# CLONES OF POLYNOMIALS

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# Clones of polynomials

We represent the functions in a **clone** by polynomials $f \in \mathbb{K}[x_1, \ldots, x_n, \ldots]$ over a **field** $\mathbb{K}$.

**Goal:**

- Use the structure of $f \in \mathbb{K}[x_1, \ldots, x_n]$ to get better information on the clone.

**Usefulness:**

- Bounding the supernilpotency degree and the free spectrum.
- Solving systems of equations over supernilpotent algebras.

# Clones of polynomials

For $A, B \subseteq \mathbb{K}[x_i \mid i \in \mathbb{N}] = \bigcup_{n\in\mathbb{N}} \mathbb{K}[x_1, \ldots, x_n]$, we define (following [Couceiro, Foldes 2009])

$$AB = \{p(q_1, \ldots, q_n) \,|\, n \in \mathbb{N}, p \in A \cap \mathbb{K}[x_1, \ldots, x_n], q_1, \ldots, q_n \in B\}.$$

$C \subseteq \mathbb{K}[x_i \mid i \in \mathbb{N}]$ is a **clone of polynomials** if for each $i \in \mathbb{N}$, $x_i \in C$ and $CC \subseteq C$.

A polynomial $f$ is **homovariate** if all of its monomials contain the same variables.

- $5x_1x_2^3x_4 - 2x_1^{17}x_2x_4^3 + x_1^6x_2^3x_4^{20}$, $x_2 + 6x_2^4$, and $2$ are all homovariate.
- None of $x_1 + x_2$, $1 + 3x_1^3 + x_1^5$ is homovariate.

# Clones of polynomials

The function defined by

$$f(x_1, x_2, x_4) := 5x_1x_2^3x_4 - 2x_1^{17}x_2x_4^3 + x_1^6x_2^3x_4^{20}$$

is **absorbing**, meaning that $f(0, y, z) = f(x, 0, z) = f(x, y, 0) = 0$ for all $x, y, z$.

**Theorem**

Let $\mathbb{K}$ be a field, let $F \subseteq \mathbb{K}[x_i \mid i \in \mathbb{N}]$, $\mathrm{totdeg}(f) \leq n$ for all $f \in F$. Let $L := \mathrm{Clop}(\{x_1 + x_2, -x_1, 0\})$. Then there exists a set $H \subseteq \mathbb{K}[x_1, \ldots, x_n]$ of homovariate polynomials such that

$$L\ \mathrm{Clop}(H) = \mathrm{Clop}(F \cup \{x_1 + x_2, -x_1, 0\})$$

and $\mathrm{totdeg}(h) \leq n$ for all $h \in H$.

# Nilpotency and Supernilpotency

Let $C$ be a clone of polynomials on $\mathbb{K}$ that contains $x_1 + x_2$ and $-x_1$. Let $H \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be such that all $h \in H$ are homovariate, and $L\ \mathrm{Clop}(H) = C$.

- If the algebra $\mathbf{K} = (\mathbb{K}, \overline{C})$ is $k$-**nilpotent**, then each function in $\overline{\mathrm{Clop}(H)}$ depends on $\leq n^{k-1}$ arguments.
- The algebra $\mathbf{K} = (\mathbb{K}, \overline{C})$ is $s$-**supernilpotent** if each absorbing polynomial function of $\mathbf{K}$ depends on $\leq s$ arguments.

# On the implication nilpotent $\Rightarrow$ supernilpotent

Let $C$ be a clone of polynomials on $\mathbb{K}$ that contains $x_1 + x_2$ and $-x_1$.
Let $H \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be such that all $h \in H$ are homovariate, and $L\ \mathrm{Clop}(H) = C$.

Then:

$\mathbf{K} = (\mathbb{K}, \overline{C})$ is $k$-nilpotent
$\Rightarrow$ each function in $\overline{\mathrm{Clop}(H)}$ depends on $\leq n^{k-1}$ arguments
$\Rightarrow$ each absorbing polynomial function of $\mathbf{K} = (K, \overline{L\ \mathrm{Clop}(H)})$ depends on $\leq n^{k-1}$ arguments
$\Rightarrow$ $\mathbf{K}$ is $n^{k-1}$-supernilpotent.

# Expansions of additive groups of fields

### Theorem

Let $(A, +, *)$ be a field, and let $\mathbf{A} = (A, +, -, 0, (f_i)_{i \in I})$ be an algebra. Assume

- For each $i \in I$, $\operatorname{totdeg}(f_i) \leq n$,
- $\mathbf{A}$ is nilpotent of class at most $k$.

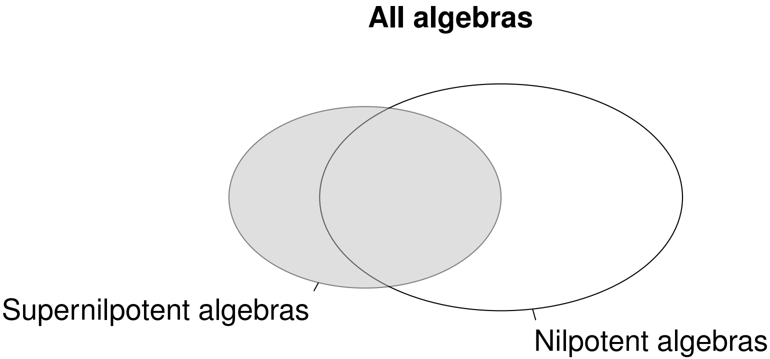
Then all absorbing polynomial functions of $\mathbf{A}$ are of essential arity at most $n^{k-1}$.

### Corollary

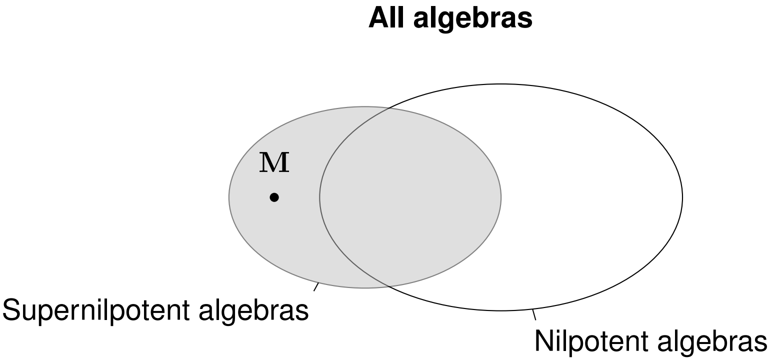
Let $\mathbb{A} = (A, +, *)$ be a field, and let $\mathbf{A} = (A, +, -, 0, (f_i)_{i \in I})$ be an expansion of $(A, +)$ with polynomial functions of $\mathbb{A}$ of total degree $\leq n$. Then:

- If $\mathbf{A}$ is $k$-nilpotent, it is $n^{k-1}$-supernilpotent.

# Nilpotency vs. supernilpotency

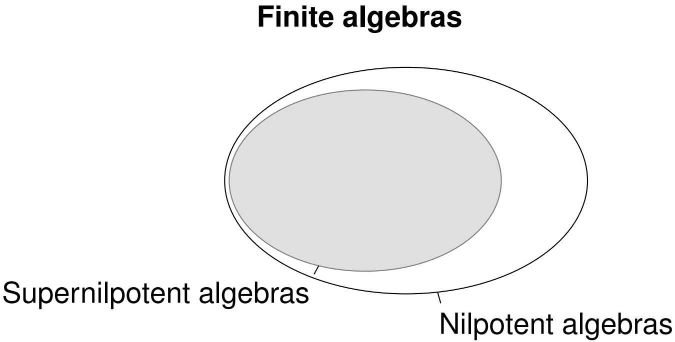


# Nilpotency vs. supernilpotency




M . . . [Moore Moorhead 2018]

# Nilpotency vs. supernilpotency

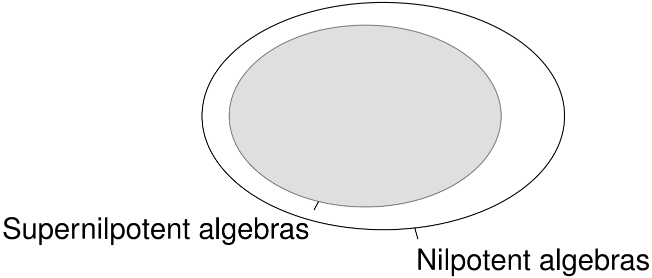



**Theorem – announced by [Kearnes Szendrei 2018]**

Every finite supernilpotent algebra is nilpotent.

# Nilpotency vs. supernilpotency

**Algebras in congruence modular varieties**

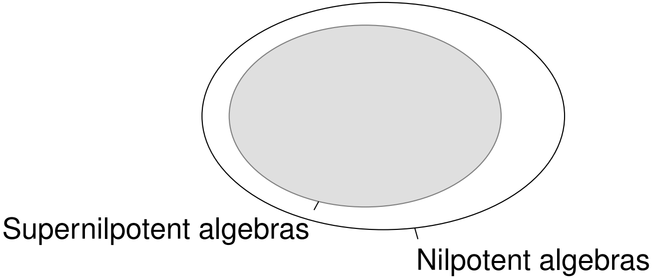



**Theorem [Wires 2019]**

Every supernilpotent algebra in a congruence modular variety is nilpotent.

# Nilpotency vs. supernilpotency

**Algebras in congruence permutable varieties**

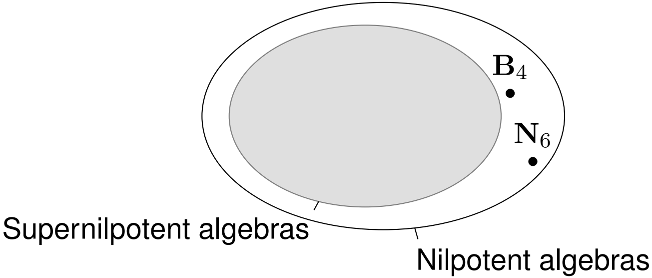



**Theorem [EA Mudrinski 2010]**

Every supernilpotent algebra in a congruence permutable variety is nilpotent.

# Nilpotency vs. supernilpotency

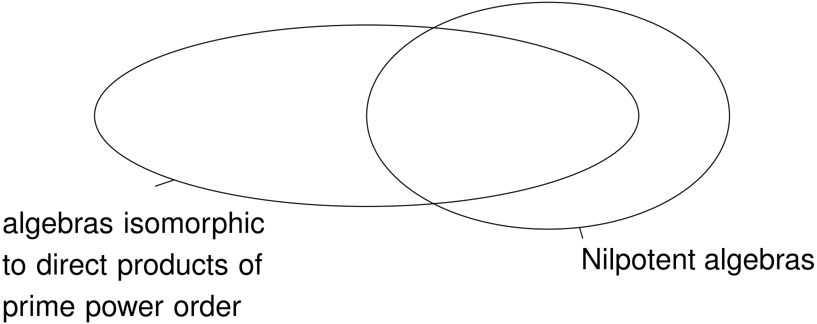
**Algebras in congruence permutable varieties**




$\mathbf{B}_4 = (\mathbb{Z}_4, +, 2x_1x_2, 2x_1x_2x_3, \ldots)$
$\mathbf{N}_6 = (\mathbb{Z}_6, +, (-1)^x)$.

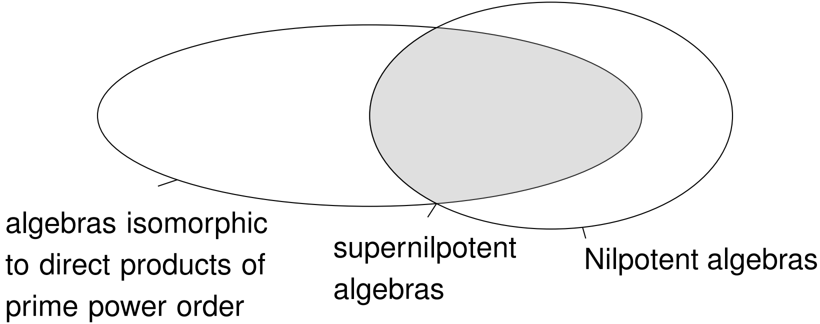
# Nilpotency vs. supernilpotency

**Algebras in cong. mod. varieties with fin. many basic operations**



# Nilpotency vs. supernilpotency

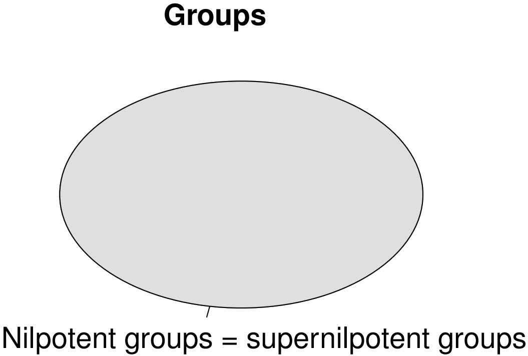
**Algebras in cong. mod. varieties with fin. many basic operations**



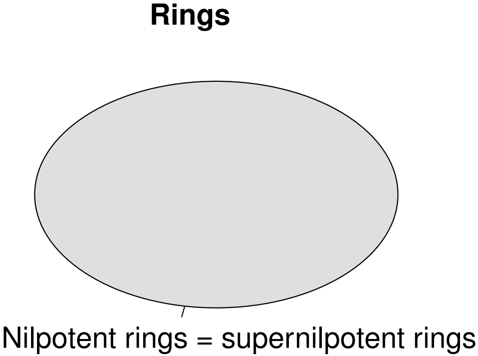

**Theorem [Kearnes 1999], [Berman Blok 1987]**

$\mathbf{A}$ in a cm variety, finitely many basic operations. Then $\mathbf{A}$ is supernilpotent $\Longleftrightarrow$ $\mathbf{A}$ is nilpotent and isomorphic to a product of algebras of prime power order.

# Nilpotency vs. supernilpotency



# Nilpotency vs. supernilpotency



# Coordinatization

We have seen a result on the structure of **nilpotent expansions of** $((\mathbb{Z}_p)^n, +)$.

It would be nice to have a result on **nilpotent algebras of prime power order in congruence modular varieties**.

To this end, we will expand such algebras with a group operation.

# Coordinatization

**Theorem.** Let $\mathbf{A} = (A, (f_i)_{i\in\mathbb{N}})$ be a nilpotent algebra in a congruence modular variety, $|A| = p^n$ with $p$ prime.

Then there exists $+ : A \times A \to A$ and $* : A \times A \to A$ such that

- $(A, +, *)$ is a field and hence $(A, +) \cong (\mathbb{Z}_p^n, +)$.
- $\mathbf{A}' = (A, (f_i)_{i\in\mathbb{N}}, +)$ is nilpotent.

# Structure of nilpotent algebras

**Theorem**

Let $\mathbf{A}$ be a finite nilpotent algebra in a congruence modular variety that is a direct product of algebras of prime power order, with all fundamental operations of arity at most $m$, $|A| > 1$. Let

$$s := \left(m(|A| - 1)\right)^{(\log_2(|A|) - 1)}.$$

Then $\mathbf{A}$ is $s$-supernilpotent and there is a polynomial $p \in \mathbb{R}[x]$ of degree $\leq s$ such that the free spectrum satisfies

$$f_{\mathbf{A}}(n) = \mathrm{Clo}_n(\mathbf{A}) = 2^{p(n)} \text{ for all } n \in \mathbb{N}.$$

# Solving systems of equations

**Theorem**

Let $\mathbf{A}$ be supernilpotent in a cm variety with all basic operations of arity $\leq \mu$. Let $F : A^n \to A^t$ with $F \in \mathrm{Pol}_n(\mathbf{A})^t$ be a polynomial map, and let $z \in A$.

Then

$\forall \boldsymbol{a} \in A^n \ \exists \boldsymbol{y} \in A^n$ such that

$$F(\boldsymbol{y}) = F(\boldsymbol{a}) \text{ and } \#\{j \in \underline{n} : \boldsymbol{y}(j) \neq z\} \leq t|A|^{\log_2(\mu)+\log_2(|A|)+1}.$$

Hence systems of $t$ polynomial equations over supernilpotent algebras can be solved in polynomial time.

For one equation: [Kompatscher, 2018] with a different bound.

Theorem was proved by extending [Károlyi and Szabó, 2015] from nilpotent rings to supernilpotent algebras in cmv.