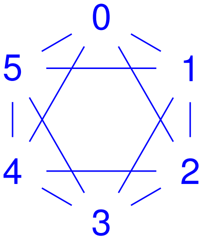
We use this to prove coNP-completeness for QUASIIDVAL.



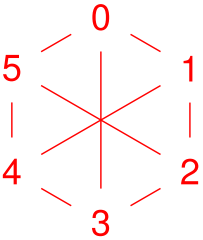
H_0



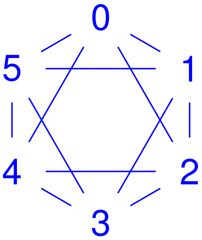
H_1



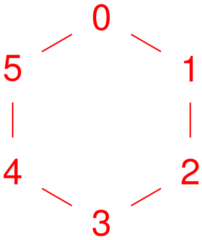
H_2



H_3



H_4



H_5

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  iff $\exists z \neq 0 : G \preceq H_z \wedge H_2 \preceq H_z$
  iff $\Phi(E, z) \wedge \Phi(\rho_2, z) \wedge z \neq 0$ is satisfiable, where $\Phi(E, z)$ is $\bigwedge_{(u,v)\in E} y_{(u,v)}(x_u - x_v)$

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This reduces H_2-COLORING to COQUASIIDVAL$(\mathbb{Z}_6, +, -, \cdot, 0)$.

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- H_2-COLORING is NP-complete.
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This reduces H_2-COLORING to COQUASIIDVAL$(\mathbb{Z}_6, +, -, \cdot, 0)$.
Therefore QUASIIDVAL$(\mathbb{Z}_6, +, -, \cdot, 0) \in \mathrm{coNPC}$.

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- To define analogue of $y(a - b) = z$ for Mal'cev algebras:
  $\Longrightarrow$ Use commutator theory over Mal'cev algebras. Commutator theory explains what abelian, nilpotent, solvable mean for Mal'cev algebras.

Generalization to Mal'cev Algebras

In particular, for groups and rings we obtain:

Theorem

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Theorem

Let $\mathbf{G} = (G, \cdot)$ be a finite group. Then $\textsc{QuasiIdVal}(\mathbf{G})$ is in P if $\mathbf{G}$ is abelian, and coNP-complete otherwise.