A remark on the composition of polynomial functions over algebraically closed fields

Erhard Aichinger and Stefan Steinerberger

Universität Linz and Universität Bonn

AAA81 Salzburg, February 2011

Compositions that are polynomial functions

Question

Let $\mathbb{K}$ be a field, and let $f, g : \mathbb{K} \to \mathbb{K}$. We assume

- $f \circ g$ is polynomial,
- g is polynomial.

Can we conclude that

$$f \text{ is a polynomial on the range of } g?$$

Some observations

Obvious fact

Let $\mathbb{K}$ be a finite field, and let $f, g : \mathbb{K} \to \mathbb{K}$. If

$$f \circ g \text{ and } g \text{ are polynomial,}$$

then

$$f \text{ is polynomial.}$$

The real case

On the reals, let $f(x) := \sqrt[3]{x}$, $g(x) := x^3$. Then

$$f \circ g(x) = x \text{ for all } x \in \mathbb{R},$$

but f is not polynomial.

The complex numbers

Theorem

Let $f, g : \mathbb{C} \to \mathbb{C}$, g nonconstant polynomial, $f \circ g$ polynomial. Then f is polynomial.

Sketch of the proof:

- By complex analysis arguments, f is holomorphic [Rudin, 1966, Chapter 10, p.221, Exercise 20].
- If, for large $|x|$, $|g| \sim |x^m|$, $|f \circ g| \sim |x^n|$, then $|f| \sim |x^{\frac{n}{m}}|$, and hence (Liouville) f is polynomial.

Some prerequisites

Observation

Let $\mathbb{K}$ be a field, $f, g, h : \mathbb{K} \to \mathbb{K}$ such that

$$h = f \circ g.$$

Then for all $a, b \in \mathbb{K}$, we have

$$g(a) = g(b) \Longrightarrow h(a) = h(b).$$

Hilbert's Nullstellensatz

Let $\mathbb{A}$ be an algebraically closed field, and let $f_1, \ldots, f_m, g \in \mathbb{A}[x_1, \ldots, x_n]$. TFAE:

- For all $\mathbf{a} \in \mathbb{A}^n$: $f_1^{\mathbb{A}}(\mathbf{a}) = \cdots = f_m^{\mathbb{A}}(\mathbf{a}) = 0 \Longrightarrow g^{\mathbb{A}}(\mathbf{a}) = 0$.
- $\exists r \in \mathbb{N}_0 \, \exists b_1, \ldots, b_m \in \mathbb{A}[x_1, \ldots, x_n]$:

$$g^r = b_1 \cdot f_1 + \cdots + b_m \cdot f_m.$$

Fried and MacRae's Theorem

Theorem [Fried and MacRae, 1969]

Let $\mathbb{K}$ be a field, $p, q, f, g \in \mathbb{K}[t]$, $\deg(p) > 0$, $\deg(q) > 0$. TFAE:

- $p(x) - q(y) \mid f(x) - g(y)$ in $\mathbb{K}[x, y]$.
- $\exists h \in \mathbb{K}[t] : f = h(p(t))$ and $g = h(q(t))$.

Proofs:

- Original proof: field of algebraic functions over some curve.
- Elementary algebraic proofs by E.A. and F. Binder [Binder, 1996].
- [Schicho, 1995]: J. Schicho provides a proof from category theory that suggests many generalizations.

From $\mathbb{C}$ to algebraically closed fields

Theorem

Let $\mathbb{A}$ be an algebraically closed field, let $f, g : \mathbb{A} \to \mathbb{A}$. If

$$g \text{ is polynomial}, f \circ g \text{ is polynomial}, g' \neq 0,$$

then

$$f \text{ is polynomial.}$$

A multivariate generalization

Theorem

Let $\mathbb{A}$ be an algebraically closed field, $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathbb{A}[t]$, and let f be a function from $\mathbb{A}^n$ to $\mathbb{A}$. We assume that

$$g : \mathbb{A}^n \to \mathbb{A}, (a_1, \ldots, a_n) \mapsto f(p_1^{\mathbb{A}}(a_1), \ldots, p_n^{\mathbb{A}}(a_n))$$

is a polynomial function, and that for each $i \in \{1, \ldots, n\}$, the derivative $p_i' \neq 0$. Then

$$f \text{ is a polynomial function.}$$

A proof by reduction to the unary case

Theorem [Prager and Schwaiger, 2009]

Let $\mathbb{K}$ be a field with $|\mathbb{K}| > \aleph_0$, and let $f : \mathbb{K}^n \to \mathbb{K}$. If for all $i \in \{1, \ldots, n\}$ and all $b_1, \ldots, b_n \in \mathbb{K}$,

$$x \mapsto f(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n)$$

is a polynomial function, then

$$f \text{ is a polynomial function.}$$

A generalization of Fried and MacRae's Theorem

Theorem (cf. [Schicho, 1995])

Let $\mathbb{K}$ be a field, $n \in \mathbb{N}$, let $p_1, \ldots, p_n, q_1, \ldots, q_n$ be nonconstant polynomials in $\mathbb{K}[t]$, and let $f, g \in \mathbb{K}[t_1, \ldots, t_n]$. Then the following are equivalent:

- $$f(x_1, \ldots, x_n) - g(y_1, \ldots, y_n) \in \langle p_i(x_i) - q_i(y_i) \mid i \in \{1, \ldots, n\} \rangle_{\mathbb{K}[\mathbf{x}, \mathbf{y}]}.$$

- There is $h \in \mathbb{K}[t_1, \ldots, t_n]$ such that

$$\begin{aligned} f(x_1, \ldots, x_n) &= h(p_1(x_1), \ldots, p_n(x_n)) \\ g(x_1, \ldots, x_n) &= h(q_1(x_1), \ldots, q_n(x_n)). \end{aligned}$$

A generalization of "$p(x) - p(y)$ is squarefree".

Lemma

Let $\mathbb{A}$ be an algebraically closed field, and let $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathbb{A}[t]$ with $p_i' \neq 0$ and $q_i' \neq 0$ for all i. Then

$$\langle p_i(x_i) - q_i(y_i) \,|\, i \in \{1, \ldots, n\} \rangle_{\mathbb{A}[x,y]}$$

is a radical ideal.

Remark: "Algebraically closed" can be dropped.

Binder, F. (1996).
Fast computations in the lattice of polynomial rational function fields.
In Lakshman, Y. N., editor, *Proceedings of the 1996 international symposium on symbolic and algebraic computation, ISSAC '96, Zuerich, Switzerland, July 24–26, 1996. New York, NY: ACM Press. 43-48. [ISBN 0-89791-796-0/pbk]*.

Fried, M. D. and MacRae, R. E. (1969).
On curves with separated variables.
Math. Ann., 180:220–226.

Prager, W. and Schwaiger, J. (2009).
Generalized polynomials in one and in several variables.
Math. Pannon., 20(2):189–208.

Rudin, W. (1966).
Real and complex analysis.
McGraw-Hill Book Co., New York.

Schicho, J. (1995).
A note on a theorem of Fried and MacRae.
Arch. Math. (Basel), 65(3):239–243.