Independence of algebras

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Outline

We will study:

- relation between $\mathrm{Clo}_k(\mathbf{A}), \mathrm{Clo}_k(\mathbf{B})$ and $\mathrm{Clo}_k(\mathbf{A} \times \mathbf{B})$.
- relation between $\mathbf{F}_{V(\mathbf{A})}(k) \times \mathbf{F}_{V(\mathbf{B})}(k)$ and $\mathbf{F}_{V(\mathbf{A}\times\mathbf{B})}(k)$.
- relation between $V(\mathbf{A}), V(\mathbf{B})$ and $V(\mathbf{A}) \vee V(\mathbf{B})$.

Term functions on direct products

Question

How do the term functions of $\mathbf{A} \times \mathbf{B}$ depend on the term functions of $\mathbf{A}$ and $\mathbf{B}$?

Proposition

Let $\mathbf{A}$, $\mathbf{B}$ be similar algebras, $k \in \mathbb{N}$, and define

$$\begin{array}{rclcl} \phi & : & \mathrm{Clo}_k(\mathbf{A} \times \mathbf{B}) & \longrightarrow & \mathrm{Clo}_k(\mathbf{A}) \times \mathrm{Clo}_k(\mathbf{B}) \\ & & t^{\mathbf{A}\times\mathbf{B}} & \longmapsto & (t^{\mathbf{A}},\ t^{\mathbf{B}}). \end{array}$$

Then ϕ is a subdirect embedding.

Proposition

$\mathbf{A}$, $\mathbf{B}$ from a cp variety, $k \in \mathbb{N}$. Then for all k-ary terms s, t:

$$(s^{\mathbf{A}}, t^{\mathbf{B}}) \in \mathrm{Im}(\phi) \iff V(\mathbf{A}) \cap V(\mathbf{B}) \models s \approx t.$$

Disjoint varieties

$$\begin{array}{rcccl} \phi & : & \mathrm{Clo}_k(\mathbf{A}\times\mathbf{B}) & \longrightarrow & \mathrm{Clo}_k(\mathbf{A})\times\mathrm{Clo}_k(\mathbf{B}) \\ & & t^{\mathbf{A}\times\mathbf{B}} & \longmapsto & (t^{\mathbf{A}},\, t^{\mathbf{B}}). \end{array}$$

If **A**, **B** are from a cp variety, then

$$\begin{array}{rcl} (s^{\mathbf{A}}, t^{\mathbf{B}}) \in \mathrm{Im}(\phi) & \Leftrightarrow & \exists u : u^{\mathbf{A}} = s^{\mathbf{A}} \text{ and } u^{\mathbf{B}} = t^{\mathbf{B}} \\ & \Leftrightarrow & V(\mathbf{A}) \cap V(\mathbf{B}) \models s \approx t. \end{array}$$

Definition

V_1 and V_2 are *disjoint* if $V_1 \cap V_2 \models x \approx y$.

Corollary

A, **B** from a cp variety, $k \geq 2$. Then ϕ is an isomorphism from $\mathrm{Clo}_k(\mathbf{A}\times\mathbf{B})$ to $\mathrm{Clo}_k(\mathbf{A})\times\mathrm{Clo}_k(\mathbf{B}) \Longleftrightarrow V(\mathbf{A})$ and $V(\mathbf{B})$ are disjoint.

History (1955 – 1969)

Definition [Foster, 1955]

A sequence $(V_1, \ldots, V_n)$ of subvarieties of W is *independent* if there is a term $t(x_1, \ldots, x_n)$ such that $\forall i \in [n] : V_i \models t(x_1, \ldots, x_n) \approx x_i$.

Example [Grätzer et al., 1969]

$$
\begin{aligned}
V_0 &:= \{ (G,\ f_0(x,y) = x \cdot y,\ f_1(x,y) = x \cdot y^{-1}) \mid \\
&\qquad (G, \cdot, {}^{-1}, 1) \text{ is a group}\} \\
V_1 &:= \{ (L,\ f_0(x,y) = x \vee y,\ f_1(x,y) = x \wedge y) \mid \\
&\qquad (L, \vee, \wedge) \text{ is a lattice}\}, \\
t(x,y) &:= f_1(f_0(x,y), y).
\end{aligned}
$$

Then

- $V_0 \models f_1(f_0(x,y),y) = (x \cdot y) \cdot y^{-1} \approx x$ and
- $V_1 \models f_1(f_0(x,y),y) = (x \vee y) \wedge y \approx y$.

History (1969)

Theorem [Grätzer et al., 1969]

Let V_0 and V_1 be independent subvarieties of W. Then every $\mathbf{A} \in V_0 \vee V_1$ is isomorphic to a direct product $\mathbf{A}_0 \times \mathbf{A}_1$ with $\mathbf{A}_0 \in V_0$ and $\mathbf{A}_1 \in V_1$.

Consequence

Let V_0 and V_1 be independent. Then $(V_0 \vee V_1)_{SI} = (V_0)_{SI} \cup (V_1)_{SI}$.

History (1971)

Theorem [Hu and Kelenson, 1971]

Let $(V_1, \dots, V_n)$ be a sequence of subvarieties of a cp variety W. If for all $i \neq j$, $V_i \cap V_j \models x \approx y$ (V_i and V_j are disjoint), then $(V_1, \dots, V_n)$ is independent.

Proof for $n = 2$:

- Goal: construct $t(x_1, x_2)$ with $V_1 \models t(x_1, x_2) \approx x_1$ and $V_2 \models t(x_1, x_2) \approx x_2$.
- $\phi : \mathbf{F}_{V_1 \vee V_2}(x, y) \to \mathbf{F}_{V_1}(x, y) \times \mathbf{F}_{V_2}(x, y)$, $t/{\sim_{V_1 \vee V_2}} \mapsto (t/{\sim_{V_1}}, t/{\sim_{V_2}})$.
- $\mathrm{Im}(\phi) \leq_{sd} \mathbf{F}_{V_1}(x, y) \times \mathbf{F}_{V_2}(x, y)$.
- Fleischer's Lemma yields $\mathbf{D}$, $\alpha_1 : \mathbf{F}_{V_1}(x, y) \twoheadrightarrow \mathbf{D}$, $\alpha_2 : \mathbf{F}_{V_2}(x, y) \twoheadrightarrow \mathbf{D}$ with

$$\mathrm{Im}(\phi) = \{(f, g) \,|\, \alpha_1(f) = \alpha_2(g)\}.$$

- $|\mathbf{D}| = 1$, hence ϕ is surjective.
- Thus $(x/{\sim_{V_1}}, y/{\sim_{V_2}}) \in \mathrm{Im}(\phi)$, which yields t.

History (2004 – 2013)

Theorem [Jónsson and Tsinakis, 2004]

The join of two independent finitely based varieties is finitely based.

Theorem [Kowalski et al., 2013]

Let V_1, V_2 be disjoint subvarieties of W. Then V_1 and V_2 are independent iff $\exists q(x, y, z) : V_1 \models q(x, x, y) \approx y$ and $V_2 \models q(x, y, y) \approx x$.

Product subalgebras

Definition

$\mathbf{C} \leq \mathbf{E} \times \mathbf{F}$ is a *product subalgebra* if $\mathbf{C} = \pi_{\mathbf{E}}(\mathbf{C}) \times \pi_{\mathbf{F}}(\mathbf{C})$.

Proposition

$\mathbf{C} \leq \mathbf{E} \times \mathbf{F}$ is a product subalgebra iff for all a, b, c, d:
$(a, b) \in C$ and $(c, d) \in C \Longrightarrow (a, d) \in C$.

Definition

$\alpha \in \mathrm{Con}(\mathbf{E} \times \mathbf{F})$ is a *product congruence* if $\alpha = \beta \times \gamma$ for some $\beta \in \mathrm{Con}(\mathbf{E})$ and $\gamma \in \mathrm{Con}(\mathbf{F})$.

Product subalgebras of powers

Theorem [Aichinger and Mayr, 2015]

Let $\mathbf{A}, \mathbf{B}$ be algebras in a cp variety. We assume that

1. all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras, and
2. for all $\mathbf{E} \leq \mathbf{A}$ and $\mathbf{F} \leq \mathbf{B}$, all congruences of $\mathbf{E} \times \mathbf{F}$ are product congruences.

Then for all $m, n \in \mathbb{N}_0$, all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras.

Product subalgebras of powers

Theorem [Aichinger and Mayr, 2015]

Let $k \geq 2$, let $\mathbf{A}, \mathbf{B}$ be algebras in a variety with k-edge term. We assume that

1. for all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and
2. for all $\mathbf{E} \leq \mathbf{A}$ and $\mathbf{F} \leq \mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.

Then for all $m, n \in \mathbb{N}_0$, every subalgebra of $\mathbf{A}^m \times \mathbf{B}^n$ is a product subalgebra.

Direct products and independence

Definition

$\mathbf{A}, \mathbf{B} \in W$ are *independent* $:\Longleftrightarrow$ $V(\mathbf{A})$ and $V(\mathbf{B})$ are independent.

Independence in cp varieties

Proposition

Let **A** and **B** be similar algebras. TFAE:

1. **A** and **B** are independent.
2. For all sets I, J with $|I| \leq |A|^2$ and $|J| \leq |B|^2$, all subalgebras of $\mathbf{A}^I \times \mathbf{B}^J$ are product subalgebras.

If **A** and **B** lie in a cp variety, then these two items are furthermore equivalent to

3. $V(\mathbf{A})$ and $V(\mathbf{B})$ are disjoint.

Theorem (EA, Mayr, 2015)

Let **A**, **B** be finite algebras in a cp variety. TFAE:

1. **A** and **B** are independent.
2. All subalgebras of $\mathbf{A} \times \mathbf{B}$ are product subalgebras, and all congruences of all subalgebras of $\mathbf{A} \times \mathbf{B}$ are product congruences.
3. All subalgebras of $\mathbf{A}^2 \times \mathbf{B}^2$ are product subalgebras.
4. $HS(\mathbf{A}^2) \cap HS(\mathbf{B}^2)$ contains only one element algebras.

Independence for algebras with edge term

Theorem [Aichinger and Mayr, 2015]

Let $k \geq 2$, and let **A**, **B** be finite algebras in a variety with k-edge term. Then the following are equivalent:

1. **A** and **B** are independent.
2. For all $r, s \in \mathbb{N}$ with $r + s \leq \max(2, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra, and for all $E \leq \mathbf{A}, F \leq \mathbf{B}$, every tolerance of $\mathbf{E} \times \mathbf{F}$ is a product tolerance.
3. For all $r, s \in \mathbb{N}$ with $r + s \leq \max(4, k-1)$, every subalgebra of $\mathbf{A}^r \times \mathbf{B}^s$ is a product subalgebra.

Example - infinite groups

Let p, q be primes, $p \neq q$,
$\mathbf{A} := C_{p^\infty} = \{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : z^{p^n} = 1\}$, $\mathbf{B} := C_{q^\infty}$. Then all subalgebras of $\mathbf{A}^m \times \mathbf{B}^n$ are product subalgebras, but **A** and **B** are not independent.

Application to polynomial functions

Theorem

Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras in a variety with a 3-edge term, and let $k \in \mathbb{N}$. We assume that every tolerance of $\mathbf{A} \times \mathbf{B}$ is a product tolerance. Let $\psi : \mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B}) \to (A \times B)^{(A \times B)^k}$ be the mapping defined by

$$\psi(f, g)\,((a_1, b_1), \ldots, (a_k, b_k)) := (f(\mathbf{a}), g(\mathbf{b}))$$

for $f \in \mathrm{Pol}_k(\mathbf{A}), g \in \mathrm{Pol}_k(\mathbf{B})$, $\mathbf{a} \in A^k$, and $\mathbf{b} \in B^k$. Then ψ is a bijection from $\mathrm{Pol}_k(\mathbf{A}) \times \mathrm{Pol}_k(\mathbf{B})$ to $\mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$.

Application to polynomial functions

Corollary

Let **A** and **B** be algebras in the variety V, and let $k \in \mathbb{N}$. If either

1. V has a majority term, or
2. V is cp, and every congruence of $\mathbf{A} \times \mathbf{B}$ is a product congruence,

then for all polynomial functions $f \in \mathrm{Pol}_k(\mathbf{A})$ and $g \in \mathrm{Pol}_k(\mathbf{B})$, there is a polynomial function $h \in \mathrm{Pol}_k(\mathbf{A} \times \mathbf{B})$ with $h((a_1, b_1), \ldots, (a_k, b_k)) = (f(\mathbf{a}), g(\mathbf{b}))$ for all $\mathbf{a} \in A^k$ and $\mathbf{b} \in B^k$.

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