Polynomial Completeness in Expanded Groups

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First polynomial completeness results

Theorem

Let $\mathbb{F}$ be a finite field, $n \in \mathbb{N}$. Every mapping from $\mathbb{F}^n \to \mathbb{F}$ is a polynomial function.

Theorem [A. Fröhlich, 1958]

Let G be a finite simple nonabelian group, let $f : G \to G$ be such that $f(1_G) = 1_G$.
Then there are $n \in \mathbb{N}$ and sequences $(g_1, \ldots, g_n)$ from G and $(e_1, \ldots, e_n)$ from $\mathbb{Z}$ such that for all $x \in G$:

$$f(x) = g_1 x^{e_1} g_1^{-1} g_2 x^{e_2} g_2^{-1} \ldots g_n x^{e_n} g_n^{-1}.$$

Anticipation of further completeness results

From Fröhlich's paper (1958)

The problem of extending the results of this note appropriately to wider classes of groups does not seem intractable [. . .].
In the first place we have characterized R as the near-ring of all mappings transforming normal subgroups of Ω into themselves [. . .].
In this case one will have to consider also induced mappings on quotient groups $\Delta_1 - \Delta_2$, where Δ_1, Δ_2 are Ψ-invariant subgroups of Ω and $\Delta_1 \supseteq \Delta_2$.

A generalization

Theorem [K. Kaarli 1978]

Let $(G, +)$ be a group, $\mathrm{Inn}(G) \subseteq E \subseteq \mathrm{End}(G)$, R the near-ring generated by E. Let

$$R' := \{f : G \to G \mid f(0) = 0, \forall A \trianglelefteq_R G \; \forall g_1, g_2 \in G : \\ g_1 - g_2 \in A \Rightarrow f(g_1) - f(g_2) \in A\}.$$

If every submodule of ${}_R G$ coincides with its R-commutator subgroup, then R is a dense subnear-ring of R'.

Corollary

Let G be a finite group. Suppose that every normal subgroup N of is perfect, i.e., $[N, N] = N$. Then every unary congruence preserving function of G is a polynomial function.

Classifying functions

Let $\mathbf{A} = (A, f_1, f_2, \ldots)$ be an algebraic structure. A function $g : A^n \to A$ is:

- a term function of $\mathbf{A}$ if it can be written in the form $g(\mathbf{x}) = f_1(x_1, f_2(f_1(x_3, x_1)))$.
- a polynomial function of $\mathbf{A}$ if it can be written in the form $g(\mathbf{x}) = f_1(a_2, f_1(x_1, f_2(f_1(x_3, a_1))))$.

Let ρ be a binary relation on A. Then

- g preserves ρ if

$$(a_1, b_1) \in \rho, \ldots, (a_n, b_n) \in \rho \Rightarrow (g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \in \rho.$$

- g is congruence preserving if it preserves all congruence relations of $\mathbf{A}$.

Connections

- Every polynomial function is congruence preserving.
- Every term function preserves all subalgebras of $\mathbf{A} \times \mathbf{A}$.
- Every term function preserves all subalgebras of $\mathbf{A}^n$.
  Note: f preserves $\rho \subseteq A^n \Leftrightarrow \rho$ is a subalgebra of $(A, f)^n$.
- $\mathbf{A}$ finite, $f : A^n \to A$, f preserves all subalgebras of $\mathbf{A}^{|A|^n} \Rightarrow$ f is a term function.

Completeness Properties

Definition

An algebra **A** is affine complete if every finitary congruence preserving function is polynomial.
A is k-affine complete if every k-ary congruence preserving function is polynomial.

Problem [G. Grätzer 1978]

Characterize affine complete algebras.

Universal Algebra Results

Theorem [Hagemann & Herrmann 1982]

Let **A** be a finite algebra in a congruence permutable variety. Then the following are equivalent:

1. Every homomorphic image of **A** is affine complete.
2. For all $\alpha \in \mathsf{Con}(\mathbf{A})$, we have $[\alpha, \alpha] = \alpha$.

Proof of Hagemann's and Herrmann's Theorem:

We prove: If $\mathrm{Con}(\mathbf{A}) \models [\alpha, \alpha] = \alpha$, then every congruence preserving function is polynomial.

1. Let $f : A \to A$ be congruence preserving.
2. We interpolate f by polynomials on finite subsets T.
3. Case $T = \{a, b\}$:

   $$\beta := \{(p(a), p(b)) \mid p \in \mathrm{Pol}_1(\mathbf{A})\}$$

   is a congruence relation containing (a, b).
4. Thus $(f(a), f(b)) \in \Theta_{\mathbf{A}}(a, b) \subseteq \beta$.
5. Hence $\exists p : (p(a), p(b)) = (f(a), f(b))$.

Proof of Hagemann's and Herrmann's Theorem:

1. Case $T = \{a, b, c\}$.
2. $\mathrm{Pol}_1(\mathbf{A}) \leq \mathbf{A}^A$ has distributive congruences.
3. Define congruences α, β, γ on $\mathrm{Pol}_1(\mathbf{A})$ by

$$\begin{aligned} p\,\alpha\,q &\;:\Leftrightarrow\; p(a) = q(a), \\ p\,\beta\,q &\;:\Leftrightarrow\; p(b) = q(b), \\ p\,\gamma\,q &\;:\Leftrightarrow\; p(c) = q(c). \end{aligned}$$

4. Solve

$$p \equiv f(a) \pmod{\alpha}, \;\; p \equiv f(b) \pmod{\beta}, \;\; p \equiv f(c) \pmod{\gamma}.$$

5. Use Chinese Remainder Theorem.

Affine complete groups

Theorem [Hagemann and Herrmann, 1982]

$\mathbf{G}$ finite group. Every homomorphic image of $\mathbf{G}$ is affine complete $\Leftrightarrow \forall N \trianglelefteq \mathbf{G} : [N, N] = N$.

Theorem [Kaarli, 1983, Hagemann and Herrmann, 1982]

$\mathbf{G}$ finite group, $\mathrm{Con}(\mathbf{G})$ distributive. Then $\mathbf{G}$ is affine complete $\Leftrightarrow \forall N \trianglelefteq \mathbf{G} : [N, N] = N$.

Theorem [Nöbauer, 1976]

$\mathbf{A}$ finite abelian group. $\mathbf{A}$ is affine complete $\Leftrightarrow \exists \mathbf{B}, \mathbf{C} : \mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ and $\exp(\mathbf{B}) = \exp(\mathbf{C})$.

Affine complete groups

Theorem [Kaarli 1982]

An abelian group $\mathbf{A}$ is affine complete $\Leftrightarrow$

1. $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbf{A}$, or
2. $\mathbb{Z} \hookrightarrow \mathbf{A}$ and $\exp(T(\mathbf{A})) = \infty$, or
3. $\mathbf{A} \cong \prod_{i=1}^{m} \mathbb{Z}_{p_i^{\alpha_i}} \times \mathbb{Z}_{p_i^{\alpha_i}} \times \mathbf{B}_i$ with $p_1, \ldots, p_m$ different primes, $\exp(\mathbf{B}_i) \mid p_i^{\alpha_i}$.

Theorem [M. Saks 1983]

A finite nonabelian Hamiltonian group is never affine complete.

Theorem [Ecker 2006]

Let $\mathbf{A}$ be a finite abelian group, $\mathbf{A} = \mathbf{PQ}$ with $\mathbf{P}$ a 2-group and $\mathbf{Q}$ of odd order. Then $\mathrm{Dih}(\mathbf{A}) = \mathbf{A} \rtimes \mathbb{Z}_2$ is affine complete iff $\exp(\mathbf{P}) = 2$ and $\mathbf{Q}$ is affine complete.

Affine complete groups

Given: a finite group **G**.
Asked: Is **G** affine complete?

Example

$\mathbf{G} := ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_4$.

Ask a computer (SONATA)

```
gap> RequirePackage("sonata");
# SONATA by Aichinger, Binder, Ecker, Mayr, Noebauer
# loaded.
gap>  C3 := Group ((1,2,3));
gap>  C3xC3 := DirectProduct (C3, C3);
gap>  a := GroupHomomorphismByImages (C3xC3, C3xC3,
              [(1,2,3), (4,5,6)], [(1,3,2),(4,6,5)]);
gap>  A := Group (a);  IsGroupOfAutomorphisms (A);
gap>  C3xC3_C2 := SemidirectProduct (A, C3xC3);
gap>  G := DirectProduct (C3xC3_C2, CyclicGroup (4));
gap> IdGroup (G);
[ 72, 32 ]
gap> StructureDescription (G);
"C4 x ((C3 x C3) : C2)"
gap> p := Size (PolynomialNearRing (G));
23328
gap> c := Size (CompatibleFunctionNearRing (G));
23328
```

Affine complete groups

Hence **G** $= ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) \times \mathbb{Z}_4 = G(72, 32)$ is 1-affine complete.

But is it 2-affine complete? Is it 3-affine complete? Is it 4-affine complete? . . . Is it 70-affine complete?

Proving Affine Completeness

Theorem [EA, 2001]

$(\mathbb{Z}_4 \times \mathbb{Z}_2, +, 2x_1x_2 \dots x_k)$ is k-affine complete and not $(k+1)$-affine complete.

Theorem [EA, Ecker, 2006]

G k-nilpotent and $(k+1)$-affine complete $\Rightarrow$ **G** is affine complete.

Theorem [EA, 2018]

Let **A** be a finite nilpotent algebra in cp variety with all fundamental operations of arity $\leq m$. We assume that **A** is a product of prime power order algebras. Let

$$s := (m|A|)^{\log_2(|A|)}.$$

Then **A** s-affine complete $\Rightarrow$ **A** affine complete.

Disproving Affine Completeness

Theorem

Let $\mathbf{A}$ be a finite algebra with finitely many fundamental operations. If the clone $\mathsf{Comp}(\mathbf{A})$ is not finitely generated, then $\mathbf{A}$ is not affine complete.

Lemma

$\mathbf{A}$ finite algebra.

- $\mathbf{A}$ simple $\Rightarrow$ $\mathsf{Comp}(\mathbf{A})$ f.g.
- $\mathbf{A}$ has permuting congruences, $\mathsf{Con}(\mathbf{A})$ distributive $\Rightarrow$ $\mathsf{Comp}(\mathbf{A})$ f.g.

Finite generation of c.p. functions - Examples

Examples of abelian groups

1. $\mathrm{Comp}(\mathbb{Z}_2)$ is f.g.
2. $\mathrm{Comp}(\mathbb{Z}_4)$ is f.g.
3. $\mathrm{Comp}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not f.g.
4. $\mathrm{Comp}(\mathbb{Z}_4 \times \mathbb{Z}_4)$ is f.g.
5. $\mathrm{Comp}((\mathbb{Z}_2 \times \mathbb{Z}_4)^2)$ is f.g.

Consequence

For finite abelian groups $\mathbf{A}, \mathbf{B}$, the triple

$$(\mathrm{Comp}(\mathbf{A}) \text{ is f.g.},\ \mathrm{Comp}(\mathbf{B}) \text{ is f.g.},\ \mathrm{Comp}(\mathbf{A} \times \mathbf{B}) \text{ is f.g.})$$

can take all 8 possible combinations of truth values.

Finite generation of c.p. functions

Lemma

Let $\mathbf{A}$ be a finite abelian group. Then $\mathsf{Comp}(\mathbf{A})$ is f.g. $\Longleftrightarrow$ $\mathsf{Comp}(\mathbf{S})$ is f.g. for every Sylow subgroup $\mathbf{S}$ of $\mathbf{A}$.

Theorem [EA, Lazić, Mudrinski (2016)]

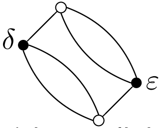
Let $p \in \mathbb{P}$, and let $\mathbf{S}$ be an abelian p-group. Then $\mathsf{Comp}(\mathbf{S})$ is f.g. $\Longleftrightarrow$ $\mathbf{S}$ is affine complete or cyclic.

Finite generation of c.p. functions

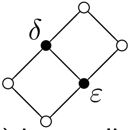
For an arbitrary group **G**, finite generation of $\mathsf{Comp}(\mathbf{G})$ can be described considering the lattice $\mathsf{Con}(\mathbf{G})$.

Definition

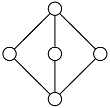
A bounded lattice $\mathbb{L}$ splits if there are $\delta < 1$ and $\varepsilon > 0$ such that $\mathbb{L} = \mathbb{I}[0, \delta] \cup \mathbb{I}[\varepsilon, 1]$.



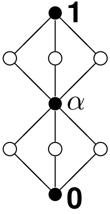
(δ, ε) is a splitting pair



(δ, ε) is a splitting pair



$\mathbb{M}_3$ does not split



$\mathbf{0}, \alpha, \mathbf{1}$ each cut the lattice

Finite generation of c.p. functions - the use of splitting

Theorem [EA, Mudrinski 2013]

$\mathbf{A}$ finite Mal'cev algebra s.t. $\mathrm{Con}(\mathbf{A})$ does not split. Then $\mathrm{Comp}(\mathbf{A})$ is f.g.

Finite generation of c.p. functions - the use of splitting

Lemma

Let $\mathbf{A}$ be an algebra such that $\mathrm{Con}(\mathbf{A})$ splits with splitting pair (δ, ε). Then every $f : A^n \to A$ with

1. $\forall \mathbf{a}, \mathbf{b} : (f(\mathbf{a}), f(\mathbf{b})) \in \varepsilon$,
2. $\forall \mathbf{a}, \mathbf{b} : \mathbf{a} \equiv_\delta \mathbf{b} \Rightarrow f(\mathbf{a}) = f(\mathbf{b})$

is congruence preserving. There are at least 2^{2^n} such functions.

Theorem

$\mathbf{A}$ finite algebra with a Mal'cev term, $\mathbb{L} := \mathrm{Con}(\mathbf{A})$. If

1. $\mathbb{L}$ is simple, and $|\mathbb{L}| \geq 3$, and
2. $\mathbb{L}$ splits,

then $\mathrm{Comp}(\mathbf{A})$ is not f.g.

Finite generation of c.p. functions - the use of splitting

Proof:

Assume

1. $\mathbb{L}$ is simple, and $|\mathbb{L}| \geq 3$,
2. $\mathbb{L}$ splits.
3. $\mathrm{Comp}(\mathbf{A})$ is f.g. by F.

Then

- (A, F) is nilpotent, prime power order, of finite type.
- Hence (A, F) is supernilpotent.
- Hence $(A, \mathrm{Pol}(F)) = (A, \mathrm{Comp}(\mathbf{A}))$ is supernilpotent.
- Hence "absorbing" c.p. functions have bounded essential arity.
- From splitting, construct c.p. functions of arbitrary finite ess. arity.

Finite generation of c.p. functions

Lemma

The clone of congruence preserving functions of a finite nilpotent group is finitely generated if and only if the clone of congruence preserving functions of every Sylow subgroup is finitely generated.

Theorem (EA, Lazić, Mudrinski 2016)

Let **G** be a finite p-group, let $\mathbb{L}$ be the lattice of normal subgroups of **G**, and let $\{e\} = N_0 < \cdots < N_n = G$ be the set of those normal subgroups that cut the lattice $\mathbb{L}$. Then the following are equivalent:

1. The clone of congruence preserving functions of **G** is finitely generated.
2. For each $i \in \{0, \ldots, n-1\}$, the interval $\mathbb{I}[N_i, N_{i+1}]$ of the lattice of normal subgroups of **G** either contains exactly 2 elements, or $\mathbb{I}[N_i, N_{i+1}]$ does not split.

Affine complete groups

Small p-groups:

- **G** non abelian p-group, $|G| \leq 32$: the normal subgroup lattice splits, hence **G** is not affine complete.
- $G(16, 11) = \mathbb{Z}_2 \times D_8$, $G(16, 12) = \mathbb{Z}_2 \times Q_8$, $G(32, 27)$, $G(32, 34) = \mathrm{Dih}(\mathbb{Z}_4 \times \mathbb{Z}_4)$, $G(32, 46) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_8$, $G(32, 47) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8$ are 1-affine complete.

Theorem [Saxinger, 2015]

The groups $G(64, 73)$ and $G(64, 76)$ are affine complete.

All other groups of order 64 are abelian or have splitting congruence lattice.

Small affine complete groups

Theorem

The six non-abelian affine complete groups of order ≤ 100 are:

- $G(36,13) = \mathrm{Dih}(\mathbb{Z}_2 \times {\mathbb{Z}_3}^2)$
- A_5
- $G(64,73)$
- $G(64,76)$
- $G(72,49) = \mathrm{Dih}({\mathbb{Z}_2}^2 \times {\mathbb{Z}_3}^2)$
- $G(100,15) = \mathrm{Dih}(\mathbb{Z}_2 \times {\mathbb{Z}_5}^2)$.

Open problems on affine complete groups

Open problems

1. Is the direct product of finite affine complete groups affine complete?
2. Is there an algorithm to decide whether a given finite group is affine complete?

Affine completeness of direct products

Theorem (Kaarli & Mayr 2010)

Let $\mathbf{A}, \mathbf{B}$ be affine complete finite algebras in the variety V. If V has a majority term, or V has a Mal'cev term and every congruence of $\mathbf{A} \times \mathbf{B}$ is a product congruence, then $\mathbf{A} \times \mathbf{B}$ is affine complete.

Decidability of affine completeness

Lemma

Let **A** be an algebra.

1. If $\mathrm{Comp}(\mathbf{A})$ is generated by its k-ary members, and **A** is k-affine complete, then **A** is affine complete.
2. If $\mathrm{Pol}(\mathbf{A})$ is determined by a set $\mathcal{R}$ of relations such that $\forall R \in \mathcal{R} : |R| \leq r$, and **A** is $(r+1)$-affine complete, then **A** is affine complete.

Theorem (EA 2010)

Let **A** a finite algebra with Mal'cev term. Then there is $n \in \mathbb{N}$ and $\rho \subseteq A^n$ such that $\mathrm{Pol}(\mathbf{A})$ consists of exactly those functions preserving ρ.

Consequence

If n can be found algorithmically, affine completeness in cp varieties is decidable.

Other concepts of completeness

General method

- Polynomial functions on a Mal'cev algebra $\mathbf{A}$ preserve certain relations:
  - congruence relations = subalgebras of $\mathbf{A} \times \mathbf{A}$ containing Δ,
  - congruence relations and abelian/nonabelian type of prime sections in the congruence lattice,
  - congruences and commutators, encoded by certain subalgebras of $\mathbf{A}^4$.
- For a subset $\mathcal{R}$ of these relations, call the algebra $\mathcal{R}$-complete if every $\mathcal{R}$-preserving function is a polynomial.

Polynomially rich algebras

Definition
Let **V** be an expanded group, and let $k \in \mathbb{N}$. Then **V** is polynomially rich if every function on **V** that preserves congruences and the types of prime sections in the congruence lattice is a polynomial function.

Theorem
A finite abelian *p*-group is polynomially rich if and only if it is affine complete or simple.

Polynomially rich algebras

Theorem (EA, Mudrinski 2009)

A finite dimensional vector-space V over a finite field is polynomially rich if and only if $\dim(V) \neq 1$ or $|V|$ is prime.

More on polynomial richness

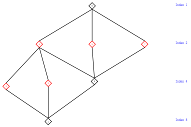
For finite expanded groups with distributive congruence lattice (or with congruence lattice satisfying (APMI)), there is a characterization of polynomial richness (EA, Mudrinski, 2009).

Lattices with (APMI)

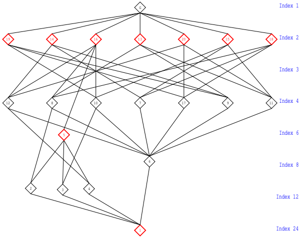
Definition

$\mathbb{L}$ lattice. $\mathbb{L}$ has *adjacent projective meet irreducibles* : $\Leftrightarrow$
$\forall$ meet irreducible $\alpha, \beta \in \mathbb{L}$:

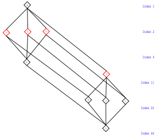
$$\mathbb{I}[\alpha, \alpha^+] \leftrightsquigarrow \mathbb{I}[\beta, \beta^+] \Rightarrow \alpha^+ = \beta^+.$$



$\mathrm{Con}(C_2 \times C_4)$ does not have (APMI).



$\mathrm{Con}(S_3 \times C_2 \times C_2)$ has (APMI).



$\mathrm{Con}(C_{11} \times C_2 \times C_2)$ has (APMI).

Algebras with (APMI) congruence lattices

Algebras that have (APMI) congruence lattices

- All $\mathbf{A}_i$ similar finite simple algebras with Mal'cev term. Then $\mathrm{Con}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$ has (APMI).
- Every finite distributive lattice has (APMI).
- $\mathbf{G}$ finite group, $\mathbf{G} \in \mathcal{V}(S_3)$ Then $\mathrm{Con}(\mathbf{G})$ has (APMI).
- $\mathbf{A}$ satisfies (SC1) $\Rightarrow$ $\mathrm{Con}(\mathbf{A})$ satisfies (APMI) [Idziak and Słomczyńska, 2001].

Definition [Idziak and Słomczyńska, 2001]

$\mathbf{A}$ with Mal'cev term. $\mathbf{A}$ has (SC1) $:\Leftrightarrow \forall \mathbf{B} \in \mathbb{H}_{\mathrm{SI}}(\mathbf{A})$:

$$\forall \alpha \in \mathrm{Con}(\mathbf{B}) : [\alpha, \mu_{\mathbf{B}}] = 0 \Rightarrow \alpha \leq \mu_{\mathbf{B}}.$$

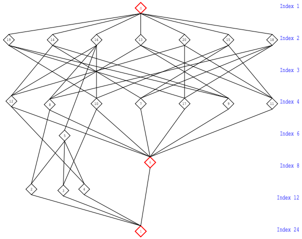
Structure of (APMI)-lattices

Theorem [Aichinger and Mudrinski, 2009]

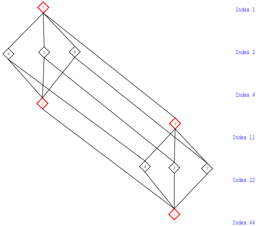
$\mathbb{L}$ finite modular lattice with (APMI), $|\mathbb{L}| > 1$. Then $\exists m \in \mathbb{N}$, $\exists \beta_0, \ldots, \beta_m \in D(\mathbb{L})$ such that

1. $0 = \beta_0 < \beta_1 < \cdots < \beta_m = 1$,
2. each $\mathbb{I}[\beta_i, \beta_{i+1}]$ is a simple complemented modular lattice.

Pictures of (APMI)-lattices



$\mathrm{Con}(S_3 \times C_2 \times C_2)$



$\mathrm{Con}(A_5 \times C_2 \times C_2)$

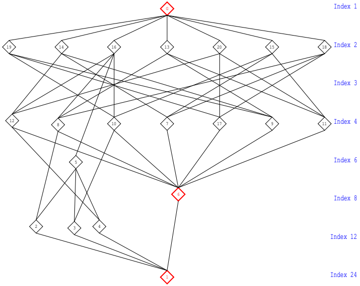
Affine completeness of congruence-(APMI)-algebras

Theorem [Aichinger and Mudrinski, 2009]

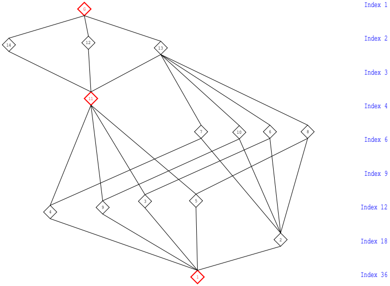
V finite expanded group, congruence-(APMI).
$U_0 < U_1 < \ldots < U_n$ maximal chain in $D(\mathbf{Id}\,(\mathbf{V}))$. Then **V** is affine complete $\Leftrightarrow$

1. **V** has (SC1),
2. $\forall i \in \{0, \ldots, n-1\}$: $[U_{i+1}, U_{i+1}]_{\mathbf{V}} \leq U_i \Rightarrow \mathbb{I}[U_i, U_{i+1}]$ is not a 2-element chain.

Examples of congruence-(APMI)-groups



$S_3 \times C_2 \times C_2$ is not affine complete



$\mathrm{Dih}(C_2 \times C_3 \times C_3)$ is affine complete (cf. [Ecker, 2006])

The clone of congruence preserving functions of (APMI)-algebras

Theorem [Aichinger and Mudrinski, 2009]

$\mathbf{V}$ finite expanded group, congruence-(APMI). Then the clone $\mathsf{Comp}(\mathbf{V})$ is generated by $\mathsf{Comp}_2(\mathbf{V})$.

Corollary

$\mathbf{V}$ finite expanded group, congruence-(APMI). $\mathbf{V}$ is affine complete if and only if $\mathsf{Comp}_2(\mathbf{V}) = \mathsf{Pol}_2(\mathbf{V})$.

Polynomial richness of congruence-(APMI) algebras

Definition - polynomial richness [Idziak and Słomczyńska, 2001]

$\mathbf{A} = (A, F)$ is *polynomially rich* if every finitary f that preserves:

1. all congruences
2. all TCT-types of prime quotients in $\mathrm{Con}(\mathbf{A})$

is a polynomial.

Theorem [Aichinger and Mudrinski, 2009]

$\mathbf{V}$ finite expanded group, congruence-(APMI).
$U_0 < U_1 < \ldots < U_n$ maximal chain in $D(\mathbf{Id}\,(\mathbf{V}))$. Then $\mathbf{V}$ is polynomially rich $\Leftrightarrow$

1. $\mathbf{V}$ has (SC1),
2. $\forall i \in \{0, \ldots, n-1\}$: $[U_{i+1}, U_{i+1}]_{\mathbf{V}} \leq U_i \Rightarrow \mathbb{I}[U_i, U_{i+1}]$ is not a 2-element chain or the module ${}_{P_0(\mathbf{V})}(U_{i+1}/U_i)$ is pol.equiv. to a simple module over the full matrix ring over a field of prime order.

A natural occurrence of the condition (APMI)

Theorem (Kaarli 1983)

A a finite algebra. TFAE:

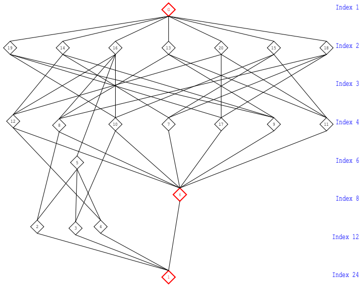
1. Every partial finitary congruence preserving function is the restriction of a total congruence preserving function.
2. $\mathrm{Con}(\mathbf{A})$ is arithmetical.

Theorem (EA Mudrinski 2009)

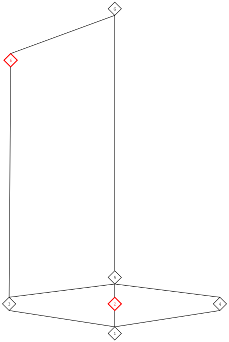
V finite expanded group. TFAE:

1. Every unary partial congruence preserving function is the restriction of a total congruence preserving function.
2. **V** is congruence-(APMI), and
   $\forall \alpha, \beta \in D(\mathrm{Con}(\mathbf{V})), \gamma \in \mathrm{Con}(\mathbf{V})$:
   $\alpha \prec_{D(\mathrm{Con}(\mathbf{V}))} \beta, \alpha \prec_{\mathrm{Con}(\mathbf{V})} \gamma < \beta \Rightarrow |0/\gamma| = 2 * |0/\alpha|$.

Unary compatible function extension property



The group $S_3 \times C_2 \times C_2$ has the unary CFEP.



The group $\mathrm{SL}(2,5) \times \mathrm{C}_2$ is not congruence-(APMI), hence (CFEP) fails.

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