Polynomial completeness properties

Erhard Aichinger

Department of Algebra
Johannes Kepler University Linz, Austria

June 2012, AAA84

Polynomials

Definition

$\mathbf{A} = \langle A, F \rangle$ an algebra, $n \in \mathbb{N}$. $\mathrm{Pol}_k(\mathbf{A})$ is the subalgebra of

$$\mathbf{A}^{A^k} = \langle \{f : A^k \to A\}, \text{“}F \text{ pointwise”} \rangle$$

that is generated by

- $(x_1, \dots, x_k) \mapsto x_i \ (i \in \{1, \dots, k\})$
- $(x_1, \dots, x_k) \mapsto a \ (a \in A)$.

Proposition

$\mathbf{A}$ be an algebra, $k \in \mathbb{N}$. Then $\mathbf{p} \in \mathrm{Pol}_k(\mathbf{A})$ iff there exists a term t in the language of $\mathbf{A}$, $\exists m \in \mathbb{N}$, $\exists a_1, a_2, \dots, a_m \in A$ such that

$$\mathbf{p}(x_1, x_2, \dots, x_k) = \mathbf{t}^{\mathbf{A}}(a_1, a_2, \dots, a_m, x_1, x_2, \dots, x_k)$$

for all $x_1, x_2, \dots, x_k \in A$.

Function algebras – Clones

$\mathcal{O}(A) := \bigcup_{k \in \mathbb{N}} \{f \mid f : A^k \to A\}$.

Definition of Clone

$\mathcal{C} \subseteq \mathcal{O}(A)$ is a clone on A iff

1. $\forall k, i \in \mathbb{N}$ with $i \leq k$: $\big((x_1, \dots, x_k) \mapsto x_i\big) \in \mathcal{C}$,
2. $\forall n \in \mathbb{N}, m \in \mathbb{N}, f \in \mathcal{C}^{[n]}, g_1, \dots, g_n \in \mathcal{C}^{[m]}$:

$$f(g_1, \dots, g_n) \in \mathcal{C}^{[m]}.$$

$\mathcal{C}^{[n]}$ ...the n-ary functions in $\mathcal{C}$.

$\mathrm{Pol}(\mathbf{A}) := \bigcup_{k \in \mathbb{N}} \mathrm{Pol}_k(\mathbf{A})$ is a clone on A.

Functional Description of Clones

$\mathbf{A}$ algebra.

$\mathrm{Pol}(\mathbf{A}) \ldots$ the smallest clone on A that contains all projections, all constant operations, all basic operations of $\mathbf{A}$.

Clones of polynomial functions

Definition

A clone is *constantive* or *a polynomial clone* if it contains all unary constant functions.

Proposition

Every constantive clone is the set of polynomial functions of some algebra.

Relational Description of Clones

Definition

I a finite set, $\rho \subseteq A^I$, $f : A^n \to A$. f preserves ρ ($f \rhd \rho$) if $\forall v_1, \ldots, v_n \in \rho$:

$$\langle f(v_1(i), \ldots, v_n(i)) \,|\, i \in I \rangle \in \rho.$$

Remark

$f \rhd \rho \iff \rho$ is a subuniverse of $\langle A, f \rangle^I$.

Definition (Polymorphisms)

Let R be a set of finitary relations on A, $\rho \in R$.

$$\begin{aligned} \mathrm{Polym}(\{\rho\}) &:= \{f \in \mathcal{O}(A) \,|\, f \rhd \rho\}, \\ \mathrm{Polym}(R) &:= \textstyle\bigcap_{\rho \in R} \mathrm{Polym}(\{\rho\}). \end{aligned}$$

Finite Description of Clones

Definition

A clone is finitely generated if it is generated by a finite set of finitary functions.

Definition

A clone $\mathcal{C}$ is finitely related if there is a finite set of finitary relations R with $\mathcal{C} = \mathrm{Polym}(R)$.

Polynomial completeness properties

Questions

Given: $\mathbf{A}$ finite algebra with Mal'cev term.

1. Asked: ρ such that $\text{Pol}(\mathbf{A}) = \text{Polym}(\{\rho\})$.
2. $\text{Pol}(\mathbf{A}) = \mathcal{O}(\mathbf{A})$? Is $\mathbf{A}$ polynomially complete = functionally complete?
3. $\text{Pol}(\mathbf{A}) = \text{Polym}(\text{Con}(\mathbf{A}))$? Is $\mathbf{A}$ affine complete?
4. Other polynomial completeness properties: polynomially rich, weakly polynomially rich.

Functionally complete algebras

Theorem (cf. [Hagemann and Herrmann, Coll.Math.Soc.J.Bolyai, 1982]), forerunner in [Istinger, Kaiser, Pixley, Coll.Math., 1979]

Let $\mathbf{A}$ be a finite algebra, $|A| \geq 2$. Then $\mathrm{Pol}(\mathbf{A}) = \mathcal{O}(\mathbf{A})$ if and only if $\mathrm{Pol}_3(\mathbf{A})$ contains a Mal'cev operation, and $\mathbf{A}$ is simple and nonabelian.

$\mathbf{A}$ is *nonabelian* iff $[1_A, 1_A] \neq 0_A$. Here, $[.,.]$ is the *term condition commutator*.
This describes finite algebras with

$$\mathrm{Pol}(\mathbf{A}) = \mathrm{Polym}(\emptyset).$$

Descriptions of affine completeness

Proposition

$\mathbf{A} = \langle A, F \rangle$ algebra with Mal'cev term. TFAE

1. $\mathbf{A}$ is affine complete, i.e., $\mathrm{Pol}(\mathbf{A}) = \mathrm{Comp}(\mathbf{A})$. $(\mathrm{Comp}(\mathbf{A}) := \mathrm{Polym}(\mathrm{Con}(\mathbf{A})))$.
2. $\forall k \in \mathbb{N}$, $\forall f : A^k \to A$ with
$$\forall (a_1, \ldots, a_k), (b_1, \ldots, b_k) \in A^k, \forall \alpha \in \mathrm{Con}(\mathbf{A}) : \Big((a_1, b_1) \in \alpha, \ldots, (a_k, b_k) \in \alpha\Big) \Rightarrow (f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \alpha.$$
we have $f \in \mathrm{Pol}_k(\mathbf{A})$.
3. $\forall f : \Big(\mathrm{Con}(\langle A, F \cup \{f\}\rangle) = \mathrm{Con}(\langle A, F\rangle)\Big) \Longrightarrow f \in \mathrm{Pol}(\mathbf{A})$.
4. Every finitary operation on A that can be interpolated at each 2-element subset of its domain by a polynomial function is a polynomial function.

Computing polynomial functions of groups

```
elgar{erhard}: gap

gap> RequirePackage("sonata");
# SONATA by Aichinger, Binder, Ecker, Mayr, Noebauer
# loaded.

gap> G := SymmetricGroup (3);
Sym( [ 1 .. 3 ] )

gap> P := PolynomialNearRing (G);
PolynomialNearRing( Sym( [ 1 .. 3 ] ) )
gap> Size (P);
324

gap> G1 := GroupReduct (P);;

gap> Size (PolynomialNearRing (G1)); time;
4251528
176
```

Computing polynomial functions on groups

```
gap> G := AlternatingGroup (5);
Alt( [ 1 .. 5 ] )

gap> Size (PolynomialNearRing (G));
48873677980689257489322752273774603865660
85017600000000000000000000000000000000000
00000000000000000000000000
gap> time;
3708

gap> 60^60;
48873677980689257489322752273774603865660
85017600000000000000000000000000000000000
00000000000000000000000000
```

Searching affine complete groups

```
gap> G := SymmetricGroup (3);;

gap> P := PolynomialNearRing (SymmetricGroup (3));;
gap> Size (P);
324

gap> C := LocalInterpolationNearRing (P, 2);
LocalInterpolationNearRing( PolynomialNearRing(
   Sym( [ 1 .. 3 ] ) ), 2 )

gap> Size (C);
2916
```

Searching affine complete groups

Conclusion

There is a unary congruence preserving function on S_3 that is not a polynomial function. Hence S_3 is not affine complete.

Searching affine complete groups

We try $D_4 \times C_2 \cong \mathrm{Dih}(C_4 \times C_2)$.

```
gap> P := PolynomialNearRing (
             Group ((1,2,3,4), (1,2)(3,4), (5,6)));
PolynomialNearRing(
   Group([ (1,2,3,4), (1,2)(3,4), (5,6) ]) )

gap> Size (P);
256

gap> C := CompatibleFunctionNearRing(
             Group ((1,2,3,4), (1,2)(3,4), (5,6)));
< transformation nearring with 7 generators >

gap> Size (C);
256
```

Searching affine complete groups

```
gap> C1 := LocalInterpolationNearRing (P, 2);
LocalInterpolationNearRing(
PolynomialNearRing( Group(
[ (1,2,3,4), (1,2)(3,4), (5,6) ]) ), 2 )

gap> time;
45363

gap> Size (C1);
256
```

Searching affine complete groups

Conclusion

Every unary congruence preserving function of $D_4 \times C_2$ is polynomial.

Questions

1. Binary congruence preserving functions = binary polynomial functions?
2. 3-ary?
3. 4-ary?
4. Is affine completeness an algorithmically decidable property of a finite group?

Searching affine complete groups

Answers

1. [Ecker, CMB, 2006]: there are binary congruence preserving functions on $D_4 \times C_2$ that are not polynomials.
2. Hence: no.
3. Hence: no.
4. Open. No example of a finite group $\mathbf{G}$ known with $\mathrm{Comp}_2(\mathbf{G}) = \mathrm{Pol}_2(\mathbf{G})$ and $\mathbf{G}$ not affine complete. Decidable for nilpotent groups [EA and Ecker, IJAC, 2006]; also decidable if $\mathrm{Con}(\mathbf{G})$ is distributive.

Results on affine complete groups

Theorem
[Hagemann and Herrmann, Coll.Math.Soc.J.Bolyai, 1982]

$\mathbf{G}$ finite group. Every homomorphic image of $\mathbf{G}$ is affine complete $\Leftrightarrow \forall N \trianglelefteq \mathbf{G} : [N, N] = N$.

Theorem [Kaarli, AU17, 1983,
Hagemann and Herrmann, Coll.Math.Soc.J.Bolyai, 1982]

$\mathbf{G}$ finite group, Con($\mathbf{G}$) distributive. Then $\mathbf{G}$ is affine complete $\Leftrightarrow \forall N \trianglelefteq \mathbf{G} : [N, N] = N$.

Remark: Both results hold if $\mathbf{G}$ is a finite algebra with Mal'cev term.

Theorem [Nöbauer, Monatsh. Math., 1976]

$\mathbf{A}$ finite abelian group. $\mathbf{A}$ is affine complete $\Leftrightarrow$
$\exists$ groups $\mathbf{B}, \mathbf{C} : \mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ and $\exp(\mathbf{B}) = \exp(\mathbf{C})$.

Results on affine complete groups

Theorem [Ecker, CMB, 2006]

$\mathbf{A}$ finite abelian group. $\mathrm{Dih}(\mathbf{A}) = \mathbf{A} \rtimes C_2$ is affine complete $\Leftrightarrow$ $\exists$ groups $\mathbf{B}, \mathbf{C}$: $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, $\exp(\mathbf{B}) = 2$, $|\mathbf{C}|$ odd, $\mathbf{C}$ is affine complete.

Theorem [EA, Acta Szeged, 2002, Ecker, CMB, 2006]

$\mathbf{A}, \mathbf{B}$ nilpotent affine complete groups, $\mathbf{G} = \mathbf{A} \rtimes \mathbf{B}$, $\{x \mapsto b^{-1} \cdot x \cdot b \,|\, b \in B\}$ is a non-trivial fixed-point-free subgroup of $\langle \mathrm{Aut}(\mathbf{A}) \cap \mathrm{Pol}(\mathbf{A}), \circ \rangle$. Then $\mathbf{G}$ is affine complete.

Example

$\mathbf{A} := C_3 \times C_3$, $\mathbf{B} := C_2 \times C_2$, $\mathbf{G} = C_2 \times \mathrm{Dih}(C_3 \times C_3)$. Then $\mathbf{G}$ is affine complete.

Results on affine completeness by investigating the clone of polynomial functions

Theorem [Scott, Monatsh. Math., 1969]

Let $\mathbf{A}$, $\mathbf{B}$ be finite groups such that $\mathbf{A} \times \mathbf{B}$ has no skew-congruences. Then "$\mathrm{Pol}(\mathbf{A} \times \mathbf{B}) = \mathrm{Pol}(\mathbf{A}) \times \mathrm{Pol}(\mathbf{B})$".

Remark: Holds also for $\mathbf{A}, \mathbf{B}$ finite expanded groups [EA, Proc. Edinburgh MS, 2001], and finite algebras with Mal'cev term [Kaarli and Mayr, Monatsh.Math., 2010].

Corollary

$\mathbf{A}$, $\mathbf{B}$ finite algebras in a cp variety, $\mathbf{A}, \mathbf{B}$ affine complete, $\mathbf{A} \times \mathbf{B}$ has no skew congruences. Then $\mathbf{A} \times \mathbf{B}$ is affine complete.

The clone of polynomial functions

Theorem [Higman, Proc.Int.Conf.Th.Groups, 1967]

$\mathbf{G}$ finite nilpotent group of class k. Then $\exists p \in \mathbb{R}[t] : \deg(p) = k$ and $\mathrm{Pol}_n(\mathbf{G}) = 2^{p(n)}$.

Theorem [Berman and Blok, AU24, 1987]

$\mathbf{A}$ finite nilpotent algebra of finite type and prime power order in cm variety. Then $\exists p : \mathrm{Pol}_n(\mathbf{A}) = 2^{p(n)}$.

The clone of congruence preserving functions

Definition – Congruence preserving functions

$\mathbf{A}$ algebra. $\mathrm{Comp}(\mathbf{A}) := \mathrm{Polym}_A(\mathrm{Con}(\mathbf{A}))$.

Theorem (cf. [EA, AU44, 2000])

$\mathbf{A}$ finite algebra, cd and cp (as a single algebra). Then $\mathrm{Comp}(\mathbf{A})$ is generated by its 3-ary members.

Corollary

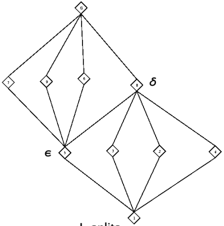
$\mathbf{A}$ finite algebra, cd and cp, $\mathrm{Comp}_3(\mathbf{A}) = \mathrm{Pol}_3(\mathbf{A})$. Then $\mathbf{A}$ is affine complete.

Splitting lattices

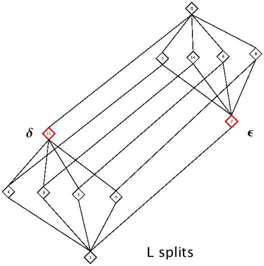
Definition

$\mathbb{L}$ lattice. $\mathbb{L}$ *splits* $:\Leftrightarrow \exists \varepsilon, \delta \in \mathbb{L}$: $0 < \varepsilon$ and $\delta < 1$ and

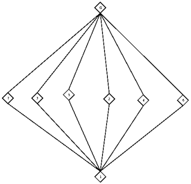
$$\forall \alpha \in \mathbb{L} : \alpha \geq \varepsilon \text{ or } \alpha \leq \delta.$$



L splits



L splits


L does not split

Clones with splitting congruence lattices

Theorem
A finite algebra, Con(**A**) splits. Then $|\mathrm{Comp}_n(\mathbf{A})| \geq 2^{2^n}$.

Theorem
G finite nilpotent group, Con(**G**) splits. Then **G** is not affine complete.

Corollary
All affine complete 2-groups of order ≤ 32 are abelian.

$|\mathbf{G}| = 32$, $\mathbf{G} \ncong C_4 \times C_4 \times C_2$, $\mathbf{G} \ncong (C_2)^5 \Longrightarrow$ Con(**G**) splits.

Clones with splitting congruence lattices

Theorem - a consequence of [Nöbauer, Monatsh. Math., 1976]

A finite abelian group is affine complete if and only if its congruence lattice does not split.

Comp($\mathbf{A}$) not finitely generated

Theorem

$\mathbf{A}$ finite algebra with Mal'cev term, Con($\mathbf{A}$) a simple lattice, $|\mathrm{Con}(\mathbf{A})| > 2$. TFAE:

1. Comp($\mathbf{A}$) is finitely generated.
2. Con($\mathbf{A}$) does not split.

Theorem [EA, AU47, 2002]

$\mathbf{G} := \langle C_{p^2} \times C_p, +\rangle$, p prime, $k \in \mathbb{N}$. Then $\overline{\mathbf{G}} := \langle G, \mathrm{Comp}_k(\mathbf{G})\rangle$ satisfies $\mathrm{Pol}_k(\overline{\mathbf{G}}) = \mathrm{Comp}_k(\overline{\mathbf{G}})$, but $\overline{\mathbf{G}}$ is not affine complete.

Decidability of affine completeness for groups

Theorem

A finite, cd and cp. Then **A** is affine complete iff $\mathrm{Pol}_3(\mathbf{A}) = \mathrm{Comp}_3\mathbf{A}$.

Theorem [EA and Ecker, IJAC, 2006]

G finite group, nilpotent of class k. Then **G** is affine complete iff $\mathrm{Pol}_{k+1}(\mathbf{G}) = \mathrm{Comp}_{k+1}(\mathbf{G})$.

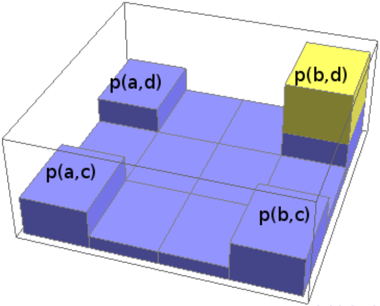
Remark: holds if **G** is a k-supernilpotent algebra with a Mal'cev term.

What does supernilpotent mean?

Description of binary commutators [EA and Mudrinski, AU63, 2010]

A algebra with Mal'cev term, $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$. Then $[\alpha, \beta]$ is the congruence generated by

$$\{(p(a,c), p(b,d)) \mid (a,b) \in \alpha, (c,d) \in \beta, p \in \mathrm{Pol}_2(\mathbf{A}),$$
$$p(a,c) = p(a,d) = p(b,c)\}.$$



Binary commutators for expanded groups

Description of binary commutators [Scott, Proc.Near-ring conference, 1997]

V expanded group, A, B ideals of **V**. Then $[A, B]$ is the ideal generated by

$$\{p(a,b) \,|\, a \in A, b \in B, p \in \mathrm{Pol}_2(\mathbf{V}),$$
$$p(0,0) = p(a,0) = p(0,b) = 0\}.$$

$[A, B]$ is also the ideal generated by

$$\{p(a,b) \,|\, a \in A, b \in B, q \in \mathrm{Pol}_2(\mathbf{V}),$$
$$q(x,0) = q(0,x) \text{ for all } x \in V\}.$$

Remark: $q(x,y) := p(x,y) - p(x,0) + p(0,0) - p(0,y)$.

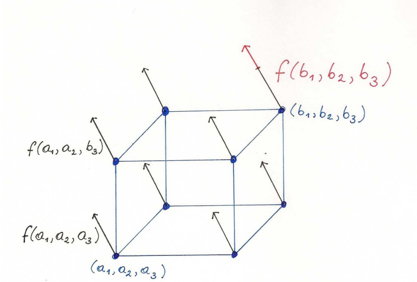
Definition of higher commutators [Bulatov, CTGA13, 2001]

- For $n \in \mathbb{N}$, n-ary commutators were defined in [Bulatov, CTGA13, 2001] by using a term condition similar to the definition of binary commutators.
- In [Mudrinski, Diss, 2009, EA and Mudrinski, AU63, 2010], properties of these higher commutators in cp varieties were investigated.

Description of higher commutators [Mudrinski, Diss, 2009], [EA and Mudrinski, AU63, 2010, Corollary 6.10]

A algebra with Mal'cev term, $\alpha_1, \dots, \alpha_n \in \mathrm{Con}(\mathbf{A})$. Then $[\alpha_1, \dots, \alpha_n]$ is the congruence generated by

$$\{(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \mid (a_1, b_1) \in \alpha_1, \dots, (a_n, b_n) \in \alpha_n, \\ f \in \mathrm{Pol}_n(\mathbf{A}), f(\mathbf{x}) = f(a_1, \dots, a_n) \text{ for all} \\ \mathbf{x} \in (\{a_1, b_1\} \times \dots \times \{a_n, b_n\}) \setminus \{(b_1, \dots, b_n)\}.\}$$



Higher commutators for expanded groups

Description of higher commutators for expanded groups

$\mathbf{V}$ expanded group, $A_1, \ldots, A_n \trianglelefteq \mathbf{V}$. Then $[A_1, \ldots, A_n]$ is the ideal generated by

$$\begin{aligned}\{f(a_1, \ldots, a_n) \,|\, \forall i : a_i \in A_i, f \in \mathrm{Pol}_n(\mathbf{V}), \\ \forall x_1, \ldots, x_n : f(0, x_2, x_3, \ldots, x_{n-1}, x_n) = \cdots \\ = \cdots = f(x_1, x_2, x_3, \ldots, x_{n-1}, 0) = 0\}\end{aligned}$$

Example

$\langle G, * \rangle$ group, $A, B, C \trianglelefteq \mathbf{G}$. Then
$[A, B, C] = [[A, B], C] * [[A, C], B] * [[B, C], A]$.

Example

$\mathbf{R}$ commutative ring with unit, $A, B, C \trianglelefteq \mathbf{R}$. Then
$[A, B, C] = \{\sum_{i=1}^{n} a_i b_i c_i \mid n \in \mathbb{N}_0, \forall i : a_i \in A, b_i \in B, c_i \in C\}$.

Example

$\mathbf{V} := \langle \mathbb{Z}_4, +, 2xyz \rangle$. Then $[[V, V], V] = 0$ and $[V, V, V] = \{0, 2\}$.

Nilpotency

Definition of the lower central series

$\gamma_1(\mathbf{A}) := 1_A$, $\gamma_n(\mathbf{A}) := [1_A, \gamma_{n-1}(\mathbf{A})]$ for $n \geq 2$.

Nilpotency

A algebra with Mal'cev term. **A** is *nilpotent of class k* $:\Leftrightarrow$ $\gamma_k(\mathbf{A}) \neq 0_A$, $\gamma_{k+1}(\mathbf{A}) = 0_A$.

The "lower superseries"

$\sigma_n(\mathbf{A}) := [\underbrace{1_A, \dots, 1_A}_{n}]$.

Supernilpotency

A algebra with Mal'cev term. **A** is *supernilpotent of class k* $:\Leftrightarrow$ $\sigma_k(\mathbf{A}) \neq 0_A$, $\sigma_{k+1}(\mathbf{A}) = 0_A$.

Properties of supernilpotency

Varieties

$\mathcal{V}$ cp variety, $n \in \mathbb{N}$.

$$S_n(\mathcal{V}) := \{\mathbf{A} \in \mathcal{V} \,|\, \mathbf{A} \text{ supernilpotent of class } \leq n\}.$$

Then $S_n(\mathcal{V})$ is a subvariety of $\mathcal{V}$.

A non-property of supernilpotency

Example [EA and Mudrinski, manuscr., 2012]

$\mathbf{V} := \langle (\mathbb{Z}_7)^3, +,\ f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ g_1, g_2 \rangle$ with g_1, g_2 bilinear such that

$$g_1(e_i, e_j, e_k) := \begin{cases} e_1 \text{ if } i,j,k \geq 2, \\ 0 \text{ else.} \end{cases} \qquad g_2(e_i, e_j, e_k) := \begin{cases} e_2 \text{ if } i,j,k = 3, \\ 0 \text{ else.} \end{cases}$$

$$\mathbf{V}_1 := \langle V, +, f, g_1 \rangle, \qquad \mathbf{V}_2 := \langle V, +, f, g_2 \rangle.$$

Then $[1,1,1]_{\mathbf{V}_1} = [1,1,1]_{\mathbf{V}_2} = [1,[1,1]_{\mathbf{V}_1}]_{\mathbf{V}_1} = [1,[1,1]_{\mathbf{V}_2}]_{\mathbf{V}_2} = 0$ and

$$[1,1,1]_{\mathbf{V}} > 0, \ \ [1,[1,1]_{\mathbf{V}}]_{\mathbf{V}} > 0.$$

Conclusion

Functions that preserve the nilpotency class or the supernilpotency class need not form a clone.

Supernilpotency and the free spectrum

Supernilpotency via absorbing polynomials

V expanded group. Then **V** is supernilpotent of class $\leq k$ $:\Leftrightarrow$
The 0-map is the only $f \in \mathrm{Pol}_{k+1}(\mathbf{V})$ with

$$\forall x_1, \ldots, x_{k+1} : 0 \in \{x_1, \ldots, x_{k+1}\} \Rightarrow f(x_1, \ldots, x_{k+1}) = 0.$$

Theorem (cf. [Berman and Blok, AU24, 1987])

A finite algebra in cp and congruence uniform variety, $k \in \mathbb{N}$.
TFAE:

1. $\exists p \in \mathbb{R}[t] : \deg(p) = k$ and $|\mathbf{F}_{\mathcal{V}(\mathbf{A})}(n)| \leq 2^{p(n)}$ for all $n \in \mathbb{N}$.
2. **A** is supernilpotent of class $\leq k$.

Assumption ”congruence uniform” can be dropped by
[Hobby and McKenzie, Cont.Math.76, 1988, Lemma 12.4].
For expanded groups, one can generalise Higman’s proof
[Higman, Proc.Int.Conf.Th.Groups, 1967] for groups.

Connections between nilpotency and supernilpotency

Supernilpotency implies Nilpotency

A algebra with a Mal'cev term. Then **A** supernilpotent of class $k \Rightarrow$ **A** nilpotent of class $\leq k$.

Follows easily from [Mudrinski, Diss, 2009].

Examples

- $\mathbf{FZ}_6 := \langle \mathbb{Z}_6, +, f \rangle$ with $f(0) = f(3) = 3$, $f(1) = f(2) = f(4) = f(5) = 0$ is nilpotent of class 2 and not supernilpotent.
- $\langle \mathbb{Z}_4, +, 2x_1x_2, 2x_1x_2x_3, 2x_1x_2x_3x_4, \ldots \rangle$ is nilpotent of class 2 and not supernilpotent.

Deeper connections between nilpotency and supernilpotency

Theorem [Berman and Blok, AU24, 1987], [Kearnes, AU42, 1999]

A finite, finite type, with Mal'cev term. TFAE

1. **A** is nilpotent and isomorphic to a direct product of algebras of prime power order.
2. **A** is supernilpotent.

Theorem

G group, $k \in \mathbb{N}$. **G** is nilpotent of class $k \Leftrightarrow$ **G** is supernilpotent of class k.

Proof: Commutator calculus from group theory.

Connections between Nilpotency and Supernilpotency

Theorem [EA and Mudrinski, manuscr., 2012]

$\mathbf{V} = \langle V, +, -, 0, g_1, g_2, \ldots \rangle$ expanded group, $m \geq 2$ such that

1. all g_i have arity $\leq m$,
2. all mappings $x \mapsto g_i(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_{m_i})$ are endomorphisms of $\langle V, + \rangle$ (multilinearity),
3. $\mathbf{V}$ is nilpotent of class k.

Then $\mathbf{V}$ is supernilpotent of class $\leq m^{k-1}$.

Idea of the proof: expand using multilinearity and then use commutator calculus.

Lattices forcing supernilpotency

Theorem [EA and Mudrinski, arXiv, 2012]

$\mathbf{A}$ finite algebra with Mal'cev term. If $\mathrm{Con}(\mathbf{A})$ does not split, then $\mathbf{A}$ is supernilpotent of class k with $k \leq$ (number of atoms of $\mathrm{Con}(\mathbf{A})$) $- 1$.

Corollary

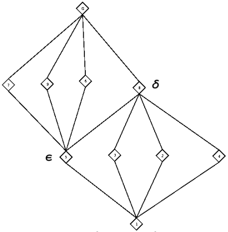
The congruence lattice of a finite non-nilpotent algebra with Mal'cev term splits.

Lattices that do not split strongly

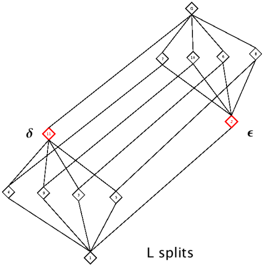
Definition

$\mathbb{L}$ lattice. $\mathbb{L}$ *splits strongly* $:\Leftrightarrow \exists \varepsilon, \delta \in \mathbb{L}$: $0 < \varepsilon \leq \delta < 1$ and

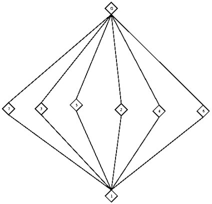
$$\forall \alpha \in \mathbb{L} : \alpha \geq \varepsilon \text{ or } \alpha \leq \delta.$$



L splits strongly



L splits


L does not split

Lattices that do not split strongly

Theorem [EA and Mudrinski, arXiv, 2012]

A finite algebra with Mal'cev term. Con(**A**) does not split strongly. Then $\exists n \in \mathbb{N}_0$, $\mathbf{B}, \mathbf{C}_1, \ldots, \mathbf{C}_n$ such that $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}_1 \times \cdots \times \mathbf{C}_n$, **B** is supernilpotent, each $\mathbf{C}_i$ is simple, and the direct product is skew-free.

Finite generation from supernilpotence

Theorem [EA and Mudrinski, AU63, 2010, Proposition 6.18]

A has Mal'cev term, **A** supernilpotent of class k. Then Pol(**A**) is generated by $\mathrm{Pol}_m(\mathbf{A})$ for $m := \max(3, k)$.

Corollary

A algebra with Mal'cev term. If Con(**A**) does not split strongly, then Comp(**A**) is generated by $\mathrm{Comp}_k(\mathbf{A})$ with $k := \max(3, (\text{number of atoms of Con}(\mathbf{A})) - 1)$.

Corollary[2]

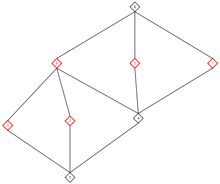
For **A** algebra with Mal'cev term s.t. Con(**A**) does not split strongly, affine completeness is an algorithmically decidable property.

Lattices with (APMI)

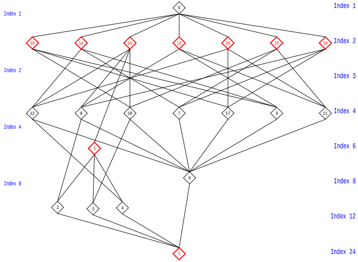
Definition

$\mathbb{L}$ lattice. $\mathbb{L}$ has *adjacent projective meet irreducibles* : $\Leftrightarrow$
$\forall$ meet irreducible $\alpha, \beta \in \mathbb{L}$:

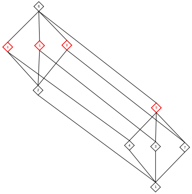
$$\mathbb{I}[\alpha, \alpha^+] \leftrightsquigarrow \mathbb{I}[\beta, \beta^+] \Rightarrow \alpha^+ = \beta^+.$$



Con($C_2 \times C_4$) does not have (APMI).



Con($S_3 \times C_2 \times C_2$) has (APMI).



Con($C_{11} \times C_2 \times C_2$) has (APMI).

Algebras with (APMI) congruence lattices

Algebras that have (APMI) congruence lattices

- All $\mathbf{A}_i$ similar finite simple algebras with Mal'cev term. Then $\mathrm{Con}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$ has (APMI).
- Every finite distributive lattice has (APMI).
- $\mathbf{G}$ finite group, $\mathbf{G} \in \mathcal{V}(S_3)$ Then $\mathrm{Con}(\mathbf{G})$ has (APMI).
- $\mathbf{A}$ satisfies (SC1) $\Rightarrow$ $\mathrm{Con}(\mathbf{A})$ satisfies (APMI) [Idziak and Słomczyńska, JPAA, 2001].

Definition [Idziak and Słomczyńska, JPAA, 2001]

$\mathbf{A}$ with Mal'cev term. $\mathbf{A}$ has (SC1) $:\Leftrightarrow \forall \mathbf{B} \in \mathbb{H}_{\mathrm{SI}}(\mathbf{A})$:

$$\forall \alpha \in \mathrm{Con}(\mathbf{B}) : [\alpha, \mu_{\mathbf{B}}] = 0 \Rightarrow \alpha \leq \mu_{\mathbf{B}}.$$

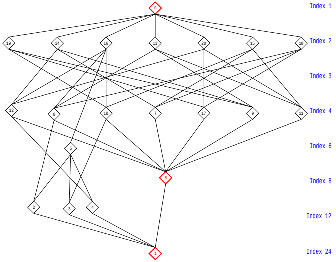
Structure of (APMI)-lattices

Theorem [EA and Mudrinski, AU60, 2009]

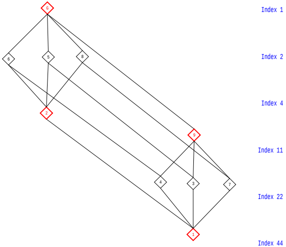
$\mathbb{L}$ finite modular lattice with (APMI), $|\mathbb{L}| > 1$. Then $\exists m \in \mathbb{N}$, $\exists \beta_0, \ldots, \beta_m \in D(\mathbb{L})$ such that

1. $0 = \beta_0 < \beta_1 < \cdots < \beta_m = 1$,
2. each $\mathbb{I}[\beta_i, \beta_{i+1}]$ is a simple complemented modular lattice.

Pictures of (APMI)-lattices



Con($S_3 \times C_2 \times C_2$)



Con($A_5 \times C_2 \times C_2$)

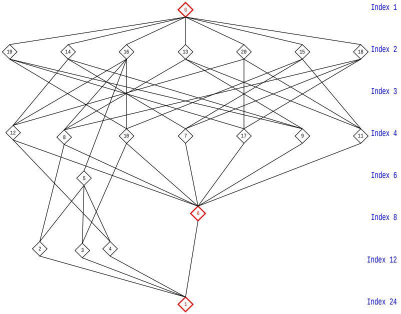
Affine completeness of of congruence-(APMI)-algebras

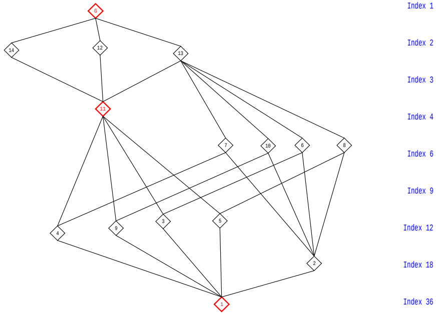
Theorem [EA and Mudrinski, AU60, 2009]

V finite expanded group, congruence-(APMI).
$U_0 < U_1 < \ldots < U_n$ maximal chain in $D(\mathbf{Id}\,(\mathbf{V}))$. Then **V** is affine complete $\Leftrightarrow$

1. **V** has (SC1),
2. $\forall i \in \{0, \ldots, n-1\}$: $[U_{i+1}, U_{i+1}]_{\mathbf{V}} \leq U_i \Rightarrow \mathbb{I}[U_i, U_{i+1}]$ is not a 2-element chain.

Examples of congruence-(APMI)-groups





$S_3 \times C_2 \times C_2$ is not affine complete

$\mathrm{Dih}(C_2 \times C_3 \times C_3)$ is affine complete (cf. [Ecker, CMB, 2006])

The clone of congruence preserving functions of (APMI)-algebras

Theorem [EA and Mudrinski, AU60, 2009]

$\mathbf{V}$ finite expanded group, congruence-(APMI). Then the clone $\mathrm{Comp}(\mathbf{V})$ is generated by $\mathrm{Comp}_2(\mathbf{V})$.

Corollary

$\mathbf{V}$ finite expanded group, congruence-(APMI). $\mathbf{V}$ is affine complete if and only if $\mathrm{Comp}_2(\mathbf{V}) = \mathrm{Pol}_2(\mathbf{V})$.

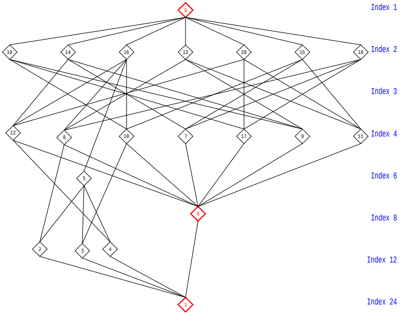
A natural occurrence of the condition (APMI)

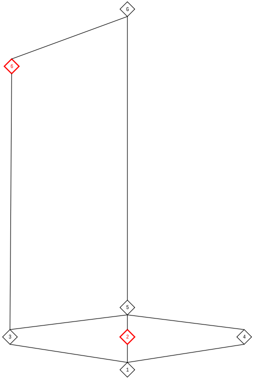
Theorem [EA and Mudrinski, AU60, 2009] (Unary compatible function extension property)

V finite expanded group. TFAE:

1. Every unary partial congruence preserving function on **V** can be extended to a total function.
2. All unary total congruence perserving functions on quotients of **V** can be lifted to **V**.
3. **V** is congruence-(APMI), and $\forall\ \alpha, \beta \in D(\mathrm{Con}(\mathbf{V}))$, $\gamma \in \mathrm{Con}(\mathbf{V}) : \alpha \prec_{D(\mathrm{Con}(\mathbf{V}))} \beta$, $\alpha \prec_{\mathrm{Con}(\mathbf{V})} \gamma < \beta \Rightarrow |0/\gamma| = 2 * |0/\alpha|$.

Unary compatible function extension property





The group $S_3 \times C_2 \times C_2$ has the unary CFEP.

The group $\mathrm{SL}(2,5)$ is not congruence-(APMI), hence (CFEP) fails.

Other concepts of polynomial completeness

Definition - polynomial richness [Idziak and Słomczyńska, JPAA, 2001]

$\mathbf{A} = \langle A, F \rangle$ is *polynomially rich* if every finitary f that preserves:

1. all congruences
2. all TCT-types of prime quotients in $\mathrm{Con}(\mathbf{A})$

is a polynomial.

Theorem [EA and Mudrinski, AU60, 2009]

V finite expanded group, congruence-(APMI).
$U_0 < U_1 < \ldots < U_n$ maximal chain in $D(\mathbf{Id}\,(\mathbf{V}))$. Then **V** is polynomially rich $\Leftrightarrow$

1. **V** has (SC1),
2. $\forall i \in \{0, \ldots, n-1\}$: $[U_{i+1}, U_{i+1}]_{\mathbf{V}} \leq U_i \Rightarrow \mathbb{I}[U_i, U_{i+1}]$ is not a 2-element chain or the module ${}_{P_0(\mathbf{V})}(U_{i+1}/U_i)$ is pol.equiv. to a simple module over the full matrix ring over a field of prime order.

Aichinger, E. (2000).
On Hagemann's and Herrmann's characterization of strictly affine complete algebras.
Algebra Universalis, 44:105–121.

Aichinger, E. (2001).
On near-ring idempotents and polynomials on direct products of Ω-groups.
Proc. Edinburgh Math. Soc. (2), 44:379–388.

Aichinger, E. (2002a).
2-affine complete algebras need not be affine complete.
Algebra Universalis, 47(4):425–434.

Aichinger, E. (2002b).
The polynomial functions on certain semidirect products of groups.
Acta Sci. Math. (Szeged), 68(1-2):63–81.

Aichinger, E. and Ecker, J. (2006).
Every $(k+1)$-affine complete nilpotent group of class k is affine complete.
Internat. J. Algebra Comput., 16(2):259–274.

Aichinger, E. and Mudrinski, N. (2009).
Types of polynomial completeness of expanded groups.
Algebra Universalis, 60(3):309–343.

Aichinger, E. and Mudrinski, N. (2010).
Some applications of higher commutators in Mal'cev algebras.
Algebra Universalis, 63(4):367–403.

Aichinger, E. and Mudrinski, N. (2012a).
On various concepts of nilpotence for expansions of groups.
Manuscript.

Aichinger, E. and Mudrinski, N. (2012b).
Sequences of commutator operations.
submitted; available on arXiv:1205.3297v3 [math.RA] 24 May 2012.

Berman, J. and Blok, W. J. (1987).
Free spectra of nilpotent varieties.
Algebra Universalis, 24(3):279–282.

Bulatov, A. (2001).
On the number of finite Mal'tsev algebras.
In *Contributions to general algebra, 13 (Velké Karlovice, 1999/Dresden, 2000)*, pages 41–54. Heyn, Klagenfurt.

Ecker, J. (2006).
Affine completeness of generalised dihedral groups.
Canad. Math. Bull., 49(3):347–357.

Hagemann, J. and Herrmann, C. (1982).
Arithmetical locally equational classes and representation of partial functions.
In *Universal Algebra, Esztergom (Hungary)*, volume 29, pages 345–360. Colloq. Math. Soc. János Bolyai.

Higman, G. (1967).
The orders of relatively free groups.
In *Proc. Internat. Conf. Theory of Groups (Canberra, 1965)*, pages 153–165. Gordon and Breach, New York.

Hobby, D. and McKenzie, R. (1988).
The structure of finite algebras, volume 76 of *Contemporary mathematics*.
American Mathematical Society.

Idziak, P. M. and Słomczyńska, K. (2001).
Polynomially rich algebras.
J. Pure Appl. Algebra, 156(1):33–68.

Istinger, M., Kaiser, H. K., and Pixley, A. F. (1979).
Interpolation in congruence permutable algebras.
Colloq. Math., 42:229–239.

Kaarli, K. (1983).
Compatible function extension property.
Algebra Universalis, 17:200–207.

Kaarli, K. and Mayr, P. (2010).
Polynomial functions on subdirect products.
Monatsh. Math., 159(4):341–359.

Kearnes, K. A. (1999).
Congruence modular varieties with small free spectra.
Algebra Universalis, 42(3):165–181.

Mudrinski, N. (2009).
On Polynomials in Mal'cev Algebras.
PhD thesis, University of Novi Sad.

Nöbauer, W. (1976).
Über die affin vollständigen, endlich erzeugbaren Moduln.
Monatshefte für Mathematik, 82:187–198.

Scott, S. D. (1969).
The arithmetic of polynomial maps over a group and the structure of certain permutational polynomial groups. I.
Monatsh. Math., 73:250–267.

Scott, S. D. (1997).
The structure of Ω-groups.
In *Nearrings, nearfields and K-loops (Hamburg, 1995)*, pages 47–137. Kluwer Acad. Publ., Dordrecht.