Higher commutators, nilpotence, and supernilpotence

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Polynomials

Definition

$\mathbf{A} = \langle A, F \rangle$ an algebra, $n \in \mathbb{N}$. $\mathrm{Pol}_k(\mathbf{A})$ is the subalgebra of

$$\mathbf{A}^{A^k} = \langle \{f : A^k \to A\}, \text{“}F \text{ pointwise”} \rangle$$

that is generated by

- $(x_1, \ldots, x_k) \mapsto x_i \ (i \in \{1, \ldots, k\})$
- $(x_1, \ldots, x_k) \mapsto a \ (a \in A)$.

Proposition

$\mathbf{A}$ be an algebra, $k \in \mathbb{N}$. Then $\mathbf{p} \in \mathrm{Pol}_k(\mathbf{A})$ iff there exists a term t in the language of $\mathbf{A}$, $\exists m \in \mathbb{N}$, $\exists a_1, a_2, \ldots, a_m \in A$ such that

$$\mathbf{p}(x_1, x_2, \ldots, x_k) = \mathbf{t}^{\mathbf{A}}(a_1, a_2, \ldots, a_m, x_1, x_2, \ldots, x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$.

§1 : Supernilpotence in expanded groups

Absorbing polynomials

Definition

$\mathbf{V} = \langle V, +, -, 0, f_1, f_2, \ldots \rangle$ expanded group, $p \in \mathrm{Pol}_n \mathbf{V}$. p is *absorbing* $:\Leftrightarrow \forall \mathbf{x} : 0 \in \{x_1, \ldots, x_n\} \Rightarrow p(x_1, \ldots, x_n) = 0$.

Examples of absorbing polynomials

- $(G, +, -, 0)$ group, $p(x, y) := [x, y] = -x - y + x + y$.
- $(G, +, -, 0)$ group, $p(x_1, x_2, x_3, x_4) := [x_1, [x_2, [x_3, x_4]]]$.
- $(R, +, \cdot, 0, 1)$ ring, $p(x_1, x_2, x_3, x_4) := x_1 \cdot x_2 \cdot x_3 \cdot x_4$.
- $\mathbf{V}$ an expanded group, $q \in \mathrm{Pol}_2(\mathbf{V})$,

$$p(x, y) := q(x, y) - q(x, 0) + q(0, 0) - q(0, y).$$

- $\mathbf{V}$ expanded group, $q \in \mathrm{Pol}_3(\mathbf{V})$,

$$p(x, y, z) := q(x, y, z) - q(x, y, 0) + q(x, 0, 0) - q(x, 0, z) + q(0, 0, z) - q(0, 0, 0) + q(0, y, 0) - q(0, y, z).$$

Supernilpotent expanded groups

Definition

V expanded group. **V** is *k-supernilpotent* $:\Leftrightarrow$ the zero-function is the only $(k+1)$-ary absorbing polynomial.

Proposition

V is an expanded group. **V** is k-supernilpotent if
$k = \max\{\text{ess. arity}(p) \mid p \in \text{Pol}(\mathbf{V}), p \text{ absorbing }\}$.

Examples

Proposition

V expanded group. **V** is

1. 1-supernilpotent iff $p(x, y) = p(x, 0) - p(0, 0) + p(0, y)$ for all $p \in \mathrm{Pol}_2(\mathbf{V})$, $x, y \in V$.
2. 2-supernilpotent iff $p(x, y, z) = p(x, y, 0) - p(x, 0, 0) + p(x, 0, z) - p(0, 0, z) + p(0, 0, 0) - p(0, y, 0) + p(0, y, z)$ for all $p \in \mathrm{Pol}_3(\mathbf{V})$, $x, y, z \in V$.

Supernilpotence class

Definition

V is supernilpotent *of class k* $:\Leftrightarrow k$ is minimal such that **V** is k-supernilpotent.

The Higman-Berman-Blok recursion

Theorem [Higman, 1967, p.154], [Berman and Blok, 1987]

$\mathbf{V}$ finite expanded group.

$$\begin{aligned} a_n(\mathbf{V}) &:= \log_2(|\{p \in \mathrm{Clo}_n(\mathbf{V}) \,|\, p \text{ is absorbing}\}|) \\ t_n(\mathbf{V}) &:= \log_2(|\mathrm{Clo}_n(\mathbf{V})|). \end{aligned}$$

Then $t_n(\mathbf{V}) = \sum_{i=0}^{n} a_i(\mathbf{V}) \binom{n}{i}$.

Proof: (17 lines).

Corollary (follows from [Berman and Blok, 1987])

$\mathbf{V}$ finite expanded group, $k \in \mathbb{N}$. TFAE:

1. $\mathbf{V}$ is supernilpotent of class k.
2. $\exists p$: $\deg(p) = k$ and $|\mathrm{Clo}_n(\mathbf{V})| = 2^{p(n)}$ for all $n \in \mathbb{N}$.

Structure of supernilpotent expanded groups

Theorem (follows from [Kearnes, 1999])

$\mathbf{V}$ finite supernilpotent expanded group. Then

$$\mathbf{V} \cong \prod_{i=1}^{k} \mathbf{W}_i,$$

all $\mathbf{W}_i$ of prime power order.

Theorem [Aichinger, 2013]

$\mathbf{V}$ supernilpotent expanded group, Con($\mathbf{V}$) of finite height. Then

$$\mathbf{V} \cong \prod_{i=1}^{k} \mathbf{W}_i,$$

all $\mathbf{W}_i$ monochromatic.

A part of the proof

- Suppose there are $A \prec B \prec C \trianglelefteq \mathbf{V}$, $\mathbb{I}[A, C] = \{A, B, C\}$, $\pi(C/B) = p \in \mathbb{P}$, $\pi(B/A) = 0$.
- Suppose $A = 0$, $[C, C] = B$, $[C, B] = 0$.
- Use $[C, C] = B$ to produce $f \in \mathrm{Pol}_1(\mathbf{V})$, $u, v \in V$ such that
  - $f(0) = 0$, $f(C) \subseteq B$,
  - $f(u + v) - f(u) \neq f(v)$,
  - f is constant on each B-coset.
- Define a $\mathbb{Z}[t]$-module

$$M \quad := \quad \{f \quad \in \quad \mathrm{Pol}_1(\mathbf{V}) \,|\, f(C) \quad \subseteq \quad B, \hat{f}(\sim_B) \quad \subseteq \quad \Delta\},$$

  $t \star m\,(x) := m(x + v)$.
- Then $(t - 1) \star f\,(u) = f(u + v) - f(u)$.

A part of the proof

- Since $\exp(C/B) = p$, $\exp(B/0) = 0$, we have

$$(t^p - 1) \star f(x) = f(x + p * v) - f(x) = f(x + b) - f(x) = 0.$$

- From $\gcd(t^p - 1, t^m - 1) = t - 1$, we obtain $(t-1)^m \star f \neq 0$ for all $m \in \mathbb{N}$.
- Define $h^{(1)} := f$, $h^{(n)}(x_1, \ldots, x_n) := h^{(n-1)}(x_1 + x_n, x_2, \ldots, x_{n-1}) - h^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + h^{(n-1)}(0, x_2, \ldots, x_{n-1}) - h^{(n-1)}(x_n, x_2, \ldots, x_{n-1})$.
- Then $h^{(n)}$ is absorbing, and $h^{(n)}(x_1, v, \ldots, v) = ((t-1)^{n-1} \star f)\ (x_1) - ((t-1)^{n-1} \star f)\ (0)$.
- If $h^{(n)} \equiv 0$, then $(t-1)^{n-1} \star f$ is constant and $(t-1)^n \star f = 0$.
- Hence $h^{(n)} \not\equiv 0$, contradicting supernilpotence.

§2 : Commutators and Higher Commutators for Algebras with Mal'cev Term.

Binary commutators

Definition ([?, ?, ?])

$\mathbf{A}$ algebra, $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$. Then $\eta := [\alpha, \beta]$ is the smallest element in $\mathrm{Con}(\mathbf{A})$ such that for all polynomials $f(\mathbf{x}, vby)$ and vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ from $\mathbf{A}$, the conditions

- $\mathbf{a} \equiv_\alpha \mathbf{b}$, $\mathbf{c} \equiv_\beta \mathbf{d}$,
- $f(\mathbf{a}, \mathbf{c}) \equiv_\eta f(\mathbf{a}, \mathbf{d})$

imply

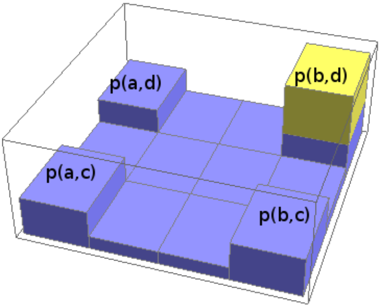
$$f(\mathbf{b}, \mathbf{c}) \equiv_\eta f(\mathbf{b}, \mathbf{d}).$$

Description of binary commutators

Proposition [Aichinger and Mudrinski, 2010]

A algebra with Mal'cev term, $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$. Then $[\alpha, \beta]$ is the congruence generated by

$$\{(p(a,c), p(b,d)) \mid (a,b) \in \alpha, (c,d) \in \beta, p \in \mathrm{Pol}_2(\mathbf{A}), \\ p(a,c) = p(a,d) = p(b,c)\}.$$



Binary commutators for expanded groups

Proposition (cf. [Scott, 1997])

V expanded group, A, B ideals of **V**. Then $[A, B]$ is the ideal generated by

$$\{p(a, b) \mid a \in A, b \in B, p \in \text{Pol}_2(\mathbf{V}), \text{p is absorbing}\}.$$

Higher commutators for expanded groups

Definition

$\mathbf{V}$ expanded group, $A_1, \ldots, A_n \trianglelefteq \mathbf{V}$. Then $[A_1, \ldots, A_n]$ is the ideal generated by

$$\{p(a_1, \ldots, a_n) \,|\, a_1 \in A_1, \ldots, a_n \in A_n, \\ p \in \mathrm{Pol}_n(\mathbf{V}), \text{p is absorbing}\}.$$

Definition [Bulatov, 2001]

$\mathbf{A}$ algebra, $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n, \beta, \delta \in \mathrm{Con}(\mathbf{A})$. Then $\alpha_1, \dots, \alpha_n$ *centralize* β *modulo* δ if for all polynomials $f(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y})$ and vectors $\mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_n, \mathbf{b}_n, \mathbf{c}, \mathbf{d}$ from $\mathbf{A}$ with

1. $\mathbf{a}_i \equiv_{\alpha_i} \mathbf{b}_i$ for all $i \in \{1, 2, \dots, n\}$,
2. $\mathbf{c} \equiv_{\beta} \mathbf{d}$, and
3. $f(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{c}) \equiv_{\delta} f(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{d})$ for all $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \{\mathbf{a}_1, \mathbf{b}_1\} \times \dots \times \{\mathbf{a}_n, \mathbf{b}_n\} \backslash \{(\mathbf{b}_1, \dots, \mathbf{b}_n)\}$,

we have

$$f(\mathbf{b}_1, \dots, \mathbf{b}_n, \mathbf{c}) \equiv_{\delta} f(\mathbf{b}_1, \dots, \mathbf{b}_n, \mathbf{d}).$$

Abbreviation: $C(\alpha_1, \dots, \alpha_n, \beta; \delta)$.

The definition of higher commutators

Definition [Bulatov, 2001]

$\mathbf{A}$ algebra, $n \geq 2$, $\alpha_1, \ldots, \alpha_n \in \mathrm{Con}(\mathbf{A})$. Then $[\alpha_1, \ldots, \alpha_n]$ is smallest congruence δ such that $C(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n; \delta)$.

Properties of higher commutators

Lemma [Mudrinski, 2009, Bulatov, 2001]

A algebra.

- $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_i \alpha_i$.
- $\alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n \Rightarrow [\alpha_1, \ldots, \alpha_n] \leq [\beta_1, \ldots, \beta_n]$.
- $[\alpha_1, \ldots, \alpha_n] \leq [\alpha_2, \ldots, \alpha_n]$.

Theorem [Mudrinski, 2009, Aichinger and Mudrinski, 2010]

A Mal'cev algebra.

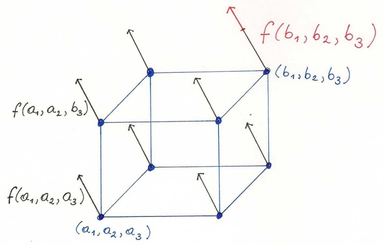
- $[\alpha_1, \ldots, \alpha_n] = [\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}]$ for all $\pi \in S_n$.
- $\eta \leq \alpha_1, \ldots, \alpha_n \Rightarrow [\alpha_1/\eta, \ldots, \alpha_n/\eta] = ([\alpha_1, \ldots, \alpha_n] \vee \eta)/\eta$.
- $[., \ldots, .]$ is join distributive in every argument.
- $[\alpha_1, \ldots, \alpha_i, [\alpha_{i+1}, \ldots, \alpha_n]] \leq [\alpha_1, \ldots, \alpha_n]$.

Proofs: $\sim$25 pages. (AU 63, p.371-395).

Theorem [Mudrinski, 2009], [Aichinger and Mudrinski, 2010, Corollary 6.10]

$\mathbf{A}$ algebra with Mal'cev term, $\alpha_1, \dots, \alpha_n \in \mathrm{Con}(\mathbf{A})$. Then $[\alpha_1, \dots, \alpha_n]$ is the congruence generated by

$$\{(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \mid (a_1, b_1) \in \alpha_1, \dots, (a_n, b_n) \in \alpha_n,$$
$$f \in \mathrm{Pol}_n(\mathbf{A}), f(\mathbf{x}) = f(a_1, \dots, a_n) \text{ for all}$$
$$\mathbf{x} \in (\{a_1, b_1\} \times \dots \times \{a_n, b_n\}) \setminus \{(b_1, \dots, b_n)\}.\}$$



Examples of Higher Commutators

Example

$\langle G, * \rangle$ group, $A, B, C \trianglelefteq \mathbf{G}$. Then
$[A, B, C] = [[A, B], C] * [[A, C], B] * [[B, C], A]$.

Example

$\mathbf{R}$ commutative ring with unit, $A, B, C \trianglelefteq \mathbf{R}$. Then
$[A, B, C] = \{\sum_{i=1}^{n} a_i b_i c_i \,|\, n \in \mathbb{N}_0, \forall i : a_i \in A, b_i \in B, c_i \in C\}$.

Example

$\mathbf{V} := \langle \mathbb{Z}_4, +, 2xyz \rangle$. Then $[[V, V], V] = 0$ and $[V, V, V] = \{0, 2\}$.

Remarks on the definition of higher commutators

Scope of Higher Commutators

- Higher commutators are defined for arbitrary algebras.
- Commutativity, join distributivity hold for Mal'cev algebras.
- For Mal'cev algebras, there are various descriptions of higher commutators in [Aichinger and Mudrinski, 2010].
- For expanded groups, higher commutators can easily be described using absorbing polynomials.
- Little is known for higher commutators outside c.p. varieties.

§2 : Supernilpotence for arbitrary algebras

Definition of Supernilpotence

Definition

A is k-supernilpotent $:\Leftrightarrow$

$$[\underbrace{1,\dots,1}_{k+1}] = 0.$$

A is supernilpotent of class k $:\Leftrightarrow [\underbrace{1,\dots,1}_{k+1}] = 0,\ [\underbrace{1,\dots,1}_{k}] > 0.$

Relation of supernilpotence to the free spectrum

Theorem (cf. [Berman and Blok, 1987])

$\mathbf{A}$ finite algebra in cp and congruence uniform variety, $k \in \mathbb{N}$.
TFAE:

1. $\exists p \in \mathbb{R}[t] : \deg(p) = k$ and $|\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)| \leq 2^{p(n)}$ for all $n \in \mathbb{N}$.
2. $\mathbf{A}$ is supernilpotent of class $\leq k$.

Assumption ”congruence uniform” can be dropped by [**?**, Lemma 12.4].

Relation of supernilpotence to commutator terms

Theorem

A finite Mal'cev algebra. TFAE:

1. **A** generates a congruence uniform variety and has a finite bound on the length of its commutator terms.
2. **A** is supernilpotent.

Nilpotence

Definition of the lower central series

$\gamma_1(\mathbf{A}) := 1_A$, $\gamma_n(\mathbf{A}) := [1_A, \gamma_{n-1}(\mathbf{A})]$ for $n \geq 2$.

Nilpotence

$\mathbf{A}$ algebra with Mal'cev term. $\mathbf{A}$ is *nilpotent of class k* $:\Leftrightarrow$
$\gamma_k(\mathbf{A}) \neq 0_A$, $\gamma_{k+1}(\mathbf{A}) = 0_A$.

The "lower superseries"

$$\sigma_n(\mathbf{A}) := [\underbrace{1_A, \dots, 1_A}_{n}].$$

Supernilpotence

$\mathbf{A}$ algebra with Mal'cev term. $\mathbf{A}$ is *supernilpotent of class k* $:\Leftrightarrow$
$\sigma_k(\mathbf{A}) \neq 0_A$, $\sigma_{k+1}(\mathbf{A}) = 0_A$.

Connections between nilpotency and supernilpotency

Supernilpotency implies Nilpotency

A algebra with a Mal'cev term. Then **A** supernilpotent of class $k \Rightarrow$ **A** nilpotent of class $\leq k$.

Idea in the proof: $[\alpha_1, [\alpha_2, \alpha_3]] \leq [\alpha_1, \alpha_2, \alpha_3]$.

Examples

- $\mathbf{N}_6 := \langle \mathbb{Z}_6, +, f \rangle$ with $f(0) = f(3) = 3$, $f(1) = f(2) = f(4) = f(5) = 0$ is nilpotent of class 2 and not supernilpotent.
- $\langle \mathbb{Z}_4, +, 2x_1x_2, 2x_1x_2x_3, 2x_1x_2x_3x_4, \ldots \rangle$ is nilpotent of class 2 and not supernilpotent.

Deeper connections between nilpotence and supernilpotence

Theorem [Berman and Blok, 1987], [Kearnes, 1999]

A finite, finite type, with Mal'cev term. TFAE:

1. **A** is nilpotent and isomorphic to a direct product of algebras of prime power order.
2. **A** is supernilpotent.

Theorem

G group, $k \in \mathbb{N}$. **G** is nilpotent of class k $\Leftrightarrow$ **G** is supernilpotent of class k.

Proof: Commutator calculus from group theory.

Connections between Nilpotence and Supernilpotence

Theorem [Aichinger and Mudrinski, 2012]

$\mathbf{V} = \langle V, +, -, 0, g_1, g_2, \ldots \rangle$ expanded group, $m \geq 2$ such that

1. all g_i have arity $\leq m$,
2. all mappings $x \mapsto g_i(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{m_i})$ are endomorphisms of $\langle V, + \rangle$ (multilinearity),
3. $\mathbf{V}$ is nilpotent of class k.

Then $\mathbf{V}$ is supernilpotent of class $\leq m^{k-1}$.

Idea of the proof: expand using multilinearity and then use commutator calculus.

A non-property of supernilpotency

Example [Aichinger and Mudrinski, 2012]

$\mathbf{V} := \langle (\mathbb{Z}_7)^3, +, \ f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ g_1, g_2 \rangle$ with g_1, g_2 bilinear such that

$$g_1(e_i, e_j, e_k) := \begin{cases} e_1 \text{ if } i, j, k \geq 2, \\ 0 \text{ else.} \end{cases} \quad g_2(e_i, e_j, e_k) := \begin{cases} e_2 \text{ if } i, j, k = 3, \\ 0 \text{ else.} \end{cases}$$

$$\mathbf{V}_1 := \langle V, +, f, g_1 \rangle, \qquad \mathbf{V}_2 := \langle V, +, f, g_2 \rangle.$$

Then $[1,1,1]_{\mathbf{V}_1} = [1,1,1]_{\mathbf{V}_2} = [1,[1,1]_{\mathbf{V}_1}]_{\mathbf{V}_1} = [1,[1,1]_{\mathbf{V}_2}]_{\mathbf{V}_2} = 0$ and

$$[1,1,1]_{\mathbf{V}} > 0, \ \ [1,[1,1]_{\mathbf{V}}]_{\mathbf{V}} > 0.$$

Conclusion

Functions that preserve the nilpotency class or the supernilpotency class need not form a clone.

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