Lattices allowing only nilpotent commutator operations

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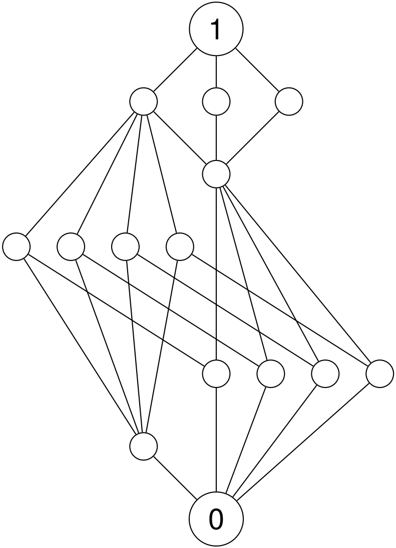
Does $\mathbb{L}$ force nilpotency?

Question

- **Given:** A modular lattice $\mathbb{L}$.
- **Asked:** Is there an algebra $\mathbf{A}$ in a congruence modular variety with $\mathrm{Con}(\mathbf{A}) \cong \mathbb{L}$ such that $\mathbf{A}$ is not nilpotent?



Towards a purely lattice theoretic viewpoint

What a non-nilpotent algebra does to a finite lattice

If there is a non-nilpotent **A** in a cm variety with Con(**A**) $= \mathbb{L}$, then the binary commutator operation of **A**

$$[.,.] : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$$

satisfies

- $\forall x, y \; : \; [x, y] = [y, x] \leq x \wedge y$,
- $\forall x, y, z \; : \; [x, y \vee z] = [x, y] \vee [x, z]$

and there is a nilpotency killer $\rho \in \mathbb{L}$ with

- $\rho > 0$.
- $[1, \rho] = \rho$.

Lattice theoretic question

An obvious dichotomy

Given a lattice **L**,

- there exists a commutative, join distributive, "subintersective" binary operation $[.,.]$ that has a $\rho \in \mathbb{L}$ with $[1,\rho] = \rho > 0$, or
- there is no such operation.

Definition

A finite lattice $\mathbb{L}$ forces nilpotent type if there are no $[.,.]$ and ρ such that

- $[.,.]$ is commutative, join distributive, subintersective (i.e., $[.,.]$ is a commutator multiplication), and
- $[1,\rho] = \rho > 0$.

Lattice theoretic question

Goal

Characterize those finite modular lattices that force nilpotent type.

Very short history

- G. Birkhoff (1948) defined *commutation lattices* $(\mathbb{L}, \vee, \wedge, (xy))$. Proved: if lower central series is finite, then the upper central series has the same length.
- J. Czelakowski (2008) defined *commutator lattices* $(\mathbb{L}, \vee, \wedge, [x, y])$ and investigated the relation of $[x, y]$ with

$$(a : b) = \text{largest } c \text{ with } [c, b] \leq a.$$

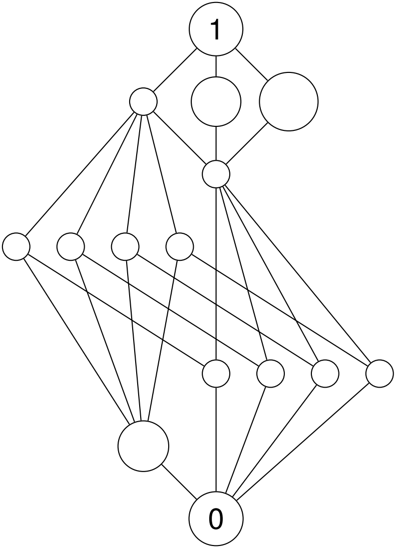
- At AAA92 (2016), we saw a condition (C) such that every finite modular lattice with (C) forces nilpotent type.
- Today, we prove the converse and thereby finish the characterization for finite modular lattices.

Construction of a commutator multiplication

Task

- **Given:** $\mathbb{L}$.
- **Asked:** A multiplication $[.,.]$ and a nilpotency killer ρ.

Finding $[.,.]$ and ρ



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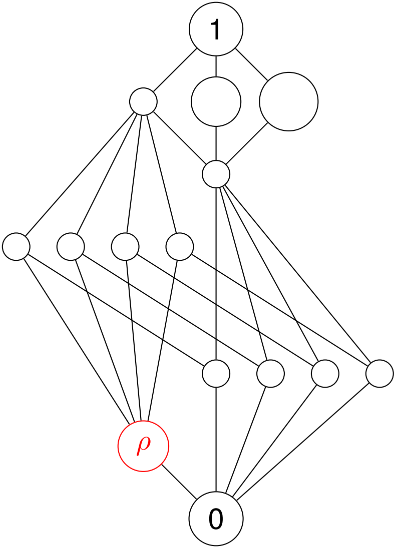

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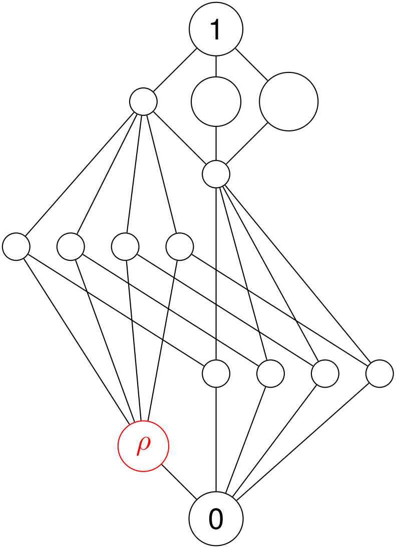

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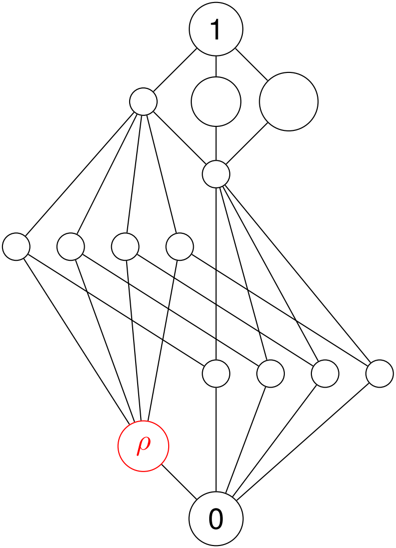

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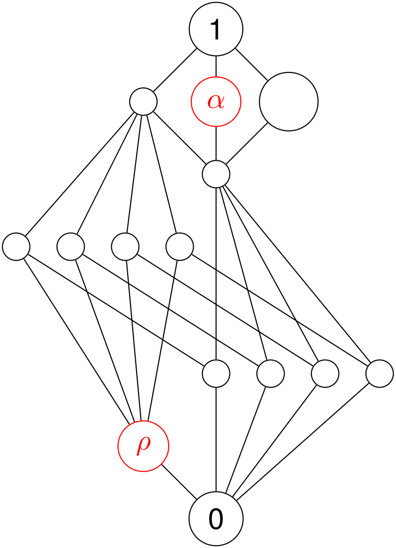

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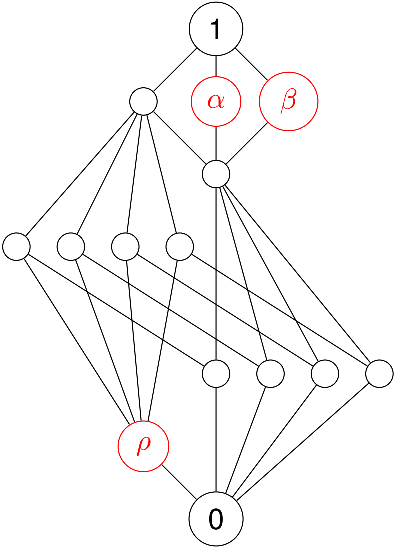
- We try this ρ.
- We want a multiplication $[.,.]$ with $[1,\rho] = \rho$.
- Not possible because: $[\alpha,\rho] \leq \alpha \wedge \rho = 0$ and $[\beta,\rho] \leq \beta \wedge \rho = 0$ and hence $[1,\rho] = [\alpha \vee \beta, \rho] = [\alpha,\rho] \vee [\beta,\rho] = 0 \vee 0 = 0$.



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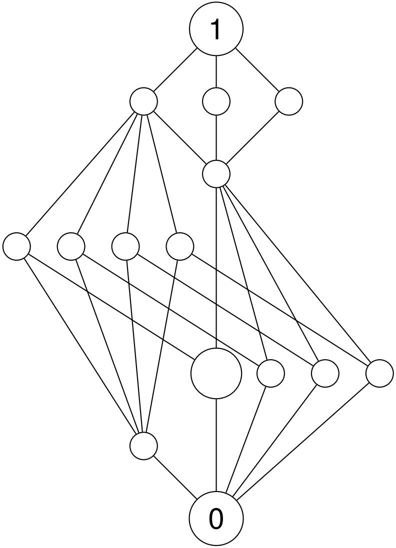

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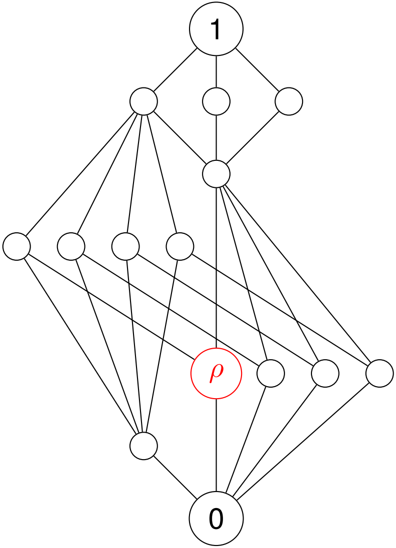

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- We try this ρ.

Construction of a commutator multiplication

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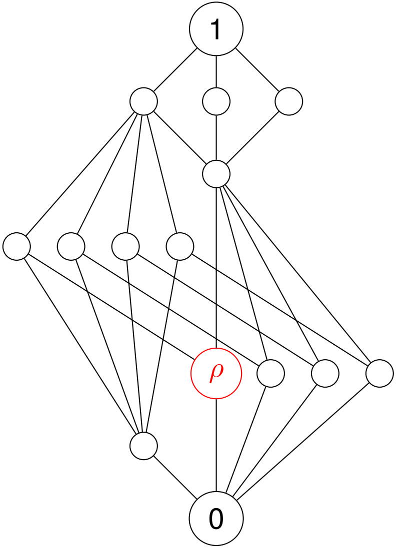
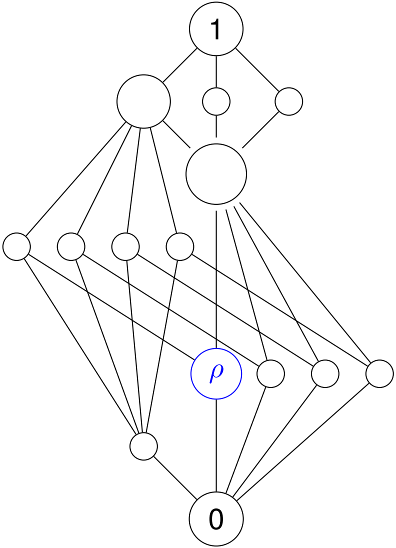
Finding $[.,.]$ and ρ

- We try this ρ.
- Now we will succeed!



Construction of the multiplication

- Find all intervals projective to $I[0, \rho]$.



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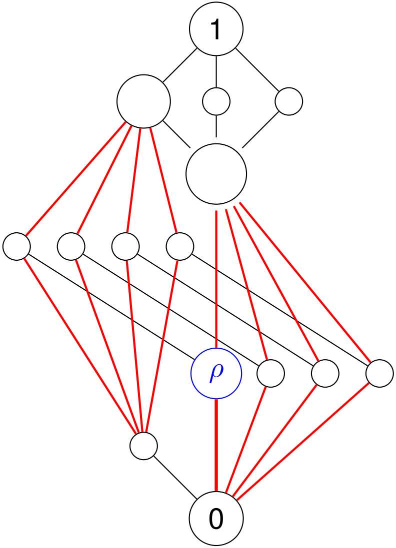

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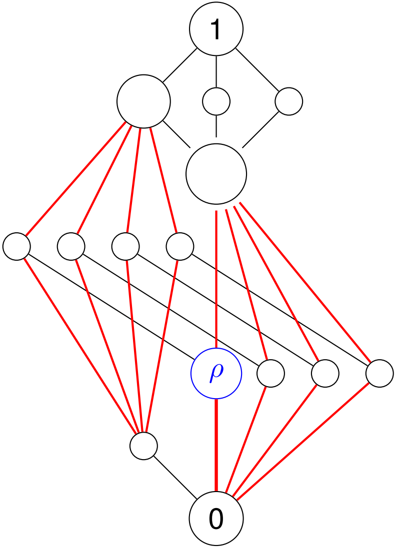

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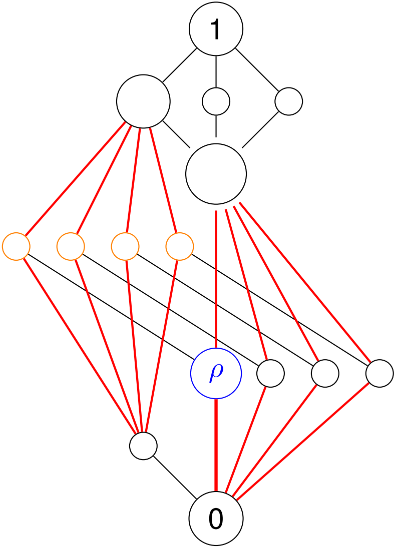

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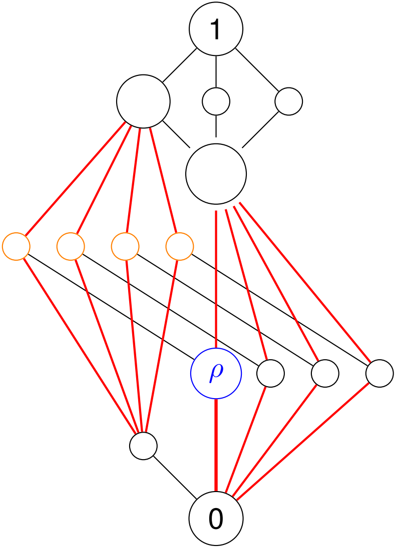

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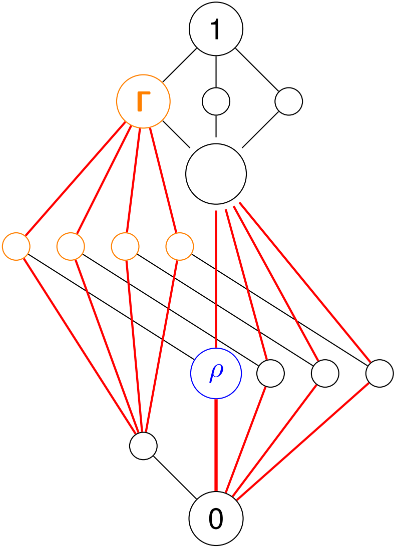

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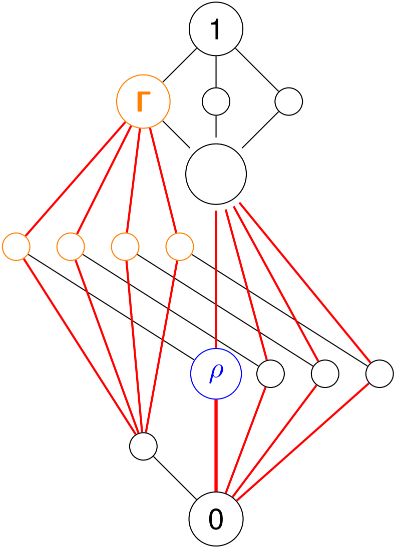

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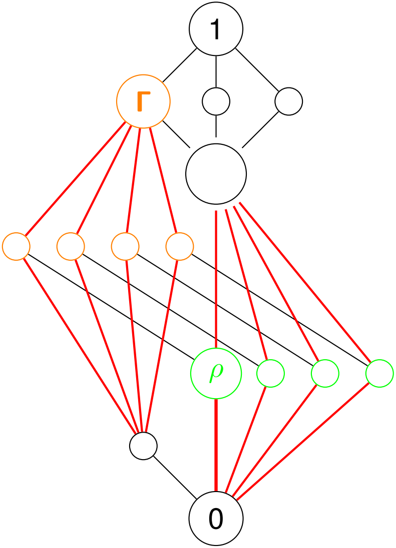

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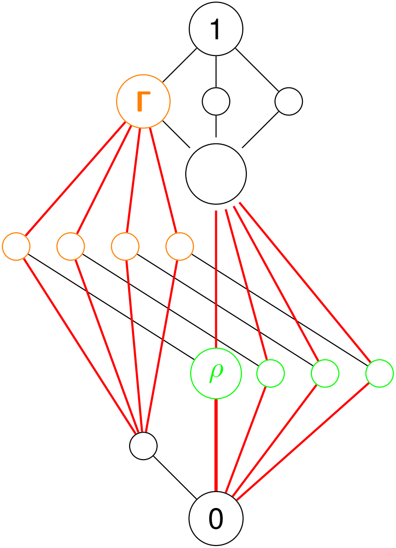

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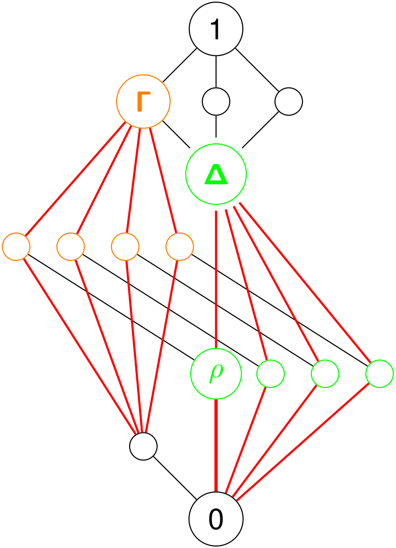

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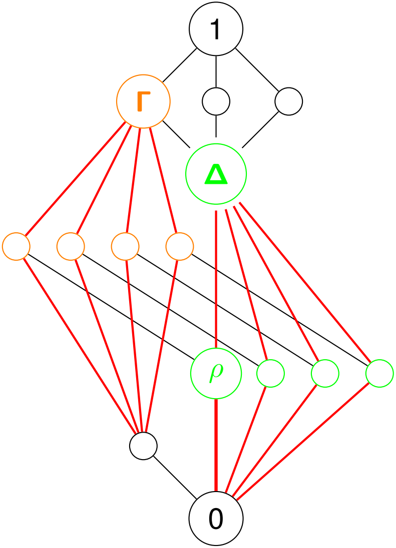

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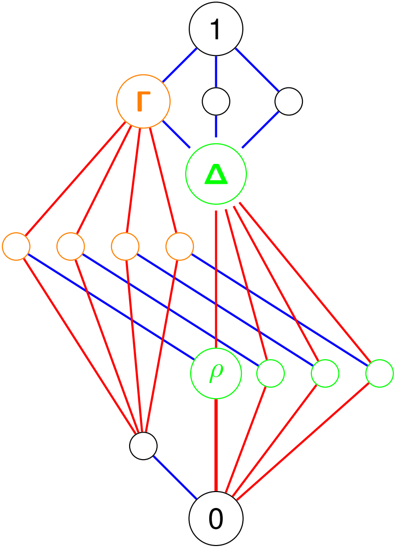

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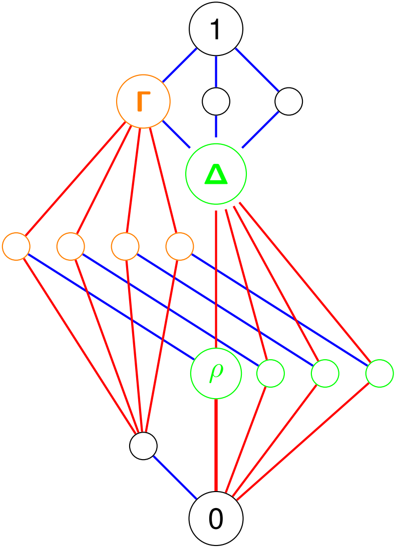

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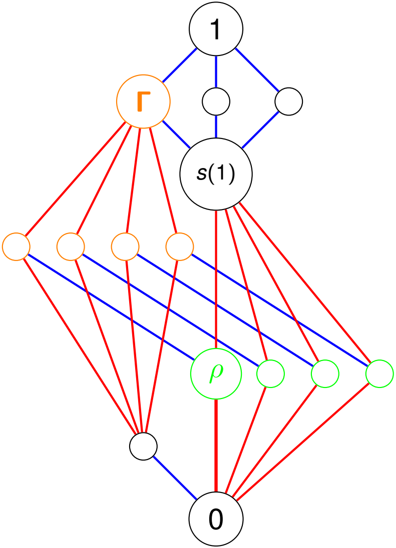

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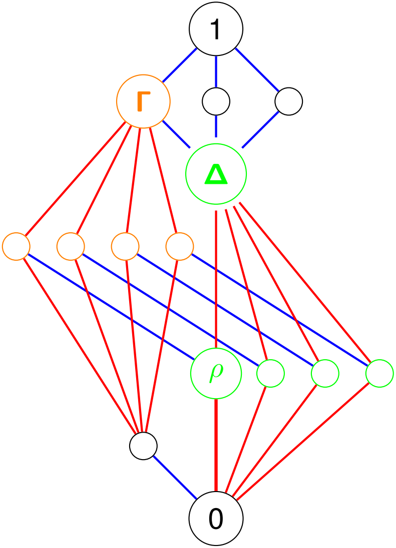
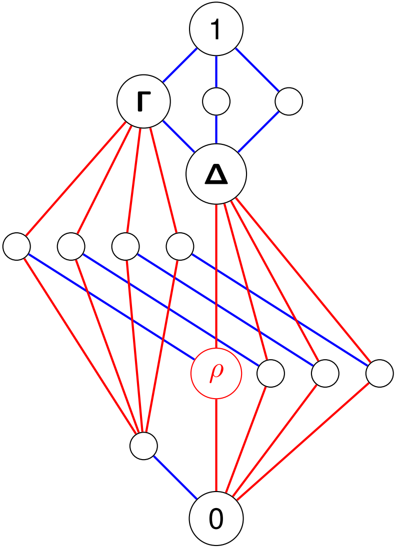


Construction of the multiplication

The multiplication



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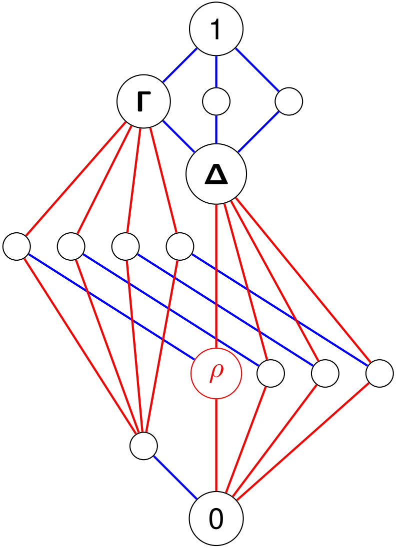
- Define $[x, y] := 0$ if $x \leq \mathbf{\Gamma}$ and $y \leq \mathbf{\Gamma}$.



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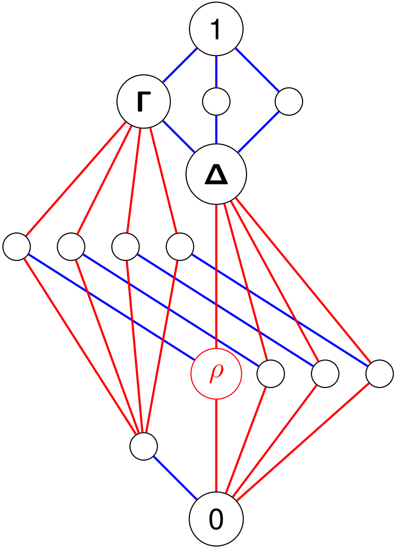

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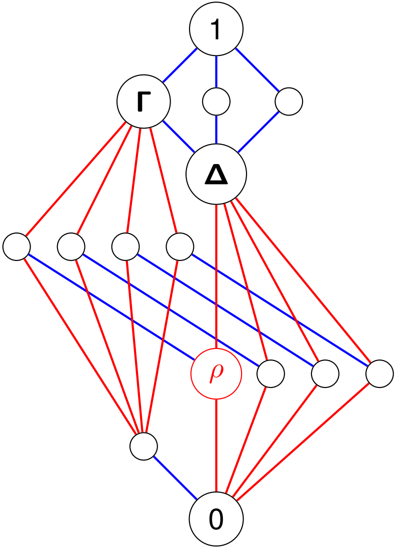
Properties of the multiplication

- $[.,.]$ is commutative, below meet.



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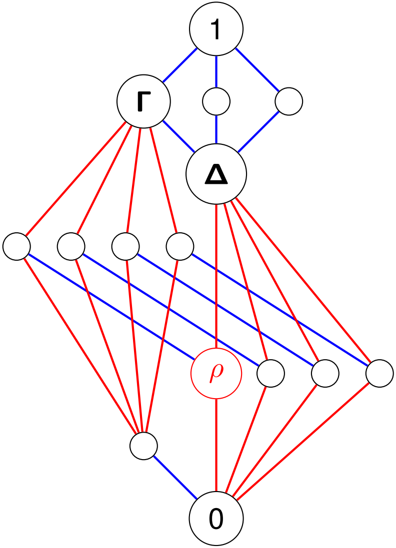

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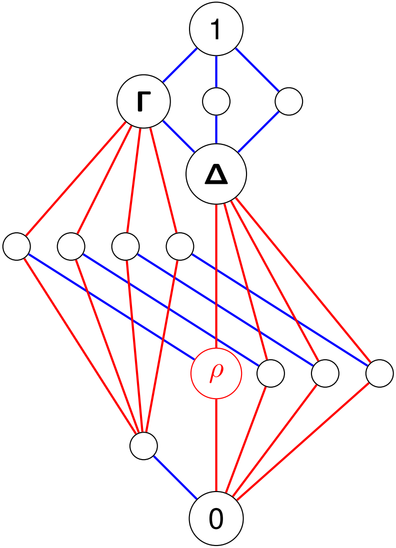
- $[., .]$ is join distributive.
- We have $[1, \rho] = \rho$.



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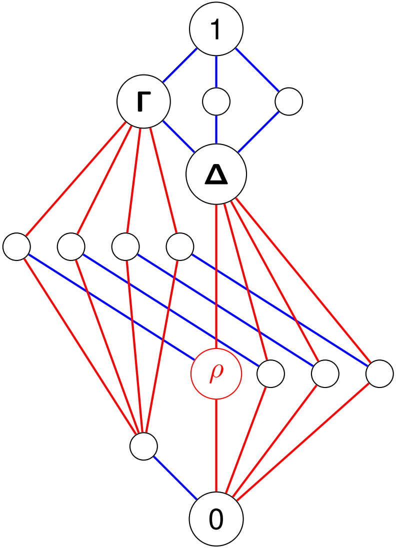
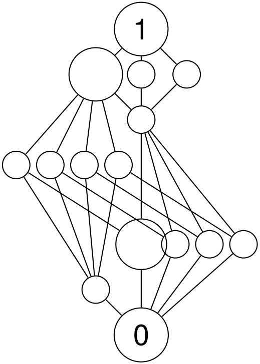
Properties of the multiplication

- $[.,.]$ is commutative, below meet.
- $[.,.]$ is join distributive.
- We have $[1, \rho] = \rho$.
- **Conclusion:** $\mathbb{L}$ does not force nilpotent type.



The general content

This construction was possible because there were:



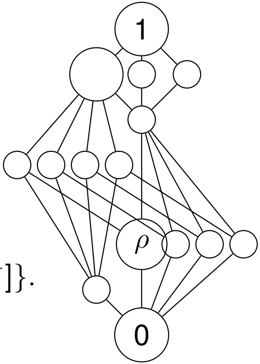
Lattices with only nilpotent commutator multiplications

The general content

This construction was possible because there were:

- a join irreducible $\rho \in \mathbb{L}$ with $\mathbf{\Gamma} = \mathbf{\Gamma}(\rho^-, \rho) < 1$, where

$$\mathbf{\Gamma} = \bigvee\{\eta \mid \eta \text{ m.i. and } I[\rho^-, \rho] \leftrightsquigarrow I[\eta, \eta^+]\}.$$



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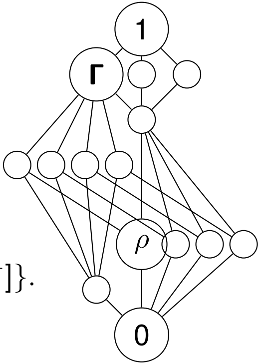

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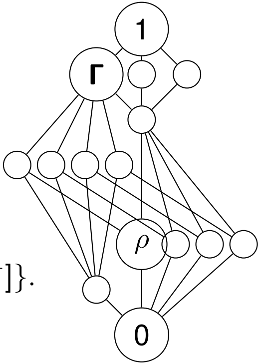


Theorem

Let $\mathbb{L}$ be a modular lattice of finite height. TFAE:

- $\mathbb{L}$ allows a non-nilpotent commutator multiplication.
- $\exists \alpha \prec \beta \in \mathbb{L}$ such that $\mathbf{\Gamma}(\alpha, \beta) < 1$.

The largest commutator multiplication

Lemma (Czelakowski)

The join of commutator multiplications is again a commutator multiplication. Hence on a given lattice, there is one largest commutator multiplication.

Czelakowski: "*The characterization of the operation $\bullet_\Omega$ in modular algebraic lattices is an open and challenging problem.*"

Description of the largest commutator multiplication

Let $\lceil .,. \rceil$ denote the largest commutator multiplication on $\mathbb{L}$.

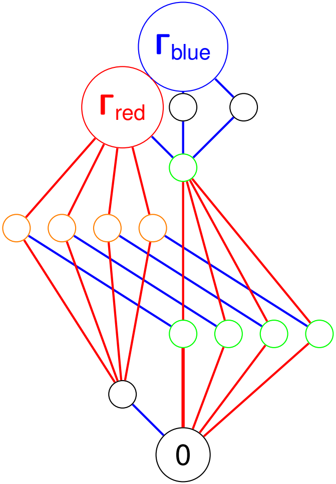
- We have no description of $\lceil x, y \rceil$ yet.
- We have no description of the associated residuation $(x : y) = \bigvee\{z \mid \lceil z, y \rceil \leq x\}$ either.
- We can describe $(x : y)$ if $x \prec y$!

The largest commutator operation

Theorem

Let $\mathbb{L}$ be a bialgebraic modular lattice, and let $(x : y)$ be the residuation operation associated with the largest commutator multiplication. Let $\alpha, \beta \in \mathbb{L}$ be such that $\alpha \prec \beta$. Then

$$(\alpha : \beta) = \Gamma(\alpha, \beta).$$



Open problem

Definition

$\mathbb{L}$ *forces abelian type* if $[x, y] = 0$ is the only commutator multiplication on $\mathbb{L}$.

Problem

Characterize those modular lattices of finite height that force abelian type.

Theorem

Let $\mathbb{L}$ be a complete lattice. If $\mathbb{L}$ has a complete $(0, 1)$-sublattice $\mathbb{K}$ that is algebraic, modular, simple, complemented, and has at least 3 elements, then $\mathbb{L}$ forces abelian type.

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- G. Birkhoff, *Lattice Theory*, AMS, editions 1948 and 1967.
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- E. Aichinger, *Congruence lattices forcing nilpotency*, arXiv, to appear in Journal of Algebra and its Applications.